ADVANCED MATHEMATICAL METHODS FOR SCIENTISTS AND ENGINEERS

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CHAPTER

NINE

BOUNDARY-LAYER THEORY

His career has been an extraordinary one. He is a man of good birth and excellent education, endowed by nature with a phenomenal mathematical faculty. At the age of twenty-one he wrote a treatise upon the binomial theorem, which has had a European vogue. On the strength of it he won the mathematical chair at one of our smaller universities, and had, to all appearances, a most brilliant career before him. But the man had hereditary tendencies of the most diabolical kind. A criminal strain ran in his blood, which, instead of being modified, was increased and rendered infinitely more dangerous by his extraordinary mental powers.

—Sherlock Holmes, *The Final Problem*
Sir Arthur Conan Doyle

(E) 9.1 INTRODUCTION TO BOUNDARY-LAYER THEORY

In this and the next chapter we discuss perturbative methods for solving a differential equation whose highest derivative is multiplied by the perturbing parameter ε. The most elementary of these methods is called boundary-layer theory.

A *boundary layer* is a narrow region where the solution of a differential equation changes rapidly. By definition, the thickness of a boundary layer must approach 0 as $\varepsilon \to 0$. In this chapter we will be concerned with differential equations whose solutions exhibit only *isolated* (well-separated) narrow regions of rapid variation. It is possible for a solution to a perturbation problem to undergo rapid variation over a thick region (one whose thickness does *not* vanish with ε). However, such a region is not a boundary layer. We will consider such problems in Chap. 10.

Here is a simple boundary-value problem whose solution exhibits boundary-layer structure.

Example 1 *Exactly soluble boundary-layer problem.* Consider the differential equation

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0, \qquad y(0) = 0,\ y(1) = 1. \tag{9.1.1}$$

The exact solution of this equation is

$$y(x) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}. \tag{9.1.2}$$

In the limit $\varepsilon \to 0+$, this solution becomes discontinuous at $x = 0$, as is shown in Fig. 9.1. For very small ε the solution $y(x)$ is slowly varying for $\varepsilon \ll x \le 1$. However, on the small interval $0 \le x \le O(\varepsilon)$ $(\varepsilon \to 0+)$ it undergoes an abrupt and rapid change. This small interval of rapid change is called a *boundary layer*. [The notation $0 \le x \le O(\varepsilon)$ means that the thickness of the boundary layer is proportional to ε as $\varepsilon \to 0+$.] The region of slow variation of $y(x)$ is called the *outer* region and the boundary-layer region is called the *inner* region.

Boundary-layer theory is a collection of perturbation methods for solving differential equations whose solutions exhibit boundary-layer structure. When the solution to a differential equation is slowly varying except in isolated boundary layers, then it may be relatively easy to obtain a leading-order approximation to that solution for small ε without directly solving the differential equation.

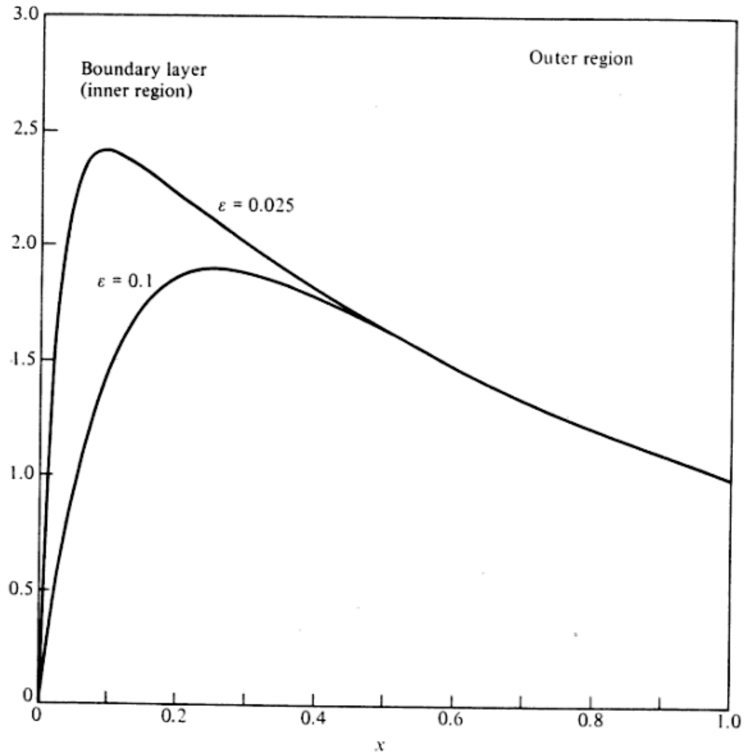

Figure 9.1 A plot of $y(x) = (e^{-x} - e^{-x/\varepsilon})/(e^{-1} - e^{-1/\varepsilon})$ $(0 \le x \le 1)$ for $\varepsilon = 0.1$ and 0.025. Note that $y(x)$ is slowly varying for $\varepsilon \ll x \le 1$ $(\varepsilon \to 0+)$. However, on the interval $0 \le x \le O(\varepsilon)$, $y(x)$ rises abruptly from 0 and becomes discontinuous in the limit $\varepsilon \to 0+$. This narrow and isolated region of rapid change is called a boundary layer.

There are two standard approximations that one makes in boundary-layer theory. In the outer region (away from a boundary layer) $y(x)$ is slowly varying, so it is valid to neglect any derivatives of $y(x)$ which are multiplied by ε. Inside a boundary layer the derivatives of $y(x)$ are large, but the boundary layer is so narrow that we may approximate the coefficient functions of the differential equation by constants. Thus, we can replace a single differential equation by a sequence of much simpler approximate equations in each of several inner and outer regions. In every region the solution of the approximate equation will contain one or more unknown constants of integration. These constants are then determined from the boundary or initial conditions using the technique of asymptotic matching which was introduced in Sec. 7.4.

The following initial-value problem illustrates these ideas.

Example 2 *First-order nonlinear boundary-layer problem.* From the initial-value problem

$$(x - \varepsilon y)y' + xy = e^{-x}, \qquad y(1) = 1/e, \tag{9.1.3}$$

we wish to determine a leading-order perturbative approximation to $y(0)$ as $\varepsilon \to 0+$.

Although this is only a first-order differential equation, it is nonlinear and is much too difficult to solve in closed form. However, in regions where y and y' are not large (such regions are called outer regions), it is valid to neglect $\varepsilon yy'$ compared with e^{-x}. Thus, in outer regions we approximate the solution to (9.1.3) by the solution to the outer equation

$$xy'_{\text{out}} + xy_{\text{out}} = e^{-x}.$$

This equation is easy to solve because it is linear. The solution which satisfies $y_{\text{out}}(1) = 1/e$ is

$$y_{\text{out}} = (1 + \ln x)e^{-x}. \tag{9.1.4}$$

Note that it is valid to impose the initial condition $y(1) = 1/e$ on $y_{\text{out}}(x)$ because $x = 1$ lies in an outer region; $x = 1$ is in an outer region because (9.1.3) implies that $y'(1) = 0$, so $y(1)$ and $y'(1)$ are of order 1 as $\varepsilon \to 0+$.

As $x \to 0+$, both $y_{\text{out}}(x)$ and $y'_{\text{out}}(x)$ become larger. Thus, near $x = 0$ the term $\varepsilon yy'$ is no longer negligible compared with e^{-x}. From the outer solution we can estimate that the thickness δ of the region in which $\varepsilon yy'$ is not small is given by

$$\delta/\ln \delta = O(\varepsilon), \qquad \varepsilon \to 0+.$$

Thus, $\delta \to 0+$ as $\varepsilon \to 0+$ [in fact, $\delta = O(\varepsilon \ln \varepsilon)$ as $\varepsilon \to 0+$] (see Prob. 9.1), and there is a boundary layer of thickness δ at $x = 0$.

In the boundary layer (the inner region), x is small so it is valid to approximate e^{-x} by 1. Furthermore, since y varies rapidly in the narrow boundary layer, we may neglect xy compared with xy'. Hence, in the inner region we approximate the solution to (9.1.3) by the solution to the inner equation

$$(x - \varepsilon y_{\text{in}})y'_{\text{in}} = 1.$$

This is a linear equation if we regard x as the dependent variable. Its solution is

$$x = \varepsilon(y_{\text{in}} + 1) + Ce^{y_{\text{in}}}, \tag{9.1.5}$$

where C is an unknown constant of integration. Since $x = 0$ is in the inner region, we may use (9.1.5) to find an approximation to $y(0)$.

C is determined by asymptotically matching the outer and inner solutions (9.1.4) and (9.1.5). Take x small but not as small as δ, say $x = O(\varepsilon^{1/2})$. Then (9.1.4) implies that $y_{out} \sim 1 + \ln x$ as $\varepsilon \to 0+$ and (9.1.5) implies that $x \sim Ce^{y_{in}}$ as $\varepsilon \to 0+$. Thus, $C = 1/e$ and a leading-order implicit equation for $y_{in}(0)$ is

$$0 = \varepsilon[y_{in}(0) + 1] + e^{y_{in}(0) - 1}. \qquad (9.1.6)$$

When $\varepsilon = 0.1$ and 0.01, the numerical solutions of (9.1.6) are $y_{in}(0) \doteq -1.683$ and $y_{in}(0) \doteq -2.942$, respectively. These results compare favorably with the numerical solution to (9.1.3) which gives $y(0) \doteq -1.508$ when $\varepsilon = 0.1$ and $y(0) \doteq -2.875$ when $\varepsilon = 0.01$. For both values of ε the relative error between the perturbative and the numerical solutions for $y(0)$ is about $\frac{1}{2}\varepsilon \ln \varepsilon$. Figures 9.2 and 9.3 compare the inner and outer perturbative approximations to $y(x)$ with the numerical solution.

Boundary-layer theory can also be a very powerful tool for determining the behavior of solutions to higher-order equations.

Example 3 *Second-order linear boundary-value problem.* Let us find an approximate solution to the boundary-value problem

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = 0, \qquad 0 \le x \le 1,\ y(0) = A,\ y(1) = B, \qquad (9.1.7)$$

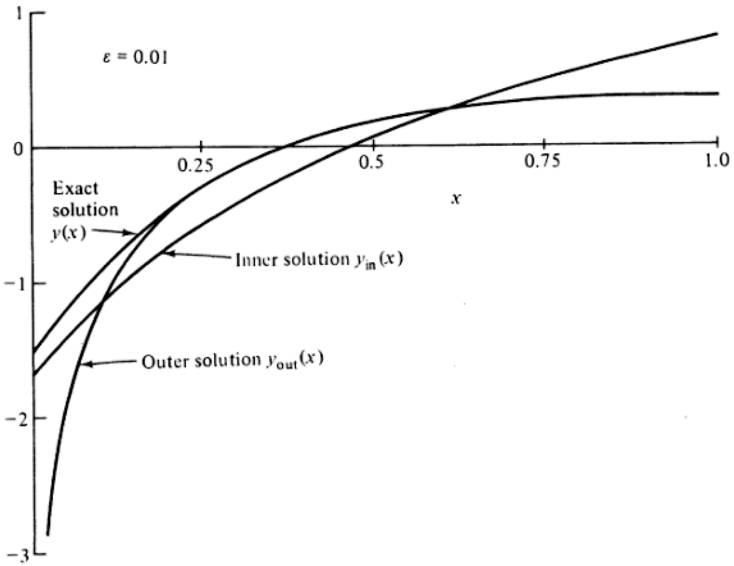

Figure 9.2 A comparison for $\varepsilon = 0.1$ of the exact solution $y(x)$ to the nonlinear differential equation (9.1.3) and the inner and outer approximations to $y(x)$ using boundary-layer theory. The integration constant in y_{out} is determined from the initial condition $y(1) = 1/e$. The integration constant in y_{in} is determined from asymptotic matching. A measure of the accuracy of the boundary-layer approximation is the magnitude of the error in the predicted value of $y(0)$. When $\varepsilon = 0.1$, $y(0) \doteq -1.508$ and $y_{in}(0) \doteq -1.683$, an error of about 10 percent.

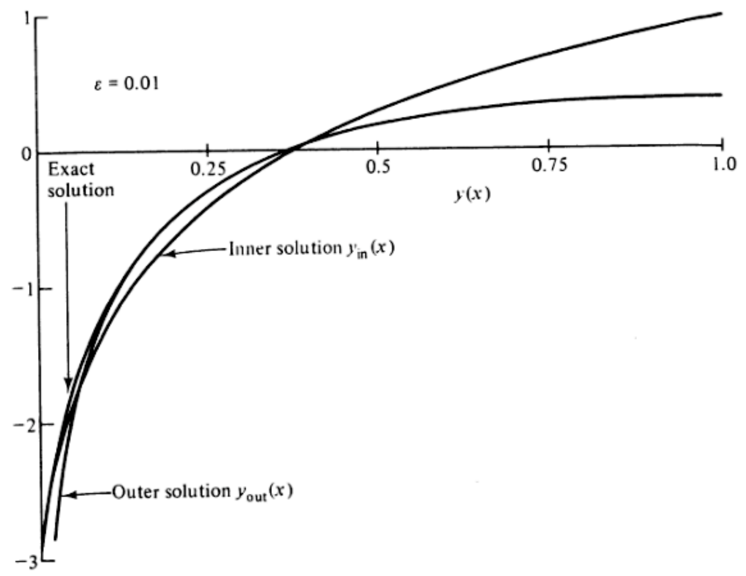

Figure 9.3 Same as Fig. 9.2 with $\varepsilon = 0.1$ replaced by $\varepsilon = 0.01$. Here, $y(0) \doteq -2.875$ and $y_{in}(0) \doteq -2.942$, an error of about 2 percent. Observe that as $\varepsilon \to 0+$, the inner and outer approximations $y_{in}(x)$ and $y_{out}(x)$ hug the exact solution $y(x)$ more closely. The error appears to be of order $\frac{1}{2}\varepsilon \ln \varepsilon$ (see Prob. 9.5).

as $\varepsilon \to 0+$. We assume for reasons to be made clear later that $a(x) \ne 0$ for $0 \le x \le 1$, and for definiteness we choose $a(x) > 0$; otherwise $a(x)$ and $b(x)$ are arbitrary continuous functions.

We shall analyze the behavior of $y(x)$ as $\varepsilon \to 0+$ by assuming that in this limit the solution $y(x)$ develops an isolated boundary layer in the neighborhood of $x = 0$ and that there are no other regions of rapid change of $y(x)$ ($\varepsilon \to 0+$). We will then justify these assumptions by showing that no other possibility is mathematically consistent.

The outer region is characterized by the absence of rapid variation of $y(x)$: $y(x)$, $y'(x)$, and $y''(x)$ are all of order 1 (assuming that A and B are finite) as $\varepsilon \to 0+$. Thus, in the outer region a good approximation to (9.1.7) is the first-order linear equation

$$a(x)y_{out}'(x) + b(x)y_{out}(x) = 0. \qquad (9.1.8)$$

Observe that the outer approximation has reduced the order of the differential equation, thereby making it soluble. The solution to (9.1.8) is $y_{out}(x) = K \exp\left[\int_x^1 b(t)/a(t)\,dt\right]$, where K is an integration constant. In general, it is not possible for $y_{out}(x)$ to satisfy both boundary conditions $y(0) = A$ and $y(1) = B$. However, we have assumed that $x = 1$ lies within the outer region and that $x = 0$ does not. Thus, we should require that $y_{out}(1) = B$, but not $y_{out}(0) = A$. It follows that $K = B$:

$$y_{out}(x) = B \exp\left[\int_x^1 b(t)/a(t)\,dt\right]. \qquad (9.1.9)$$

The outer solution (9.1.9) is a uniform approximation to the solution $y(x)$ as $\varepsilon \to 0+$ on the subinterval $\delta \ll x \le 1$ of $[0, 1]$, where $\delta(\varepsilon)$ is the thickness of the boundary layer. It is now becoming clear why we have assumed that $a(x_0) \ne 0$ for $0 \le x_0 \le 1$. If $a(x_0) = 0$ for some x_0 on this interval, then $y_{\text{out}}(x)$ would be singular at x_0, assuming that $b(x_0) \ne 0$. This would violate the assumption that y, y', and y'' are all of order 1.

The outer solution $y_{\text{out}}(x)$ is not valid in the neighborhood of $x = 0$ unless $y_{\text{out}}(0) = A$, in which case $y_{\text{out}}(x)$ is a uniformly valid leading-order approximation to $y(x)$ for $0 \le x \le 1$. However, since A is arbitrary, in general $y_{\text{out}}(0) \ne A$. Thus, the boundary condition $y(0) = A$ must be achieved through a boundary layer at $x = 0$. In other words, the outer solution $y_{\text{out}}(x)$ is approximately equal to $y(x)$ as x approaches 0 from above until $x = O(\delta)$. At this point $y_{\text{out}}(x)$ is approaching and already very close to $y_{\text{out}}(0)$, while the actual solution $y(x)$ rapidly veers off and approaches $y(0) = A$ (see Fig. 9.4).

To determine the behavior of $y(x)$ when $x = O(\delta)$, we may approximate the functions of $a(x)$ and $b(x)$ in the original differential equation (9.1.7) by $a(0) = \alpha \ne 0$ and $b(0) = \beta$ because δ vanishes as $\varepsilon \to 0$. Also, in the inner region, y is much smaller than y' because y is rapidly varying. Therefore, we may neglect y compared with y'. Thus, the inner approximation to (9.1.7) is the constant coefficient differential equation

$$\varepsilon y''_{\text{in}} + \alpha y'_{\text{in}} = 0, \tag{9.1.10}$$

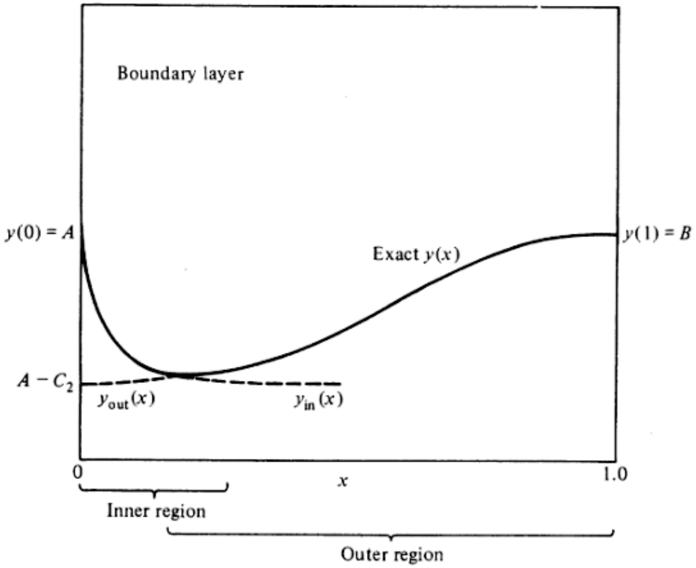

Figure 9.4 A schematic plot of the solution to the boundary-value problem $\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = 0[0 \le x \le 1; a(x) > 0]$ with $y(0) = A$, $y(1) = B$, in (9.1.7). The exact solution satisfies the boundary conditions $y(0) = A$ and $y(1) = B$ and has a boundary layer (region of rapid variation) of thickness $O(\varepsilon)$ at $x = 0$. The outer solution $y_{\text{out}}(x)$ is a good approximation to $y(x)$ in the outer region, but $y_{\text{out}}(0) = A - C_2$. The inner solution is a good approximation to $y(x)$ in the inner region. The asymptotic match of $y_{\text{in}}(x)$ and $y_{\text{out}}(x)$ occurs in the overlap of the inner and outer regions; in the overlap region $y_{\text{in}}(x)$ and $y_{\text{out}}(x)$ both approach the constant $A - C_2$.

which is soluble. The most general solution to (9.1.10) is

$$y_{\text{in}}(x) = C_1 + C_2 e^{-\alpha x/\varepsilon}.$$

Finally, we must require that $y_{\text{in}}(0) = y(0) = A$. Thus,

$$y_{\text{in}}(x) = A + C_2(e^{-\alpha x/\varepsilon} - 1). \tag{9.1.11}$$

The remaining constant of integration C_2 will be determined by asymptotic matching.

Since $y_{\text{in}}(x)$ in (9.1.11) varies rapidly when $x = O(\varepsilon)$, we conclude that the boundary-layer thickness δ is of order ε. The asymptotic match of the inner and outer solutions takes place between the rightmost edge of the inner region and the leftmost edge of the outer region, say for values of $x = O(\varepsilon^{1/2})$. For such values of x,

$$y_{\text{in}}(x) \sim A - C_2, \qquad \varepsilon \to 0+,$$

and

$$y_{\text{out}}(x) \sim y_{\text{out}}(0) = B \exp\left[\int_0^1 b(t)/a(t)\,dt\right], \qquad \varepsilon \to 0+.$$

Thus, if $y_{\text{in}}(x)$ and $y_{\text{out}}(x)$ are to be good approximations to $y(x)$ in the overlap of the inner and outer regions, then we must require that $C_2 = A - y_{\text{out}}(0)$.

To summarize, the boundary-layer approximation is

$$\begin{aligned} y(x) &\sim B \exp\left[\int_x^1 b(t)/a(t)\,dt\right], && 0 < x \le 1, \varepsilon \to 0+; \\ y(x) &\sim Ae^{-a(0)x/\varepsilon} + B(1 - e^{-a(0)x/\varepsilon}) \exp\left[\int_0^1 b(t)/a(t)\,dt\right], && x = O(\varepsilon), \varepsilon \to 0+. \end{aligned} \tag{9.1.12}$$

We may proceed further by combining the above two expressions into a single, *uniform* approximation y_{unif}, valid for all $0 \le x \le 1$. A suitable expression is

$$y_{\text{unif}}(x) = y_{\text{out}}(x) + y_{\text{in}}(x) - y_{\text{match}}(x),$$

where $y_{\text{match}} = A - C_2$. Hence,

$$y_{\text{unif}}(x) = B \exp\left[\int_x^1 b(t)/a(t)\,dt\right] + \left\{A - B \exp\left[\int_0^1 b(t)/a(t)\,dt\right]\right\} e^{-a(0)x/\varepsilon}. \tag{9.1.13}$$

To verify that $y_{\text{unif}}(x) \sim y(x)$ $(\varepsilon \to 0+)$, one must examine it for values of x in the inner and outer regions and check that it reduces to the two expressions in (9.1.12). Equation (9.1.13) is a uniform approximation in the sense that the difference between $y(x)$ and $y_{\text{unif}}(x)$ is uniformly $O(\varepsilon)$ $(0 \le x \le 1, \varepsilon \to 0+)$ (see Prob. 9.2).

We conclude this example with several observations. First, if $a(x) < 0$ throughout $[0, 1]$, then no match is possible with the boundary layer solution (9.1.11) at $x = 0$ because $y_{\text{in}}(x)$ grows exponentially with x/ε unless $C_2 = 0$. On the other hand, if the boundary layer occurs at $x = 1$, then matching is possible if $a(x) < 0$ (see Prob. 9.3). If $a(x) > 0$, it is impossible to match to a boundary layer at $x = 1$ for the same reason that a match cannot be made at $x = 0$ when $a(x) < 0$.

Second, there can be no boundary layer at an internal point x_0 $(0 < x_0 < 1)$ if $a(x_0) \ne 0$. If a boundary layer did exist at x_0, then within this narrow layer we could approximate the original differential equation (9.1.7) by $\varepsilon y''_{\text{in}} + a(x_0)y'_{\text{in}} = 0$. The general solution to this equation is

$$y_{\text{in}} = C_1 + C_2 e^{-a(x_0)(x - x_0)/\varepsilon}.$$

If $a(x_0) > 0$ (<0), then no asymptotic match is possible at the left (right) edge of the boundary layer unless $C_2 = 0$ because the approximation must remain finite. Thus, the matching conditions require that $C_2 = 0$. Hence, the outer solutions to the left and right of the boundary layer both approach the same constant C_1 as $x \to x_0$ from below and above. Thus, the outer solutions approach each other and there is no internal region of rapid change.

In summary, then, when $a(x)$ in (9.1.7) satisfies $a(x) > 0$ for $0 \le x \le 1$ the boundary layer always lies at $x = 0$ and when $a(x) < 0$ for $0 \le x \le 1$ the boundary layer always lies at $x = 1$.

This completes our heuristic introduction to boundary-layer theory. Our purpose in this section was to show how to convert difficult differential equations into easy ones by seeking approximate rather than exact solutions. However, several questions must be answered before the ideas of boundary-layer theory can really be applied with confidence. For example, how can one know *a priori* whether the solution to a differential equation has boundary-layer structure? How can one predict the locations of the boundary layers? How does one estimate δ, the thickness of the boundary layer? How can we be sure that there is an overlap region between the inner and outer regions on which to perform asymptotic matching? Is it useful to decompose a solution into its inner and outer parts if one is seeking a high-order approximation to the exact answer? These questions will be answered in the next two sections.

(E) 9.2 MATHEMATICAL STRUCTURE OF BOUNDARY LAYERS: INNER, OUTER, AND INTERMEDIATE LIMITS

Having demonstrated the power and broad applicability of boundary-layer analysis in Sec. 9.1, it is now appropriate to formalize and restate more carefully some of the rather loosely defined concepts. This section deals with the questions about boundary-layer theory that were raised at the end of the previous section.

To keep our presentation as concrete as possible we will use Example 1 of Sec. 9.1 as a model boundary-layer problem and will analyze its mathematical structure in detail. You will recall that the function

$$y(x) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}, \tag{9.2.1}$$

which is the exact solution of the boundary-value problem

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0, \qquad y(0) = 0,\ y(1) = 1, \tag{9.2.2}$$

has a boundary layer at $x = 0$ when $\varepsilon \to 0+$. The function $y(x)$ has two components: e^{-x}, a slowly varying function on the entire interval [0, 1], and $e^{-x/\varepsilon}$, a rapidly varying function in the boundary layer $x \le O(\delta)$, where $\delta = O(\varepsilon)$ is the thickness of the boundary layer.

In boundary-layer theory we treat the solution y of the differential equation as a function of two independent variables, x and ε. The goal of the analysis is to find a global approximation to y as a function of x; this is achieved by performing a local analysis of y as $\varepsilon \to 0+$.

To explain the appearance of the boundary layer we introduce the notion of

an inner and outer limit of the solution. The *outer limit* of the solution (9.2.1) is obtained by choosing a fixed x *outside* the boundary layer, that is, $\delta \ll x \le 1$, and allowing $\varepsilon \to 0+$. Thus, the outer limit is

$$y_{\text{out}}(x) \equiv \lim_{\varepsilon \to 0+} y(x) = e^{1-x}. \tag{9.2.3}$$

The difference between the outer limit of the exact solution and the exact solution itself, $|y(x) - y_{\text{out}}(x)|$, is exponentially small in the limit $\varepsilon \to 0$ when $\delta \ll x$.

Similarly, we can formally take the outer limit of the differential equation (9.2.2); the result of keeping x fixed and letting $\varepsilon \to 0+$ is simply

$$y'_{\text{out}} + y_{\text{out}} = 0, \tag{9.2.4}$$

which is satisfied by (9.2.3). Because the outer limit of (9.1.2) is a *first*-order differential equation, its solution cannot in general be required to satisfy both boundary conditions $y(0) = 0$ and $y(1) = 1$; the outer limit of (9.2.1) satisfies $y(1) = 1$ but not $y(0) = 0$.

In other words, the small-ε limit of the solution is *not* everywhere close to the solution of the unperturbed differential equation (9.2.4) [the differential equation (9.2.2) with $\varepsilon = 0$]. Thus, the problem (9.2.2) is a singular perturbation problem. The singular behavior [the appearance of a discontinuity in $y(x)$ as $\varepsilon \to 0+$] occurs because the highest-order derivative in (9.2.2) disappears when $\varepsilon = 0$.

The exact solution satisfies the boundary condition $y(0) = 0$ by developing a boundary layer in the neighborhood of $x = 0$. To examine the nature of this boundary layer, we consider the *inner limit* in which $\varepsilon \to 0+$ with $x \le O(\varepsilon)$. In this case x lies inside the boundary layer at $x = 0$. For this limit it is convenient to let $x = \varepsilon X$ with X fixed and finite. X is called an *inner* variable. X is a better variable than x to describe y in the boundary layer because, as $\varepsilon \to 0+$, y varies rapidly as a function of x but slowly as a function of X. It follows from (9.2.1) that

$$y_{\text{in}}(x) = Y_{\text{in}}(X) = \lim_{\varepsilon \to 0+} y(\varepsilon X) = e - e^{1-X}. \tag{9.2.5}$$

Similarly, the inner limit of the differential equation is obtained by rewriting (9.2.2) as

$$\frac{1}{\varepsilon}\frac{d^2Y}{dX^2} + \left(\frac{1}{\varepsilon} + 1\right)\frac{dY}{dX} + Y = 0, \tag{9.2.6}$$

where we define $Y(X) \equiv y(x)$ and use

$$\varepsilon\frac{dy}{dx} = \frac{dY}{dX}, \qquad \varepsilon^2\frac{d^2y}{dx^2} = \frac{d^2Y}{dX^2}.$$

Thus, taking the inner limit, $\varepsilon \to 0+$, X fixed, gives

$$\frac{d^2Y_{\text{in}}(X)}{dX^2} + \frac{dY_{\text{in}}(X)}{dX} = 0. \tag{9.2.7}$$

Observe that the inner limit function (9.2.5) does satisfy (9.2.7) together with the boundary condition $Y_{\text{in}}(0) = 0$.

Boundary-layer analysis is extremely useful because it allows one to construct an approximate solution to a differential equation, even when an exact answer is not attainable. This is because the inner and outer limits of an insoluble differential equation are often soluble. Once y_{in} and y_{out} have been determined, they must be asymptotically matched. This asymptotic match occurs on the overlap region which is defined by the *intermediate* limit $x \to 0$, $X = x/\varepsilon \to \infty$, $\varepsilon \to 0+$. For example, if $x = \varepsilon^{1/2} z$ with z fixed as $\varepsilon \to 0$, then an intermediate limit is obtained. A glance at (9.2.3) and (9.2.5) shows that the intermediate limits of $y_{\text{out}}(x)$ and $y_{\text{in}}(x) = Y_{\text{in}}(X)$ agree: $\lim_{x \to 0} y_{\text{out}}(x) = \lim_{X \to \infty} Y_{\text{in}}(X) = e$. This common limit verifies the asymptotic match between the inner and outer solutions. (It is *not* generally the case in boundary-layer theory that the intermediate limit is a *number* independent of x and X, as we will shortly see.) The above match condition provides the second boundary condition for the solution of (9.2.7): $Y_{\text{in}}(\infty) = e$. Observe that although the x region is finite, $0 \le x \le 1$, the size of the matching region in terms of the inner variable is infinite. As we emphasized in Chap. 7, the extent of the matching region must always be infinite.

A very subtle aspect of boundary-layer theory is the question of whether or not an overlap region for any given problem actually exists. Since one's ability to construct a matched asymptotic expansion depends on the presence of this overlap region, its existence is crucial to the solution of the problem. How did we know, for example, that the intermediate limits of y_{out} and Y_{in} would agree? That is, how did we know that the inner and outer limits of the differential equation (9.2.2) have a common region of validity?

To answer these questions we will perform a complete perturbative solution of (9.2.2) to all orders in powers of ε, and not just to leading order. First, we examine the outer solution. We seek a perturbation expansion of the outer solution of the form

$$y_{\text{out}}(x) \sim \sum_{n=0}^{\infty} y_n(x)\varepsilon^n, \qquad \varepsilon \to 0+, \tag{9.2.8}$$

and restate the boundary condition $y(1) = 1$ as

$$y_0(1) = 1, \quad y_1(1) = 0, \quad y_2(1) = 0, \quad y_3(1) = 0, \quad \ldots. \tag{9.2.9}$$

Note that $y_{\text{out}}(x)$ in (9.2.8) is *not* the same as $y_{\text{out}}(x)$ in (9.2.3); $y_{\text{out}}(x)$ in (9.2.3) is the first term $y_0(x)$ in (9.2.8).

Substituting (9.2.8) into (9.2.2) and collecting powers of ε gives a sequence of differential equations:

$$y_0' + y_0 = 0, \qquad y_0(1) = 1,$$

$$y_n' + y_n = -y_{n-1}'' - y_{n-1}', \qquad y_n(1) = 0,\ n \ge 1.$$

The solution to these equations is

$$\begin{aligned} y_0 &= e^{1-x}, \\ y_n &= 0, \qquad n \ge 1. \end{aligned} \tag{9.2.10}$$

Thus, the leading-order outer solution, $y_{\text{out}} = e^{1-x}$, is correct to *all* orders in perturbation theory. This is the reason why in the outer region, $x \gg \varepsilon$, the difference between $y(x)$ and $y_{\text{out}}(x)$ is at most exponentially small (subdominant): $|y - y_{\text{out}}| = O(\varepsilon^n)$ for *all* n as $\varepsilon \to 0+$.

Next, we perform a similar expansion of the inner solution. We assume a perturbation series of the form

$$Y_{\text{in}}(X) \sim \sum_{n=0}^{\infty} \varepsilon^n Y_n(X), \qquad \varepsilon \to 0+, \tag{9.2.11}$$

and restate the boundary condition $Y_{\text{in}}(0) = y(0) = 0$ as

$$Y_n(0) = 0, \qquad n \ge 0. \tag{9.2.12}$$

Substituting (9.2.11) into (9.2.6) gives the sequence of differential equations:

$$Y_0'' + Y_0' = 0, \qquad Y_0(0) = 0,$$

$$Y_n'' + Y_n' = -Y_{n-1}' - Y_{n-1}, \qquad Y_n(0) = 0,\ n \ge 1.$$

These equations may be solved by means of the integrating factor e^X. The results are

$$\begin{aligned} Y_0(X) &= A_0(1 - e^{-X}), \\ Y_n(X) &= \int_0^X [A_n e^{-z} - Y_{n-1}(z)]\, dz, \qquad n \ge 1, \end{aligned} \tag{9.2.13}$$

where the A_n are arbitrary integration constants.

Does this inner solution match asymptotically, order by order in powers of ε, to $y_{\text{out}}(x)$? To see if this is so, we substitute $x = \varepsilon X$ into y_{out} in (9.2.10) and expand in powers of ε:

$$y_{\text{out}}(x) = e^{1-x} = e\left(1 - \varepsilon X + \frac{\varepsilon^2 X^2}{2!} - \frac{\varepsilon^3 X^3}{3!} + \cdots\right). \tag{9.2.14}$$

Returning to equation (9.2.13), we take X large ($X \to \infty$) and obtain $Y_0(X) \sim A_0$ ($X \to \infty$). Thus, comparing with the first term of (9.2.14), we have $A_0 = e$, as we already know. Now that Y_0 is known, we may compute Y_1 from (9.2.13):

$$Y_1(X) = (A_1 + A_0)(1 - e^{-X}) - eX.$$

Asymptotic matching with y_{out} [comparing $Y_1(X)$, when $X \to \infty$, with the second term of (9.2.14)] gives $A_1 = -e$, so $Y_1(X) = -eX$. Similarly, $Y_n(X) = e[(-1)^n/n!]X^n$. Hence the full inner expansion is

$$Y_{\text{in}}(X) = e \sum_{n=0}^{\infty} \varepsilon^n \frac{(-1)^n X^n}{n!} - e^{1-X} = e^{1-\varepsilon X} - e^{1-X}. \tag{9.2.15}$$

Evidently, the inner expansion is a valid asymptotic expansion not only for values of X inside the boundary layer [$X = O(1)$] but also for large values of X

$[X = O(\varepsilon^{-\alpha}),\ 0 < \alpha < 1]$. At the same time the expansion for $y_{\text{out}}(x)$ is valid for $\varepsilon \ll x \le 1$ $(\varepsilon \to 0+)$. $[y_{\text{out}}(x)$ is not valid for $x = O(\varepsilon)$ because it does not satisfy the boundary condition $y(0) = 0$; nor does it have the boundary-layer term e^{1-X} which is present in $Y_{\text{in}}(X)$.] We conclude that *to all orders in powers of* ε it is possible to match asymptotically the inner and outer expansions because they have a common region of validity: $\varepsilon \ll x \ll 1$ $(\varepsilon \to 0+)$.

We have been able to demonstrate explicitly the existence of the overlap region for this problem because it is soluble to all orders in perturbation theory. In general, however, such a calculation is too difficult. Instead, our approach will always be to assume that an overlap region exists and then to verify the consistency of this assumption by performing an asymptotic match. In the above simple boundary-value problem, we found that the size of the overlap region was independent of the order of perturbation theory. In general, however, the extent of the matching region may vary with the order of perturbation theory (see Sec. 7.4 and Example 1 of Sec. 9.3).

One final point concerns the construction of the uniform approximation to $y(x)$. The formula used in the previous section to construct a uniform approximation is $y_{\text{unif}}(x) = y_{\text{in}}(x) + y_{\text{out}}(x) - y_{\text{match}}(x)$, where $y_{\text{match}}(x)$ is the approximation to $y(x)$ in the matching region and $y_{\text{unif}}(x)$ is a uniform approximation to $y(x)$. This formula is applicable here too. For the boundary-layer solution to (9.2.2), it is easy to verify that if $y_{\text{in}}(x)$, $y_{\text{out}}(x)$, and $y_{\text{match}}(x)$ are calculated to nth order in perturbation theory, then $|y(x) - y_{\text{unif}}(x)| = O(\varepsilon^{n+1})$ $(\varepsilon \to 0+;\ 0 \le x \le 1)$.

The differential equation (9.2.2) is sufficiently simple that $y_{\text{unif}}(x)$ can be calculated to all orders in perturbation theory. It follows from (9.2.10) for $y_{\text{out}}(x)$, (9.2.15) for $y_{\text{in}}(x)$, and the result $y_{\text{match}}(x) = e^{1-x}$ that

$$y_{\text{unif}} = e^{1-x} - e^{1-X}$$

is the infinite-order uniform approximation to $y(x)$.

It is remarkable, however, that this expression, which is the result of summing up perturbation theory to infinite order, is actually *not* equal to $y(x)$ in (9.2.1). Thus, although the perturbation series for y_{unif} is *asymptotic* to $y(x)$ as $\varepsilon \to 0+$, the asymptotic series does not *converge* to $y(x)$ as n, the order of perturbation theory, tends to ∞; there is an exponentially small error, of order $e^{-1/\varepsilon}$, which remains undetermined. Boundary-layer theory is indeed a singular, and not a regular, perturbation theory.

Why is boundary-layer theory a singular perturbation theory? The singular nature of boundary-layer theory is intrinsic to both the inner and outer expansions. The outer expansion is singular because there is an abrupt change in the order of the differential equation when $\varepsilon = 0$. By contrast, the inner expansion is a regular perturbation expansion for finite X (see Example 2 of Sec. 7.1). However, since asymptotic matching takes place in the limit $X \to \infty$, the inner expansion is also singular (see Example 4 of Sec. 7.2). Another manifestation of the singular limit $\varepsilon \to 0$ is the location of the boundary layer in (9.2.1); when the limit $\varepsilon \to 0+$ is replaced by $\varepsilon \to 0-$, the boundary layer abruptly jumps from $x = 0$ to $x = 1$.

(E) 9.3 HIGHER-ORDER BOUNDARY-LAYER THEORY

In Secs. 9.1 and 9.2 we formulated the procedure for finding the leading-order boundary-layer approximation to the solution of an ordinary differential equation; i.e., to obtain outer and inner solutions and asymptotically match them in an overlap region. The self-consistency of boundary-layer theory depends on the success of asymptotic matching. Ordinarily, if the inner and outer solutions match to all orders in ε, then boundary-layer theory gives an asymptotic approximation to the exact solution of the differential equation. Accordingly, in Sec. 9.2 we showed how to use boundary-layer theory to all orders in powers of ε for a simple constant-coefficient differential equation.

In this section we give an example to illustrate how boundary-layer theory is used to construct higher-order approximations for more complicated equations. As we shall see, an interesting aspect of boundary-layer problems is that the size of the matching region depends on the order of perturbation theory.

Example 1 *Boundary-layer analysis of a variable-coefficient differential equation.* We wish to obtain an approximate solution to the boundary-value problem

$$\varepsilon y'' + (1 + x)y' + y = 0, \qquad y(0) = 1,\ y(1) = 1, \tag{9.3.1}$$

which is correct to order ε^3.

We seek an outer solution in the form of a perturbation series in powers of ε:

$$y_{\text{out}}(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots, \qquad \varepsilon \to 0+. \tag{9.3.2}$$

Since $1 + x > 0$ for $0 \le x \le 1$, we expect a boundary layer only at $x = 0$ (see Sec. 9.1). Thus, the outer solution must satisfy the boundary condition $y_{\text{out}}(1) = 1$ and we must require that

$$y_0(1) = 1, \qquad y_n(0) = 0, \qquad n \ge 1. \tag{9.3.3}$$

Next, we substitute (9.3.2) into (9.3.1) and equate coefficients of like powers of ε. This converts (9.3.1) into a sequence of first-order inhomogeneous equations:

$$(1 + x)y_0' + y_0 = 0,\ (1 + x)y_1' + y_1 = -y_0'',\ (1 + x)y_2' + y_2 = -y_1'',\ \ldots.$$

The solutions to these equations which satisfy the boundary conditions (9.3.3) are

$$y_0(x) = 2(1 + x)^{-1},$$
$$y_1(x) = 2(1 + x)^{-3} - \tfrac{1}{2}(1 + x)^{-1},$$
$$y_2(x) = 6(1 + x)^{-5} - \tfrac{1}{2}(1 + x)^{-3} - \tfrac{1}{4}(1 + x)^{-1}.$$

Thus,

$$y_{\text{out}}(x) \sim \frac{2}{1 + x} + \varepsilon\left[\frac{2}{(1 + x)^3} - \frac{1}{2(1 + x)}\right] + \varepsilon^2\left[\frac{6}{(1 + x)^5} - \frac{1}{2(1 + x)^3} - \frac{1}{4(1 + x)}\right] + \cdots, \qquad \varepsilon \to 0+. \tag{9.3.4}$$

This completes the determination of the outer solution to second order in powers of ε.

As expected, the outer solution (9.3.4) does not satisfy the boundary condition $y(0) = 1$, so a boundary layer at $x = 0$ is necessary. As in Example 3 of Sec. 9.1, we expect the thickness δ of the boundary layer to be $O(\varepsilon)$. (In Sec. 9.4 the procedure for determining the thickness of the boundary layer is explained and examples of boundary layers having thicknesses other than ε are given.) Therefore, we introduce the inner variables $X = x/\varepsilon$ and $Y_{in}(X) \equiv y_{in}(x)$. In terms of these variables (9.3.1) becomes

$$\frac{d^2 Y_{in}}{dX^2} + (1 + \varepsilon X)\frac{dY_{in}}{dX} + \varepsilon Y_{in} = 0. \tag{9.3.5}$$

If we represent $Y_{in}(X)$ as a perturbation series in powers of ε,

$$Y_{in}(X) \sim Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) + \cdots, \qquad \varepsilon \to 0+, \tag{9.3.6}$$

then the boundary condition $y(0) = 1$ translates into the sequence of boundary conditions

$$Y_0(0) = 1, \qquad Y_1(0) = 0, \qquad Y_2(0) = 0, \qquad \ldots. \tag{9.3.7}$$

Substituting (9.3.6) into (9.3.5) and equating coefficients of like powers of ε converts (9.3.5) into a sequence of second-order constant-coefficient equations:

$$\begin{aligned} &\frac{d^2 Y_0}{dX^2} + \frac{dY_0}{dX} = 0, \\ &\frac{d^2 Y_1}{dX^2} + \frac{dY_1}{dX} = -X\frac{dY_0}{dX} - Y_0, \\ &\frac{d^2 Y_2}{dX^2} + \frac{dY_2}{dX} = -X\frac{dY_1}{dX} - Y_1. \end{aligned} \tag{9.3.8}$$

Each of the solutions of (9.3.8) which satisfy the boundary conditions (9.3.7) have one new arbitrary constant of integration:

$$Y_0(X) = 1 + A_0(e^{-X} - 1), \tag{9.3.9}$$

$$Y_1(X) = -X + A_0(-\tfrac{1}{2}X^2 e^{-X} + X) + A_1(e^{-X} - 1), \tag{9.3.10}$$

$$\begin{aligned} Y_2(X) = {} & X^2 - 2X + A_0(\tfrac{1}{8}X^4 e^{-X} - X^2 + 2X) \\ & + A_1(-\tfrac{1}{2}X^2 e^{-X} + X) + A_2(e^{-X} - 1). \end{aligned} \tag{9.3.11}$$

This completes the determination of the inner solution to second order in powers of ε.

We determine the constants $A_0, A_1, A_2, \ldots$ by asymptotically matching the inner and outer solutions. The match consists of requiring that the intermediate limits $[\varepsilon \to 0+,\ x \to 0+,\ X = x/\varepsilon \to +\infty]$ of the inner and outer solutions agree.

First, we perform a leading-order (zeroth-order in ε) match. As $x \to 0+$ in the outer solution (9.3.4), $y_{out}(x) = 2 + O(\varepsilon, x)$ $(x \to 0+,\ \varepsilon \to 0+)$, where the symbol

$$O(a, b, c, \ldots) \text{ means } O(a) + O(b) + O(c) + \cdots.$$

The error term $O(\varepsilon, x)$ indicates that we have neglected powers of ε higher than zero and that we have expanded the solution in a Taylor series in powers of x and have neglected all but the first term.

On the other hand, in the limit $X \to \infty$ and $\varepsilon X = x \to 0+$ the inner solution becomes

$$Y_{in}(X) = 1 - A_0 + O(\varepsilon X), \qquad \varepsilon X \to 0+,\ X \to \infty, \tag{9.3.12}$$

where the correction of order εX arises from the term $\varepsilon Y_1(X)$ in the expansion of $Y_{in}(X)$. The inner and outer solutions are required to match to lowest order [that is, $y_{out}(x) \sim Y_{in}(X)$ in the intermediate limit]. Thus, $2 = 1 - A_0$ or $A_0 = -1$. This completes the leading-order match. Observe that the match occurs for values of x for which $1 \ll X = x/\varepsilon$ as well as $x \ll 1$. Thus, the size of the overlap region is determined to leading order as $\varepsilon \ll x \ll 1$ $(\varepsilon \to 0+)$.

Next, we match to first order in ε. We expand the outer solution, keeping terms of order ε and x but discarding terms of order ε^2, x^2, and εx. The result is

$$y_{out}(x) = 2 - 2x + \tfrac{3}{2}\varepsilon + O(\varepsilon^2, \varepsilon x, x^2), \qquad x \to 0+,\ \varepsilon \to 0+. \tag{9.3.13}$$

We also expand the inner solution for $X \gg 1$ (x outside the boundary layer), but neglect terms of order $\varepsilon^2 X^2 = x^2$, $\varepsilon^2 X = \varepsilon x$, and ε^2. The result is

$$Y_{in}(X) = 1 - A_0 - \varepsilon X + \varepsilon A_0 X - \varepsilon A_1 + O(x^2), \qquad \varepsilon X \to 0+,\ X \to \infty. \tag{9.3.14}$$

Matching (9.3.14) with (9.3.13) gives $A_0 = -1$, which we already know, and $A_1 = -3/2$.

Observe that for a successful match to first order it is necessary to neglect terms of order ε^2, x^2, and εx, compared with x and ε. If we had retained some of these terms, we would have found that there is no way to choose the constants of integration to make (9.3.13) agree with (9.3.14). Since matching now requires that $x^2 \ll \varepsilon$, the size of the overlap region is *smaller* than it was in the leading-order match. Its extent is $\varepsilon \ll x \ll \varepsilon^{1/2}$ $(\varepsilon \to 0+)$.

To perform a second-order match, we expand the inner and outer solutions and neglect terms of order ε^3, $\varepsilon^2 x$, εx^2, and x^3. The result is

$$y_{out}(x) = 2 - 2x + 2x^2 + \varepsilon(\tfrac{3}{2} - \tfrac{11}{2}x) + \tfrac{21}{4}\varepsilon^2 + O(\varepsilon^3, \varepsilon^2 x, \varepsilon x^2, x^3),$$
$$\varepsilon \to 0+,\ x \to 0+,$$

and

$$\begin{aligned} Y_{in}(X) = {} & 1 - A_0 - \varepsilon X + \varepsilon A_0 X - \varepsilon A_1 + \varepsilon^2 X^2 - 2\varepsilon^2 X - \varepsilon^2 A_0 X^2 + 2\varepsilon^2 A_0 X \\ & + \varepsilon^2 A_1 X - \varepsilon^2 A_2 + O(\varepsilon^3, \varepsilon^2 x, \varepsilon x^2, x^3), \\ & \qquad \varepsilon X \to 0+,\ X \to \infty. \end{aligned} \tag{9.3.15}$$

Matching requires $A_0 = -1$, $A_1 = -3/2$, $A_2 = -21/4$.

Since one must neglect x^3 compared with ε^2 to obtain a match in second order, it follows that the size of the matching region, $\varepsilon \ll x \ll \varepsilon^{2/3}$ $(\varepsilon \to 0+)$, is smaller than it was in the first-order calculation.

The process of asymptotic matching may be carried out to all orders in powers of ε (see Prob. 9.10). And as the order of perturbation theory increases, the size of the matching region continues to shrink. In nth order the common region of validity of the inner and outer expansions is $\varepsilon \ll x \ll \varepsilon^{n/(n+1)}$ $(\varepsilon \to 0+)$. However, the extent of the matching region in terms of the inner variable is still infinite as $\varepsilon \to 0+$: $1 \ll X \ll \varepsilon^{-1/(n+1)}$ $(\varepsilon \to 0+)$.

Once the boundary-layer solution is determined, one may construct a uniform approximation to the solution $y(x)$ using the formula $y_{unif}(x) = y_{out}(x) + y_{in}(x) - y_{match}(x)$, where $y_{match}(x)$ is the expansion of either the inner or outer approximations in the matching region. To third order in ε (see Prob. 9.12),

$$\begin{aligned} y_{unif,3}(x) = {} & \left(\frac{2}{1+x} - e^{-X}\right) + \varepsilon\left[\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} + \left(\frac{1}{2}X^2 - \frac{3}{2}\right)e^{-X}\right] \\ & + \varepsilon^2\left[\frac{6}{(1+x)^5} - \frac{1}{2(1+x)^3} - \frac{1}{4(1+x)} - \left(\frac{1}{8}X^4 - \frac{3}{4}X^2 + \frac{21}{4}\right)e^{-X}\right] \\ & + \varepsilon^3\left[\frac{30}{(1+x)^7} - \frac{3}{2(1+x)^5} - \frac{1}{4(1+x)^3} - \frac{5}{16(1+x)}\right. \\ & \left. + \left(\frac{1}{48}X^6 - \frac{3}{16}X^4 + \frac{21}{8}X^2 - \frac{1949}{72}\right)e^{-X}\right] + O(\varepsilon^4), \qquad \varepsilon \to 0+. \end{aligned} \tag{9.3.16}$$

This expression is a uniform approximation to $y(x)$ over the entire region $0 \le x \le 1$:

$$|y_{unif,3}(x) - y(x)| = O(\varepsilon^4), \qquad \varepsilon \to 0+. \tag{9.3.17}$$

Also, $y_{\text{unif}}(x)$ may be used to approximate the derivatives of $y(x)$:

$$|y'_{\text{unif},3}(x) - y'(x)| = O(\varepsilon^3), \qquad \varepsilon \to 0+, \tag{9.3.18}$$

$$|y''_{\text{unif},3}(x) - y''(x)| = O(\varepsilon^2), \qquad \varepsilon \to 0+, \tag{9.3.19}$$

for $0 \le x \le 1$ (see Prob. 9.13).

In Figs. 9.5 to 9.8 we show how well $y_{\text{unif}}(x)$ approximates the exact solution $y(x)$. We plot the percentage relative error [percentage relative error $= 100(y_{\text{unif}} - y)/y$] for the first four uniform approximations $y_{\text{unif},n}$ ($n = 0, 1, 2, 3$). $y_{\text{unif},n}$ is the uniform approximation to $y(x)$ accurate to order ε^n; for example, $y_{\text{unif},1}$ is obtained from (9.3.16) by neglecting the terms containing ε^2 and ε^3. Figures 9.5 to 9.8 suggest the asymptotic nature of the boundary-layer approximation; the uniform approximation in (9.3.16) is the first four terms of a divergent asymptotic series in powers of ε (see Prob. 9.14).

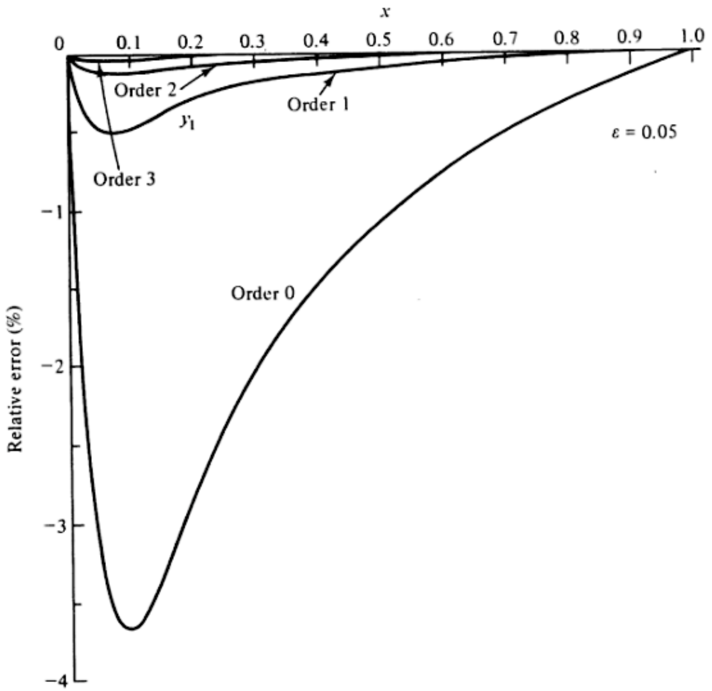

Figure 9.5 A plot of the percentage relative error between the exact solution $y(x)$ to the boundary-value problem in (9.3.1) with $\varepsilon = 0.05$ and the zeroth-order, first-order, second-order, and third-order uniform approximations to $y(x)$ obtained from boundary-layer analysis [see (9.3.16)]. The percentage relative error $= 100[y_{\text{unif}}(x) - y(x)]/y(x)$. The graphs in this plot lie below the x axis because y_{unif} underestimates $y(x)$. Observe that as the order of perturbation theory increases the relative error decreases. However, for sufficiently large order the asymptotic nature of boundary-layer theory will surface and the relative error will increase with order.

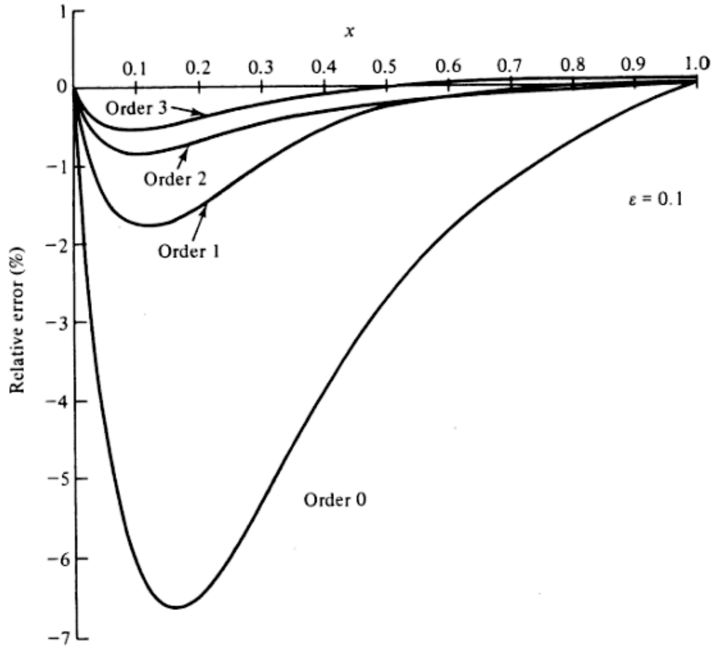

Figure 9.6 Same as in Fig. 9.5 except that $\varepsilon = 0.1$.

(I) 9.4 DISTINGUISHED LIMITS AND BOUNDARY LAYERS OF THICKNESS $\neq \varepsilon$

Until now, most of the boundary layers we have seen have had thickness $\delta = \varepsilon$. In general, however, the thickness of a boundary layer need not be of order ε as $\varepsilon \to 0+$. There are examples where $\delta = O(\varepsilon^{1/2})$, $\delta = O(\varepsilon^{2/3})$, and so on.

The determination of δ requires the notion of a *distinguished limit* which involves nothing more than a dominant-balance argument. We return to Example 3 of Sec. 9.1 to illustrate the relevant techniques. The solution of the boundary-value problem

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \qquad y(0) = A,\ y(1) = B, \tag{9.4.1}$$

has a boundary layer at $x = 0$ if $a(x) > 0$ ($0 \le x \le 1$). In the inner region we let $y(x) = Y_{\text{in}}(X)$, $X = x/\delta$, so

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\delta}\frac{dY_{\text{in}}(X)}{dX}, \\ \frac{d^2y}{dx^2} &= \frac{1}{\delta^2}\frac{d^2Y_{\text{in}}(X)}{dX^2}. \end{aligned} \tag{9.4.2}$$

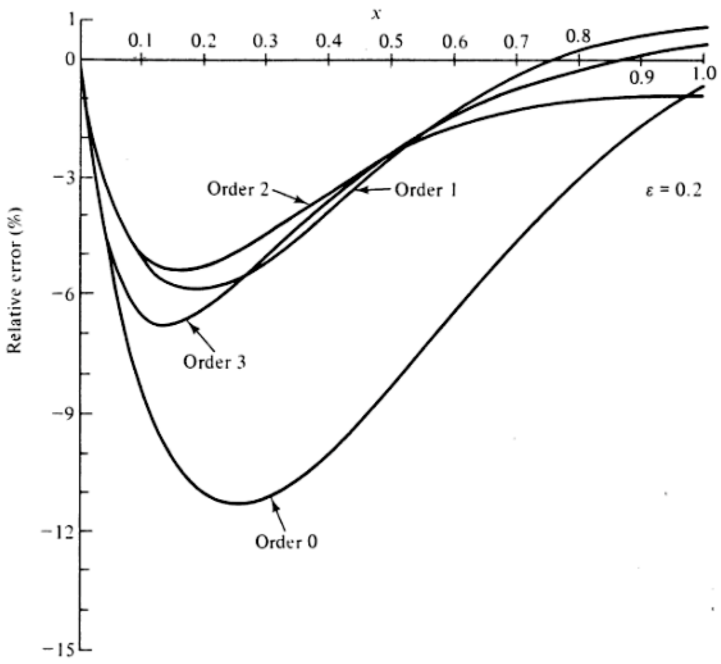

Figure 9.7 Same as in Fig. 9.5 except that $\varepsilon = 0.2$.

Thus, the differential equation (9.4.1) assumes the form

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y_{\text{in}}}{dX^2} + \frac{a(\delta X)}{\delta}\frac{dY_{\text{in}}}{dX} + b(\delta X)Y_{\text{in}} = 0. \tag{9.4.3}$$

Our task is to determine $\delta(\varepsilon)$. There are three possibilities to consider: $\delta(\varepsilon) \ll \varepsilon$, $\delta(\varepsilon) \sim \varepsilon$, and $\varepsilon \ll \delta(\varepsilon)$ as $\varepsilon \to 0$. In the first case, $\delta \ll \varepsilon$, we may approximate (9.4.3) by $d^2Y_{\text{in}}/dX^2 = 0$. Thus, $Y_{\text{in}}(X) = A + cX$, which satisfies the boundary condition $Y_{\text{in}}(0) = A$. This inner limit does not match the outer solution because $\lim_{X\to\infty} Y_{\text{in}}(X) = \infty$ unless $c = 0$, while $y_{\text{out}}(0)$ is finite and not generally equal to A.

Similarly, $\varepsilon \ll \delta$ in (9.4.3) gives $a(0)\, dY_{\text{in}}/dX = 0$, so $Y_{\text{in}}(X) = A$ because $Y_{\text{in}}(0) = A$. Again, no match is possible if $A \neq y_{\text{out}}(0)$.

Finally, the choice $\delta = \varepsilon$ in (9.4.3) gives the leading-order equation

$$\frac{d^2Y_{\text{in}}}{dX^2} + a(0)\frac{dY_{\text{in}}}{dX} = 0.$$

The choice $\delta = \varepsilon$ is called a *distinguished limit* because it involves a nontrivial relation (a dominant balance) between two or more terms of the equation (9.4.3);

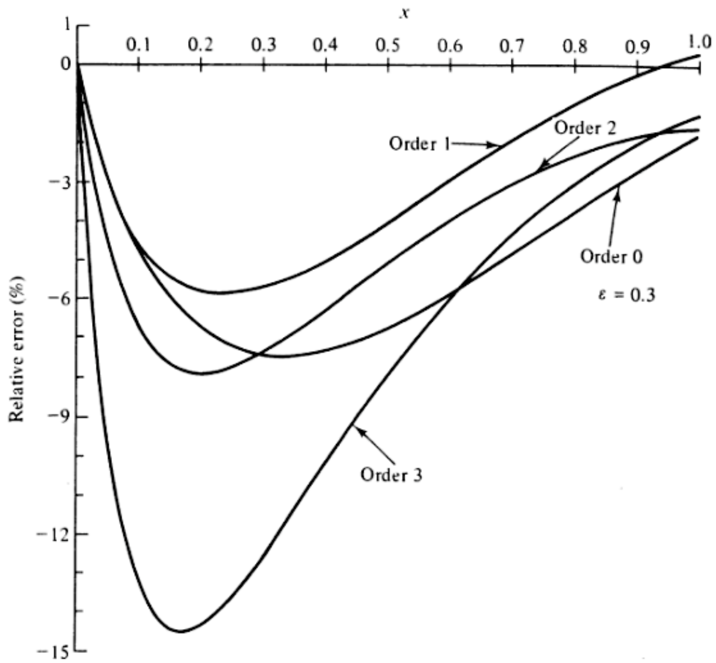

Figure 9.8 Same as in Fig. 9.5 except that $\varepsilon = 0.3$. For this value of ε the asymptotic nature of perturbation theory is evident; optimal accuracy is reached in first order and the relative error increases with the order when the order is greater than 1.

two terms are of comparable size while the third is smaller. The cases $\delta(\varepsilon) \ll \varepsilon$ and $\varepsilon \ll \delta(\varepsilon)$ as $\varepsilon \to 0$ are *undistinguished*. In general, only the distinguished limit gives a nontrivial boundary-layer structure which is asymptotically matchable to the outer solution.

In the above argument we have used the notation $\delta \sim \varepsilon$ and have neglected the possibility that $\delta \sim c\varepsilon$ where c is a constant. This is because we are only interested in the order of magnitude of the boundary-layer thickness. We will always ignore any constant of proportionality.

Example 1 *Boundary layer of thickness $\delta = \varepsilon^{1/2}$.* Consider the boundary-value problem

$$\varepsilon y'' - x^2 y' - y = 0, \qquad y(0) = y(1) = 1. \tag{9.4.4}$$

We will show that the leading-order solution to this problem has two boundary layers, one at the right boundary $x = 1$ for which $\delta = \varepsilon$ and one at the left boundary for which $\delta = \varepsilon^{1/2}$.

The leading-order outer solution satisfies $-x^2y_0' - y_0 = 0$, so

$$y_0(x) = C_0 e^{1/x}. \tag{9.4.5}$$

Next, observe that the coefficient of y' in (9.4.4) is negative at $x = 1$. Thus, on the basis of Example 3 of Sec. 9.1 we expect that a boundary layer of thickness ε will develop at $x = 1$ as $\varepsilon \to 0+$. In terms of the inner variable $X = (1 - x)/\varepsilon$, the leading behavior in this boundary layer is governed by $d^2Y_{0,\text{right}}/dX^2 + dY_{0,\text{right}}/dX = 0$. Hence,

$$Y_{0,\text{right}}(X) = A_0 + B_0 e^{-X}. \tag{9.4.6}$$

The boundary condition $y(1) = 1$ requires that $Y_{0,\text{right}}(0) = A_0 + B_0 = 1$ and matching to the outer solution (9.4.5) in the neighborhood of $x = 1$ requires that $A_0 = C_0 e$. Hence, $B_0 = 1 - C_0 e$. But this does not complete the solution because C_0 is still undetermined. Moreover, as $x \to 0+$ the outer solution (9.4.5) becomes infinite unless $C_0 = 0$, so it certainly cannot satisfy the boundary condition $y(0) = 1$.

Thus, we must have a boundary layer at $x = 0$ if we are to satisfy $y(0) = 1$. To determine the thickness of this layer, we use Z as the inner variable and substitute $x = Z\delta$ and (9.4.2) into (9.4.4) to obtain

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y_{\text{in, left}}}{dZ^2} - \delta Z^2 \frac{dY_{\text{in, left}}}{dZ} - Y_{\text{in, left}} = 0,$$

where $Y_{\text{in, left}}(Z) = y(x)$ in the left boundary layer. The distinguished limits are $\varepsilon/\delta^2 \sim \delta$, $\delta \sim 1$, or $\varepsilon/\delta^2 \sim 1$ as $\varepsilon \to 0+$. The first case, $\delta \sim \varepsilon^{1/3}$, is inconsistent because the undifferentiated term is dominant; the second case, $\delta \sim 1$, reproduces the outer limit. Thus, the only consistent choice of boundary-layer scale is $\delta = \varepsilon^{1/2}$. With this choice (9.4.4) becomes

$$\frac{d^2Y_{\text{in,left}}}{dZ^2} - Y_{\text{in,left}} = \varepsilon^{1/2}Z^2\frac{dY_{\text{in,left}}}{dZ}.$$

The leading-order inner solution $Y_{0,\text{left}}(Z)$ therefore satisfies $d^2Y_{0,\text{left}}/dZ^2 - Y_{0,\text{left}} = 0$, whose solution is

$$Y_{0,\text{left}}(Z) = D_0 e^Z + E_0 e^{-Z}. \tag{9.4.7}$$

The boundary condition $y(0) = 1$ implies that $Y_{0,\text{left}}(0) = 1$ or that $D_0 + E_0 = 1$.

Finally, we must match $Y_{0,\text{left}}(Z)$ to the outer solution $y_0(x)$. However, no match is possible unless $D_0 = 0$; otherwise the inner solution grows exponentially as $Z \to \infty$. Matching also requires that $C_0 = 0$; otherwise the outer solution $y(x)$ grows exponentially as $x \to 0+$. It follows that $A_0 = C_0 = D_0 = 0$, $B_0 = E_0 = 1$.

A uniform approximation to $y(x)$ over the entire interval $0 \le x \le 1$, including both boundary layers, is given by

$$\begin{aligned} y_{\text{unif}} &= y_0 + Y_{0,\text{left}} + Y_{0,\text{right}} - y_{\text{left match}} - y_{\text{right match}} \\ &= e^{(x-1)/\varepsilon} + e^{-x/\varepsilon^{1/2}}. \end{aligned} \tag{9.4.8}$$

Thus, outside the boundary layers at $x = 0$ and $x = 1$, the solution is exponentially small. Figures 9.9 and 9.10 compare the exact solution to (9.4.4) with a plot of the leading uniform asymptotic approximation to $y(x)$ in (9.4.8) for $\varepsilon = 0.05$ and $\varepsilon = 0.005$. Observe the close agreement.

Example 2 *Higher-order treatment of $\delta = \varepsilon^{1/2}$ boundary layer.* In this example we will obtain a leading- and higher-order approximation to the solution of the singular perturbation problem

$$\varepsilon y'' + x^2 y' - y = 0, \qquad y(0) = y(1) = 1. \tag{9.4.9}$$

Note that this differential equation differs from that of the previous example by a sign change in the one-derivative term.

This problem does not quite satisfy the assumptions of the third example of Sec. 9.1. There, we assumed that $a(x) > 0$ $(0 \le x \le 1)$; in this example $a(x) = x^2$, so this inequality is violated at the left boundary. Nevertheless, $a(x) > 0$ for all other x, so the conclusions of Sec. 9.1 (that a boundary layer could not occur at $x = 1$ or at an interior point) are still valid. By elimination,

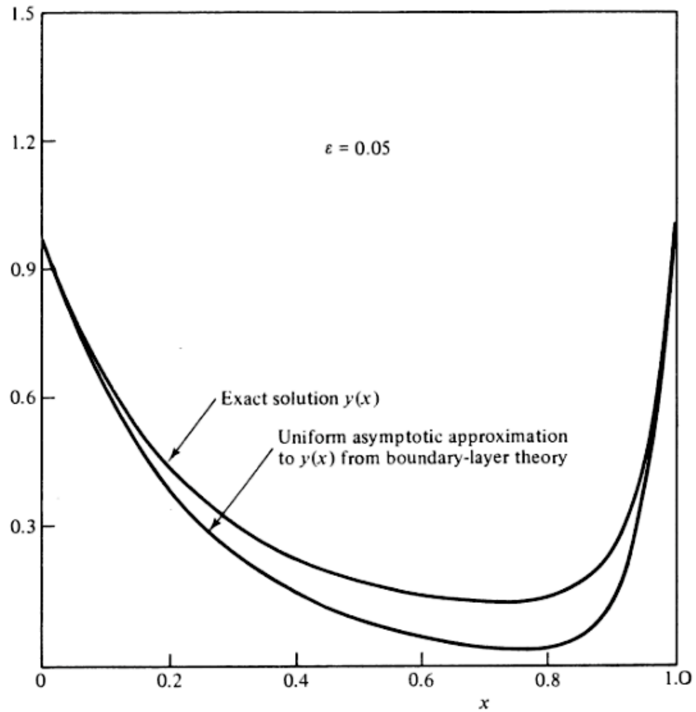

Figure 9.9 A comparison of the exact solution $y(x)$ to the boundary-value problem in (9.4.4) with the leading uniform asymptotic approximation to $y(x)$ from boundary-layer theory (9.4.8). For this plot $\varepsilon = 0.05$.

then, a boundary layer can and indeed does occur at $x = 0$. As in the previous example, the boundary layer at $x = 0$ has thickness $\delta \sim \varepsilon^{1/2}$.

In general, whenever a boundary-layer thickness is not of order ε, the matching of the inner and outer solutions is affected in a peculiar way. Recall that in the first example of Sec. 9.3, where the boundary-layer thickness was ε, the first term of the inner solution and the first term of the outer solution were matched. Next, we matched the first two terms of the inner and outer solutions and, finally, we matched the first three terms of each solution. In general, an nth-order match in this problem consists of matching the first $(n + 1)$ terms of the inner and outer solutions.

In the present example, as in the previous example, the boundary layer is comparatively thick ($\varepsilon \ll \delta = \varepsilon^{1/2}$ as $\varepsilon \to 0+$), so many more terms of the inner solution are required to describe its behavior in the matching region. Specifically, when the boundary-layer thickness is $\varepsilon^{1/2}$, $(2n - 1)$ terms of the inner expansion are required to match to n terms of the outer expansion. This and the next example will illustrate this phenomenon.

We begin the analysis by assuming an outer expansion of the form

$$y_{\text{out}}(x) \sim y_0(x) + \varepsilon y_1(x) + \cdots, \qquad \varepsilon \to 0+.$$

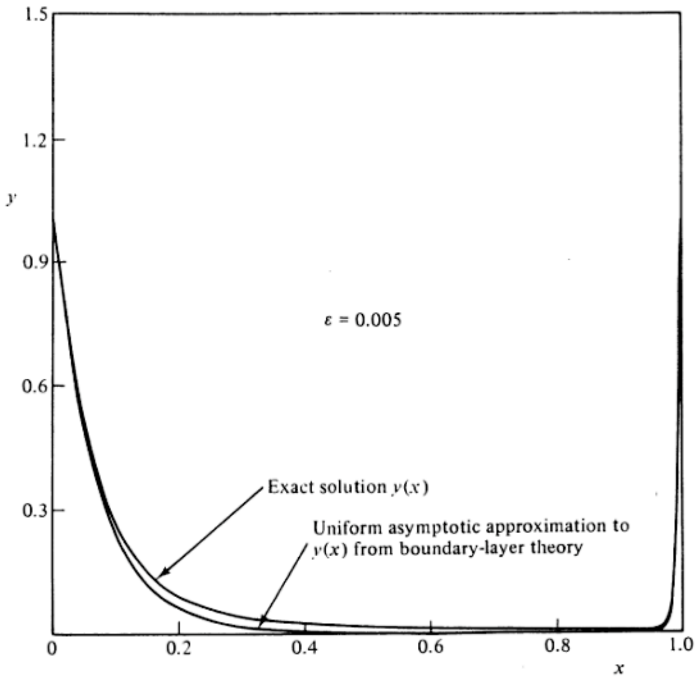

Figure 9.10 Same as in Fig. 9.9 except that $\varepsilon = 0.005$. Observe that the left boundary layer, whose thickness is of order $\varepsilon^{1/2}$, is much larger than the right boundary layer, whose thickness is of order ε.

This series leads to the sequence of equations

$$x^2 y_0' - y_0 = 0, \qquad x^2 y_n' - y_n = -y_{n-1}'', \qquad n \ge 1,$$

whose solutions are

$$\begin{aligned} y_0(x) &= C_0 e^{-1/x}, \\ y_1(x) &= C_0(\tfrac{1}{5}x^{-5} - \tfrac{1}{2}x^{-4})e^{-1/x} + C_1 e^{-1/x}, \end{aligned} \tag{9.4.10}$$

and so on. Since the coefficient of y' in (9.4.9) is positive at $x = 1$, we conclude from Example 3 of Sec. 9.1 that there cannot be a boundary layer at $x = 1$ when $\varepsilon \to 0+$. Thus, $y_{\text{out}}(1) = 1$ and the constants $C_0, C_1, \ldots$ are all determined: $C_0 = e$, $C_1 = 3e/10, \ldots$.

A boundary layer is required at $x = 0$ because the outer solution does not itself satisfy the boundary condition $y(0) = 1$. To reveal the structure of this boundary layer, we introduce the inner variable $X = x/\delta$. In terms of X, (9.4.9) becomes

$$\frac{\varepsilon}{\delta^2}\frac{d^2 Y_{\text{in}}}{dX^2} + \delta X^2 \frac{dY_{\text{in}}}{dX} - Y_{\text{in}} = 0,$$

where $y_{\text{in}}(x) = Y_{\text{in}}(X)$. As in the previous example, the only consistent distinguished limit gives $\delta = \varepsilon^{1/2}$, and with this choice (9.4.9) becomes

$$\frac{d^2 Y_{\text{in}}}{dX^2} - Y_{\text{in}} = -\varepsilon^{1/2} X^2 \frac{dY_{\text{in}}}{dX}. \tag{9.4.11}$$

Since the small parameter in this equation is $\varepsilon^{1/2}$ and not ε, the appropriate perturbation expansion of the inner solution is

$$Y_{\text{in}}(X) \sim Y_0(X) + \varepsilon^{1/2} Y_1(X) + \varepsilon Y_2(X) + \cdots, \qquad \varepsilon \to 0+. \tag{9.4.12}$$

Substituting (9.4.12) into (9.4.11) gives

$$\frac{d^2 Y_0}{dX^2} - Y_0 = 0, \qquad \frac{d^2 Y_n}{dX^2} - Y_n = -X^2 \frac{dY_{n-1}}{dX}, \qquad n \ge 1.$$

The solutions to these equations for $n = 0$, 1, and 2 have the form

$$\begin{aligned} Y_0(X) &= A_0 e^X + B_0 e^{-X}, \\ Y_1(X) &= A_0(-\tfrac{1}{6}X^3 + \tfrac{1}{4}X^2 - \tfrac{1}{4}X)e^X \\ &\quad + B_0(-\tfrac{1}{6}X^3 - \tfrac{1}{4}X^2 - \tfrac{1}{4}X)e^{-X} + A_1 e^X + B_1 e^{-X}, \\ Y_2(X) &= A_0(\tfrac{1}{72}X^6 - \tfrac{1}{60}X^5 + \tfrac{1}{96}X^4 + \tfrac{1}{48}X^3 - \tfrac{1}{32}X^2 + \tfrac{1}{32}X)e^X \\ &\quad + B_0(\tfrac{1}{72}X^6 + \tfrac{1}{60}X^5 + \tfrac{1}{96}X^4 - \tfrac{1}{48}X^3 - \tfrac{1}{32}X^2 - \tfrac{1}{32}X)e^{-X} \\ &\quad + A_1(-\tfrac{1}{6}X^3 + \tfrac{1}{4}X^2 - \tfrac{1}{4}X)e^X + B_1(-\tfrac{1}{6}X^3 - \tfrac{1}{4}X^2 - \tfrac{1}{4}X)e^{-X} \\ &\quad + A_2 e^X + B_2 e^{-X}. \end{aligned} \tag{9.4.13}$$

It is not possible to match the inner and outer solutions unless $A_0 = A_1 = A_2 = \cdots = 0$; exponentially growing terms are discarded because they blow up in the intermediate limit $x \to 0$, $X \to \infty$, $\varepsilon \to 0+$. The remaining constants $B_0, B_1, \ldots$ are determined by the boundary condition $Y_{\text{in}}(0) = y(0) = 1$: $B_0 = 1$, $B_1 = B_2 = \cdots = 0$. It is rather remarkable that the outer solution (9.4.10) and the inner solution (9.4.13) match asymptotically in the overlap region $X \to \infty$, $x \to 0$, $\varepsilon \to 0+$ for *all* values of the constants $C_0, C_1, \ldots, B_0, B_1, \ldots$, so long as $A_0 = A_1 = \cdots = 0$, because both the inner and outer solutions are exponentially small. There is no interaction between the outer solution and the inner solution in this problem!

To leading order, the uniform asymptotic approximation to $y(x)$ for $0 \le x \le 1$ is

$$y_{\text{unif},0} = e^{1-1/x} + \exp(-x/\sqrt{\varepsilon}) + O(\varepsilon^{1/2}), \qquad \varepsilon \to 0+. \tag{9.4.14}$$

To first order in ε we need one additional term from the outer expansion and *two* additional terms from the inner expansion to construct the uniform expansion. The result is

$$\begin{aligned} y_{\text{unif},1} &= e^{1-1/x}\left(1 - \frac{1}{2}\varepsilon x^{-4} + \frac{1}{5}\varepsilon x^{-5} + \frac{3}{10}\varepsilon\right) \\ &\quad + \exp(-x/\sqrt{\varepsilon})\left(1 - \frac{x^3}{6\varepsilon} - \frac{x^2}{4\sqrt{\varepsilon}} - \frac{x}{4} + \frac{x^6}{72\varepsilon^2} + \frac{x^5}{60\varepsilon^{3/2}}\right. \\ &\quad \left. + \frac{x^4}{96\varepsilon} - \frac{x^3}{48\sqrt{\varepsilon}} - \frac{x^2}{32} - \frac{x\sqrt{\varepsilon}}{32}\right) + O(\varepsilon^{3/2}), \qquad \varepsilon \to 0+. \end{aligned} \tag{9.4.15}$$

In general, it is necessary to compute two new terms in the inner expansion for every new term in the outer expansion.

Figures 9.11 to 9.13 compare the above two uniform approximations to $y(x)$ with the actual numerical solution to $y(x)$ for $\varepsilon = 0.05$, 0.01, 0.001. Note that $y_{\text{unif},1}$ does not approximate $y(x)$ better than $y_{\text{unif},0}$ until ε is as small as 0.001.

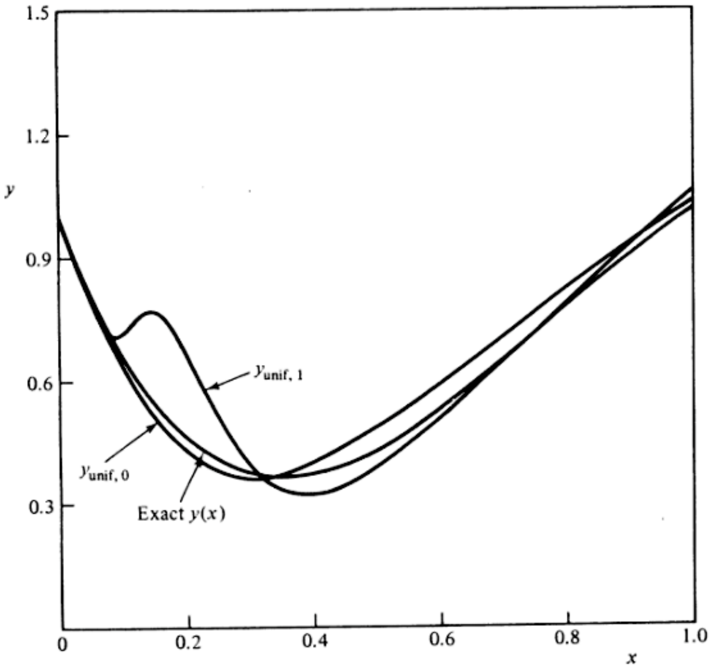

Figure 9.11 Comparison of exact and approximate solutions to $\varepsilon y'' + x^2 y' - y = 0$ [$y(0) = y(1) = 1$] for $\varepsilon = 0.05$. The two approximate solutions $y_{\text{unif},0}(x)$ and $y_{\text{unif},1}(x)$ are derived using boundary-layer theory and are given in (9.4.14) and (9.4.15). Note that even for this small value of ε, $y_{\text{unif},0}(x)$ is a better approximation to $y(x)$ than $y_{\text{unif},1}(x)$. Note that the higher-order approximation $y_{\text{unif},1}(x)$ crosses the exact solution $y(x)$ more frequently than $y_{\text{unif},0}(x)$; for very small ε, $y_{\text{unif},1}(x)$ hugs the curve $y(x)$ more closely than $y_{\text{unif},0}(x)$ (see Figs. 9.12 and 9.13).

Example 3 *Boundary-layer problem involving* ln ε. Consider the singular perturbation problem

$$\varepsilon y'' + xy' - xy = 0, \qquad y(0) = 0,\ y(1) = e. \tag{9.4.16}$$

Again, in this example there is a boundary layer at $x = 0$ whose thickness is of order $\varepsilon^{1/2}$, and not ε. However, the novelty of this example is that the inner expansion is not just a series in powers of $\varepsilon^{1/2}$. Terms containing ln ε also appear.

The outer expansion is obtained by assuming that

$$y_{\text{out}}(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots, \qquad \varepsilon \to 0+. \tag{9.4.17}$$

Substituting (9.4.17) into (9.4.16) gives $y_0' - y_0 = 0$, $xy_n' - xy_n = -y_{n-1}''$ $(n \ge 1)$. A boundary layer may appear at $x = 0$ but not at $x = 1$, so the boundary condition satisfied by $y_{\text{out}}(x)$ is $y_{\text{out}}(1) = e$. The resulting outer solution is

$$y_{\text{out}}(x) \sim e^x - \varepsilon e^x \ln x + \varepsilon^2 e^x \left[\frac{1}{2}(\ln x)^2 - \frac{2}{x} + \frac{1}{2x^2} + \frac{3}{2}\right] + \cdots, \qquad \varepsilon \to 0+. \tag{9.4.18}$$

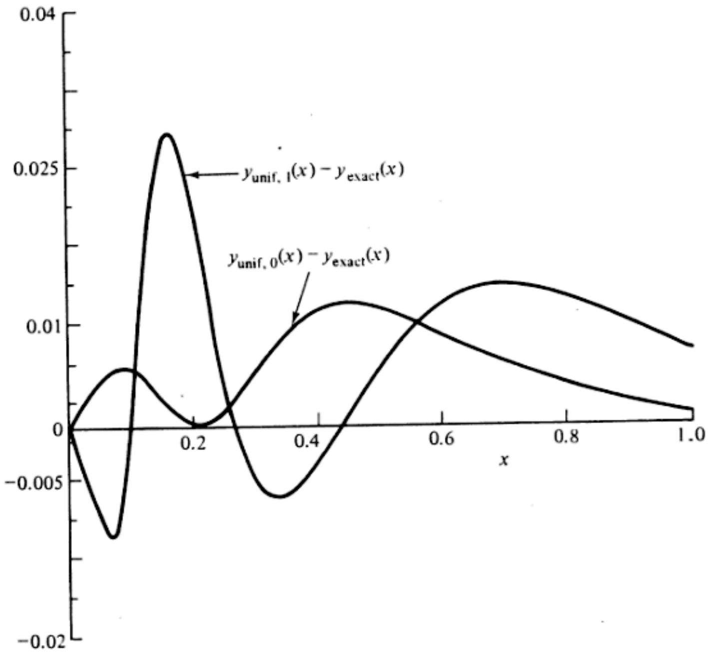

Figure 9.12 Comparison of the errors (not the relative errors) between the exact solution of $\varepsilon y'' + x^2 y' - y = 0$ [$y(0) = y(1) = 1$] for $\varepsilon = 0.01$ and the zeroth- and first-order uniform approximations to $y_{\text{exact}}(x)$ (see Fig. 9.11). Note that $\varepsilon = 0.01$ is still too large for $y_{\text{unif},1}(x)$ to be a better approximation than $y_{\text{unif},0}(x)$.

Since $y_{\text{out}}(0) \ne 0$, there must be a boundary layer in the neighborhood of $x = 0$. Substituting (9.4.2) into (9.4.16) gives

$$\frac{\varepsilon}{\delta^2}\frac{d^2 Y_{\text{in}}}{dX^2} + X\frac{dY_{\text{in}}}{dX} - \delta X Y_{\text{in}} = 0.$$

Thus, if $\delta(\varepsilon)/\varepsilon^{1/2} \to 0$ or ∞ as $\varepsilon \to 0$, matching to the boundary condition $Y_{\text{in}}(0) = 0$ and the outer solution for $X \to \infty$ would be impossible. The distinguished limit is $\delta = \varepsilon^{1/2}$, so

$$\frac{d^2 Y_{\text{in}}}{dX^2} + X\frac{dY_{\text{in}}}{dX} - \varepsilon^{1/2} X Y_{\text{in}} = 0. \tag{9.4.19}$$

We would like to represent $Y_{\text{in}}(X)$ as a perturbation series, and in (9.4.19) the small parameter is $\varepsilon^{1/2}$ (and not ε). Thus, it would seem reasonable to assume an expansion of the form

$$Y_{\text{in}}(X) \sim Y_0(X) + \varepsilon^{1/2} Y_1(X) + \varepsilon Y_2(X) + \cdots, \qquad \varepsilon \to 0+, \tag{9.4.20}$$

where the boundary condition $y(0) = 0$ becomes $Y_n(0) = 0$ $(n \ge 0)$.

Substituting the expansion (9.4.20) into (9.4.19) gives $Y_0'' + XY_0' = 0$, whose solution is

$$Y_0(X) = A_0 \int_0^X e^{-t^2/2}\, dt, \tag{9.4.21}$$

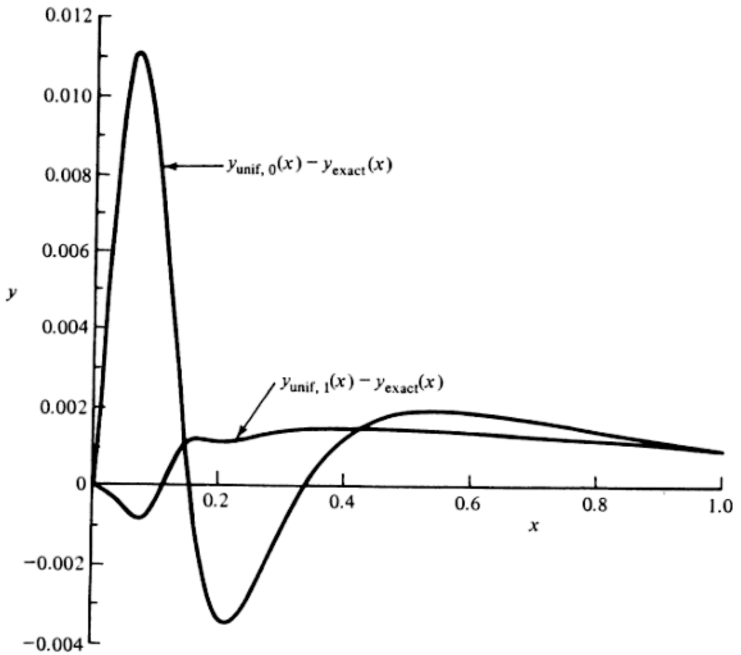

Figure 9.13 Same as in Fig. 9.12 except that $\varepsilon = 0.001$. This value of ε is sufficiently small for $y_{\text{unif},\,1}(x)$ to be a better approximation to $y_{\text{exact}}(x)$ than $y_{\text{unif},\,0}(x)$.

where A_0 is a constant to be determined by matching. Also, $Y_n'' + XY_n' = XY_{n-1}$ $(n \geq 1)$, so the relation between Y_n and Y_{n-1} is

$$Y_n(X) = A_n \int_0^X e^{-t^2/2}\, dt + \int_0^X e^{-t^2/2}\, dt \int_0^t e^{s^2/2} s Y_{n-1}(s)\, ds. \tag{9.4.22}$$

These integrals cannot be evaluated in closed form; nevertheless, we can still match the inner and outer solutions.

Matching is done by taking the intermediate limit $\varepsilon \to 0+$, $x \to 0+$, $X \to \infty$. To leading order, the outer solution (9.4.18) becomes $y_{\text{out}}(x) = 1 + \mathrm{O}(x, \varepsilon \ln x)$, $(\varepsilon/x^2 \to 0,\ x \to 0+)$, while the leading-order inner solution is $Y_0(X) \sim \sqrt{\pi/2}\, A_0$ $(\varepsilon \to 0+,\ X \to \infty)$ with exponentially small corrections. It follows that $A_0 = \sqrt{2/\pi}$. Thus, to leading order, a uniform approximation to the solution of (9.4.16) is

$$y_{\text{unif},\,0}(x) = e^x + \sqrt{\frac{2}{\pi}} \int_0^X e^{-t^2/2}\, dt - 1 = e^x - \sqrt{\frac{2}{\pi}} \int_X^\infty e^{-t^2/2}\, dt. \tag{9.4.23}$$

A comparison between this approximation and the exact solution to (9.4.16) is given in Fig. 9.14.

To second order, the intermediate limit of the outer expansion is

$$\begin{aligned} y_{\text{out}}(x) \sim{} & 1 + x + \tfrac{1}{2}x^2 - \varepsilon \ln x + \tfrac{1}{2}\varepsilon^2 x^{-2} + \tfrac{1}{6}x^3 - \varepsilon x \ln x \\ & - \tfrac{3}{2}\varepsilon^2 x^{-1} + \mathrm{O}(x^4, \varepsilon x^2 \ln x, \varepsilon^2 (\ln x)^2, \varepsilon^3/x^4), \qquad \varepsilon/x^2 \to 0+,\ x \to 0+. \end{aligned} \tag{9.4.24}$$

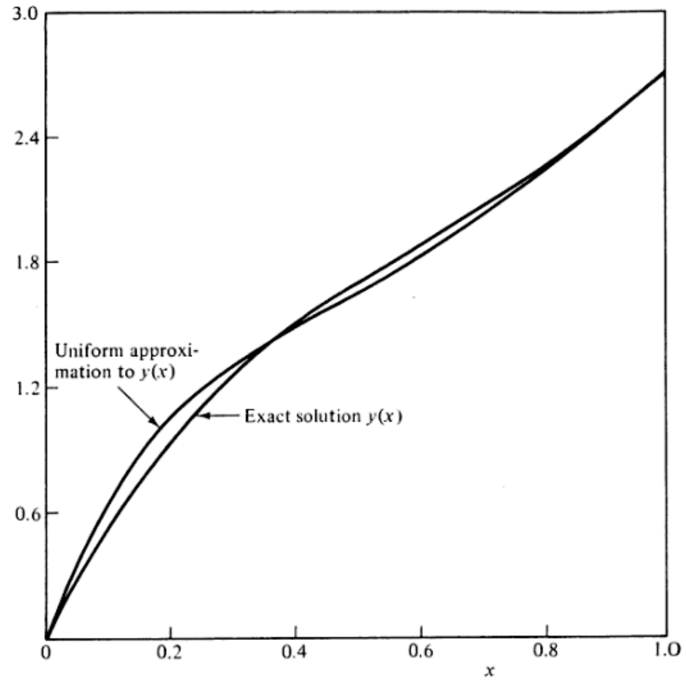

Figure 9.14 Comparison between the exact solution $y(x)$ to $\varepsilon y'' + xy' - xy = 0$ $[y(0) = 0,\ y(1) = e]$ and the lowest-order uniform approximation to $y(x)$ obtained from boundary-layer analysis [see (9.4.23)]. The value of ε is 0.05.

We must also compute the intermediate limit $X \to \infty$ of the inner expansion:

$$\frac{dY_1}{dX} = A_1 e^{-X^2/2} + e^{-X^2/2} \int_0^X e^{s^2/2} s Y_0(s)\, ds \sim \sqrt{\frac{\pi}{2}} A_0, \qquad X \to \infty,$$

to within exponentially small terms. Therefore,

$$Y_1(X) \sim \sqrt{\frac{\pi}{2}} A_0 X + C_1, \qquad X \to \infty,$$

to within exponentially small terms, where C_1 is an integration constant which subsumes the constant A_1 in (9.4.22). Proceeding similarly in next order, we find that

$$Y_2(X) \sim \sqrt{\frac{\pi}{2}} A_0(\tfrac{1}{2}X^2 - \ln X + \tfrac{1}{2}X^{-2}) + C_1 X + C_2 + \mathrm{O}(X^{-3}), \qquad X \to \infty,$$

and

$$\begin{aligned} Y_3(X) \sim{} & \sqrt{\frac{\pi}{2}} A_0(\tfrac{1}{6}X^3 - X \ln X - \tfrac{3}{2}X^{-1}) + C_1(\tfrac{1}{2}X^2 - \ln X) \\ & + C_2 X + C_3 + \mathrm{O}(X^{-2}), \qquad X \to \infty, \end{aligned}$$

where C_2 and C_3 are integration constants.

The matching condition on $y_{\text{out}}(x)$ and $Y_{\text{in}}(X)$ is therefore

$$\begin{aligned}
&1 + x + \frac{1}{2}x^2 - \varepsilon \ln x + \frac{1}{2}\varepsilon^2 x^{-2} + \frac{1}{6}x^3 - \varepsilon x \ln x - \frac{3}{2}\varepsilon^2 x^{-1} \\
&= \sqrt{\frac{\pi}{2}} A_0 \left(1 + \varepsilon^{1/2} X + \frac{1}{2}\varepsilon X^2 - \varepsilon \ln X + \frac{1}{2}\varepsilon X^{-2} + \frac{1}{6}\varepsilon^{3/2} X^3 - \varepsilon^{3/2} X \ln X - \frac{3}{2}\varepsilon^{3/2} X^{-1}\right) \\
&\quad + \varepsilon^{1/2} C_1 \left(1 + \varepsilon^{1/2} X + \frac{1}{2}\varepsilon X^2 - \varepsilon \ln X\right) + \varepsilon C_2 (1 + \varepsilon^{1/2} X) \\
&\quad + \varepsilon^{3/2} C_3 + \mathrm{O}(x^4, \varepsilon x^2 \ln x, \varepsilon^2 (\ln x)^2, \varepsilon^3/x^4), \qquad x \to 0+, \ \varepsilon/x^2 \to 0+,
\end{aligned} \tag{9.4.25}$$

with $X = x\varepsilon^{-1/2}$. However, matching is *not* possible here because, when the right side of (9.4.25) is rewritten in terms of $x = \varepsilon^{1/2}X$, terms of order $\varepsilon \ln \varepsilon$ appear!

Apparently, the assumption that the inner solution is a series in powers of $\varepsilon^{1/2}$ is naive. Instead, the inner expansion in (9.4.20) must be modified to read

$$\begin{aligned}
Y_{\text{in}}(X) &\sim Y_0(X) + \varepsilon^{1/2} Y_1(X) + \varepsilon Y_2(X) + \varepsilon \ln \varepsilon \bar{Y}_2(X) + \varepsilon^{3/2} Y_3(X) \\
&\quad + \varepsilon^{3/2} \ln \varepsilon \bar{Y}_3(X) + \cdots, \qquad \varepsilon \to 0+.
\end{aligned}$$

Terms of order $\varepsilon^2 \ln \varepsilon$, $\varepsilon^2(\ln \varepsilon)^2$, and so on, must also be included in higher order.

In the intermediate matching region the additional terms contribute to the right side of (9.4.25) as

$$\varepsilon \ln \varepsilon \sqrt{\frac{\pi}{2}} \bar{A}_2 (1 + \varepsilon^{1/2} X) + \varepsilon^{3/2} \ln \varepsilon \bar{C}_3,$$

where $\bar{C}_3$ is a new integration constant. Now the match in (9.2.25) can be accomplished. The matching conditions are $A_0 = \sqrt{2/\pi}$, $\bar{A}_2 = -\frac{1}{2}A_0$, $C_1 = C_2 = C_3 = \bar{C}_3 = 0$.

(I) 9.5 MISCELLANEOUS EXAMPLES OF LINEAR BOUNDARY-LAYER PROBLEMS

This section is a collection of six examples of linear differential-equation boundary-value problems which can be solved approximately using boundary-layer theory. We have selected these problems because they illustrate the broad spectrum of analysis that boundary-layer theory entails. The first two examples involve higher-order differential equations.

Example 1 *Third-order boundary-value problem.* Consider the third-order boundary-value problem

$$\varepsilon y'''(x) - y'(x) + xy(x) = 0, \qquad y(0) = y'(0) = y(1) = 1, \tag{9.5.1}$$

in the limit $\varepsilon \to 0+$. The novelty of this tricky third-order problem is that boundary layers occur at both $x = 0$ and $x = 1$.

In the outer region which contains no boundary layers, we assume an expansion of the form $y_{\text{out}}(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$ ($\varepsilon \to 0+$). This reduces (9.5.1) to a sequence of first-order differential equations:

$$y_n' - xy_n = \begin{cases} 0, & n = 0, \\ y_{n-1}''', & n > 0. \end{cases} \tag{9.5.2}$$

We find that

$$y_0(x) = a_0 e^{x^2/2}, \tag{9.5.3}$$

$$y_1(x) = [a_0(x^4/4 + 3x^2/2) + a_1]e^{x^2/2}, \tag{9.5.4}$$

and so on.

Next, we consider the possibility of a boundary layer of thickness δ at $x = 0$. In terms of the inner variable $X = x/\delta$ with $Y_{\text{in}}(X) = y(x)$, (9.5.1) becomes

$$\frac{\varepsilon}{\delta^3}\frac{d^3 Y_{\text{in}}}{dX^3} - \frac{1}{\delta}\frac{dY_{\text{in}}}{dX} + \delta X Y_{\text{in}} = 0. \tag{9.5.5}$$

There are two consistent distinguished limits, $\delta = \mathrm{O}(1)$ and $\delta = \mathrm{O}(\varepsilon^{1/2})$ ($\varepsilon \to 0+$). However, $\delta = \mathrm{O}(1)$ reproduces the outer limit. Thus, the only consistent choice for the boundary layer thickness is $\delta = \varepsilon^{1/2}$. For this choice (9.5.5) reduces to

$$\frac{d^3 Y_{\text{in}}}{dX^3} - \frac{dY_{\text{in}}}{dX} = -\varepsilon X Y_{\text{in}}. \tag{9.5.6}$$

The next step is to approximate Y_{in} by an appropriate inner expansion. However, as we will see, the most obvious choice for an inner expansion,

$$Y_{\text{in}}(X) \sim Y_0(X) + \varepsilon Y_1(X) + \cdots \qquad \varepsilon \to 0+, \tag{9.5.7}$$

is inadequate. Substituting (9.5.7) into (9.5.6) gives

$$\frac{d^3 Y_n}{dX^3} - \frac{dY_n}{dX} = \begin{cases} 0, & n = 0, \\ -XY_{n-1}, & n > 0, \end{cases}$$

whose solutions are

$$Y_0(X) = A_0 e^X + B_0 e^{-X} + C_0, \tag{9.5.8}$$

$$\begin{aligned}
Y_1(X) &= [-A_0(X^2/4 - 3X/4) + A_1]e^X \\
&\quad + [-B_0(X^2/4 + 3X/4) + B_1]e^{-X} + C_0 X^2/2 + C_1,
\end{aligned} \tag{9.5.9}$$

and so on. The boundary conditions $y(0) = y'(0) = 1$ become

$$Y_{\text{in}}(0) = 1, \qquad \frac{dY_{\text{in}}}{dX}(0) = \varepsilon^{1/2}. \tag{9.5.10}$$

Note that $Y_{\text{in}}(0) = 1$ implies that $A_0 + B_0 + C_0 = 1$, $A_1 + B_1 + C_1 = 0$, and so on. However, it is not possible to satisfy the condition $Y_{\text{in}}'(0) = \varepsilon^{1/2}$ for any choice of constants! Apparently, our choice of inner expansion in (9.5.7) was wrong. There is no way to represent $Y_{\text{in}}'(0) = \varepsilon^{1/2}$ by an expansion in integral powers of ε.

We therefore revise the inner expansion to read

$$Y_{\text{in}}(X) \sim Y_0(X) + \varepsilon^{1/2} Y_{1/2}(X) + \varepsilon Y_1(X) + \cdots, \qquad \varepsilon \to 0+, \tag{9.5.11}$$

where Y_0 and Y_1 are still given by (9.5.8) and (9.5.9) and

$$Y_{1/2}(X) = A_{1/2} e^X + B_{1/2} e^{-X} + C_{1/2}. \tag{9.5.12}$$

Note that the equations for Y_n with n integral and n half integral decouple because $\varepsilon^{1/2}$ does not appear explicitly in (9.5.6).

The boundary conditions (9.5.10) can now be satisfied:

$$\begin{aligned}
&Y_0(0)=1: && A_0+B_0+C_0=1,\\
&Y_0'(0)=0: && A_0-B_0=0,\\
&Y_{1/2}(0)=0: && A_{1/2}+B_{1/2}+C_{1/2}=0,\\
&Y_{1/2}'(0)=1: && A_{1/2}-B_{1/2}=1,\\
&Y_1(0)=0: && A_1+B_1+C_1=0,\\
&Y_1'(0)=0: && A_1-B_1=0.
\end{aligned} \tag{9.5.13}$$

It is also necessary to require that $A_0 = A_{1/2} = A_1 = \cdots = 0$. (Otherwise, each term in the inner expansion would contain terms that grow exponentially and this would prevent the inner and outer expansions from being matched.) Combining this requirement with (9.5.13) gives $A_0 = B_0 = A_{1/2} = A_1 = B_1 = C_1 = 0$, $C_0 = C_{1/2} = 1$, $B_{1/2} = -1$. Thus, our final expression for $Y_{\rm in}(X)$ correct to order ε is

$$Y_{\rm in}(X) \sim 1 + \sqrt{\varepsilon}(1 - e^{-X}) + \varepsilon X^2/2 + \cdots, \qquad \varepsilon \to 0+. \tag{9.5.14}$$

In the overlap region defined by $x \to 0+$, $X \to \infty$, $\varepsilon \to 0+$, we have

$$\begin{aligned}
y_{\rm out}(x) &= a_0 + a_0 x^2/2 + \varepsilon a_1 + O(x^4) + O(\varepsilon x^2) + O(\varepsilon^2), && x \to 0+,\ \varepsilon \to 0+,\\
Y_{\rm in}(X) &= 1 + \varepsilon^{1/2} + \varepsilon X^2/2 + O(\varepsilon^{3/2}X^2), && X \to \infty,\ \varepsilon^{1/2}X \to 0+
\end{aligned}$$

and again there is trouble! Matching is impossible because there is no $\varepsilon^{1/2}$ term in the expansion of $y_{\rm out}(x)$. Apparently, it is necessary to include $\varepsilon^{n/2}$ terms in the outer expansion as well as the inner expansion. The outer expansion must be generalized to

$$y_{\rm out}(x) \sim y_0(x) + \varepsilon^{1/2} y_{1/2}(x) + \varepsilon y_1(x) + \cdots, \qquad \varepsilon \to 0+,$$

and from (9.5.1) we have $y_{1/2}(x) = a_{1/2} e^{x^2/2}$. The new (and now correct) outer expansion in the matching region is

$$y_{\rm out}(x) = a_0 + a_0 x^2/2 + a_{1/2}\varepsilon^{1/2} + a_1 \varepsilon + O(x^4) + O(\varepsilon^{1/2}x^2) + O(\varepsilon^{3/2}), \qquad x \to 0+,\ \varepsilon \to 0+.$$

The inner and outer expansions now match and the matching condition is $a_0 = 1$, $a_{1/2} = 1$, $a_1 = 0$. Thus, our final expression for $y_{\rm out}(x)$ correct to order ε is

$$y_{\rm out}(x) \sim e^{x^2/2}[1 + \varepsilon^{1/2} + \varepsilon(x^4/4 + 3x^2/2) + \cdots], \qquad \varepsilon \to 0+. \tag{9.5.15}$$

This outer solution does not satisfy the boundary condition $y(1) = 1$. Thus, there is another boundary layer at $x = 1$. Again, the only distinguished limit is $\delta = \varepsilon^{1/2}$. Thus, the appropriate inner variable is $\bar{X} = (1 - x)\varepsilon^{-1/2}$. Letting $\bar{Y}_{\rm in}(\bar{X}) = y(x)$, (9.5.1) becomes

$$\frac{d^3\bar{Y}_{\rm in}}{d\bar{X}^3} - \frac{d\bar{Y}_{\rm in}}{d\bar{X}} = \varepsilon^{1/2}\bar{Y}_{\rm in} - \varepsilon\bar{X}\bar{Y}_{\rm in}.$$

This suggests an inner expansion of the form

$$\bar{Y}_{\rm in}(\bar{X}) = \bar{Y}_0(\bar{X}) + \varepsilon^{1/2}\bar{Y}_{1/2}(\bar{X}) + \varepsilon\bar{Y}_1(\bar{X}) + \cdots. \tag{9.5.16}$$

The remainder of this problem is routine. We solve for $\bar{Y}_0$, $\bar{Y}_{1/2}$, and $\bar{Y}_1$ by imposing the boundary condition $y(1) = 1$ and matching $\bar{Y}_{\rm in}(\bar{X})$ to $y_{\rm out}(x)$. The results are given in Prob. 9.21. By combining these results with (9.5.14) and (9.5.15), we obtain the following uniform asymptotic

approximation to the solution of (9.5.1) correct to terms of order ε:

$$\begin{aligned}
y_{\rm unif}(x) = {}& e^{x^2/2}[1 + \varepsilon^{1/2} + \varepsilon(x^4/4 + 3x^2/2)] - \sqrt{\varepsilon}\exp(-x/\sqrt{\varepsilon})\\
&+ \exp[-(1-x)/\sqrt{\varepsilon}][(\sqrt{e} - 1)(\tfrac{1}{8}x^2 + \tfrac{1}{4}x - \tfrac{11}{8})\\
&+ \sqrt{\varepsilon}(-9\sqrt{e} - 3 + 3x + x\sqrt{e})/8 - \tfrac{7}{4}\varepsilon\sqrt{e}].
\end{aligned} \tag{9.5.17}$$

In Fig. 9.15 we compare the exact solution to (9.5.1) with the uniform asymptotic approximation to $y(x)$ in (9.5.17).

Example 2 *Fourth-order boundary-value problem.* Consider the inhomogeneous fourth-order boundary-value problem

$$\varepsilon^2 \frac{d^4y}{dx^4} - (1 + x)\frac{d^2y}{dx^2} = 1, \qquad y(0) = y'(0) = y(1) = y'(1) = 1, \tag{9.5.18}$$

in the limit $\varepsilon \to 0+$. As in the previous example, we will see that boundary layers occur at $x = 0$ and at $x = 1$.

In the outer region the $\varepsilon^2\, d^4y/dx^4$ term is small, so we are inclined to use an outer expansion in powers of ε^2:

$$y_{\rm out}(x) \sim y_0(x) + \varepsilon^2 y_2(x) + \cdots, \qquad \varepsilon \to 0+. \tag{9.5.19}$$

However, our experience from the previous example suggests that this choice may be naive. To determine the correct form for the outer expansion, let us first examine the boundary layers at $x = 0$ and at $x = 1$.

We take the inner limit in the neighborhood of $x = 0$ by introducing the inner variable

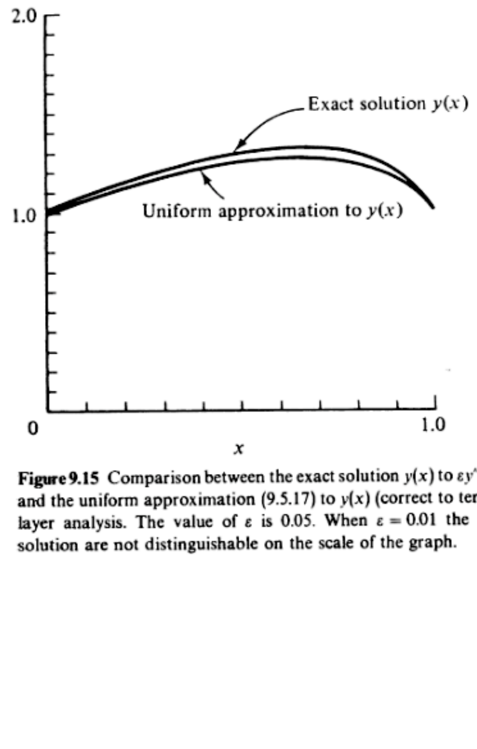

Figure 9.15 Comparison between the exact solution $y(x)$ to $\varepsilon y''' - y' + xy = 0\,[y(0) = y'(0) = y(1) = 1]$ and the uniform approximation (9.5.17) to $y(x)$ (correct to terms of order ε) obtained from boundary-layer analysis. The value of ε is 0.05. When $\varepsilon = 0.01$ the uniform approximation and the exact solution are not distinguishable on the scale of the graph.

$X = x/\delta$ and setting $Y_{\text{in}}(X) = y(x)$. In the inner region (9.5.18) becomes

$$\frac{\varepsilon^2}{\delta^4}\frac{d^4 Y_{\text{in}}}{dX^4} - \frac{1+\delta X}{\delta^2}\frac{d^2 Y_{\text{in}}}{dX^2} = 1.$$

The distinguished limits are $\delta = 1$, which reproduces the outer limit, and $\delta = \varepsilon$. Setting $\delta = \varepsilon$, we have

$$\frac{d^4 Y_{\text{in}}}{dX^4} - (1+\varepsilon X)\frac{d^2 Y}{dX^2} = \varepsilon^2, \tag{9.5.20}$$

which suggests an inner expansion in powers of ε:

$$Y_{\text{in}}(X) \sim Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) + \cdots, \qquad \varepsilon \to 0+. \tag{9.5.21}$$

We assume an inner expansion in powers of ε and not ε^2 for two reasons. First, there is a term containing ε in (9.5.20). Second, the boundary conditions at $x = 0$ when written in terms of the inner variable X read $Y_{\text{in}}(0) = 1$, $dY_{\text{in}}/dX(0) = \varepsilon$.

Substituting (9.5.21) into (9.5.20) gives the following equations and boundary conditions:

$$\frac{d^4 Y_0}{dX^4} - \frac{d^2 Y_0}{dX^2} = 0, \qquad Y_0(0) = 1,\ Y_0'(0) = 0;$$

$$\frac{d^4 Y_1}{dX^4} - \frac{d^2 Y_1}{dX^2} = X\frac{d^2 Y_0}{dX^2}, \qquad Y_1(0) = 0,\ Y_1'(0) = 1.$$

The solutions to these equations are

$$Y_0(X) = 1 + A_0(e^X - 1 - X) + B_0(e^{-X} - 1 + X),$$

$$Y_1(X) = X + \tfrac{1}{4}A_0(X^2e^X - 5Xe^X + 5X) - \tfrac{1}{4}B_0(X^2e^{-X} + 5Xe^{-X} - 5X)$$
$$+ A_1(e^X - 1 - X) + B_1(e^{-X} - 1 + X).$$

However, these equations simplify considerably because matching to the outer solution is impossible if terms growing like e^X as $X \to +\infty$ are present. Therefore, we must require that $A_n = 0$ for all n:

$$\begin{aligned} Y_0(X) &= 1 + B_0(e^{-X} - 1 + X), \\ Y_1(X) &= X - \tfrac{1}{4}B_0(X^2e^{-X} + 5Xe^{-X} - 5X) + B_1(Xe^{-X} - 1 + X). \end{aligned} \tag{9.5.22}$$

In the matching region near $x = 0$, the intermediate limit is $x \to 0+$, $X = x/\varepsilon \to +\infty$. Thus, in the matching region (9.5.22) becomes

$$\begin{aligned} Y_0(X) &\sim 1 + B_0(X - 1), && X \to +\infty, \\ Y_1(X) &\sim X + \tfrac{5}{4}B_0 X + B_1(X - 1), && X \to +\infty, \end{aligned} \tag{9.5.23}$$

with exponentially small corrections.

A similar inner expansion can be made in the neighborhood of $x = 1$. The appropriate inner variable is $\bar{X} = (1 - x)/\varepsilon$. The first two terms of the inner solution are

$$\begin{aligned} \bar{Y}_0(\bar{X}) &= 1 + \bar{B}_0[\exp(-\sqrt{2}\,\bar{X}) - 1 + \sqrt{2}\,\bar{X}], \\ \bar{Y}_1(X) &= -\bar{X} + \tfrac{1}{8}\bar{B}_0[\sqrt{2}\,\bar{X}^2\exp(-\sqrt{2}\,\bar{X}) + 5\bar{X}\exp(-\sqrt{2}\,\bar{X}) - 5\bar{X}] \\ &\quad + \bar{B}_1[\exp(-\sqrt{2}\,\bar{X}) - 1 + \sqrt{2}\,\bar{X}], \end{aligned} \tag{9.5.24}$$

where the inner expansion is $\bar{Y}_{\text{in}}(\bar{X}) \sim \bar{Y}_0 + \varepsilon \bar{Y}_1 + \cdots$ $(\varepsilon \to 0+)$. In the matching region near $x = 1$, $x \to 1-$, $\bar{X} = (1-x)/\varepsilon \to +\infty$, so

$$\begin{aligned} \bar{Y}_0(\bar{X}) &\sim 1 + \bar{B}_0(-1 + \sqrt{2}\,\bar{X}), && \bar{X} \to +\infty, \\ \bar{Y}_1(\bar{X}) &\sim -\bar{X} - \tfrac{5}{8}\bar{B}_0\bar{X} + \bar{B}_1(-1 + \sqrt{2}\,\bar{X}), && \bar{X} \to +\infty. \end{aligned} \tag{9.5.25}$$

In terms of the outer variable x, (9.5.23) becomes

$$Y_{\text{in}}(X) = \frac{B_0 x}{\varepsilon} + 1 - B_0 + x + \frac{5}{4}B_0 x + B_1 x - \varepsilon B_1 + O(\varepsilon^2) + O(\varepsilon x) + O(x^2),$$
$$x \to 0+,\ x/\varepsilon \to +\infty, \tag{9.5.26}$$

and (9.5.25) becomes

$$\bar{Y}_{\text{in}}(\bar{X}) = \bar{B}_0\sqrt{2}\,\frac{1-x}{\varepsilon} + 1 - \bar{B}_0 - (1-x) - \frac{5}{8}\bar{B}_0(1-x) + \bar{B}_1\sqrt{2}(1-x)$$
$$- \bar{B}_1\varepsilon + O(\varepsilon^2) + O[\varepsilon(1-x)] + O[(x-1)^2],$$
$$x \to 1-,\ (1-x)/\varepsilon \to +\infty. \tag{9.5.27}$$

Now we can tell if the outer expansion in (9.5.19) is valid. If we substitute (9.5.19) into (9.5.18) and equate coefficients of like powers of ε, we obtain the leading-order outer equation $-(1+x)(d^2y_0/dx^2) = 1$. The solution to this equation is

$$y_0(x) = -(1+x)\ln(1+x) + a_0 x + b_0.$$

In the intermediate limit the outer solution behaves near $x = 0$ like

$$y_0(x) \sim b_0 + a_0 x - x + O(x^2), \qquad x \to 0+. \tag{9.5.28}$$

Demanding that (9.5.28) match asymptotically with (9.5.26) implies that $B_0 = 0$, $b_0 = 1$, $B_1 = 0$, $a_0 = 2$, which is consistent. However, demanding that $y_{\text{out}}(x)$ also match asymptotically with (9.5.27) gives different values for the constants a_0 and b_0! Near $x = 1$ we have

$$y_0(x) \sim -2\ln 2 + a_0 + b_0 + (1-x)(1 - a_0 + \ln 2) + O[\varepsilon^2, (1-x)^2, \varepsilon(1-x)],$$
$$\varepsilon \to 0+,\ x \to 1-. \tag{9.5.29}$$

Matching (9.5.29) to (9.5.27) gives $\bar{B}_0 = 0$, $\bar{B}_1 = 0$, $b_0 = \ln 2 - 1$, $a_0 = 2 + \ln 2$.

Apparently the outer expansion (9.5.19) is wrong because it is overdetermined. Even though ε^2 seems to be the natural expansion parameter for (9.5.18), matching requires that the outer expansion should in fact be done in powers of ε: $y_{\text{out}}(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$ $(\varepsilon \to 0+)$. With this expansion $y_0(x)$ is the same as before but

$$y_1(x) = a_1 x + b_1.$$

With these additional terms in the outer expansion the matching conditions at $x = 0$ become $B_0 = 0$, $b_0 = 1$, $1 + B_1 = a_0 - 1$, $b_1 = -B_1$. Similarly, the matching conditions at $x = 1$ become $\bar{B}_0 = 0$, $1 = a_0 + b_0 - 2\ln 2$, $1 - a_0 + \ln 2 = \sqrt{2}\,\bar{B}_1 - 1$, $-\bar{B}_1 = a_1 + b_1$. Thus, a consistent match at both $x = 0$ and $x = 1$ is possible if $B_0 = \bar{B}_0 = 0$, $b_0 = 1$, $a_0 = 2\ln 2$, $B_1 = 2\ln 2 - 2$, $b_1 = 2 - 2\ln 2$, $\sqrt{2}\,\bar{B}_1 = 2 - \ln 2$, $a_1 = (\ln 2)/\sqrt{2} - \sqrt{2} - 2 - 2\ln 2$, and so on.

Notice that to lowest order the outer solution becomes

$$y_{\text{out}}(x) = -(1+x)\ln(1+x) + 2x\ln 2 + 1 + O(\varepsilon), \qquad \varepsilon \to 0+, \tag{9.5.30}$$

which satisfies the boundary conditions $y_{\text{out}}(0) = y_{\text{out}}(1) = 1$. However, $y'_{\text{out}}(0) \neq 1$ and $y'_{\text{out}}(1) \neq 1$. Apparently, boundary layers appear at $x = 0$ and at $x = 1$ to adjust the slope of the outer solution to the imposed boundary values. The values of y' have an $O(1)$ jump across the boundary layers whose thickness is $O(\varepsilon)$. The outer solution satisfies the boundary conditions correct to $O(\varepsilon)$ because the jump in y across the boundary layers is $O(\varepsilon)$:

$$\text{Jump in } y \text{ at boundary layer} = O\left(\int_0^{\varepsilon} y'\,dx\right) = O(\varepsilon), \qquad \varepsilon \to 0+.$$

Similarly, we observe that the boundary layer at $x = 0$ in Example 1 has an $O(1)$ jump in y' but an $O(\varepsilon^{1/2})$ jump in y [see (9.5.15)].

In the next three examples we examine the boundary-layer structure of a singular differential equation.

Example 3 *Boundary-value problem having no boundary layers.* Consider the boundary-value problem

$$\varepsilon y'' + \frac{1}{x}y' + y = 0, \qquad y(1) = e^{-1/2},\ y(0) \text{ finite}, \tag{9.5.31}$$

in the limit $\varepsilon \to 0+$. The point $x = 0$ is a regular singular point of the differential equation (see Sec. 3.3). Frobenius' method gives the indicial exponents 0 and $1 - 1/\varepsilon$, so if $\varepsilon < 1$, the condition that $y(0)$ be finite suffices to determine the solution uniquely. (Why?) However, the condition on $y(0)$ is so weak that the solution does not exhibit a boundary layer at $x = 0$ as $\varepsilon \to 0+$, even though $1/x > 0$ for $x > 0$!

Suppose there were a boundary layer of thickness δ situated at $x = 0$. Then, we could introduce the inner variables $X = x/\delta$, $Y_{\text{in}}(X) = y(x)$ and rewrite the differential equation as

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y_{\text{in}}}{dX^2} + \frac{1}{\delta^2 X}\frac{dY_{\text{in}}}{dX} + Y_{\text{in}} = 0.$$

Observe that there is *no* distinguished limit for $\delta \ll 1$! The singularity of the differential equation at $x = 0$ ensures that the solution to (9.5.31) has no boundary layers at $x = 0$, despite first appearances. Therefore, a complete approximation of the solution to the problem on the interval $0 \le x \le 1$ is given by the outer expansion

$$y_{\text{out}}(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots, \qquad \varepsilon \to 0+. \tag{9.5.32}$$

Even though there is no boundary layer at $x = 0$, (9.5.32) is not a regular perturbation expansion (see Prob. 9.24).

Substituting (9.5.32) into (9.5.31) gives

$$\frac{1}{x}y_n' + y_n = \begin{cases} 0, & n = 0, \\ -y_{n-1}'', & n > 0. \end{cases}$$

There is no boundary layer at $x = 1$ because $1/x > 0$, so we must require that $y_0(1) = e^{-1/2}$, $y_n(1) = 0$ $(n > 0)$. We obtain $y_0(x) = e^{-x^2/2}$, $y_1(x) = -\frac{1}{4}(x^2 - 1)^2 e^{-x^2/2}$, and so on. Using these equations we can predict the value of $y(0)$:

$$y(0) = 1 - \varepsilon/4 + O(\varepsilon^2), \qquad \varepsilon \to 0+. \tag{9.5.33}$$

Example 4 *Singular boundary-value problem.* Let us reexamine (9.5.31) in the limit $\varepsilon \to 0-$. With $\varepsilon < 0$ in Example 3, the effect of the coordinate singularity at $x = 0$ changes abruptly. Now, Frobenius' method gives two positive indicial exponents 0 and $1 - 1/\varepsilon$, so the condition that $y(0)$ be finite does not suffice to determine the solution. The value of $y(0)$ must be specified.

Let us change the sign of ε and pose a representative problem:

$$\varepsilon y'' - y'/x - y = 0, \qquad 0 \le x \le 1, \qquad y(0) = 1,\ y(1) = 1, \qquad \varepsilon \to 0+.$$

The boundary condition $y(0) = 1$ uniquely determines the outer solution

$$y_{\text{out}}(x) = e^{-x^2/2}[1 + \varepsilon(x^4 - 2x^2)/4 + O(\varepsilon^2)], \qquad \varepsilon \to 0+,$$

because $-1/x$ is negative for $x > 0$, so no boundary layer can exist at $x = 0$.

A boundary layer of thickness ε is required at $x = 1$. Using the matching procedures developed in this chapter, we find that the inner solution is

$$Y_{\text{in}}(X) = (1 - e^{-1/2})[1 - \tfrac{1}{2}\varepsilon(X^2 - 4X)]e^{-X} + e^{-1/2}[1 + \varepsilon X] + O(\varepsilon^2), \qquad \varepsilon \to 0+, \tag{9.5.34}$$

where $X = (1 - x)/\varepsilon$. A uniform approximation to $y(x)$ for $0 \le x \le 1$, accurate to order ε, is

$$y_{\text{unif}} = e^{-x^2/2}[1 + \tfrac{1}{4}\varepsilon(x^2 - 1)^2] + (1 - e^{-1/2})[1 - \tfrac{1}{2}\varepsilon(X^2 - 4X)]e^{-X}. \tag{9.5.35}$$

Example 5 *Imposition of boundary conditions near a singularity of the differential equation.* Suppose we change the boundary conditions in (9.5.31) slightly and formulate the new boundary-value problem

$$\varepsilon y'' + y'/x + y = 0, \qquad y(\varepsilon) = 0,\ y(1) = e^{-1/2},$$

on the restricted interval $\varepsilon \le x \le 1$. There is no coordinate singularity on the interval $\varepsilon \le x \le 1$, so boundary conditions must be imposed at both $x = \varepsilon$ and $x = 1$.

The outer solution, away from a boundary layer near $x = \varepsilon$, is given correctly to order ε^2 by

$$y_{\text{out}}(x) = e^{-x^2/2}[1 - \varepsilon(x^2 - 1)^2/4] + O(\varepsilon^2), \qquad \varepsilon \to 0+. \tag{9.5.36}$$

The interesting feature of this example is the character of the boundary layer at $x = \varepsilon$. To determine the structure of the boundary layer, we introduce the inner variables $X = (x - \varepsilon)/\delta$, $Y_{\text{in}}(X) = y(x)$. In terms of these variables (9.5.31) become

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y_{\text{in}}}{dX^2} + \frac{1}{\varepsilon + \delta X}\frac{1}{\delta}\frac{dY_{\text{in}}}{dX} + Y_{\text{in}} = 0.$$

The only distinguished limit with $\delta \ll 1$ is $\delta = \varepsilon^2$. For this choice the differential equation becomes

$$\frac{d^2Y_{\text{in}}}{dX^2} + \frac{1}{1 + \varepsilon X}\frac{dY_{\text{in}}}{dX} + \varepsilon^3 Y_{\text{in}} = 0.$$

The solution to this equation in the narrow boundary layer at $x = \varepsilon$ must satisfy the boundary condition $y(\varepsilon) = Y_{\text{in}}(0) = 0$ and match asymptotically with (9.5.36). We find that (see Prob. 9.22)

$$Y_{\text{in}}(X) = 1 - e^{-X} - \varepsilon[(\tfrac{1}{2}X^2 + X - \tfrac{1}{4})e^{-X} + \tfrac{1}{4}] + O(\varepsilon^2), \qquad \varepsilon \to 0+. \tag{9.5.37}$$

The next example is somewhat contrived, but it illustrates the remarkable phenomenon of a boundary-layer structure inside a boundary layer.

Example 6 *Nested boundary layers.* Consider the differential equation

$$\varepsilon^3 xy'' + x^2 y' - y(x^3 + \varepsilon) = 0, \tag{9.5.38}$$

subject to the boundary conditions

$$y(0) = 1, \qquad y(1) = \sqrt{e}. \tag{9.5.39}$$

This differential equation has a regular singular point at $x = 0$ with indicial exponents 0 and 1. Therefore, both linearly independent solutions at $x = 0$ are finite and it is consistent to impose the boundary conditions in (9.5.39).

First, we determine the outer solution by assuming an outer expansion of the form $y_{\text{out}} \sim y_0 + \varepsilon y_1 + \cdots$ $(\varepsilon \to 0+)$. Substituting this expansion into (9.5.38) gives the equations for y_0 and y_1:

$$x^2 y_0' - y_0 x^3 = 0, \qquad x^2 y_1' - x^3 y_1 = y_0.$$

There is no boundary layer possible at $x = 1$ because the coefficient of y' is positive. Therefore, y_0 and y_1 satisfy the boundary conditions $y_0(1) = \sqrt{e}$, $y_1(1) = 0$. Solving the above pair of differential equations and imposing the boundary conditions at $x = 1$ gives the first two terms in the outer expansion:

$$y_{\text{out}}(x) = e^{x^2/2} + \varepsilon(1 - 1/x)e^{x^2/2} + \cdots, \qquad \varepsilon \to 0+. \tag{9.5.40}$$

Even though $y_0(0) = 1$, a boundary layer is required at $x = 0$ because the second term becomes infinite at $x = 0$. To determine the thickness of the boundary layer we set $X = x/\delta$, $Y_{\text{in}}(X) = y(x)$, and obtain

$$\frac{\varepsilon^3}{\delta} X \frac{d^2 Y_{\text{in}}}{dX^2} + \delta X^2 \frac{dY_{\text{in}}}{dX} - Y(\delta^3 X^3 + \varepsilon) = 0. \tag{9.5.41}$$

There are two distinguished limits for which $\delta \ll 1$: $\delta = \varepsilon$ and $\delta = \varepsilon^2$.

We examine the case $\delta = \varepsilon$ first because the outer solution becomes large when x is of order ε. Setting $\delta = \varepsilon$ in (9.5.41) gives

$$X^2 \frac{dY_{\text{in}}}{dX} - Y_{\text{in}} = -\varepsilon X \frac{d^2 Y_{\text{in}}}{dX^2} + \varepsilon^2 X^3 Y_{\text{in}}.$$

Assuming an inner expansion of the form $Y_{\text{in}}(X) \sim Y_0(X) + \varepsilon Y_1(X) + \cdots$ $(\varepsilon \to 0+)$ gives

$$X^2 \frac{dY_0}{dX} - Y_0 = 0,$$

whose solution is

$$Y_0 = \alpha_0 e^{-1/X},$$

and

$$X^2 \frac{dY_1}{dX} - Y_1 = -X \frac{d^2 Y_0}{dX^2},$$

whose solution is

$$Y_1 = \alpha_1 e^{-1/X} + \alpha_0 \left(\frac{2}{3X^3} - \frac{1}{4X^4} \right) e^{-1/X}.$$

To determine the values of α_0 and α_1 we match $Y_{\text{in}}(X)$ to $y_{\text{out}}(x)$. Writing Y_0 and Y_1 in terms of the outer variable x and taking the intermediate limit $x \to 0$, $X \to \infty$ gives $Y_{\text{in}}(X) \sim \alpha_0(1 - \varepsilon/x) + \varepsilon\alpha_1 + O(\varepsilon^2) + O(\varepsilon^2/x^2)$ $(\varepsilon \to 0+,\ x/\varepsilon \to +\infty)$. This expansion matches asymptotically with that in (9.5.40) in the limit $x \to 0+$ and we obtain the matching conditions $\alpha_0 = \alpha_1 = 1$. Thus,

$$Y_{\text{in}}(X) = e^{-1/X}\left[1 + \varepsilon\left(1 + \frac{2}{3X^3} - \frac{1}{4X^4}\right)\right] + O(\varepsilon^2), \qquad \varepsilon \to 0+. \tag{9.5.42}$$

So far our analysis has been straightforward. However, now we observe that it is still not possible to satisfy the boundary condition $y(0) = 1$ with $Y_{\text{in}}(X)$ because $Y_{\text{in}}(X)$ vanishes exponentially as $X \to 0+$. Apparently, there is an additional boundary layer very near $x = 0$ which enables us to satisfy the boundary condition $y(0) = 1$. The thickness of this boundary layer must be $\delta = \varepsilon^2$ because this is the only other distinguished limit for the differential equation. Therefore we set $\delta = \varepsilon^2$ in (9.5.41),

$$\bar{X} \frac{d^2 \bar{Y}_{\text{in}}}{d\bar{X}^2} - \bar{Y}_{\text{in}} = -\varepsilon \bar{X}^2 \frac{d\bar{Y}_{\text{in}}}{d\bar{X}} + \varepsilon^5 \bar{X}\bar{Y}_{\text{in}},$$

and assume an inner-inner expansion of the form

$$\bar{Y}_{\text{in}} \sim \bar{Y}_0 + \varepsilon \bar{Y}_1 + \cdots, \qquad \varepsilon \to 0+.$$

$\bar{Y}_0$ satisfies

$$\frac{d^2 \bar{Y}_0}{d\bar{X}^2} = \frac{\bar{Y}_0}{\bar{X}}$$

whose general solution is a linear combination of modified Bessel functions (see Prob. 9.25):

$$\bar{Y}_0(\bar{X}) = \bar{\alpha}_0 \sqrt{\bar{X}}\, I_1(2\sqrt{\bar{X}}) + \bar{\beta}_0 \sqrt{\bar{X}}\, K_1(2\sqrt{\bar{X}}). \tag{9.5.43}$$

Since $I_1(2\sqrt{\bar{X}})$ grows exponentially as $\bar{X} \to \infty$, we will be unable to match $Y_{\text{in}}(X)$ unless $\bar{\alpha}_0 = 0$. $\bar{\beta}_0$ is determined by the boundary condition $\bar{Y}_0(0) = 1$. We find that $\bar{\beta}_0 = 2$ (see Prob. 9.25).

Observe that $\bar{Y}_{\text{in}}(\bar{X})$ and $Y_{\text{in}}(X)$ match asymptotically in the intermediate limit $X \to 0$, $\bar{X} \to \infty$. In the matching region they both vanish exponentially.

(D) 9.6 INTERNAL BOUNDARY LAYERS

Boundary layers (localized regions of rapid change of y) may occur in the interior as well as on the edge of an interval. However, the structure of internal layers tends to be complicated, so we confine our discussion to the leading-order behavior of solutions only.

We consider the simplest second-order differential-equation boundary-value problem that can exhibit internal boundary layers:

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \qquad y(-1) = A,\ y(1) = B. \tag{9.6.1}$$

We know that if $a(x) \neq 0$ for $-1 < x < 1$, then there are no internal boundary layers. However, suppose we now assume that $a(x)$ has a simple zero at $x = 0$,

$$a(x) \sim \alpha x, \qquad x \to 0,$$

and that $a(x)$ has no other zeros for $-1 \le x \le 1$. We also assume that $b(x) \sim \beta \neq 0$ $(x \to 0)$. There are two cases to consider.

CASE I $\alpha > 0$. Here $a(x)$ has positive slope at $x = 0$, so $a(-1) < 0$ and $a(1) > 0$. Thus, boundary layers at either $x = +1$ or $x = -1$ are not possible.

For this problem there are *two* outer solutions, one to the left and one to the right of $x = 0$. Either outer solution $y_{\text{out}}(x)$ has an expansion of the form $y_{\text{out}}(x) = y_0(x) + \varepsilon y_1(x) + \cdots$. Thus, to lowest order we have $a(x)y_0'(x) + b(x)y = 0$. Because there are no boundary layers at $x = \pm 1$, both outer solutions are determined by the boundary conditions at $x = \pm 1$:

$$y_{0,\text{right}} = B \exp\left[\int_x^1 \frac{b(t)}{a(t)}\,dt\right], \qquad x > 0, \tag{9.6.2}$$

$$y_{0,\text{left}} = A \exp\left[-\int_{-1}^x \frac{b(t)}{a(t)}\,dt\right], \qquad x < 0. \tag{9.6.3}$$

Next we investigate the neighborhood of $x = 0$. Setting $x = \delta X$, $y(x) = Y_{in}(X)$, (9.6.1) becomes

$$\frac{\varepsilon}{\delta^2}\frac{d^2 Y_{in}}{dX^2} + \frac{a(\delta X)}{\delta}\frac{dY_{in}}{dX} + b(\delta X)Y_{in} = 0.$$

To investigate the immediate neighborhood of $x = 0$, we replace $a(x)$ by αx and $b(x)$ by β. The only distinguished limit for which $\delta \ll 1$ is $\delta = \sqrt{\varepsilon}$. Thus, to leading order we have

$$\frac{d^2 Y_0}{dX^2} + \alpha X\frac{dY_0}{dX} + \beta Y_0 = 0, \tag{9.6.4}$$

where $Y_{in} = Y_0$ + higher-order terms.

The solution to the differential equation (9.6.4) cannot be expressed in terms of elementary functions. As we will now show, (9.6.4) is related to the parabolic cylinder equation, one of the standard equations in mathematical physics. To solve (9.6.4) we let

$$Y_0 = e^{-\alpha X^2/4}W. \tag{9.6.5}$$

$W(X)$ satisfies

$$\frac{d^2 W}{dX^2} + (\beta - \tfrac{1}{2}\alpha - \tfrac{1}{4}\alpha^2 X^2)W = 0.$$

Next, we let $\sqrt{\alpha}\,X = Z$ and obtain

$$\frac{d^2 W}{dZ^2} + \left(\frac{\beta}{\alpha} - \frac{1}{2} - \frac{1}{4}Z^2\right)W = 0,$$

which we recognize as the parabolic cylinder equation (see Sec. 3.5). Assuming that β/α is not a positive integer (we discuss this special case later), the general solution to this equation is an arbitrary linear combination of parabolic cylinder functions: $W(Z) = C_1 D_{\beta/\alpha-1}(Z) + C_2 D_{\beta/\alpha-1}(-Z)$. Thus,

$$Y_0(X) = e^{-\alpha X^2/4}[C_1 D_{\beta/\alpha-1}(X\sqrt{\alpha}) + C_2 D_{\beta/\alpha-1}(-X\sqrt{\alpha})]. \tag{9.6.6}$$

The intermediate limit is defined by $x \to 0\pm$, $X \to \pm\infty$. Does the inner solution Y_0 in (9.6.6) match asymptotically with the outer solutions in (9.6.2) and (9.6.3) in the intermediate limit? If it does, we hope that the constants C_1 and C_2 are determined by the matching condition. In the intermediate limit the arguments of the parabolic cylinder functions become large. Therefore, it is necessary to use the asymptotic behaviors of the parabolic cylinder function for large positive and negative arguments, which were determined in Sec. 3.8:

$$\begin{aligned} D_\nu(t) &\sim t^\nu e^{-t^2/4}, && t \to +\infty,\\ D_\nu(-t) &\sim t^{-\nu-1}e^{t^2/4}\frac{\sqrt{2\pi}}{\Gamma(-\nu)}, && t \to +\infty. \end{aligned} \tag{9.6.7}$$

Thus, as $X \to +\infty$,

$$Y_0(X) \sim C_2(X\sqrt{\alpha})^{-\beta/\alpha}\frac{\sqrt{2\pi}}{\Gamma(1 - \beta/\alpha)}, \tag{9.6.8}$$

and as $X \to -\infty$,

$$Y_0(X) \sim C_1(-X\sqrt{\alpha})^{-\beta/\alpha}\frac{\sqrt{2\pi}}{\Gamma(1 - \beta/\alpha)}, \tag{9.6.9}$$

where we have discarded exponentially small terms. Finally, in preparation for asymptotic matching, we replace the inner variable X in (9.6.8) by the outer variable x:

$$Y_0(X) \sim \frac{C_2\sqrt{2\pi}}{\Gamma(1 - \beta/\alpha)}(\sqrt{\alpha/\varepsilon})^{-\beta/\alpha}e^{-\beta(\ln x)/\alpha}, \qquad x \to 0+, \tag{9.6.10}$$

$$Y_0(X) \sim \frac{C_1\sqrt{2\pi}}{\Gamma(1 - \beta/\alpha)}(\sqrt{\alpha/\varepsilon})^{-\beta/\alpha}e^{-\beta[\ln(-x)]/\alpha}, \qquad x \to 0-. \tag{9.6.11}$$

It is now clear that (9.6.2) and (9.6.10) match as $x \to 0+$ because

$$\int_x^1 \frac{b(t)}{a(t)}\,dt \sim -\frac{\beta}{\alpha}\ln x, \qquad x \to 0+,$$

and that (9.6.3) and (9.6.11) match as $x \to 0-$ because

$$-\int_{-1}^x \frac{b(t)}{a(t)}\,dt \sim -\frac{\beta}{\alpha}\ln(-x), \qquad x \to 0-.$$

Moreover, we can determine the constants C_1 and C_2 because

$$\int_x^1 \frac{b(t)}{a(t)}\,dt + \frac{\beta}{\alpha}\ln x \sim \int_0^1 \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t}\right]dt, \qquad x \to 0+,$$

and
$$-\int_{-1}^x \frac{b(t)}{a(t)}\,dt + \frac{\beta}{\alpha}\ln(-x) \sim \int_0^{-1} \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t}\right]dt, \qquad x \to 0-,$$

where the two integrals on the right exist. Specifically,

$$C_2 = B\frac{\Gamma(1 - \beta/\alpha)}{\sqrt{2\pi}}(\sqrt{\alpha/\varepsilon})^{\beta/\alpha}\exp\int_0^1 \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t}\right]dt, \tag{9.6.12a}$$

and
$$C_1 = \frac{A\Gamma(1 - \beta/\alpha)}{\sqrt{2\pi}}(\sqrt{\alpha/\varepsilon})^{\beta/\alpha}\exp\int_0^{-1} \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t}\right]dt. \tag{9.6.12b}$$

The results in (9.6.2) and (9.6.3), (9.6.6), and (9.6.12) may be combined into a single uniform asymptotic approximation which is valid on the interval $-1 \le x \le 1$:

$$y_{\text{unif}}(x) = \frac{\Gamma(1-\beta/\alpha)}{\sqrt{2\pi}} (\sqrt{\alpha/\varepsilon})^{\beta/\alpha} e^{-\alpha x^2/4\varepsilon}$$
$$\times \left(A \exp\left\{ \int_x^{-1} \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t} \right] dt \right\} D_{\beta/\alpha - 1}(x\sqrt{\alpha/\varepsilon}) \right.$$
$$\left. + B \exp\left\{ \int_x^{1} \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t} \right] dt \right\} D_{\beta/\alpha - 1}(-x\sqrt{\alpha/\varepsilon}) \right). \quad (9.6.13)$$

This result is verified in Prob. 9.30.

Example 1 *Internal boundary layer, case I.* Consider the boundary-value problem

$$\varepsilon y'' + 2xy' + (1 + x^2)y = 0, \qquad y(-1) = 2,\ y(1) = 1, \quad (9.6.14)$$

in the limit $\varepsilon \to 0+$. For this problem $\alpha = 2$ and $\beta = 1$. Also,

$$\int_0^1 \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t} \right] dt = \int_0^{-1} \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t} \right] dt = \frac{1}{4}$$

and $C_1 = 2(e/2\varepsilon)^{1/4}$, $C_2 = (e/2\varepsilon)^{1/4}$. Therefore, the two outer solutions are

$$y_{0,\text{right}} = x^{-1/2} e^{(1-x^2)/4}, \quad \sqrt{\varepsilon} \ll x \le 1), \qquad y_{0,\text{left}} = 2(-x)^{-1/2} e^{(1-x^2)/4}, \ \sqrt{\varepsilon} \ll -x \le 1,$$

and the inner solution is

$$Y_0(X) = Y_0(x/\sqrt{\varepsilon}) = e^{-x^2/2\varepsilon}(e/2\varepsilon)^{1/4}[2D_{-1/2}(x\sqrt{2/\varepsilon}) + D_{-1/2}(-x\sqrt{2/\varepsilon})].$$

The outer and inner solutions may be combined to give a uniform approximation to $y(x)$:

$$y_{\text{unif}}(x) = e^{-x^2/2\varepsilon} \left(\frac{e^{(1-x^2)}}{2\varepsilon} \right)^{1/4} [2D_{-1/2}(x\sqrt{2/\varepsilon}) + D_{-1/2}(-x\sqrt{2/\varepsilon})]. \quad (9.6.15)$$

In Fig. 9.16 we compare this leading-order uniform approximation with the exact solution to (9.6.14).

CASE II $\alpha < 0$. Let us return to (9.6.1) and see what happens when $\alpha < 0$. Now $a(x)$ has negative slope at $x = 0$, so $a(-1) > 0$ and $a(1) < 0$. This implies that boundary layers may occur at both $x = -1$ and at $x = 1$. We are thus faced with the possibility of having three boundary layers at $x = -1, 0, 1$, and two outer solutions between -1 and 0, and 0 and 1! It is quite surprising that the solution to this problem is much less complicated than the solution in case I.

The proper way to approach this problem is to analyze the inner solution at $x = 0$ first. Using the same analysis as in case I we find that (see Prob. 9.30)

$$Y_0(X) = e^{-\alpha X^2/4}[C_1 D_{-\beta/\alpha}(\sqrt{-\alpha}\, X) + C_2 D_{-\beta/\alpha}(-\sqrt{-\alpha}\, X)]. \quad (9.6.16)$$

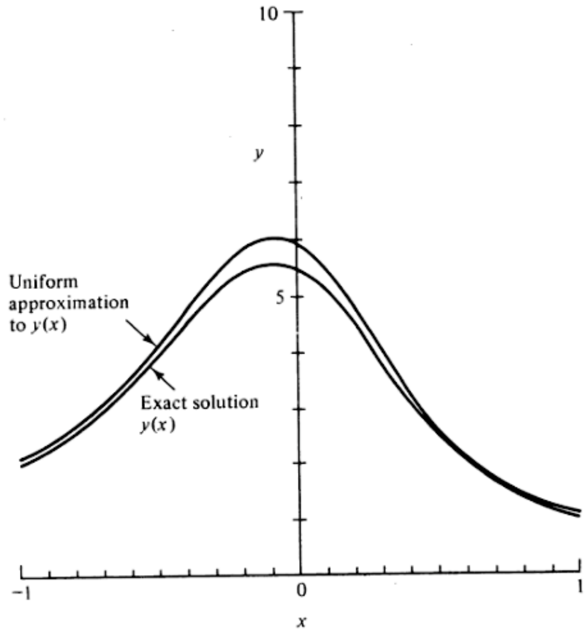

Figure 9.16 Comparison between the exact solution $y(x)$ to $\varepsilon y'' + 2xy' + (1 + x^2)y = 0$ [$y(-1) = 2$, $y(1) = 1$] and the leading-order uniform approximation (9.6.15) obtained from boundary-layer analysis. The value of ε is 0.2.

Now, α is negative, so the term $e^{-\alpha X^2/4}$ grows exponentially in both directions as $X \to \pm\infty$. (In case I it decays exponentially in both directions.) Observe that as $X \to +\infty$, $D_{-\beta/\alpha}(-\sqrt{-\alpha}\, X)$ also grows exponentially as $X \to +\infty$, assuming that $-\beta/\alpha \ne 0, 1, 2, 3, \ldots$. (We treat the integer case later.) Thus, no asymptotic match to the right outer solution is possible unless $C_2 = 0$. Similarly, as $X \to -\infty$, $D_{-\beta/\alpha}(\sqrt{-\alpha}\, X)$ grows exponentially. Thus, no asymptotic match to the left outer solution is possible unless $C_1 = 0$. We therefore obtain the very simple result that $Y_0(X) \equiv 0$. Apparently, there is *no* region of rapid change at $x = 0$ in case II.

Next, we consider the outer solutions which satisfy $a(x)y_0'(x) + b(x)y = 0$. This is a first-order homogeneous linear equation. Therefore, each outer solution is determined up to a multiplicative constant. However, requiring that the outer solutions match to $Y_0(X)$ implies that each multiplicative constant must be 0. We conclude that to leading order the solution vanishes everywhere except for boundary layers at $x = -1$ and at $x = 1$!

The leading-order boundary-layer solutions at ± 1 are easy to write down:

$$Y_{0,\text{left}} = Ae^{-a(-1)(x+1)/\varepsilon}, \tag{9.6.17}$$

$$Y_{0,\text{right}} = Be^{a(1)(1-x)/\varepsilon}. \tag{9.6.18}$$

Combining (9.6.17) and (9.6.18) we obtain the leading-order uniform asymptotic approximation

$$y_{\text{unif}}(x) = Ae^{-a(-1)(x+1)/\varepsilon} + Be^{a(1)(1-x)/\varepsilon}. \tag{9.6.19}$$

Example 2 *Case II.* Consider the boundary-value problem

$$\varepsilon y'' - 2xy' + (1 + x^2)y = 0, \qquad y(-1) = 2,\ y(1) = 1, \tag{9.6.20}$$

in the limit $\varepsilon \to 0+$. For this problem $\alpha = -2$. The leading-order uniform asymptotic approximation for this problem is

$$y_{\text{unif}} = 2e^{-2(x+1)/\varepsilon} + e^{-2(1-x)/\varepsilon}. \tag{9.6.21}$$

In Fig. 9.17 we compare the exact solution to (9.6.20) with the uniform asymptotic approximation in (9.6.21).

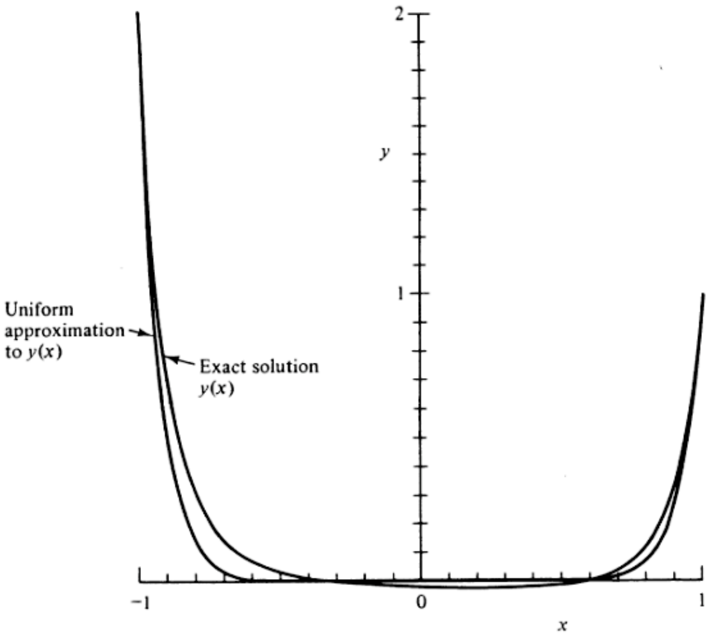

Figure 9.17 Comparison between the exact solution $y(x)$ to $\varepsilon y'' - 2xy' + (1 + x^2)y = 0$ [$y(-1) = 2$, $y(1) = 1$] and the leading-order uniform approximation (9.6.21) obtained from boundary-layer analysis. The value of ε is 0.15.

There are two exceptional cases that we have not yet discussed, namely, $\alpha > 0$ with $\beta/\alpha = 1, 2, 3, \ldots$ and $\alpha < 0$ with $\beta/\alpha = 0, -1, -2, \ldots$. In both of these cases leading-order boundary-layer analysis breaks down. Let us see why.

Case III $\alpha > 0$, $\beta/\alpha = 1, 2, 3, \ldots$. This case is special because the indices of the parabolic cylinder functions in (9.6.6) are nonnegative integers. Recall that $D_n(Z) = \mathrm{He}_n(Z)e^{-Z^2/4}$ when $n = 0, 1, 2, \ldots$, where $\mathrm{He}_n(Z)$ is a Hermite polynomial of degree n, and that $D_n(-Z) = (-1)^n D_n(-Z)$. Therefore, $D_n(Z)$ and $D_n(-Z)$ are not linearly independent and the general leading-order inner solution $Y_0(X)$ is not given by (9.6.6). Instead, as shown in Sec. 3.8, the general solution has the form

$$Y_0(X) = e^{-\alpha X^2/4}[K_1 D_{\beta/\alpha - 1}(\sqrt{\alpha}\, X) + K_2 D_{-\beta/\alpha}(i\sqrt{\alpha}\, X)]. \tag{9.6.22}$$

From (3.7.18), the leading behavior of (9.6.22) as $X \to \pm\infty$ is

$$Y_0(X) \sim K_2(i\sqrt{\alpha}\, X)^{-\beta/\alpha}, \qquad X \to +\infty. \tag{9.6.23}$$

The contribution of the terms multiplied by K_1 is exponentially small compared to those multiplied by K_2. Therefore, the coefficient K_1 plays no role in the asymptotic matching and must remain undetermined!

No boundary layers are possible at $x = \pm 1$ when $\alpha > 0$, so we must match $y_0(X)$ to the outer solution $y_{0,\text{left}} = A \exp[-\int_{-1}^{x} b(t)/a(t)\, dt]$ $(-1 \le x < 0)$ and $y_{0,\text{right}} = B \exp[\int_x^1 b(t)/a(t)\, dt]$ $(0 < x \le 1)$. Matching to $y_{0,\text{left}}$ in the intermediate limit $X \to -\infty$, $x \to 0-$ gives

$$K_2 = (i\sqrt{\alpha/\varepsilon})^{\beta/\alpha} A \exp \int_0^{-1} \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t}\right] dt.$$

On the other hand, matching to $y_{0,\text{right}}$ in the intermediate limit $X \to +\infty$, $x \to 0+$ gives

$$K_2 = (i\sqrt{\alpha/\varepsilon})^{\beta/\alpha} B \exp \int_0^{1} \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t}\right] dt.$$

Only in rare cases will the values of K_2 determined in these two different ways agree. In most cases the problem has no leading-order solution because the coefficient K_2 is *overdetermined.*

Case IV $\alpha < 0$ with $\beta/\alpha = 0, -1, -2, \ldots$. In contrast with case III, the leading-order boundary-layer solution in case IV is *underdetermined*. Here, the general leading-order inner solution $Y_0(X)$ is not given by (9.6.16). Instead it has the form:

$$Y_0(X) = e^{-\alpha X^2/4}[K_1 D_{-\beta/\alpha}(\sqrt{-\alpha}\, X) + K_2 D_{\beta/\alpha - 1}(i\sqrt{-\alpha}\, X)]. \tag{9.6.24}$$

From (3.7.18), the leading behavior of $Y_0(X)$ as $X \to \pm\infty$ is

$$Y_0(X) \sim K_1(\sqrt{-\alpha}\, X)^{-\beta/\alpha} + K_2(i\sqrt{-\alpha}\, X)^{\beta/\alpha - 1} e^{-\alpha X^2/2}, \quad X \to \pm\infty. \tag{9.6.25}$$

Since $\alpha < 0$, matching is only possible if $K_2 = 0$, so

$$Y_0 \sim K_1(\sqrt{-\alpha/\varepsilon}\, x)^{-\beta/\alpha}, \qquad x/\sqrt{\varepsilon} \to \pm\infty. \tag{9.6.26}$$

The leading-order outer solution for $0 < x < 1$ that matches to (9.6.26) when $X \to +\infty$, $x \to 0+$ is

$$y_{0,\text{right}}(x) = K_1(x\sqrt{-\alpha/\varepsilon})^{-\beta/\alpha} \exp \int_0^x \left[\frac{\beta}{\alpha t} - \frac{b(t)}{a(t)}\right] dt, \qquad x > 0, \tag{9.6.27}$$

while the leading-order outer solution for $-1 < x < 0$ that matches to (9.6.26) when $X \to -\infty$, $x \to 0-$ is

$$y_{0,\text{left}}(x) = K_1(x\sqrt{-\alpha/\varepsilon})^{-\beta/\alpha} \exp \int_x^0 \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t}\right] dt, \qquad x < 0. \tag{9.6.28}$$

Since $\alpha < 0$, the outer solution (9.6.27) can be asymptotically matched as $x \to 1-$ to a boundary-layer solution; the leading-order boundary-layer approximation is

$$Y_{0,\text{right}} = Be^{a(1)(x-1)/\varepsilon} + K_1(\sqrt{-\alpha/\varepsilon})^{-\beta/\alpha}[1 - e^{a(1)(x-1)/\varepsilon}] \times \exp \int_0^1 \left[\frac{\beta}{\alpha t} - \frac{b(t)}{a(t)}\right] dt. \tag{9.6.29}$$

The outer solution (9.6.28) can also be matched as $x \to -1+$ to a boundary-layer solution at $x = -1$ because $\alpha < 0$:

$$Y_{0,\text{left}} = Ae^{-a(-1)(x+1)/\varepsilon} + K_1(-\sqrt{-\alpha/\varepsilon})^{-\beta/\alpha}[1 - e^{-a(-1)(x+1)/\varepsilon}] \times \exp \int_{-1}^0 \left[\frac{b(t)}{a(t)} - \frac{\beta}{\alpha t}\right] dt. \tag{9.6.30}$$

The boundary-layer solutions (9.6.29) and (9.6.30), together with the outer solutions (9.6.27) and (9.6.28) and the internal layer solution (9.6.24) with $K_2 = 0$, match to leading order in ε. However, K_1 is still arbitrary, so leading-order boundary-layer theory has not determined a unique solution to the boundary-value problem!

Discussion of cases III and IV Cases III and IV have serious difficulties because $D_n(z)$ $(n = 0, 1, 2, \ldots)$ decays exponentially as $z \to +\infty$ and as $z \to -\infty$. In contrast, $D_\nu(z)$ $(\nu \neq 0, 1, 2, \ldots)$ decays exponentially only as $z \to +\infty$ and grows as $z \to -\infty$. Thus, in case II, for example, where $\beta/\alpha \neq 0, -1, -2, \ldots$ we could logically argue that $C_1 = C_2 = 0$ in order for a match to be possible. In case IV, where $\beta/\alpha = 0, -1, -2, \ldots$ and $\alpha < 0$ we can no longer argue that $K_1 = 0$.

The resolution of the difficulties in cases III and IV is not trivial. Sometimes it is possible to resolve these difficulties by carrying the boundary-layer analysis to higher order. A higher-order treatment may give rise to solutions in the internal boundary layer which do not decay exponentially as $X \to +\infty$ and as $X \to -\infty$.

When this happens, the higher-order analysis proceeds as in cases I and II and a unique solution can be determined (see Probs. 9.31 and 9.32). Higher-order boundary-layer analysis does not always resolve the difficulties discussed here. Often it is possible to determine a unique asymptotic solution to ambiguous internal boundary-layer problems using the methods of WKB theory (see Probs. 9.33 and 10.28).

(I) 9.7 NONLINEAR BOUNDARY-LAYER PROBLEMS

Boundary-layer analysis applies to nonlinear as well as to linear differential equations. This section is a collection of three illustrative examples. The first example is very elementary. We include this example merely to show that the techniques we have used to solve linear boundary-layer problems can apply equally well to nonlinear problems.

Example 1 *Leading-order analysis of a nonlinear autonomous equation.* Consider the boundary-value problem

$$\varepsilon y'' + 2y' + e^y = 0, \qquad y(0) = y(1) = 0. \tag{9.7.1}$$

If e^y were a linear function of y, there would be a boundary layer at $x = 0$ (and no boundary layer at $x = 1$) because the coefficient of y' is positive. This nonlinear problem also has just one boundary layer at $x = 0$.

The outer expansion has the form

$$y_{\text{out}}(x) \sim y_0(x) + \varepsilon y_1(x) + \cdots, \qquad \varepsilon \to 0+. \tag{9.7.2}$$

Substituting (9.7.2) into (9.7.1) gives $2y_0' + e^{y_0} = 0$, whose solution is $y_0 = \ln 2/(x + c_0)$. Assuming that there is no boundary layer at $x = 1$ (a boundary layer at $x = 1$ leads to a contradiction; see Prob. 9.37), we impose the boundary condition $y_0(1) = 0$. Thus,

$$y_0(x) = \ln \frac{2}{1 + x}. \tag{9.7.3}$$

The boundary layer has thickness $\delta = \varepsilon$. (Why?) Therefore, setting $X = x/\varepsilon$, $Y_{\text{in}}(X) = y(x)$ in (9.7.1) gives

$$\frac{d^2 Y_{\text{in}}}{dX^2} + 2\frac{dY_{\text{in}}}{dX} = -\varepsilon e^{Y_{\text{in}}}.$$

Assuming an inner expansion of the form $Y_{\text{in}} = Y_0 + \varepsilon Y_1 + \cdots$ gives in leading order

$$Y_0 = A_0 + B_0 e^{-2X}. \tag{9.7.4}$$

The constants A_0 and B_0 are determined by the boundary condition at $x = 0$, $Y_0(0) = 0$, which gives $A_0 + B_0 = 0$ and the asymptotic match of (9.7.4) with (9.7.3) in the intermediate limit $x \to 0+$, $X \to +\infty$, which gives $A_0 = \ln 2$. Thus,

$$Y_0(X) = (1 - e^{-2X}) \ln 2. \tag{9.7.5}$$

We can combine (9.7.5) and (9.7.3) into a single uniform asymptotic approximation:

$$y_{\text{unif}}(x) = \ln \frac{2}{1 + x} - e^{-2x/\varepsilon} \ln 2. \tag{9.7.6}$$

A comparison between (9.7.6) and the exact solution to (9.7.1) is given in Fig. 9.18. For a higher-order treatment of (9.7.1) see Prob. 9.38.

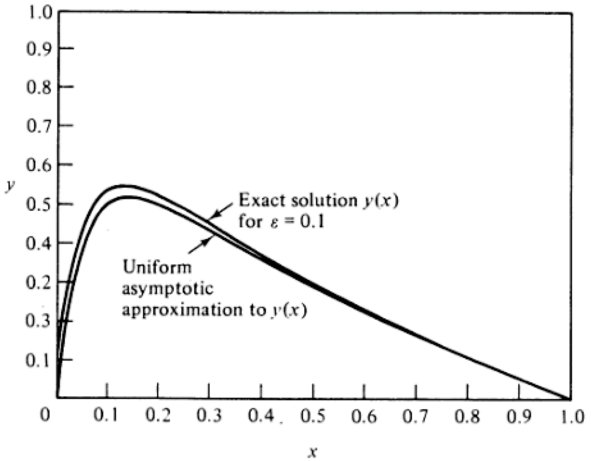

Figure 9.18 Comparison between the exact solution to the boundary-value problem $\varepsilon y'' + 2y' + e^y = 0$ $[y(0) = y(1) = 0]$ for $\varepsilon = 0.1$ in (9.7.1) and $y_{\text{unif}}(x) = \ln [2/(1 + x)] - e^{-2x/\varepsilon} \ln 2$, the leading-order uniform asymptotic approximation to $y(x)$ obtained using boundary-layer theory.

Our treatment of Example 1 was very straightforward. However, one should not be misled. Similar nonlinear boundary-layer problems can be extraordinarily difficult (see Prob. 9.39). In the next example we study a boundary-value problem that has many solutions; some can be predicted by boundary-layer theory while others are beyond the scope of boundary-layer methods.

Example 2 *Boundary-layer analysis of a nonlinear problem of Carrier.* An extremely beautiful and intricate nonlinear boundary-value problem of a type first proposed by Carrier is

$$\varepsilon y'' + 2(1 - x^2)y + y^2 = 1, \qquad y(-1) = y(1) = 0. \tag{9.7.7}$$

If we attempt a leading-order boundary-layer analysis of (9.7.7), we are immediately surprised to find that the outer equation obtained by setting $\varepsilon = 0$ is an algebraic equation rather than a differential equation:

$$y_{\text{out}}^2 + 2(1 - x^2)y_{\text{out}} - 1 = 0.$$

Because this equation is quadratic, it has two solutions

$$y_{\text{out},\pm}(x) = x^2 - 1 \pm \sqrt{1 + (1 - x^2)^2}. \tag{9.7.8}$$

In Fig. 9.19 we plot the two outer solutions. Observe that neither one satisfies the boundary conditions at $x = \pm 1$. Therefore, there must be boundary layers at $x = -1$ and at $x = +1$ which allow the boundary conditions to be satisfied. The question is, Which of the two outer solutions can be joined to inner solutions which satisfy the boundary conditions?

Let us examine the boundary layer at $x = 1$. If we substitute the inner variables $X = (1 - x)/\delta$, $Y_{\text{in}}(X) = y(x)$ into (9.7.7), we obtain in leading order

$$\frac{d^2 Y_{\text{in}}}{dX^2} + \frac{\delta^2}{\varepsilon}(Y_{\text{in}}^2 - 1) = 0.$$

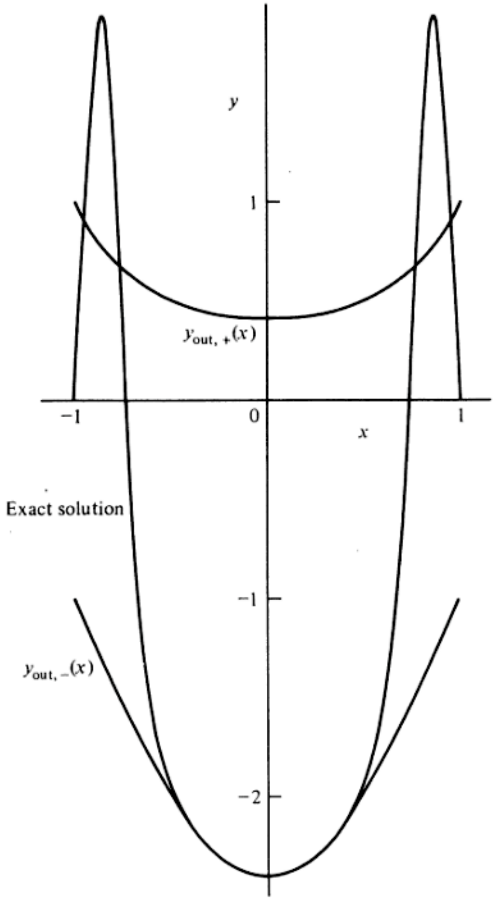

Figure 9.19 Exact solution to the nonlinear boundary-value problem in (9.7.7), $\varepsilon y'' + 2(1 - x^2)y + y^2 = 1[y(-1) = y(1) = 0]$ for $\varepsilon = 0.01$. Also shown are the two outer approximations $y_{\text{out},\pm}$ in (9.7.8). As predicted by boundary-layer analysis, the lower outer approximation is an extremely good approximation to $y(x)$ away from the boundary layers at $x = \pm 1$. Observe that $y(x)$ has a local maximum in both boundary layers. y_{unif} in (9.7.12) with the lower choice of signs is an accurate approximation to $y(x)$ for $-1 \le x \le 1$; it predicts that the maximum value of y is 2.

Thus, the distinguished limit is $\delta = \sqrt{\varepsilon}$. The solution to the leading-order inner equation

$$\frac{d^2 Y_{\text{in}}}{dX^2} + Y_{\text{in}}^2 - 1 = 0 \tag{9.7.9}$$

must satisfy the boundary condition $Y_{\text{in}}(0) = 0$ and must match asymptotically with one (or both) of the outer solutions. That is, Y_{in} must approach either ± 1 as $X \to +\infty$.

Is it possible for Y_{in} to approach 1 as $X \to +\infty$? Suppose we let $Y_{in} = 1 + W(X)$. If $Y_{in} \to 1$, then $W(X) \to 0$ and we can replace (9.7.9) with the approximate linear equation $W'' + 2W = 0$. However, solutions to this equation oscillate as $X \to +\infty$ and do not approach 0. This simple analysis shows that it is not possible for Y_{in} to match to $y_{\text{out},+}$ in (9.7.8).

Fortunately, the same argument suggests that it *is* possible for Y_{in} to match to $y_{\text{out},-}$. Let $Y_{in} = -1 + W(X)$. Now if $Y_{in} \to -1$ then $W \to 0$ and we can replace (9.7.9) by the approximate linear equation $W'' - 2W = 0$. Since this equation has a solution which decays to 0 exponentially, it is at least consistent to assume that Y_{in} matches asymptotically with $y_{\text{out},-}$.

Having established this much, let us solve the inner equation exactly. Substituting $Y_{in} = -1 + W(X)$ into (9.7.9) gives the autonomous equation

$$W'' + W^2 - 2W = 0, \tag{9.7.10}$$

subject to the boundary conditions $W(\infty) = 0$, $W(0) = 1$. Also, since we expect W to decay exponentially as $X \to +\infty$, we may assume that $W'(\infty) = 0$. To solve (9.7.10) we multiply by $W'(X)$, integrate the equation once, and determine the integration constant by setting $X = \infty$. We obtain $\frac{1}{2}(W')^2 + \frac{1}{3}W^3 - W^2 = 0$, which is a separable first-order equation: $dW/W\sqrt{2 - 2W/3} = \pm dX$. Integrating this equation gives

$$-\sqrt{2}\tanh^{-1}\sqrt{1 - W/3} = \pm X + C.$$

The integration constant is determined by the requirement that $W = 1$ at $X = 0$. Apparently, there are *two* solutions:

$$Y_{in}(X) = -1 + 3\,\mathrm{sech}^2\left(\pm\frac{X}{\sqrt{2}} + \tanh^{-1}\sqrt{2/3}\right). \tag{9.7.11}$$

There are also *two* inner solutions at $x = -1$ which satisfy the boundary condition $y(-1) = 0$ and match to the lower outer solution $y_{\text{out},-}$ in (9.7.8).

We can combine the outer with the two inner solutions to form a single uniform approximation valid over the entire interval $-1 \le x \le 1$:

$$\begin{aligned} y_{\text{unif}}(x) = x^2 - 1 - \sqrt{1 + (1 - x^2)^2} &+ 3\,\mathrm{sech}^2\left(\pm\frac{1 - x}{\sqrt{2\varepsilon}} + \tanh^{-1}\sqrt{2/3}\right) \\ &+ 3\,\mathrm{sech}^2\left(\pm\frac{1 + x}{\sqrt{2\varepsilon}} + \tanh^{-1}\sqrt{2/3}\right). \end{aligned} \tag{9.7.12}$$

Notice that the solution in (9.7.12) is not unique. There are actually four different solutions depending on the two choices of plus or minus signs in the boundary layer. For one choice of sign, $y_{\text{unif}}(x)$ in the boundary layer rapidly descends from its boundary value $y(\pm 1) = 0$ until it joins onto the outer solution $y_{\text{out},-}$. For the other choice of sign, $y_{\text{unif}}(x)$ rises rapidly until it reaches a maximum and then descends and joins onto the outer solution. It is easy to see that this maximum value of y_{unif} is 2 because the maximum value of sech is 1. It is a glorious triumph of boundary-layer theory that all four solutions actually exist and are extremely well approximated by the leading-order uniform approximation in (9.7.12)! See Figs. 9.19 to 9.21.

The analysis does not end here, however. The existence of four solutions to (9.7.7) may lead one to wonder if there are still more solutions. One may begin by asking whether there can be any internal boundary layers. We will now show that internal boundary layers are consistent.

Assume there is an internal boundary layer at $x = 0$. The thickness of such a boundary layer is $\delta = \sqrt{\varepsilon}$. (Why?) The leading-order equation is

$$Y''_{in}(X) + 2Y_{in} + Y_{in}^2 = 1. \tag{9.7.13}$$

Since $y_{\text{out},-}(0) = -1 - \sqrt{2}$, the boundary conditions on Y_{in} in (9.7.13) are $\lim_{X \to \pm\infty} Y_{in}(X) = -1 - \sqrt{2}$. The exact solution to (9.7.13) which satisfies these boundary conditions contains an arbitrary parameter A:

$$Y_{in} = 3\sqrt{2}\,\mathrm{sech}^2\,(2^{-1/4}x/\sqrt{\varepsilon} + A) - 1 - \sqrt{2}.$$

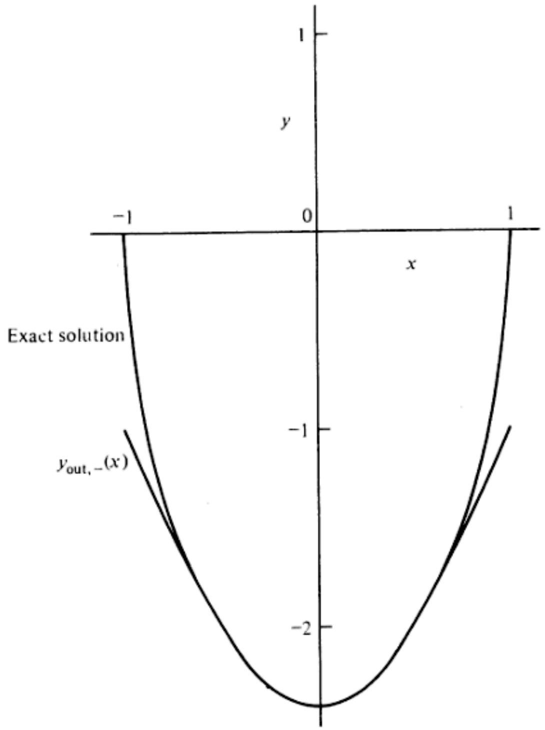

Figure 9.20 Same differential equation as in Fig. 9.19, but a different solution. y_{unif} becomes a good approximation to the plotted solution for the upper choice of signs.

Note that if $A = \pm\infty$ then there is no internal boundary-layer structure. However, for all finite values of A there is a narrow region in which Y rises abruptly to a sharp peak at which it attains a maximum value of $2\sqrt{2} - 1 \doteq 1.8$. Do you believe from this analysis that there are actual solutions to (9.7.7) having an internal boundary layer at $x = 0$? In fact, in Figs. 9.22 to 9.24 we see that for each solution in Figs. 9.19 to 9.21 there is another solution which is almost identical except that it exhibits a boundary layer at $x = 0$! What is more, the maximum in the boundary layer is close to 1.8.

Now that we have observed solutions having one internal boundary layer we may ask whether there exist solutions having multiple internal boundary layers. It is here that boundary-layer theory is no longer useful. Boundary-layer analysis shows (see Prob. 9.42) that it is consistent to have any number of internal layers at any location. However, boundary-layer theory cannot predict the number or the location of these boundary layers. In fact, for a given positive value of ε there are exactly $4(N + 1)$ solutions which have from 0 to N internal boundary layers at definite locations, where N is a finite number depending on ε. To determine N it is necessary to use some rather advanced phase-plane analysis to establish a kind of WKB quantization condition (see the References). In Figs. 9.25 to 9.28 we give some examples of solutions having several internal boundary layers. In Figs. 9.19 to 9.28 $\varepsilon = 0.01$.

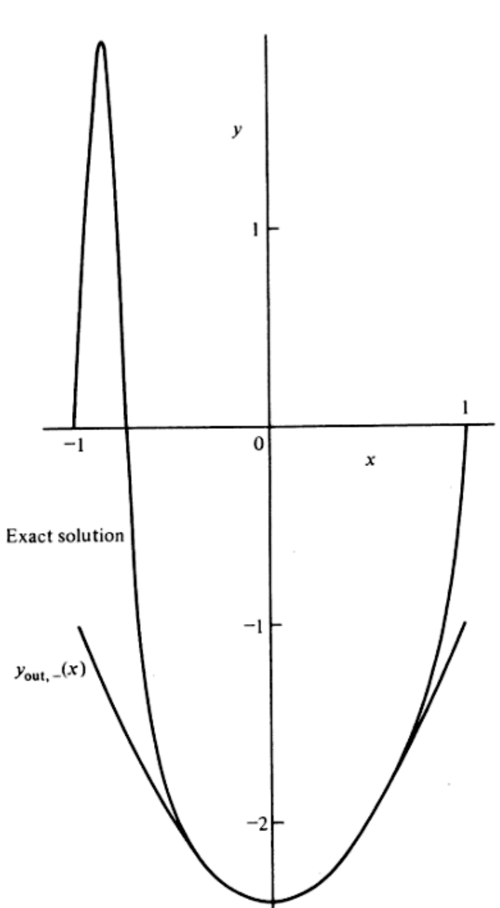

Figure 9.21 Same differential equation as in Fig. 9.19. y_{unif} in (9.7.12) is a good approximation to the plotted solution for one upper sign and one lower sign. There is also another solution which is the reflection about the y axis of the one shown here.

In the next example we use boundary-layer theory to study the approximate shape of a limit cycle in the phase plane.

Example 3 *Limit cycle of the Rayleigh oscillator.* The Rayleigh equation is

$$\varepsilon\frac{d^2y}{dt^2} - \left[\frac{dy}{dt} - \frac{1}{3}\left(\frac{dy}{dt}\right)^3\right] + y = 0. \tag{9.7.14}$$

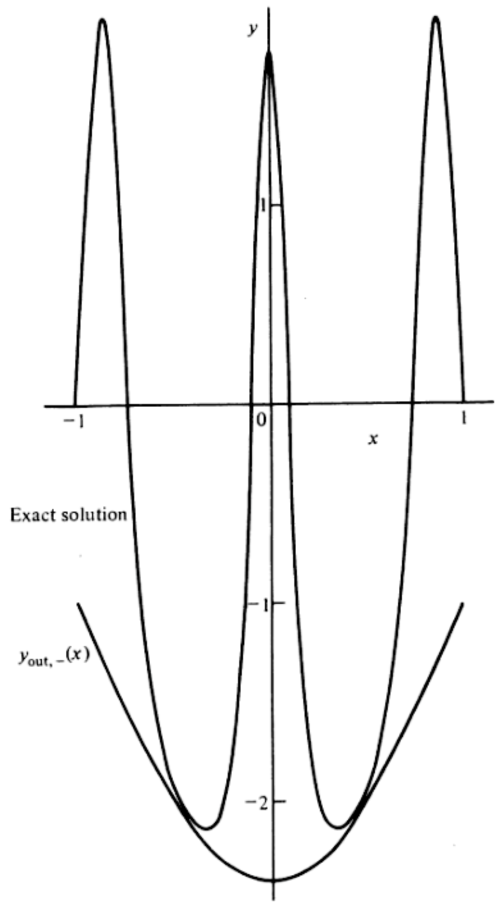

Figure 9.22 An exact solution to the boundary-value problem in (9.7.7). Apart from the internal boundary layer at $x = 0$, this solution is nearly identical to the solution in Fig. 9.19. The outer approximation $y_{\text{out},-}(x)$ in (9.7.8) is a good approximation to $y(x)$ between the boundary layers. The function $y_{\text{in}} = 3\sqrt{2}\,\text{sech}^2\,(2^{-1/4}x/\sqrt{\varepsilon}) - 1 - \sqrt{2}$ gives a good description of y in the internal boundary layer. It predicts that the maximum value of y_{in} is $2\sqrt{2} - 1 \doteq 1.8$, a result which is verified by this graph.

This autonomous equation can be rewritten as the system

$$\frac{dy}{dt} = z, \tag{9.7.15}$$

$$\varepsilon\frac{dz}{dt} = z - \frac{1}{3}z^3 - y, \tag{9.7.16}$$

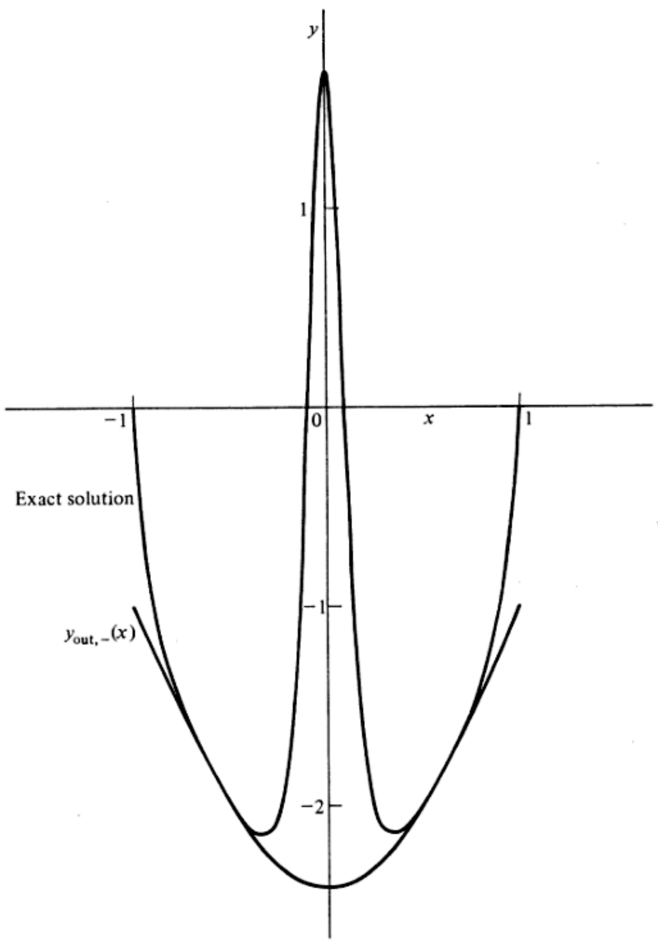

Figure 9.23 An exact solution to the boundary-value problem in (9.7.7). Apart from the internal boundary layer at $x = 0$, this solution is nearly identical to that in Fig. 9.20.

whose trajectories can be studied in the phase plane (y, z). The Rayleigh equation is an interesting model because, for any initial conditions and any $\varepsilon > 0$, $y(t)$ approaches a periodic solution as $t \to +\infty$. This periodic solution corresponds to a limit cycle in the phase plane (see Prob. 9.45). In Sec. 11.3 we study the approach to this periodic solution when ε is large using multiple-scale perturbation theory. In this example we consider the opposite limit $\varepsilon \to 0+$ and use boundary-layer theory to determine the shape of the limit cycle in the phase plane.

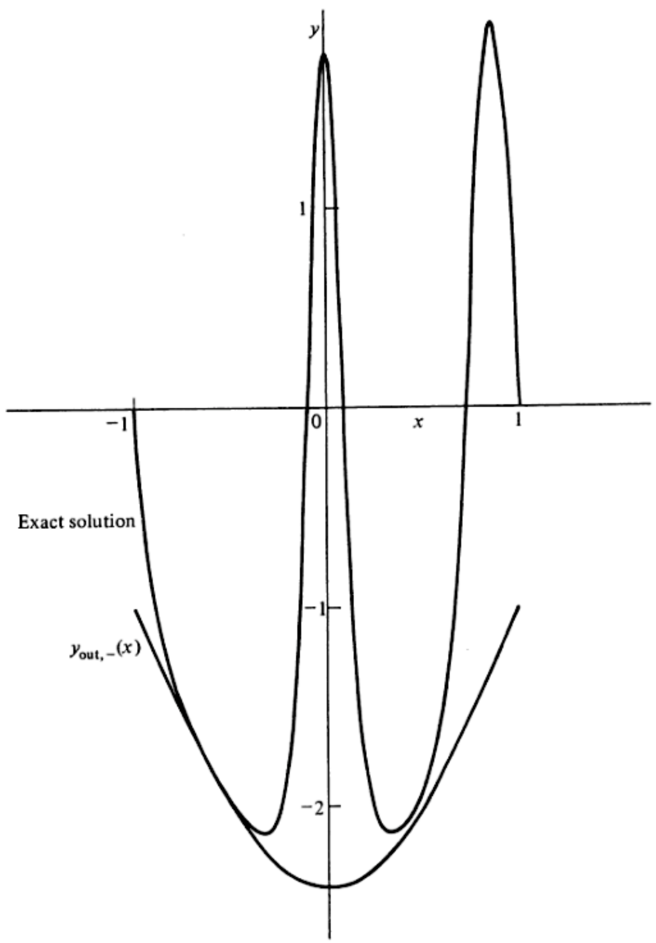

Figure 9.24 An exact solution to the boundary-value problem in (9.7.7). Apart from the internal boundary layer at $x = 0$, this solution is nearly identical to that in Fig. 9.21 reflected about the y axis.

We begin by dividing (9.7.16) by (9.7.15) to obtain

$$\varepsilon \frac{dz}{dy} = \frac{z - z^3/3 - y}{z} \tag{9.7.17}$$

and treat z as a function of the independent variable y. First, we look at the leading-order approximation to (9.7.17) as $\varepsilon \to 0+$. The outer limit is obtained by simply setting $\varepsilon = 0$. This

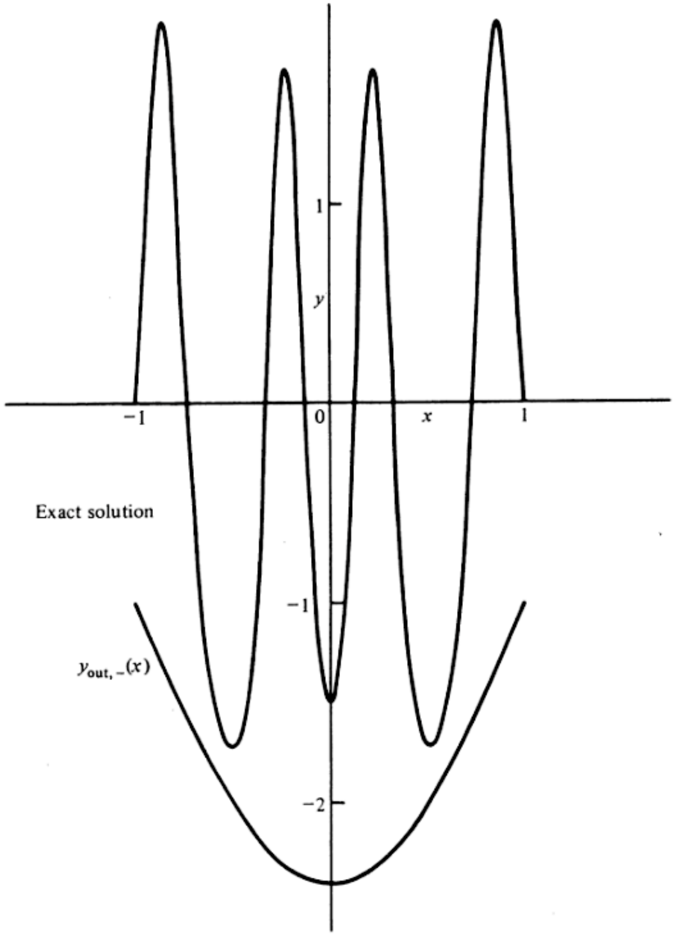

Figure 9.25 An exact solution to (9.7.7) having two internal boundary layers. Apart from the internal boundary layers, this solution is nearly identical to that in Fig. 9.19.

gives the algebraic equation

$$z_{\text{out}} - \tfrac{1}{3}z_{\text{out}}^3 = y. \tag{9.7.18}$$

This curve is plotted as a dashed line in Fig. 9.29. Observe that for $|y| < \frac{2}{3}$, there are three possible values of z_{out}. We will show that it is consistent to have a boundary-layer solution which joins the point $A(y = \frac{2}{3}, z = 1)$ to the point $B(y = \frac{2}{3}, z = -2)$ by the almost vertical line shown in Fig. 9.29 and to have a second boundary-layer solution which joins the point $C(y = -\frac{2}{3}, z = -1)$

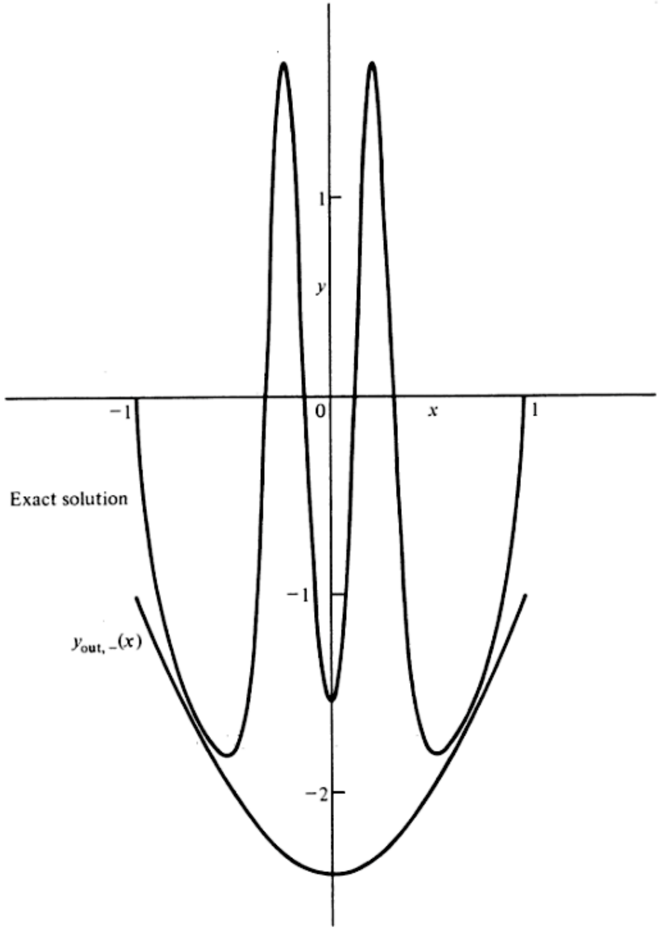

Figure 9.26 An exact solution to (9.7.7) analogous to that in Fig. 9.23 except that it has two internal boundary layers.

to the point $D(y = -\frac{2}{3}, z = 2)$. The limit cycle consists of the four segments DA and BC satisfying the outer equation (9.7.18) and AB and CD given by boundary-layer approximations.

To obtain the boundary-layer approximation joining the outer solutions DA and BC from A to B, we introduce the inner variable $Y = (y - \frac{2}{3})/\delta$ and obtain a distinguished limit $\delta = \varepsilon$:

$$\frac{dZ}{dY} = \frac{Z - Z^3/3 - \frac{2}{3} - \varepsilon Y}{Z}, \tag{9.7.19}$$

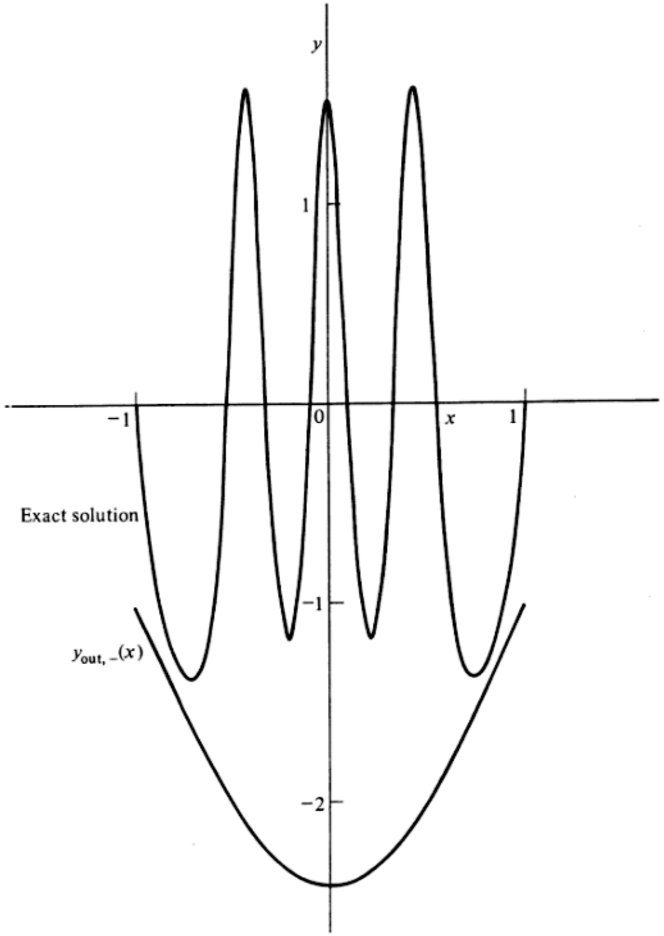

Figure 9.27 An exact solution to (9.7.7) analogous to that in Fig. 9.26 except that it has three internal boundary layers.

where $Z(Y) = z(y)$. The leading-order approximation Z_0 to Z satisfies

$$\frac{dZ_0}{dY} = \frac{Z_0 - Z_0^3/3 - \frac{2}{3}}{Z_0}. \tag{9.7.20}$$

The solution of this separable equation is

$$-\frac{2}{9}\ln|Z_0 + 2| + \frac{2}{9}\ln|Z_0 - 1| - \frac{1}{3(Z_0 - 1)} = -\frac{1}{3}(Y + c), \tag{9.7.21}$$

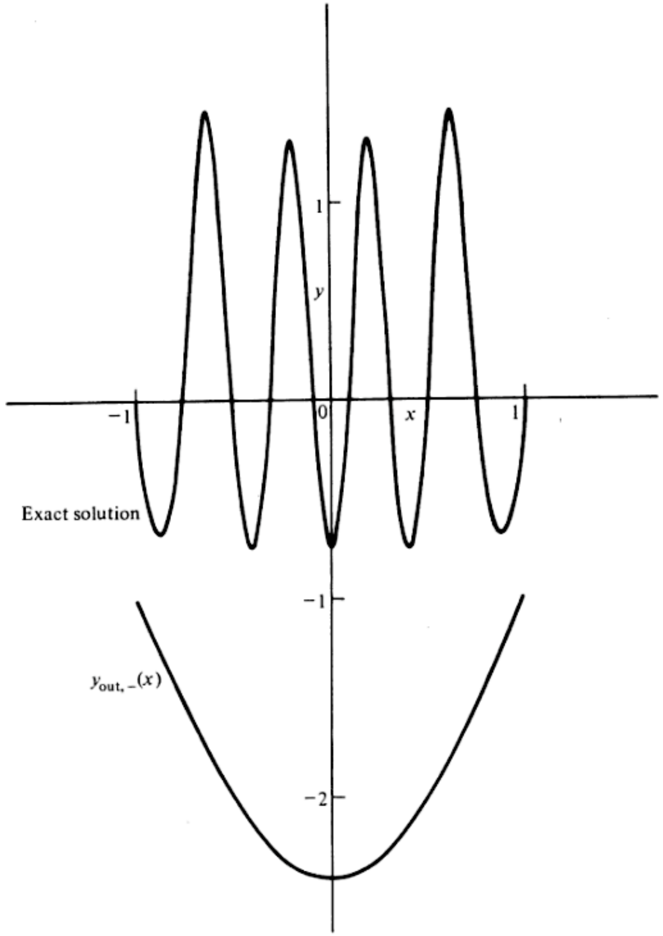

Figure 9.28 An exact solution to (9.7.7) analogous to that in Fig. 9.27 except that it has four internal boundary layers. Observe that as the number of internal boundary layers increases, the outer solution $y_{\text{out},-}(x)$ becomes a poorer approximation to $y(x)$ between boundary layers. Indeed, in this figure the regions of rapid variation of $y(x)$ are no longer localized in isolated boundary layers. Rather, individual boundary layers are so close together that $y(x)$ exhibits rapid variation on a global scale. This configuration is similar to that in Fig. 7.3. Boundary-layer analysis is useful only when the regions of rapid variation are localized.

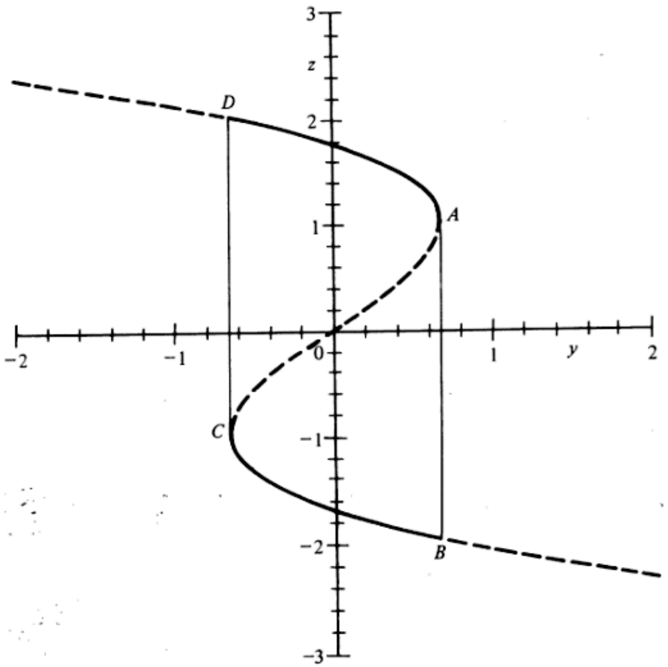

Figure 9.29 A plot of the leading-order outer solution in (9.7.18) to the Rayleigh oscillator (9.7.14). The outer solution is given by the solid lines connecting the points D to A and B to C. The dashed lines are additional segments of the graph of the algebraic functon in (9.7.18). Boundary-layer solutions connect the points A to B and C to D along the curves $y = \pm\frac{2}{3}$, respectively.

where c is a constant of integration. In the boundary-layer region this solution is a double-valued function of Y. As Z_0 decreases from 1, Y increases from $-\infty$; Y reaches a maximum at $Z_0 = 0$, $Y = 3c + \frac{2}{3}\ln 2 - 1$. As Z_0 decreases from 0 to -2, Y decreases from its maximum to $-\infty$. Asymptotic matching to lowest order is accomplished by the intermediate limits $(y, z_{out}) \to (\frac{2}{3}-, 1)$, $(Y, Z_0) \to (-\infty, 1)$ and $(y, z_{out}) \to (\frac{2}{3}-, -2)$, $(Y, Z_0) \to (-\infty, -2)$. These asymptotic matches complete the lowest-order boundary-layer analysis. Note that the constant c remains undetermined in leading order.

In the same way, it is possible to join the points C and D in Fig. 9.29 by a boundary-layer approximation; this approximation is given by (9.7.21) with the signs of Z_0 and Y reversed. The solution obtained in this way is periodic in t because the trajectory is closed in phase space. In Fig. 9.30 we plot the exact limit-cycle solution to the Rayleigh oscillator (9.7.14) in the (y, z) phase plane. Note how well leading-order boundary-layer theory predicts the shape of the limit cycle (see Fig. 9.29). In Fig. 9.31 we plot y and z versus t for the limit cycle in Fig. 9.30.

The leading-order boundary-layer solutions (9.7.18) and (9.7.21) can be used to compute a leading-order approximation to the period of the limit cycle of the Rayleigh equation as $\varepsilon \to 0+$. Since $z = dy/dt$, the period T of the limit cycle is given by

$$T = \oint \frac{dy}{z}, \tag{9.7.22}$$

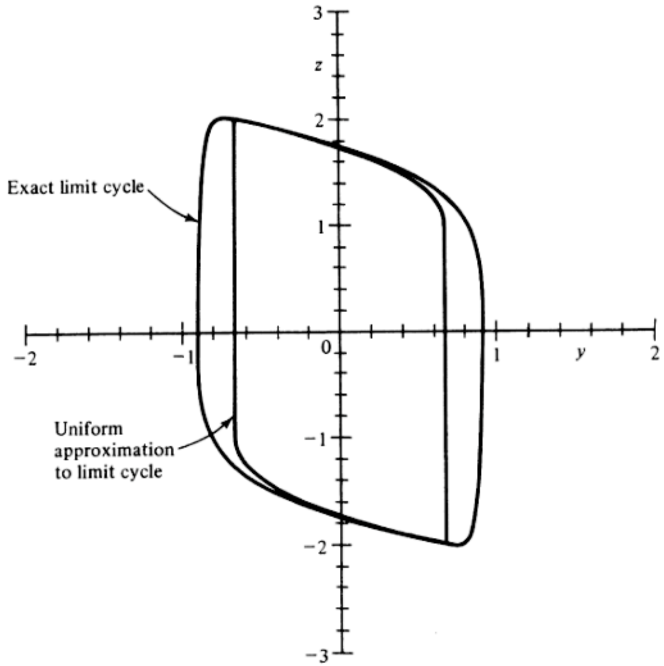

Figure 9.30 Comparison between the exact limit cycle of the Rayleigh oscillator (9.7.14) and the leading-order uniform approximation plotted in Fig. 9.29. The value of ε is 0.05.

where the integral is taken over the full limit cycle. The leading-order approximation to the period T is dominated by the contributions to the integral (9.7.22) from the outer solutions (9.7.18). The contribution to T from the boundary layer goes to 0 as $\varepsilon \to 0+$ because the width of the boundary layers in y is $O(\varepsilon)$. Therefore,

$$T \sim \int_{DA} \frac{dy}{z} + \int_{BC} \frac{dy}{z} = \int_2^1 \frac{(1-z^2)}{z}\,dz + \int_{-1}^{-2} \frac{(1-z^2)}{z}\,dz = 3 - 2\ln 2, \qquad \varepsilon \to 0+. \tag{9.7.23}$$

The full asymptotic expansion of the period $T(\varepsilon)$ of the limit cycles of (9.7.14) is very difficult to obtain. Dorodnicyn showed that

$$T(\varepsilon) \sim 3 - 2\ln 2 + 3\alpha\varepsilon^{2/3} + \tfrac{1}{6}\varepsilon \ln \varepsilon + \cdots, \qquad \varepsilon \to 0+, \tag{9.7.24}$$

where α is the smallest zero of Ai $(-t)$ ($\alpha \doteq 2.3381$). We check the accuracy of this expansion in Table 9.1. One may wonder why the series (9.7.24) for $T(\varepsilon)$ is so complicated when the leading-order boundary-layer analysis is so simple. The reason for the complexity of (9.7.24) becomes apparent when we attempt a higher-order boundary-layer analysis.

The next-order corrections to the outer solution (9.7.18) are found by expanding $z_{out} = z_0 + z_1 + \cdots$. Here the leading-order outer approximation satisfies (9.7.18), $z_0 - \frac{1}{3}z_0^3 = y$, and from (9.7.17), the equation for z_1 is $z_0\, dz_0/dy = (1 - z_0^2)z_1$. Thus,

$$z_1 = \frac{z_0}{(1 - z_0^2)^2}. \tag{9.7.25}$$

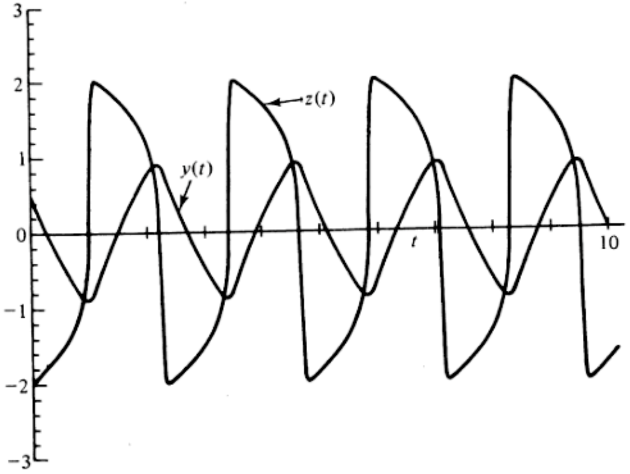

Figure 9.31 A plot of $y(t)$ and $z(t)$ versus t for the limit cycle of the Rayleigh oscillator (9.7.14) plotted in the phase plane in Fig. 9.30. The value of ε is 0.05.

To perform asymptotic matches at points A and B (see Fig. 9.29), we will need the expansions of z_{out} in the neighborhood of the points A and B:

$$z_{\text{out}} \sim 1 + \sqrt{\frac{2}{3} - y} + \frac{1}{6}\left(y - \frac{2}{3}\right) + \frac{\varepsilon}{4(\frac{2}{3} - y)} + \cdots, \qquad y \to \frac{2}{3}-, \varepsilon \to 0+, \tag{9.7.26}$$

$$z_{\text{out}} \sim -2 - \frac{1}{3}\left(y - \frac{2}{3}\right) + \cdots, \qquad y \to \frac{2}{3}-, \varepsilon \to 0+ \tag{9.7.27}$$

(see Prob. 9.46). The coefficient of $\sqrt{\frac{2}{3} - y}$ in (9.7.26) is positive because $z > z_0$ along DA; if z were less than z_0 along DA, then (9.7.25) would imply that $dz/dy > 0$ which is false.

The next-order boundary-layer approximation behaves in the neighborhood of the point A like

$$Z \sim 1 + \frac{1}{Y + c} + \frac{2 \ln |Y + c|}{3(Y + c)^2} + \frac{2 \ln 3}{3(Y + c)^2}$$

$$+ \varepsilon\left(-\frac{1}{4}Y^2 + \frac{1}{9}Y \ln Y\right) + \cdots, \qquad Y \to -\infty, \varepsilon Y^2 \to 0+, \tag{9.7.28}$$

and in the neighborhood of the point B like

$$Z \sim -2 - \tfrac{1}{3}\varepsilon Y, \qquad Y \to -\infty, \varepsilon Y \to 0+. \tag{9.7.29}$$

There is no trouble performing the asymptotic match at B; $Y = (y - \frac{2}{3})/\varepsilon$, so (9.7.27) and (9.7.29) match to first order in ε. On the other hand, (9.7.26) and (9.7.28) do *not* match at the point A; in particular, the term $\sqrt{\frac{2}{3} - y}$ in (9.7.26) is not present in (9.7.28).

This failure of boundary-layer analysis in higher order is remedied as in Example 6 of Sec. 9.5 by introducing a new scale and a new boundary layer in which the higher-order match near A can be accomplished. Surprisingly, it turns out that this new boundary-layer scale near A must be

Table 9.1 Comparison between the exact period $T(\varepsilon)$ of the Rayleigh oscillator and the asymptotic approximation (9.7.24) to $T(\varepsilon)$, $T(\varepsilon) \sim 3 - 2 \ln 2 + 3\alpha\varepsilon^{2/3} + O(\varepsilon \ln \varepsilon)$ $(\varepsilon \to 0+)$, for various values of ε

ε	Exact value of $T(\varepsilon)$	$3 - 2 \ln 2$	$3 - 2 \ln 2 + 3\alpha\varepsilon^{2/3}$
1	6.687	1.6137	8.6280
0.25	3.8155	1.6137	4.3973
0.04	2.3211	1.6137	2.4341
0.01	1.9155	1.6137	1.9393
0.0025	1.7355	1.6137	1.7429

thicker than the lowest-order boundary layer of width ε. To seek a new distinguished limit near the point A we set $y = \frac{2}{3} + \delta\hat{Y}$, $z = 1 + \eta\hat{Z}$, with $\hat{Y}$ and $\hat{Z}$ both of order 1 in the boundary layer.

Substituting into (9.7.17), we obtain

$$\frac{\varepsilon\eta}{\delta}\frac{d\hat{Z}}{d\hat{Y}} = -\frac{(\eta\hat{Z})^2 + (\eta\hat{Z})^3/3 + \delta\hat{Y}}{1 + \eta\hat{Z}}.$$

A new distinguished limit is obtained by choosing $\varepsilon\eta/\delta = \eta^2 = \delta$, so $\delta = \varepsilon^{2/3}$, $\eta = \varepsilon^{1/3}$. Thus, the thickness of the new intermediate boundary layer is $\varepsilon^{2/3}$. This explains why there is a term of order $\varepsilon^{2/3}$ in the expansion (9.7.24) for $T(\varepsilon)$. (See Prob. 9.48.)

PROBLEMS FOR CHAPTER 9

Section 9.1

(E) **9.1** Show that the solution to the asymptotic relation $\delta/\ln \delta = O(\varepsilon)$ $(\varepsilon \to 0+)$ satisfies $\delta = O(\varepsilon \ln \varepsilon)$ $(\varepsilon \to 0+)$.

(E) **9.2** Verify that $y_{\text{unif}}(x) - y(x) = O(\varepsilon)$ $(\varepsilon \to 0+;\ 0 \le x \le 1)$, where $y_{\text{unif}}(x)$ is given in (9.1.13) and $y(x)$ is the solution to (9.1.7).

(I) **9.3** (*a*) Show that if $a(x) < 0$ for $0 \le x \le 1$, then the solution to (9.1.7) has a boundary layer at $x = 1$.

(*b*) Find a uniform approximation with error $O(\varepsilon)$ to the solution (9.1.7) when $a(x) < 0$ for $0 \le x \le 1$.

(*c*) Show that if $a(x) > 0$, it is impossible to match to a boundary layer at $x = 1$.

(E) **9.4** Find leading-order uniform asymptotic approximations to the solutions of

(*a*) $\varepsilon y'' + (\cosh x)y' - y = 0$, $y(0) = y(1) = 1$ $(0 \le x \le 1)$,

(*b*) $\varepsilon y'' + (x^2 + 1)y' - x^3 y = 0$, $y(0) = y(1) = 1$ $(0 \le x \le 1)$, in the limit $\varepsilon \to 0+$.

(D) **9.5** Estimate the error between $y(0)$ and $y_{\text{in}}(0)$ in the limit as $\varepsilon \to 0+$ where $y(x)$ is the exact solution to (9.1.3) and $y_{\text{in}}(0)$ is given by (9.1.6).

(I) **9.6** Consider the initial-value problem $y' = (1 + \frac{1}{100}x^{-2})y^2 - 2y + 1$ $[y(1) = 1]$ on the interval $0 \le x \le 1$.

(*a*) Formulate this problem as a perturbation problem by introducing a suitable small parameter ε.

(*b*) Find an outer approximation correct to order ε (with errors of order ε^2). Where does this approximation break down?

(*c*) Introduce an inner variable and find the inner solution valid to order 1 (with errors of order ε). By matching to the outer solution find a uniformly valid approximation to $y(x)$ on the interval $0 \le x \le 1$. Estimate the accuracy of this approximation.

(*d*) Find the inner solution correct to order ε (with errors of order ε^2) and show that it matches to the outer solution correct to order ε.

(E) **9.7** How does the solution to (9.1.1) behave in the limit $\varepsilon \to 0-$?

Section 9.2

(I) **9.8** Use boundary-layer theory to find a uniform approximation with error of order ε^2 for the problem $\varepsilon y'' + y' + y = 0$ $[y(0) = e,\ y(1) = 1]$. Notice that there is no boundary layer in leading order, but one does appear in next order. Compare your solution with the exact solution to this problem.

(I) **9.9** Use boundary-layer methods to find an approximate solution to the initial-value problem $\varepsilon y'' + ay' + by = 0$ with $y(0) = 1$, $y'(0) = 1$, and $a > 0$. Show that the leading-order uniform approximation satisfies $y(0) = 1$ but not $y'(0) = 1$ for arbitrary b. Compare the leading-order uniform approximation with the exact solution to the problem when $a(x)$ and $b(x)$ are constants.

Section 9.3

(I) **9.10** Show that the matching region in nth-order boundary-layer theory for the problem discussed in Example 1 of Sec. 9.3 is $\varepsilon \ll x \ll \varepsilon^{n/(n+1)}$ $(\varepsilon \to 0+)$.

(I) **9.11** Obtain a uniform approximation accurate to order ε^2 as $\varepsilon \to 0+$ for the problem $\varepsilon y'' + (1 + x)^2 y' + y = 0$ $[y(0) = 1,\ y(1) = 1]$.

(I) **9.12** Verify (9.3.16).

(I) **9.13** Verify (9.3.18) and (9.3.19).

(I) **9.14** Show that the series (9.3.16) diverges like the series $\sum n!\, x^n$.

Clue: Try to find a recursion relation for the coefficient of $(1 + x)^{-2n-1}$ in the outer solution.

(I) **9.15** Find first-order uniform approximations valid as $\varepsilon \to 0+$ for the solutions of the differential equations given in Prob. 9.4.

(D) **9.16** Find second-order uniform approximations valid as $\varepsilon \to 0+$ for the solutions of the differential equations given in Prob. 9.4.

Section 9.4

(I) **9.17** For what real values of the constant α does the singular perturbation problem $\varepsilon y''(x) + y'(x) - x^{\alpha} y(x) = 0$, $y(0) = 1$, $y(1) = 1$ $(0 \le x \le 1)$ have a solution with a boundary layer near $x = 0$ as $\varepsilon \to 0+$?

Clue: Perform a local analysis to decide if the problem has a solution. Show that if $\alpha \le -2$, there is no solution that behaves like $y(x) \sim 1$ $(x \to 0+)$. Also show that if $-2 < \alpha < -1$, the thickness of the boundary layer at $x = 0$ is $\varepsilon^{1/(2+\alpha)}$ and that if $\alpha > -1$, the boundary layer at $x = 0$ has thickness ε.

(D) **9.18** Consider the problem (discussed by Cole) $\varepsilon y'' + \sqrt{x}\, y' - y = 0$ $[y(0) = 0,\ y(1) = e^2]$.

(*a*) Show that there is a boundary layer of thickness $\varepsilon^{2/3}$ at $x = 0$.

(*b*) Show that a leading-order uniform approximation to $y(x)$ is $y_{\text{unif},0}(x) = \exp(2\sqrt{x}) - 1 + [(\frac{3}{2})^{1/3}/\Gamma(\frac{2}{3})] \int_0^{x\varepsilon^{-2/3}} e^{-2t^{3/2}/3}\, dt$.

(*c*) Show that the next correction to the outer solution is $y_{\text{out}}(x) \sim \exp(2\sqrt{x}) + \varepsilon(-1/2x + 2/\sqrt{x} - \frac{3}{2}) \exp(2\sqrt{x}) + \cdots$ $(\varepsilon \to 0+)$.

(*d*) Find integral representations for the first four terms of the inner expansion in powers of $\varepsilon^{1/3}$. That is, calculate $Y_{\text{in}} \sim Y_0 + \varepsilon^{1/3} Y_1 + \varepsilon^{2/3} Y_2 + \varepsilon Y_3 + \cdots$ $(\varepsilon \to 0+)$.

(*e*) Show that the first two terms in the outer expansion match with the first four terms in the inner expansion.

(I) **9.19** Find a lowest-order uniform approximation to the boundary-value problem $\varepsilon y'' + y' \sin x + y \sin(2x) = 0$ $[y(0) = \pi,\ y(\pi) = 0]$.

(I) **9.20** Consider the problem $\varepsilon y'' + x^{\alpha} y' + y = 0$ with $y(0) = y(1) = 1$ as $\varepsilon \to 0+$. For what values of α is there a boundary layer at $x = 0$? What is the thickness of the boundary layer?

Section 9.5

(I) **9.21** Complete Example 1 of Sec. 9.5.

(*a*) Show that

$$\frac{d^3 \bar{Y}_n}{d\bar{X}^3} - \frac{d\bar{Y}_n}{d\bar{X}} = \begin{cases} 0, & n = 0, \\ \bar{Y}_0, & n = \frac{1}{2}, \\ \bar{Y}_{n-1/2} - \bar{X}\bar{Y}_{n-1}, & n > \frac{1}{2}. \end{cases}$$

(*b*) Next, show that in terms of the inner variable X, the outer expansion (9.5.15) becomes $y_{\text{out}} = \sqrt{e} + \varepsilon^{1/2}\sqrt{e}(1 - \bar{X}) + \varepsilon\sqrt{e}\,\bar{X}^2 + O(\varepsilon\bar{X}, \varepsilon^{3/2}\bar{X}^3)$.

(*c*) Finally, show that the first three solutions to these equations which satisfy $y(1) = 1$ and which match asymptotically to $y_{\text{out}}(x)$ are

$$\bar{Y}_0(\bar{X}) = (1 - \sqrt{e})e^{-\bar{X}} + \sqrt{e},$$

$$\bar{Y}_{1/2}(\bar{X}) = [-\sqrt{e} - \tfrac{1}{2}(\sqrt{e} - 1)\bar{X}]e^{-\bar{X}} - \sqrt{e}\,\bar{X} + \sqrt{e}.$$

$$\bar{Y}_1(\bar{X}) = [\tfrac{1}{8}(\sqrt{e} - 1)\bar{X}^2 - \tfrac{1}{8}(\sqrt{e} + 3)\bar{X} - \tfrac{7}{4}\sqrt{e}]e^{-\bar{X}} + \sqrt{e}\,\bar{X}^2 - \sqrt{e}\,\bar{X} + \tfrac{7}{4}\sqrt{e}.$$

(*d*) Verify (9.5.17).

(I) **9.22** Verify (9.5.37).

(I) **9.23** Consider the boundary-value problems:

(*a*) $\varepsilon y'' + y'/x + y = 0$ $[y(-1) = 2e^{-1/2},\ y(1) = e^{-1/2}]$,

(*b*) $\varepsilon y'' + y'/x^2 + y = 0$ $[y(0) = 0,\ y(1) = e^{-1/3}]$,

as $\varepsilon \to 0+$. Do these problems have a solution? If so, find a leading-order approximation to the solution.

(D) **9.24** Show that the outer expansion (9.5.32) diverges.

Clue: One way to do this is to study the exact solution to (9.5.31), which is $y = e^{-1/2} x^{(\varepsilon-1)/2\varepsilon} J_{1/2-1/2\varepsilon}(x/\sqrt{\varepsilon})/J_{1/2-1/2\varepsilon}(1/\sqrt{\varepsilon})$.

(I) **9.25** (*a*) Verify (9.5.43).

(*b*) Show that $\bar{\alpha}_0 = 0$ and $\bar{\beta}_0 = 2$ in (9.5.43).

(I) **9.26** (*a*) Find a leading-order uniform approximation to the solution to the problem $\varepsilon(x + \varepsilon^2)y'' + xy' + y = 0$ $[y(0) = y(1) = 1]$ as $\varepsilon \to 0+$.

(*b*) Show that the term $\varepsilon^3 y''$ is always a small perturbation, even when $x \ll \varepsilon^2$ $(\varepsilon \to 0+)$.

(D) **9.27** Find a uniform approximation accurate to order ε for the problem $\varepsilon y'' + (1 + 2\varepsilon/x + 2\varepsilon^3/x^2)y' + 2y/x = 0$ $[y(0) = y(1) = 1]$ as $\varepsilon \to 0+$. Show that:

(*a*) $y(x) \sim \alpha + \beta e^{-2\varepsilon^2/x}$ $(x \to 0+)$, so that $y(0)$ is finite and it is appropriate to specify a nonzero value for $y(0)$;

(*b*) $y_{\text{out}}(x) \sim x^{-2} + 2\varepsilon(x^{-3} - x^{-2}) + \cdots$ $(\varepsilon \to 0+)$;

(*c*) distinguished limits are $\delta = \varepsilon$ and $\delta = \varepsilon^2$;

(*d*) inner and inner-inner expansions exist at $x = 0$ which match to each other; the inner-inner expansion satisfies the boundary condition at $x = 0$ and the inner expansion matches to $y_{\text{out}}(x)$.

(I) **9.28** Use boundary-layer theory to solve $\varepsilon y'' + a(x)y' + b(x)y = \delta(x)$ $[y(x) = 0;\ x < 0]$ to leading order in ε as $\varepsilon \to 0+$, where $a(x) > 0$ for $x \ge 0$.

Section 9.6

(I) **9.29** Find leading-order uniform approximations to the solutions of the following problems in th limit $\varepsilon \to 0+$:

(*a*) $\varepsilon y'' - 2(\tan x)y' + y = 0$ $[y(\pm 1) = 1]$;

(*b*) $\varepsilon y'' + 2(\tan x)y' - y = 0$ $[y(\pm 1) = 1]$;

(*c*) $\varepsilon y'' + (\sinh x)y' + J_0(x)y = 0$ $[y(\pm 1) = 0]$;

(*d*) $\varepsilon y'' + xy' = x \cos x$ $[y(\pm 1) = 2]$;

(*e*) $\varepsilon y'' - xy' - (3 + x)y = 0$ $[y(\pm 1) = 1]$;

(*f*) $\varepsilon y'' + (\ln x)y' - x(\ln x)y = 0$ $[y(\frac{1}{2}) = y(\frac{3}{2}) = 1]$.

(I) **9.30** (a) Verify (9.6.13).
(b) Verify (9.6.16).

(D) **9.31** Use first-order boundary-layer theory to find a uniform asymptotic approximation to the solution of $\varepsilon y'' + (x^2 - 1)y' + (x^2 - 1)^2 y = 0$ [$y(-2) = A$, $y(0) = B$].

(a) This problem is an example of case IV because $\alpha = -2$, $\beta = 0$ at $x = -1$. To leading order in the internal layer near $x = -1$, let $X = (x + 1)/\sqrt{\varepsilon}$ to obtain $d^2 Y_{in}/dX^2 - 2X\, dY_{in}/dX = 0$. Show that $Y_{in} = C_1 + C_2 \int_0^X e^{t^2}\, dt$, so that matching to the outer solutions requires that $C_2 = 0$ but leaves C_1 undetermined.

(b) Now consider the next-order correction to the inner equation: $d^2 Y_{in}/dX^2 - 2X\, dY_{in}/dX + 4\varepsilon X^2 Y_{in} = 0$. Show that an approximate solution to this equation is $Y_{in} = e^{X^2/2}\{C_1 D_\varepsilon[X\sqrt{2}\,(1 - \varepsilon)] + C_2 D_\varepsilon[-X\sqrt{2}\,(1 - \varepsilon)]\}$. Use this result to argue that both constants C_1 and C_2 may now be determined by asymptotic matching, just as in case II.

(D) **9.32** Examples of singular perturbation problems where higher-order boundary-layer theory resolves the ambiguity of case IV of Sec. 9.6 are

(1) $\varepsilon y'' - xy' + (n + \beta x)y = 0$ [$y(-1) = A$, $y(1) = B$],
(2) $\varepsilon y'' - xy' + (n + \beta x^2)y = 0$ [$y(-1) = A$, $y(1) = B$],

where $\varepsilon \to 0+$, $\beta \neq 0$, and $n = 0, 1, 2, \ldots$.

(a) For each of these differential equations, show that the leading-order approximation to the internal layer solution $Y(X)$, where $X = x/\sqrt{\varepsilon}$, is $Y_0(X) = e^{X^2/4}[K_1 D_n(X) + K_2 D_{-n-1}(iX)]$. The difficulty with the leading-order boundary-layer analysis given in Sec. 9.6 is that when $K_2 = 0$, $Y_0(X)$ grows algebraically with X as $X \to \pm\infty$, so matching is possible for any value of K_1.

(b) Show that there is a higher-order approximation to $Y(X)$ that grows exponentially as $|X| \to \infty$ if either of K_1 or K_2 is not zero. In what order of perturbation theory does $Y(X)$ first grow exponentially?

(c) We may conclude that in order for a match to be possible, it is necessary that $K_1 = K_2 = 0$. Show that the boundary-layer analysis of these problems then reverts back to that of case II of Sec. 9.6.

(D) **9.33** Examples of case IV of Sec. 9.6 that are not resolved by higher-order boundary-layer analysis are

(1) $\varepsilon y'' - x(1 + x)y' + xy = 0$ [$y(-\frac{1}{2}) = A$, $y(\frac{1}{2}) = B$],
(2) $\varepsilon y'' - x(1 - x^2)y' - x(x + 1)y = 0$ [$y(-\frac{1}{2}) = A$, $y(\frac{1}{2}) = B$].

Show that higher-order corrections to the leading-order internal-layer solution $Y_0(X) = 1$ do not grow faster than algebraically as $X \to \pm\infty$. Thus, conclude that boundary-layer theory remains ambiguous to all orders in ε^n as $\varepsilon \to 0+$. The resolution of this dilemma will be given in Prob. 10.28 using WKB analysis.

(I) **9.34** Use boundary-layer techniques to find a leading-order uniform approximation to the solution of $\varepsilon y''(x) + 2xy'(x) - 4x^2 y(x) = 0$ $(-1 \le x \le 1)$, with $y(-1) = 0$, $y(+1) = e$ in the limit $\varepsilon \to 0+$.

(I) **9.35** Find leading-order uniform approximations to the solutions of the following problems:

(a) $\varepsilon y'' - (x + x^3)y' - 2y = 0$ [$y(-1) = A$, $y(1) = B$],
(b) $\varepsilon y'' + (x + x^3)y' + 2y = 0$ [$y(-1) = A$, $y(1) = B$],
(c) $\varepsilon y'' + (x^2 - 1)y' + (x^2 - 1)^2 y = 0$ [$y(0) = A$, $y(2) = B$],

in the limit $\varepsilon \to 0+$.

(I) **9.36** Consider the boundary-value problem $\varepsilon y'' - xy' + y = 0$ [$y(-1) = -1$, $y(1) = 1$].

(a) Show that there is a one-parameter family of solutions determined by boundary-layer theory.
(b) Solve the original differential equation exactly and show that the solution is actually unique.

Section 9.7

(E) **9.37** Show that it is inconsistent to have a boundary layer at $x = 1$ in (9.7.1) as $\varepsilon \to 0+$.

(I) **9.38** Solve (9.7.1) correct to first order in ε.

9.39 Consider the boundary-value problem $\varepsilon y'' - y' + e^y = 0$ [$y(0) = A$, $y(1) = 0$].

(I) (a) Find a leading-order uniform approximation to the solution when $A < 0$.

(D) (b) Discuss what happens when $A \ge 0$. Is there a solution? If so, can you find it using boundary-layer theory?

(D) **9.40** Discuss the qualitative nature of the solutions to:

(a) $\varepsilon y'' - y' + y^2 = 0$ [$y(0) = y(1) = 1$] in the limit $\varepsilon \to 0+$;
(b) $\varepsilon y'' - y' + 1/y = 0$ [$y(0) = \sqrt{2}$, $y(1) = 0$] in the limit $\varepsilon \to 0+$.

(D) **9.41** Do the boundary-value problems,

(a) $\varepsilon y'' - (y')^2 + e^y = 0$ [$y(0) = y(1) = 1$],
(b) $\varepsilon y'' - (y')^2 + y^2 = 0$ [$y(0) = y(1) = 1$],

have solutions when $\varepsilon \to 0+$? If they do, find a leading-order uniform approximation. Note that these problems can be solved exactly using the methods of Sec. 1.7.

Clue: Perform a boundary-layer analysis in the phase plane (y, y').

(I) **9.42** Show that it is consistent for the solution to (9.7.7) to exhibit an internal boundary layer at any value of x $(-1 < x < 1)$.

(D) **9.43** Find a leading-order approximation to the solution of $\varepsilon\, d/dx[(2 - x)^3 y\, dy/dx] - (2 - x)\, dy/dx + y = 0$, with $y(0) = y(1) = 1$ as $\varepsilon \to 0+$.

(D) **9.44** Use the approach of Example 2 of Sec. 9.7 to study the solutions of $\varepsilon y'' - xy - y^2 = 0$ [$y(1) = A$, $y(0) = 0$] in the limit $\varepsilon \to 0+$ for various values of A.

(D) **9.45** Use phase-plane arguments to show that there exists a unique limit cycle of the Rayleigh equation (9.7.14) which attracts trajectories satisfying arbitrary initial conditions.

(I) **9.46** Derive the expansions in (9.7.26) and (9.7.27).

(D) **9.47** Show that although boundary layers are possible near any value of y with $|y| < \frac{2}{3}$, it is not possible to construct a closed trajectory of (9.7.14) except for the limit cycle discussed in Example 3 of Sec. 9.7.

(D) **9.48** Compute the term proportional to $\varepsilon^{2/3}$ in the expansion (9.7.24) for the period $T(\varepsilon)$ of the limit cycle of the Rayleigh equation.

Clue: Perform an asymptotic match of the outer solution to the intermediate layer of width $\varepsilon^{2/3}$ and then match the intermediate layer to the inner boundary layer of width ε. Show that the leading-order intermediate-layer solution satisfies a Riccati differential equation whose solution is a ratio of Airy functions. Argue that the coefficient of Bi is zero by matching to the outer solution [because the coefficient of $\sqrt{\frac{2}{3} - y}$ in (9.7.26) is positive]. Then match to the inner boundary layer as $\bar{Y} \to \alpha$, where α is the smallest solution of $\mathrm{Ai}(-t) = 0$. Finally, compute the $\varepsilon^{2/3}$ term in (9.7.24) by suitably evaluating the integral (9.7.22) for $T(\varepsilon)$.

(D) **9.49** Perform an asymptotic boundary-layer analysis of the limit cycle of the Van der Pol equation $\varepsilon\, d^2 z/dt^2 - (1 - z^2)\, dz/dt + z = 0$ by obtaining appropriate inner, outer, and intermediate expansions directly from the differential equation (not transformed to the phase plane). Show that the period $T(\varepsilon)$ of this limit cycle satisfies $T(\varepsilon) \sim 3 - 2\ln 2 + 3\alpha\varepsilon^{2/3}$ $(\varepsilon \to 0+)$, where α is the smallest zero of $\mathrm{Ai}(-t)$. The period of the limit cycles of the Van der Pol and Rayleigh equations are the same because the substitution $z = dy/dt$ converts the Rayleigh equation into the Van der Pol equation.

(D) **9.50** Consider the nonlinear perturbation problem $y'' + 2y'/x + \varepsilon y y' = 0$ [$y(1) = 0$, $y(+\infty) = 1$] as $\varepsilon \to 0+$.

(a) Find the form of the outer solution accurate to order ε. Show that the problem is a singular perturbation problem even though ε does not multiply the highest derivative term. The problem is singular because the domain is infinite.

(b) Argue that there must be an "inner" expansion near $x = \infty$. Find its scale by setting $X = \delta x$ and seeking a dominant balance. Express the inner solution to order ε in terms of exponential integrals $E_n(t) = \int_t^\infty e^{-s} s^{-n}\, ds$.

(c) Try to perform an asymptotic match to order ε by taking the intermediate limit $x \to +\infty$, $X \to 0+$. The asymptotic expansion (6.2.11) of $E_n(t) \equiv \Gamma(1 - n, t)$ as $t \to 0+$ is helpful. Show that no match is possible.

(d) Argue that terms of order $\varepsilon \ln(1/\varepsilon)$ must be included in the outer expansion for matching to succeed. Introduce this intermediate-order term and show that the asymptotic match between inner and outer expansions can now be completed.

CHAPTER
TEN
WKB THEORY

When you follow two separate chains of thought, Watson, you will find some point of intersection which should approximate the truth.

—Sherlock Holmes, *The Disappearance of Lady Francis Carfax*
Sir Arthur Conan Doyle

(E) 10.1 THE EXPONENTIAL APPROXIMATION FOR DISSIPATIVE AND DISPERSIVE PHENOMENA

WKB theory is a powerful tool for obtaining a global approximation to the solution of a linear differential equation whose highest derivative is multiplied by a small parameter ε; it contains boundary-layer theory as a special case.

The WKB approximation to a solution of a differential equation has a simple structure. The exact solution may be some unknown function of overwhelming complexity; yet, the WKB approximation, order by order in powers of ε, consists of exponentials of elementary integrals of algebraic functions, and well-known special functions, such as the Airy function or parabolic cylinder function. WKB approximation is suitable for linear differential equations of any order, for initial-value and boundary-value problems, and for eigenvalue problems. It may also be used to evaluate integrals of the solution of a differential equation. The limitation of conventional WKB techniques is that they are only useful for linear equations.

Dissipative and Dispersive Phenomena

In our study of boundary-layer theory we have shown how to construct an approximate solution to a differential equation containing a small parameter ε. This construction requires one to match slowly varying outer solutions to rapidly varying inner solutions. [In perturbation theory a *slowly varying function* changes its value by $O(1)$ over an interval of size $O(1)$ as $\varepsilon \to 0+$ while a *rapidly varying function* changes its value by $O(1)$ over an interval whose size approaches 0 as $\varepsilon \to 0+$.]

An outer solution remains smooth if we allow ε to approach $0+$. But in this limit an inner solution becomes discontinuous across the boundary layer because the thickness of the boundary layer tends to 0. We thus say that the solution

suffers a *local breakdown* at the boundary layer as $\varepsilon \to 0+$. A local breakdown occurs where the approximate solution is exponentially increasing or decreasing. This kind of behavior is called *dissipative* because the rapidly varying component of the solution decays exponentially (dissipates) away from the point of local breakdown. The solution of a differential equation having a strong positive or negative damping term (like ay' in $\varepsilon y'' + ay' + by = 0$) typically exhibits dissipative behavior.

Some differential equations with small parameters have solutions which exhibit a *global breakdown*. For example, the boundary-value problem

$$\varepsilon y'' + y = 0, \qquad y(0) = 0,\ y(1) = 1, \tag{10.1.1}$$

has the exact solution

$$y(x) = \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}, \qquad \varepsilon \neq (n\pi)^{-2}, \tag{10.1.2}$$

which becomes rapidly oscillatory for small ε (see Fig. 7.3) and discontinuous when $\varepsilon \to 0+$. The breakdown is global because it occurs throughout the finite interval $0 \le x \le 1$. A global breakdown is typically associated with rapidly oscillatory, or *dispersive*, behavior. A dispersive solution is wavelike with very small and slowly changing wavelengths and slowly varying amplitudes as functions of x.

Boundary-layer techniques are not powerful enough to handle dispersive phenomena. To see why, let us try to solve (10.1.1) using boundary-layer methods. Setting $\varepsilon = 0$ in (10.1.1) gives the outer solution $y_{\text{out}}(x) = 0$, which is obviously a terrible approximation to the actual solution in (10.1.2). The actual solution in Fig. 7.3 looks like a sequence of internal boundary layers with no outer solution at all. Even for this very simple problem, boundary-layer analysis is insufficient.

From our understanding of Chap. 9 we can intuit that it is the absence of a one-derivative term which causes the global breakdown of the solution to (10.1.1). In Sec. 9.6 we showed that internal boundary layers may occur in the solution of $\varepsilon y'' + a(x)y' + b(x)y = 0$ $[y(0) = A,\ y(1) = B]$ at isolated points for which $a(x) = 0$. When $a(x) \equiv 0$ on an interval, it is not surprising to find that the solution is rapidly varying on the entire interval. Fortunately, WKB theory provides a simple and general approximation method for linear differential equations which treats dissipative and dispersive phenomena equally well.

The Exponential Approximation

Dissipative and dispersive phenomena are both characterized by exponential behavior, where the exponent is real in the former case and imaginary in the latter case. Thus, for a differential equation that exhibits either or both kinds of behavior, it is natural to seek an approximate solution of the form

$$y(x) \sim A(x)e^{S(x)/\delta}, \qquad \delta \to 0+. \tag{10.1.3}$$

The *phase* $S(x)$ is assumed nonconstant and slowly varying in a breakdown region. When S is real, there is a boundary layer of thickness δ; when S is imaginary, there is a region of rapid oscillation characterized by waves having wavelength of order δ. When $S(x)$ is constant, the behavior of $y(x)$, which is characteristic of an outer solution in boundary-layer theory, is expressed by the slowly varying *amplitude* function $A(x)$.

The exponential approximation in (10.1.3) is conventionally known as a *WKB approximation*, named after Wentzel, Kramers, and Brillouin who popularized the theory. However, credit should also be given to many others including Rayleigh and Jeffreys who contributed to its early development.

Formal WKB Expansion

The exponential approximation in (10.1.3) is not in a form most suitable for deriving asymptotic approximations because the amplitude and phase functions $A(x)$ and $S(x)$ depend implicitly on δ. It is best to represent the dependences of these functions on δ explicitly by expanding $A(x)$ and $S(x)$ as series in powers of δ. We can then combine these two series in a single exponential power series of the form

$$y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right], \qquad \delta \to 0. \tag{10.1.4}$$

This expression is the starting formula from which all WKB approximations are derived.

Example 1 *Approximate solution to a Schrödinger equation.* A second-order homogeneous linear differential equation is in Schrödinger form if the y' term is absent. The approximate solutions to the Schrödinger equation

$$\varepsilon^2 y'' = Q(x)y, \qquad Q(x) \neq 0, \tag{10.1.5}$$

are easy to find using WKB analysis when ε is small. We merely substitute (10.1.4) into (10.1.5). Differentiating (10.1.4) twice gives

$$\begin{aligned} y' &\sim \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'\right)\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n\right), && \delta \to 0,\\ y'' &\sim \left[\frac{1}{\delta^2}\left(\sum_{n=0}^{\infty}\delta^n S_n'\right)^2 + \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''\right]\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n\right), && \delta \to 0. \end{aligned} \tag{10.1.6}$$

Next, we substitute (10.1.6) into (10.1.5) and divide off the exponential factors:

$$\frac{\varepsilon^2}{\delta^2}S_0'^2 + \frac{2\varepsilon^2}{\delta}S_0'S_1' + \frac{\varepsilon^2}{\delta}S_0'' + \cdots = Q(x). \tag{10.1.7}$$

The largest term on the left side of (10.1.7) is $\varepsilon^2 S_0'^2/\delta^2$. By dominant balance this term must have the same order of magnitude as $Q(x)$ on the right side. (Here we have used the assumption that $Q \neq 0$.) Thus, δ is proportional to ε and for simplicity we choose $\delta = \varepsilon$. As in boundary-layer theory, the small scale parameter δ is determined by a *distinguished limit* (see Sec. 9.3).

Setting $\delta = \varepsilon$ in (10.1.7) and comparing powers of ε gives a sequence of equations which determine $S_0, S_1, S_2, \ldots$:

$$S_0'^2 = Q(x), \tag{10.1.8}$$

$$2S_0'S_1' + S_0'' = 0, \tag{10.1.9}$$

$$2S_0'S_n' + S_{n-1}'' + \sum_{j=1}^{n-1} S_j'S_{n-j}' = 0, \qquad n \geq 2. \tag{10.1.10}$$

The equation for S_0 (10.1.8) is called the *eikonal* equation; its solution is

$$S_0(x) = \pm\int^x \sqrt{Q(t)}\,dt. \tag{10.1.11}$$

The equation for S_1 (10.1.9) is called the *transport* equation; its solution, apart from an additive constant, is

$$S_1(x) = -\tfrac{1}{4}\ln Q(x). \tag{10.1.12}$$

Combining (10.1.11) and (10.1.12) gives a pair of approximate solutions to the Schrödinger equation (10.1.5), one for each sign of S_0. The general solution is a linear combination of the two:

$$\begin{aligned} y(x) \sim{}& c_1 Q^{-1/4}(x)\exp\left[\frac{1}{\varepsilon}\int_a^x dt\sqrt{Q(t)}\right]\\ &+ c_2 Q^{-1/4}(x)\exp\left[-\frac{1}{\varepsilon}\int_a^x dt\sqrt{Q(t)}\right], \qquad \varepsilon \to 0, \end{aligned} \tag{10.1.13}$$

where c_1 and c_2 are constants to be determined from initial or boundary conditions and a is an arbitrary but fixed integration point. This expression is the leading-order WKB approximation to the solution of (10.1.5); it differs from the exact solution by terms of order ε in regions where $Q(x) \neq 0$.

A more accurate approximation to $y(x)$ may be constructed from the higher terms in the WKB series. The next four terms, as computed from (10.1.10) by repeated differentiation, are

$$S_2 = \pm\int^x \left[\frac{Q''}{8Q^{3/2}} - \frac{5(Q')^2}{32Q^{5/2}}\right]dt, \tag{10.1.14}$$

$$S_3 = -\frac{Q''}{16Q^2} + \frac{5Q'^2}{64Q^3}, \tag{10.1.15}$$

$$S_4 = \pm\int^x \left[\frac{d^4Q/dx^4}{32Q^{5/2}} - \frac{7Q'Q'''}{32Q^{7/2}} - \frac{19(Q'')^2}{128Q^{7/2}} + \frac{221Q''(Q')^2}{256Q^{9/2}} - \frac{1{,}105(Q')^4}{2{,}048Q^{11/2}}\right]dt, \tag{10.1.16}$$

$$S_5 = -\frac{d^4Q/dx^4}{64Q^3} + \frac{7Q'Q'''}{64Q^4} + \frac{5(Q'')^2}{64Q^4} - \frac{113(Q')^2Q''}{256Q^5} + \frac{565(Q')^4}{2{,}048Q^6}. \tag{10.1.17}$$

A discussion of the structure and properties of these expressions is given in Sec. 10.7.

Example 2 *Solution of $\varepsilon y'' + y = 0$ $[y(0) = 0,\ y(1) = 1]$ as $\varepsilon \to 0+$.* Even though it is not possible to solve this problem using boundary-layer theory, the WKB approximation in (10.1.13), with ε replaced by $\sqrt{\varepsilon}$, leads to the exact solution. For this problem $Q(x) = -1$. Thus, (10.1.13) reduces to $y(x) = c_1 \exp(ix/\sqrt{\varepsilon}) + c_2 \exp(-ix/\sqrt{\varepsilon})$.

Imposing the boundary conditions $y(0) = 0$ and $y(1) = 1$ reproduces the exact solution in (10.1.2):

$$y(x) = \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}.$$

Example 3 *WKB solution of an initial-value problem.* To solve the initial-value problem $\varepsilon^2 y'' = Q(x)y$ [$y(0) = A$, $y'(0) = B$] we set $a = 0$ in (10.1.13) and differentiate (10.1.13) to obtain simultaneous equations for c_1 and c_2:

$$[Q(0)]^{-1/4}(c_1 + c_2) = A,$$
$$-\tfrac{1}{4}Q'(0)[Q(0)]^{-5/4}(c_1 + c_2) + (c_1 - c_2)[Q(0)]^{1/4}/\varepsilon = B.$$

For example, when $A = 0$ and $B = 1$ we obtain $c_1 = -c_2 = \tfrac{1}{2}\varepsilon[Q(0)]^{-1/4}$. Thus, the approximate solution to this initial-value problem is

$$y(x) \sim \varepsilon[Q(x)Q(0)]^{-1/4} \sinh\left[\int_0^x dt\sqrt{Q(t)}/\varepsilon\right], \qquad \varepsilon \to 0.$$

If we now take $Q(x) = (1 + x^2)^2$, then a uniform approximation (valid for all x) to the solution of

$$\varepsilon^2 y'' = (1 + x^2)^2 y, \qquad y(0) = 0,\ y'(0) = 1, \tag{10.1.18}$$

is

$$y(x) \sim \frac{\varepsilon}{\sqrt{1 + x^2}} \sinh\left[\frac{1}{\varepsilon}(x + x^3/3)\right], \qquad \varepsilon \to 0. \tag{10.1.19}$$

In Fig. 10.1 we compare (10.1.19) with the exact solution to (10.1.18) from a computer for three values of ε. Note that ε need not be very small for (10.1.19) to be a good approximation to the exact solution (see Prob. 10.3).

Example 4 *Rederivation of a boundary-layer approximation.* Here we show that the WKB approximation contains boundary-layer theory as a special case. Consider the familiar boundary-value problem

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \qquad y(0) = A,\ y(1) = B, \tag{10.1.20}$$

where we assume that $a(x) > 0$ for $0 \le x \le 1$ and $\varepsilon \to 0+$.

We begin by substituting equations (10.1.6) into (10.1.20) and neglecting terms which vanish as $\delta \to 0$:

$$\frac{\varepsilon}{\delta^2}S_0'^2 + 2\frac{\varepsilon}{\delta}S_0'S_1' + \frac{\varepsilon}{\delta}S_0'' + \frac{1}{\delta}S_0'a + S_1'a + b + \cdots = 0. \tag{10.1.21}$$

The largest of the first three terms is $\varepsilon\delta^{-2}S_0'^2$ and the largest of the next three (assuming that $a \ne 0$) is $\delta^{-1}S_0'a$. By dominant balance these two terms must be of equal magnitude, so $\varepsilon\delta^{-2} = O(\delta^{-1})$. Therefore, δ is proportional to ε and for simplicity we choose $\delta = \varepsilon$. As in Example 1, the small scale parameter δ is again determined by a *distinguished* limit. [There is another distinguished limit possible in (10.1.21); namely, $\delta = 1$. However, this limit reproduces the outer solution below. Why?]

Next we return to (10.1.21) and identify the coefficients of ε^{-1} and ε^0. The resulting equations

$$S_0'^2 + S_0'a = 0 \tag{10.1.22}$$

and

$$2S_0'S_1' + S_0'' + S_1'a + b = 0 \tag{10.1.23}$$

are easy to solve. Equation (10.1.22) yields two solutions for S_0':

$$S_0' = 0 \text{ and } S_0' = -a.$$

When $S_0' = 0$, (10.1.23) becomes $S_1'a + b = 0$. Thus, $S_1 = -\int^x [b(t)/a(t)]\,dt$, and one WKB approximation to the original differential equation in (10.1.20) is

$$y_1(x) \sim c_1 \exp\left[-\int_0^x \frac{b(t)}{a(t)}dt\right], \qquad \varepsilon \to 0+,$$

where c_1 is a constant which includes the term $e^{S_0/\varepsilon}$. This is the outer solution of boundary layer theory.

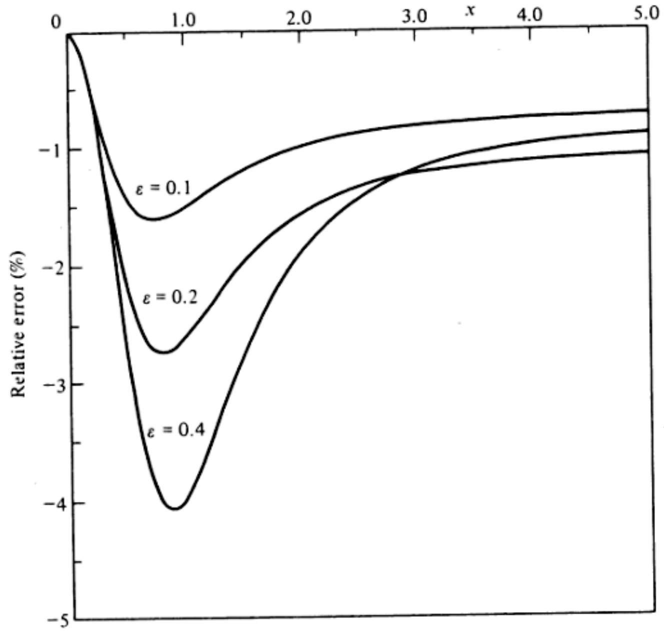

Figure 10.1 A plot of the relative error between the exact solution to the initial-value problem $\varepsilon^2 y'' = (1 + x^2)^2 y$ [$y(0) = 0$, $y'(0) = 1$] in (10.1.18) and the leading-order WKB approximation to $y(x)$, $y(x) \sim \varepsilon(1 + x^2)^{-1/2} \sinh[(x + x^3/3)/\varepsilon]$ ($\varepsilon \to 0$), in (10.1.19) for three values of ε. The relative error is defined as (WKB approximation − exact solution)/(exact solution).

When $S_0' = -a$, (10.1.23) becomes $aS_1' + a' = b$. Thus, $S_1 = -\ln a + \int_0^x [b(t)/a(t)]\,dt$, and another WKB approximation to the original differential equation in (10.1.20) is

$$y_2(x) \sim c_2 \frac{1}{a(x)} \exp\left[\int_0^x \frac{b(t)}{a(t)}dt - \frac{1}{\varepsilon}\int_0^x a(t)\,dt\right], \qquad \varepsilon \to 0+,$$

where c_2 is another constant. This is the inner solution of boundary-layer theory.

The general solution is a linear combination of y_1 and y_2. We must now impose the boundary conditions to determine c_1 and c_2. The boundary condition at $x = 0$ gives

$$A = c_1 + c_2/a(0) \tag{10.1.24}$$

and the boundary condition at $x = 1$ gives

$$B = c_1 \exp\left[-\int_0^1 \frac{b(t)}{a(t)}dt\right], \tag{10.1.25}$$

where we have neglected the exponentially small term containing $\exp[-\varepsilon^{-1}\int_0^1 a(t)\,dt]$. Solving (10.1.24) and (10.1.25) simultaneously gives

$$y(x) \sim B \exp\left[\int_x^1 \frac{b(t)}{a(t)}dt\right] + \frac{a(0)}{a(x)}\left[A - B\exp\int_0^1 \frac{b(t)}{a(t)}dt\right]\exp\left[\int_0^x \frac{b(t)}{a(t)}dt - \frac{1}{\varepsilon}\int_0^x a(t)\,dt\right].$$

This expression simplifies because the second term contributes only when $x = O(\varepsilon)$ $(\varepsilon \to 0+)$; it is negligible for larger values of x. Thus,

$$y(x) \sim B \exp\left[\int_x^1 \frac{b(t)}{a(t)}\,dt\right] + \left[A - B\exp\int_0^1 \frac{b(t)}{a(t)}\,dt\right] e^{-a(0)x/\varepsilon}. \tag{10.1.26}$$

Equation (10.1.26) is precisely the *uniformly* valid lowest-order boundary-layer solution that we obtained in (9.1.13). Notice that (10.1.26) was obtained without ever having to perform an asymptotic match!

Example 5 *WKB analysis of a Sturm-Liouville problem.* We know from our discussion of the Sturm-Liouville eigenvalue problem in Sec. 1.8 that the boundary-value problem

$$y''(x) + EQ(x)y(x) = 0, \qquad Q(x) > 0,\ y(0) = y(\pi) = 0, \tag{10.1.27}$$

has an infinite number of nontrivial solutions: the eigenvalues $E_1, E_2, E_3, \ldots$ are discrete, nondegenerate (eigenvalues associated with different eigenfunctions are unequal), and are all positive real numbers; the nth eigenvalue E_n is associated with the eigenfunction $y_n(x)$; eigenfunctions associated with different eigenvalues are orthogonal with respect to the weight function $Q(x)$:

$$\int_0^\pi dx\ y_n(x)y_m(x)Q(x) = 0, \qquad n \neq m. \tag{10.1.28}$$

This orthogonality property is easy to prove using integration by parts twice. See Prob. 10.7(b).

Since the differential equation and boundary conditions in (10.1.27) are homogeneous, the eigenfunctions $\{y_n\}$ are determined only up to an arbitrary multiplicative constant. It is conventional to choose the normalization of y_n so that

$$\int_0^\pi [y_n(x)]^2 Q(x)\,dx = 1. \tag{10.1.29}$$

Now the eigenfunctions form a complete orthonormal set.

WKB theory may be used to find approximate formulas for E_n and $y_n(x)$ when n is large. As the WKB theory itself will verify, E_n is approximately proportional to n^2 as $n \to \infty$; thus, the leading-order WKB approximation to the solution of $\varepsilon y''(x) + Q(x)y(x) = 0$, where $\varepsilon = 1/E_n$, is accurate for large n because $\varepsilon \to 0$ as $n \to \infty$.

The leading-order WKB approximation (10.1.13) to the general solution of $y''(x) + EQ(x)y(x) = 0$ is a linear combination of $Q^{-1/4}(x)\sin\left[\sqrt{E}\int_0^x \sqrt{Q(t)}\,dt\right]$ and $Q^{-1/4}(x)\cos\left[\sqrt{E}\int_0^x \sqrt{Q(t)}\,dt\right]$. (Recall that all the eigenvalues E are positive real numbers; also, we fix the sign of $\sqrt{E}$ to be positive.) The boundary condition $y(0) = 0$ implies that

$$y(x) \sim CQ^{-1/4}(x)\sin\left[\sqrt{E}\int_0^x \sqrt{Q(t)}\,dt\right], \qquad E \to \infty, \tag{10.1.30}$$

where C is an arbitrary normalization constant.

The boundary condition $y(\pi) = 0$ determines the eigenvalues

$$E_n \sim \left[\frac{n\pi}{\int_0^\pi \sqrt{Q(t)}\,dt}\right]^2, \qquad n \to \infty. \tag{10.1.31}$$

Next we determine the eigenfunctions. The normalization integral in (10.1.29) fixes C in (10.1.30); substituting (10.1.30) into (10.1.29) gives

$$1 \sim \int_0^\pi dx\ Q(x)C_n^2 \frac{1}{\sqrt{Q(x)}}\sin^2\left[\sqrt{E_n}\int_0^x dt\ \sqrt{Q(t)}\right], \qquad n \to \infty.$$

The change of variable $u = \sqrt{E_n}\int_0^x dt\sqrt{Q(t)}$ gives $1 \sim (C_n^2/\sqrt{E_n})\int_0^{n\pi} du\ \sin^2 u$ $(n \to \infty)$, whence

$$C_n^2 \sim \frac{2}{\int_0^\pi \sqrt{Q(t)}\,dt}, \qquad n \to \infty. \tag{10.1.32}$$

Thus, the eigenfunctions are

$$y_n(x) \sim \left(\int_0^\pi \frac{\sqrt{Q(t)}}{2}\,dt\right)^{-1/2} Q^{-1/4}(x)\sin\left[n\pi\frac{\int_0^x \sqrt{Q(t)}\,dt}{\int_0^\pi \sqrt{Q(t)}\,dt}\right], \qquad n \to \infty. \tag{10.1.33}$$

Note that if $Q(x) \equiv 1$, then the right side of (10.1.33) reduces to $\sqrt{2/\pi}\sin(nx)$, which is the exact solution to $y'' + y = 0$ $[y(0) = y(\pi) = 0]$.

To demonstrate the accuracy of our results, we choose $Q(x) = (x + \pi)^4$. Then the approximate eigenvalues and eigenfunctions are given by

$$E_n \sim \frac{9n^2}{49\pi^4}, \qquad n \to \infty, \tag{10.1.34}$$

and

$$y_n(x) \sim \sqrt{\frac{6}{7\pi^3}}\,\frac{\sin[n(x^3 + 3x^2\pi + 3\pi^2 x)/7\pi^2]}{(\pi + x)}, \qquad n \to \infty. \tag{10.1.35}$$

We have checked these results numerically by computer. The comparisons between the approximate analytical and the computer solutions are given in Table 10.1 and Figs. 10.2 and 10.3.

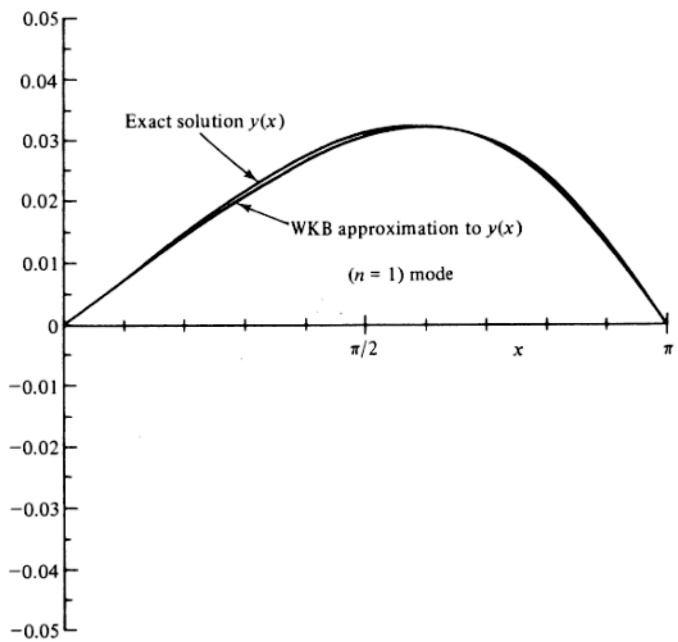

Figure 10.2 Comparison of the exact solution to $y''(x) + E_n(x + \pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$, with the WKB approximation to this solution as given in (10.1.35) for the lowest $(n = 1)$ mode. Although WKB becomes exact as $n \to \infty$, this plot shows that even when $n = 1$ the WKB approximation is extraordinarily accurate.

Table 10.1 A comparison of the exact eigenvalues E_n of the Sturm-Liouville problem $y''(x) + E(x+\pi)^4 y(x) = 0$ [$y(0) = y(\pi) = 0$] with the leading-order WKB prediction [see (10.1.34)] for these eigenvalues $E_n \sim 9n^2/49\pi^2$ ($n \to \infty$)

As expected, this prediction becomes more accurate as n increases. The relative error is defined as (approximate − exact)/(exact)

n	E_n(WKB)	E_n(exact)	Relative error, %
1	0.00188559	0.00174401	8.1
2	0.00754235	0.00734865	2.6
3	0.0169703	0.0167524	1.3
4	0.0301694	0.0299383	0.77
5	0.0471397	0.0469006	0.51
10	0.188559	0.188305	0.13
20	0.754235	0.753977	0.035
40	3.01694	3.01668	0.009

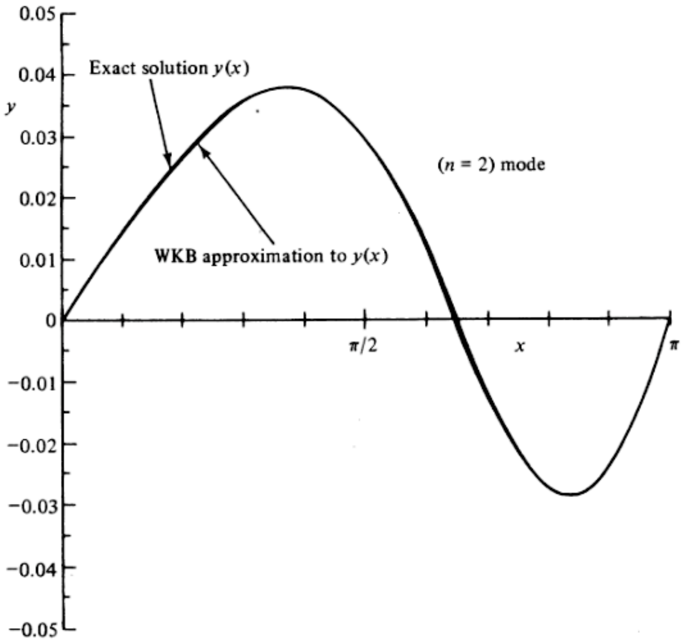

Figure 10.3 Same as in Fig. 10.2 except that $n = 2$. The exact eigenfunction and the WKB approximation are almost indistinguishable.

The preceding examples demonstrate the power and efficiency of the exponential approximation (10.1.4). However, we have already seen that rapidly varying exponentials appear naturally when one attempts to approximate solutions to linear equations. In Chap. 3 the exponential approximation was found to be useful for finding the leading behaviors of solutions near irregular singular points [see (3.4.6)]. In Chap. 6 the rapid variation of exponentials of the form (10.1.3) led to the principal ideas of approximation methods for integrals such as Laplace's method, the method of stationary phase, and the method of steepest descents. Exponentials also appear in the equations of boundary-layer theory. Thus, we are not terribly surprised when exponentials resurface in the context of WKB theory as the basis of the WKB approximation.

(E) 10.2 CONDITIONS FOR VALIDITY OF THE WKB APPROXIMATION

WKB theory is a singular perturbation theory because it is used to solve a differential equation whose highest derivative is multiplied by a small parameter (when the small parameter vanishes, the order of the differential equation changes abruptly). The singular nature of WKB theory is clearly evident in the $1/\delta$ term in the exponential approximation (10.1.4):

$$y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right], \qquad \delta \to 0. \tag{10.2.1}$$

Unless $S_0(x) \equiv 0$, the approximation ceases to exist when $\delta = 0$. The singular nature of this approximation also surfaces in a more subtle way—the WKB series $\sum \delta^n S_n$ usually diverges. (The series converges if it truncates, but this is rare. See Prob. 10.2.) This is why we use the asymptotic notation $\sim$ rather than $=$. Nevertheless, even though the WKB series diverges, we know from the numerical examples in Sec. 10.1 that it can give an extremely accurate approximation to $y(x)$.

This section develops criteria for predicting when the WKB approximation will be useful. These criteria are quantitative; i.e., they specify how small δ must be for the WKB series in (10.2.1) to approximate $y(x)$ to some prescribed relative error.

In order that the WKB approximation (10.2.1) be valid on an interval, it is necessary that the series $\sum \delta^{n-1} S_n(x)$ be an asymptotic series in δ as $\delta \to 0$ *uniformly for all x on the interval.* This requires that the asymptotic relations

$$\begin{aligned} S_1(x) &\ll \frac{1}{\delta} S_0(x), && \delta \to 0,\\ \delta S_2(x) &\ll S_1(x), && \delta \to 0,\\ &\vdots \\ \delta^n S_{n+1}(x) &\ll \delta^{n-1} S_n(x), && \delta \to 0, \end{aligned} \tag{10.2.2}$$

hold uniformly in x. These conditions are equivalent to the requirement that each of the functions $S_{n+1}(x)/S_n(x)$ ($n = 0, 1, 2, 3, \ldots$) be bounded functions of x on the interval (although these bounds may be arbitrary functions of n). If the series $\sum \delta^{n-1}S_n(x)$ is *uniformly* asymptotic in x as $\delta \to 0$, the optimal truncation rule suggests that truncating the series before the smallest term $\delta^N S_{N+1}(x)$ gives an approximation to $\ln y$ with uniformly small relative error throughout the x interval.

However, because the WKB series appears in the *exponent* in (10.2.1), the asymptotic conditions in (10.2.2) are not sufficient to ensure that $\exp[\sum \delta^{n-1}S_n(x)]$ will be a good approximation to $y(x)$. For the WKB series truncated at the term $\delta^{N-1}S_N(x)$ to be a good approximation to $y(x)$, the next term must be small compared with 1 for all x in the interval of approximation:

$$\delta^N S_{N+1}(x) \ll 1, \qquad \delta \to 0. \tag{10.2.3}$$

If this relation holds, then $\exp[\delta^N S_{N+1}(x)] = 1 + O[\delta^N S_{N+1}(x)]$ ($\delta \to 0$). Thus, the relative error between $y(x)$ and the WKB approximation is small:

$$\frac{y(x) - \exp[1/\delta \sum_{n=0}^{N} \delta^n S_n(x)]}{y(x)} \sim \delta^N S_{N+1}(x), \qquad \delta \to 0.$$

Both conditions (10.2.2) and (10.2.3) must be satisfied for WKB to be useful.

Geometrical and Physical Optics

If we retain only the first term in the WKB series, we are making the approximation of *geometrical optics*: $e^{S_0(x)/\delta}$. However, while this expression may faithfully reflect the structure of $y(x)$, it does *not* constitute an asymptotic approximation to $y(x)$ because $S_1(x)$, the next term in the WKB series, is not small compared with 1 as $\delta \to 0$ (it does not depend on δ) and condition (10.2.3) is not satisfied.

The first two terms in the WKB series constitute the approximation of *physical optics*:

$$y(x) \sim e^{S_0(x)/\delta + S_1(x)}, \qquad \delta \to 0. \tag{10.2.4}$$

The relative error between y and the approximation of physical optics is of order $\delta S_2(x)$, which vanishes uniformly with δ if $S_2(x)$ is bounded. Thus, the approximation in (10.2.4) is an asymptotic relation. For example, if it is required that the physical-optics approximation be accurate to a relative error of 2 percent on an interval, we must then choose δ so small that $|\delta S_2(x)| \le 0.02$ for all x on that interval.

Usually, the approximation of physical optics expresses the leading asymptotic behavior of $y(x)$ while the approximation of geometrical optics contains just the controlling factor (the most rapidly varying component) of the leading behavior.

Example 1 *Behavior of Airy functions as $x \to +\infty$.* The Airy equation $y'' = xy$ is a Schrödinger equation with $Q(x) = x$ and $\varepsilon = 1$. Thus, from (10.1.11), (10.1.12), and (10.1.14) we have $S_0 = \pm\frac{2}{3}x^{3/2}$, $S_1 = -\frac{1}{4}\ln x$, $S_2 = \pm\frac{5}{48}x^{-3/2}$. We observe that even when $\varepsilon = 1$, the asymptotic inequalities $\varepsilon S_2 \ll S_1 \ll S_0/\varepsilon$, $\varepsilon S_2 \ll 1$ ($x \to +\infty$) hold. We conclude that for fixed ε the physical-optics approximation is valid as $x \to +\infty$. Indeed, we have just rederived the leading behaviors of

solutions to the Airy equation as $x \to +\infty$ as well as the first correction to the leading behaviors [see (3.5.21)]:

$$y(x) \sim c_\pm x^{-1/4} e^{\pm 2x^{3/2}/3}\left(1 \pm \tfrac{5}{48}x^{-3/2}\right),$$

where $c_\pm$ is a constant. Note that the rapidly varying exponential factors $e^{\pm 2x^{3/2}/3}$ come from the geometrical-optics approximation.

Example 2 *Behavior of parabolic cylinder functions as $x \to +\infty$.* WKB theory also gives the large-x behavior of the solutions to the parabolic cylinder equation $y'' = (\frac{1}{4}x^2 - \nu - \frac{1}{2})y$. Here, again, even though $\varepsilon = 1$, the physical-optics approximation is valid as $x \to +\infty$ because $S_1 \ll S_0/\varepsilon$, $\varepsilon S_2 \ll S_1$, $\varepsilon S_2 \ll 1$ ($x \to +\infty$).

The components of the physical-optics approximation are calculated from $Q(x) = \frac{1}{4}x^2 - \nu - \frac{1}{2}$ as follows:

$$S_0 = \pm\int^x \sqrt{Q(t)}\,dt \sim \pm\int^x \frac{t}{2}\left(1 - \frac{2\nu+1}{t^2}\right)dt \sim \pm\left[\frac{x^2}{4} - \left(\nu + \frac{1}{2}\right)\ln x\right], \qquad x \to +\infty,$$

where we have used the binomial expansion and $S_1 = -\frac{1}{4}\ln Q \sim -\frac{1}{4}\ln(\frac{1}{4}x^2)$ ($x \to +\infty$). Hence, the physical-optics approximation gives

$$y \sim c_\pm e^{S_0 + S_1} \sim \begin{cases} c_+ x^{-\nu-1} e^{x^2/4}, & x \to +\infty, \\ c_- x^{\nu} e^{-x^2/4}, \end{cases}$$

which are the leading behaviors of solutions to the parabolic cylinder equation [see (3.5.12)]. Again, the controlling factor $e^{\pm x^2/4}$ arises from the geometrical-optics approximation.

Example 3 *Accuracy of physical optics.* How small must we make ε to be sure that the physical-optics approximation to the exponentially decaying solution of $\varepsilon^2 y'' = \sqrt{x}\,y$ is accurate to 5 percent when $x \ge 1$?

The physical-optics approximation to $y(x)$ is $y(x) \sim cx^{-1/8}e^{-4x^{5/4}/5\varepsilon}$ ($\varepsilon \to 0$). The relative error between this asymptotic approximation and the exact solution is εS_2, which from (10.1.14) is $|\varepsilon S_2| = 9\varepsilon x^{-5/4}/160$. This equation shows that we must choose $\varepsilon \le 0.9$ to make $|\varepsilon S_2| \le 5$ percent for all $x > 1$.

Example 4 *Violation of criteria for validity of WKB.* Is it valid to use WKB theory to predict the large-x behavior of the solutions to

$$y''(x) = \left(\frac{\ln x}{x}\right)^2 y(x)? \tag{10.2.5}$$

First, we determine the behavior of $y(x)$ for large x without using WKB. We transform (10.2.5) by letting $s = \ln x$:

$$\frac{d^2}{ds^2}y(s) - \frac{d}{ds}y(s) = s^2 y(s).$$

The substitution $y(s) = e^{s/2}z(s)$ eliminates the dy/ds term: $(d^2/ds^2)z(s) = (s^2 + \frac{1}{4})z(s)$. Finally, the change of variable $t = \sqrt{2}\,s$ gives a parabolic cylinder equation

$$\frac{d^2}{dt^2}z(t) = \left(\frac{t^2}{4} + \frac{1}{8}\right)z(t)$$

whose solutions are $D_{-5/8}(t)$ and $D_{-5/8}(-t)$. When t is large and positive, the behavior of parabolic cylinder functions is given by [see (3.5.14)]:

$$D_\nu(t) \sim t^\nu e^{-t^2/4}\left[1 - \frac{\nu(\nu-1)}{2^1 1!\, t^2} + \cdots\right], \qquad t \to \infty.$$

$$D_\nu(-t) \sim \frac{\sqrt{2\pi}}{\Gamma(-\nu)} t^{-\nu-1} e^{t^2/4}\left[1 + \frac{(\nu+1)(\nu+2)}{2^1 1!\, t^2} + \cdots\right], \qquad t \to \infty.$$

Thus, the behavior of $y(x)$ for large x is

$$y(x) \sim c_+ \sqrt{x}\, (\ln x)^{-3/8} e^{(\ln x)^2/2}[1 - \tfrac{15}{256}(\ln x)^{-2} + \cdots], \qquad x \to +\infty, \tag{10.2.6a}$$

or

$$y(x) \sim c_- \sqrt{x}\, (\ln x)^{-5/8} e^{-(\ln x)^2/2}[1 - \tfrac{65}{256}(\ln x)^{-2} + \cdots], \qquad x \to +\infty. \tag{10.2.6b}$$

Now we examine the predictions of WKB. From (10.2.5) we see that $Q(x) = (\ln x)^2 x^{-2}$. Thus,

$$S_0 = \pm \int^x \sqrt{Q(t)}\, dt = \pm \tfrac{1}{2}(\ln x)^2,$$

$$S_1 = -\tfrac{1}{4} \ln Q(x) = \tfrac{1}{2} \ln x - \tfrac{1}{2} \ln (\ln x),$$

$$S_2 = \pm \int^x \left(\frac{Q''}{8Q^{3/2}} - \frac{5Q'^2}{32Q^{5/2}} \right) dt = \pm \tfrac{1}{8} \ln (\ln x) \pm \tfrac{3}{16} (\ln x)^{-2}.$$

From these formulas, it is clear that $S_2 \ll S_1 \ll S_0$ $(x \to \infty)$, but that $1 \ll S_2$ $(x \to \infty)$. Hence, the condition in (10.2.2) is satisfied, but that in (10.2.3) with $N = 1$ is violated. Thus, while the geometrical-optics approximation e^{S_0} gives the correct controlling factors in (10.2.6), we are not surprised that the physical-optics approximation $e^{S_0+S_1}$ gives the wrong leading behavior.

Next, let us calculate S_3. From (10.1.15) we have

$$S_3(x) = -\frac{Q''}{16Q^2} + \frac{5Q'^2}{64Q^3} = \frac{3}{16}(\ln x)^{-4} - \frac{1}{16}(\ln x)^{-2}. \tag{10.2.7}$$

Here, we are gratified to find that $S_3 \ll S_2$, $S_3 \ll 1$ $(x \to +\infty)$. Thus, we expect the leading behaviors in (10.2.6) to be given by the first *three* terms in the WKB series: $y(x) \sim e^{S_0+S_1+S_2}$ $(x \to +\infty)$. Indeed, we find that

$$e^{S_0+S_1+S_2} = c_\pm e^{\pm(\ln x)^2/2} x^{1/2} (\ln x)^{-1/2 \pm 1/8}.$$

In Prob. 10.10 you are asked to verify that

$$e^{S_0+S_1+S_2+S_3+S_4} \tag{10.2.8}$$

reproduces the asymptotic formulas in (10.2.6).

We emphasize that in this example we have taken $\varepsilon = 1$. If we apply WKB analysis to the equation

$$\varepsilon^2 y''(x) = [(\ln x)/x]^2 y(x)$$

then the physical-optics approximation $e^{S_0/\varepsilon + S_1}$ is the leading asymptotic approximation to $y(x)$ as $\varepsilon \to 0$ for each fixed $x > 0$. However, this physical-optics approximation is not uniformly asymptotic to $y(x)$ as $\varepsilon \to 0$ for all $x > 0$. To obtain the leading behavior of $y(x)$ as $x \to \infty$ for fixed ε it is necessary to use $e^{S_0/\varepsilon + S_1 + \varepsilon S_2}$. See Prob. 10.10(*b*).

Physical Optics for Higher-Order Equations

WKB analysis is not sensitive to the order of a differential equation. It is very easy to show (see Prob. 10.11) that for the nth-order equation

$$\varepsilon^n \frac{d^n}{dx^n} y(x) = Q(x) y, \tag{10.2.9}$$

the WKB approximation has the form $e^{(S_0 + \varepsilon S_1 + \cdots)/\varepsilon}$, where

$$S_0 = \omega \int^x [Q(t)]^{1/n}\, dt, \qquad \omega^n = 1, \tag{10.2.10}$$

and

$$S_1 = \frac{1-n}{2n} \ln Q. \tag{10.2.11}$$

Compare this result with that in (3.4.28). See Prob. 10.11(*b*).

Example 5 *Behavior of solutions to $d^4y/dx^4 = xy$ for large x.* For the hyperairy equation of order 4, $d^4y/dx^4 = xy$, $n = 4$, $\varepsilon = 1$, and $Q = x$. The physical-optics approximation is

$$e^{(S_0+S_1)/\varepsilon} = cQ^{-3/8} \exp\left(\omega \int^x Q^{1/4}\, dt\right) = cx^{-3/8} \exp\left(\omega \frac{4}{5} x^{5/4}\right),$$

where c is a constant and $\omega = \pm 1$ or $\pm i$, which agrees with the leading behavior in (3.5.23).

Turning Points

Equations (10.2.10) and (10.2.11) show that the condition (10.2.2) for the validity of the WKB series on an interval is violated if $Q(x)$ vanishes on that interval. Specifically, the asymptotic relation $S_0(x)/\varepsilon \gg S_1(x) = [(1-n)/2n] \ln Q(x)$ $(\varepsilon \to 0)$ breaks down at a zero of Q because S_1 becomes singular. Points where Q vanishes are called *turning points*.

The expression "turning point" comes from the Schrödinger equation which describes a quantum mechanical particle in a potential $V(x)$:

$$\left(-\varepsilon^2 \frac{d^2}{dx^2} + V(x) - E \right) y(x) = 0.$$

$V(x)$ is the potential energy of the particle and E is the total energy of the particle. For this equation $Q(x) = V(x) - E$, so $Q(x)$ vanishes at points where $V(x) = E$. The classical orbit of a particle in the potential $V(x)$ is confined to regions where $V(x) \le E$. The particle moves until it reaches a point where $V = E$ and then it stops, turns around, and moves off in the opposite direction.

The physical-optics approximation is clearly invalid at a turning point because

$$e^{S_0/\varepsilon + S_1} = Q^{(1-n)/2n} \exp\left(\frac{\omega}{\varepsilon} \int^x Q^{1/n}\, dt \right)$$

is infinite at such points. On the other hand, the theory of linear differential equations asserts that if $Q(x)$ is analytic, then the exact solutions of the differential equation (10.2.9) are regular! We resolve this puzzle and use asymptotic matching to construct approximate solutions of differential equations with turning points in Sec. 10.4.

(E) 10.3 PATCHED ASYMPTOTIC APPROXIMATIONS: WKB SOLUTION OF INHOMOGENEOUS LINEAR EQUATIONS

From the discussion of Sec. 10.1 one might conclude that WKB theory is useful only for homogeneous linear equations; it would seem that unless a differential equation is homogeneous, it would not be possible to divide off the exponential

WKB series to obtain equations for $S_0, S_1, S_2, \ldots$. However, there is no real difficulty with an inhomogeneous equation because one can solve the associated homogeneous equations in terms of WKB approximations and then use the method of variation of parameters (see Sec. 1.5) to generate the solution of the full inhomogeneous equation. The only possible drawback with this procedure is that imposing the boundary conditions can involve evaluating cumbersome integrals. In this section we propose a simple and general method for solving the inhomogeneous Schrödinger equation in terms of a Green's function which neatly incorporates the boundary conditions at an early stage of the problem. (Green's functions are discussed in Sec. 1.5.)

We will solve the general inhomogeneous Schrödinger equation of the form

$$\varepsilon^2 y'' - Q(x)y(x) + R(x) = 0, \tag{10.3.1}$$

subject to the homogeneous boundary conditions $y(\pm\infty) = 0$. We will assume that $Q(x) > 0$ for all x (so that the homogeneous differential equation has no turning points) and that $Q(x) \gg x^{-2}$ as $x \to \pm\infty$ so that the physical-optics approximation to the solution of the homogeneous equation is valid for all x. We must also assume that the inhomogeneous term $R(x)$ is such that a solution to (10.3.1) which satisfies $y(\pm\infty) = 0$ actually exists.

One way to analyze this problem is simply to take the limit $\varepsilon \to 0+$ in (10.3.1) to obtain

$$y(x) \sim \frac{R(x)}{Q(x)}, \qquad \varepsilon \to 0+, \tag{10.3.2}$$

which is like the outer limit of boundary-layer theory. In fact, if $R(x)$ and $Q(x)$ are smooth and there are no turning points, then (10.3.2) is valid everywhere. Thus, in order that there exist a solution to (10.3.1) satisfying $y(\pm\infty) = 0$, it is necessary that $R(x) \ll Q(x)$ as $x \to \pm\infty$ (see Prob. 10.12).

The outer behavior (10.3.2) breaks down at points of discontinuity of $R(x)$ and at turning points of $Q(x)$. At such points, the solution $y(x)$ to (10.3.1) is continuous but $R(x)/Q(x)$ is discontinuous. The WKB analysis given below yields a uniformly valid approximation to $y(x)$ even in the neighborhood of discontinuities of $R(x)$. Our WKB analysis can also be extended to obtain a uniform approximation to $y(x)$ when there are turning points (see Prob. 10.24).

Our approach here consists of solving the Green's function equation

$$\varepsilon^2 \frac{\partial^2 G}{\partial x^2}(x, x') - Q(x)G(x, x') = -\delta(x - x'), \qquad G(\pm\infty, x') = 0, \tag{10.3.3}$$

and then using the Green's function $G(x, x')$ to construct the solution to (10.3.1):

$$y(x) = \int_{-\infty}^{\infty} G(x, x')R(x')\,dx'. \tag{10.3.4}$$

If we had the exact solution to (10.3.3), then (10.3.4) would constitute an exact solution to (10.3.1) because it satisfies both the differential equation and the boundary conditions. Lacking this, we propose to use the WKB physical-optics approximation to $G(x, x')$ in place of the exact Green's function. The resulting integral (10.3.4) is then asymptotic to $y(x)$ as $\varepsilon \to 0+$.

The WKB solution of (10.3.3) will require the *patching* of two WKB solutions which are valid in their respective regions. Patching is a *local* procedure because it is done at a single point $x = x'$. By contrast, asymptotic matching, which is used in the next section to construct a solution to the one-turning-point problem, is a global procedure which is performed on an interval whose length becomes infinite as $\varepsilon \to 0+$.

Construction of the Green's Function $G(x, x')$

To solve (10.3.3) we divide the x axis into two regions: region I, where $x > x'$, and region II, where $x < x'$. In each region the differential equation is homogeneous, so the WKB physical-optics approximation may be used. In region I $G \to 0$ as $x \to +\infty$; the WKB approximation to $G(x, x')$ which incorporates this condition is

$$G_{\mathrm{I}}(x, x') = C_{\mathrm{I}}[Q(x)]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_{x'}^{x} \sqrt{Q(t)}\,dt\right], \qquad x \to x', \tag{10.3.5}$$

where C_{I} is a constant and we have chosen the lower limit of integration to lie at x'.

In region II $G \to 0$ as $x \to -\infty$; thus,

$$G_{\mathrm{II}}(x, x') = C_{\mathrm{II}}[Q(x)]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_{x}^{x'} \sqrt{Q(t)}\,dt\right], \qquad x \to x', \tag{10.3.6}$$

where C_{II} is a second constant.

The constants C_{I} and C_{II} are determined by patching. There are two patching conditions. First, at $x = x'$, the boundary of regions I and II, we require that $G(x, x')$ be continuous: $\lim_{\eta\to 0+} [G_{\mathrm{I}}(x' + \eta, x') - G_{\mathrm{II}}(x' - \eta, x')] = 0$. This condition implies that $C_{\mathrm{I}} = C_{\mathrm{II}}$.

The second condition is derived by integrating the differential equation (10.3.3) from $x' - \eta$ to $x' + \eta$ and letting $\eta \to 0+$. We obtain

$$\lim_{\eta\to 0+}\left[\frac{\partial}{\partial x}G_{\mathrm{I}}(x, x')\bigg|_{x=x'+\eta} - \frac{\partial}{\partial x}G_{\mathrm{II}}(x, x')\bigg|_{x=x'-\eta}\right] = -\frac{1}{\varepsilon^2}.$$

[Normally, the solution to a second-order differential equation has a continuous first derivative, but the delta function in (10.3.3) gives rise to a finite discontinuity in the slope of $G(x, x')$ at $x = x'$ (a cusp).] This condition implies that

$$C_{\mathrm{I}} = C_{\mathrm{II}} = \frac{1}{2\varepsilon}[Q(x')]^{-1/4}.$$

$G_{\mathrm{I}}(x, x')$ and $G_{\mathrm{II}}(x, x')$ are now completely determined and may be combined into a single expression which is a uniformly valid approximation to the solution of (10.3.3) for all x:

$$G_{\mathrm{unif}}(x, x') = \frac{1}{2\varepsilon}[Q(x)Q(x')]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\left|\int_{x'}^{x} \sqrt{Q(t)}\,dt\right|\right]. \tag{10.3.7}$$

For all x the relative error in this approximation is of order ε.

Example 1 *Comparison between exact and approximate Green's functions.* For $Q(x) = 1 + x^2$ the uniform approximation to $G(x, x')$ in (10.3.7) is astoundingly accurate. Equation (10.3.7) becomes

$$G_{\text{unif}}(x, x') = \frac{\exp\left(-\left|x\sqrt{x^2+1} - x'\sqrt{x'^2+1}\right|/2\varepsilon\right)}{2\varepsilon[(x^2+1)(x'^2+1)]^{1/4}}\left(\frac{x+\sqrt{x^2+1}}{x'+\sqrt{x'^2+1}}\right)^{(x'-x)/(2\varepsilon|x'-x|)} \tag{10.3.8}$$

In Figs. 10.4 and 10.5 we compare $G_{\text{unif}}(x, 0)$ with the exact numerical solution for $\varepsilon = 1$ and two values of x'. Observe how small the error is, even when ε is as large as 1.

Integrals of the Green's Function

The uniform approximation in (10.3.7) to the Green's function may be used to evaluate integrals of the Green's function. For example, to calculate $A = \int_{-\infty}^{\infty} G(x, x')\, dx$ we use integration by parts to determine the leading behav-

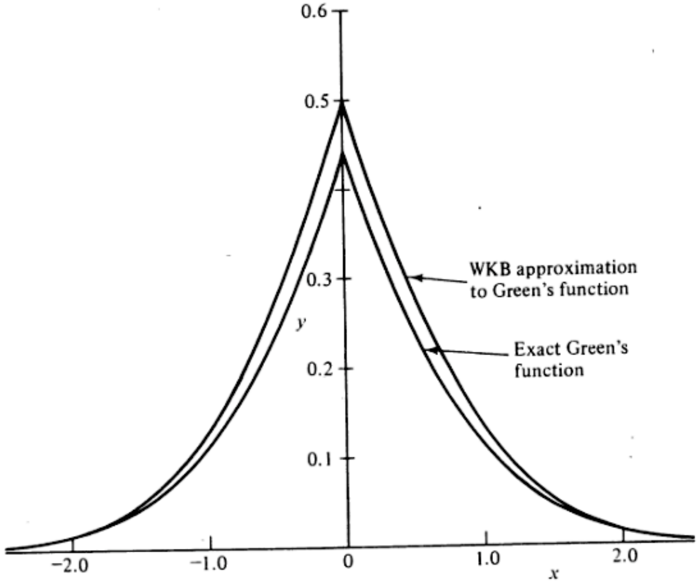

Figure 10.4 Comparison of the exact solution to the Green's function equation, $\varepsilon^2\partial^2 G/\partial x^2(x, x') - (1 + x^2)G(x, x') = -\delta(x - x')$ $[G(\pm\infty, x') = 0]$, with $x' = 0$, $\varepsilon = 1$, with the WKB physical-optics approximation to $G(x, x')$ in (10.3.8). Observe that the error is greatest at $x = 0$, the point at which the exponent in (10.3.8) is smallest. The true value of $G(0, 0)$ is 0.443 11 . . . , while the WKB formula in (10.3.8) predicts that $G(0, 0) = (2\varepsilon)^{-1} = 0.5$. Thus, the WKB formula has a maximum error of about 5 percent.

ior of this integral. The contribution from region I is given by

$$\begin{aligned} A_{\text{I}} &= \left[\frac{Q(x')}{2\varepsilon}\right]^{-1/4} \int_{x'}^{\infty} dx[Q(x)]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_{x'}^{x} \sqrt{Q(t)}\, dt\right] \\ &= \left[\frac{Q(x')}{-2}\right]^{-1/4} \int_{x'}^{\infty} dx[Q(x)]^{-3/4} \frac{d}{dx} \exp\left[-\frac{1}{\varepsilon}\int_{x'}^{x} \sqrt{Q(t)}\, dt\right] \\ &= \frac{1}{2Q(x')} + O(\varepsilon), \qquad \varepsilon \to 0+. \end{aligned}$$

The contribution from region II is identical. Thus, we obtain the simple result

$$\int_{-\infty}^{\infty} G(x, x')\, dx = \frac{1}{Q(x')} + O(\varepsilon), \qquad \varepsilon \to 0+. \tag{10.3.9}$$

This result also follows from (10.3.4) with $R(x) = 1$ since (10.3.2) implies that the solution to $\varepsilon y'' = Q(x)y - 1$ satisfies $y \sim 1/Q(x)$ $(\varepsilon \to 0+)$.

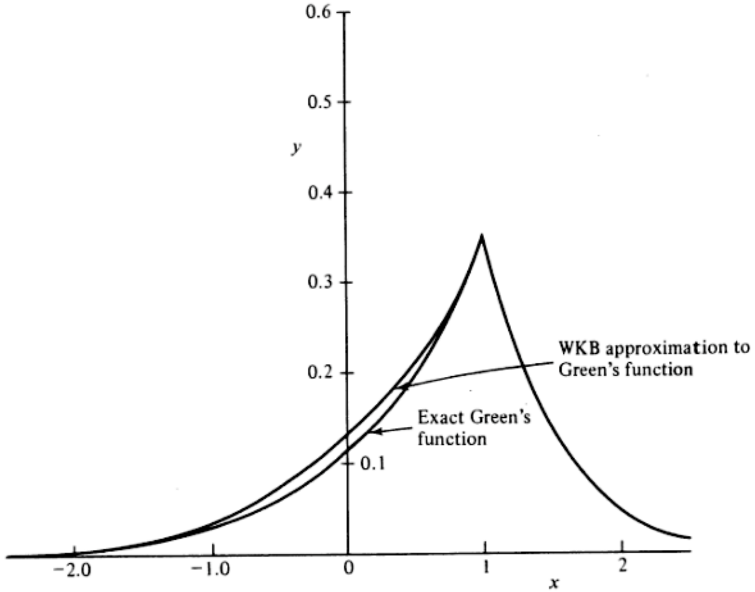

Figure 10.5 Same as in Fig. 10.4 except that $x' = 1$ instead of 0; ε is still 1. Again, the error is greatest at $x = 0$. The WKB formula in (10.3.8) predicts that when $\varepsilon = 1$, $G(1, 1) = \frac{1}{4}\sqrt{2} \doteq 0.35355$. The true value of $G(1, 1)$ is 0.349 13. . . .

Example 2 *Comparison between exact and approximate integrals.* If we choose $Q(x) = 1 + x^2$ and $\varepsilon = 1$ as we did in Example 1, we conclude from (10.3.9) that $A = \int_{-\infty}^{\infty} G(x, x')\, dx = 1$ when $x' = 0$ and $A = 0.5$ when $x' = 1$. Numerical integration of the Green's function differential equation (see Figs. 10.4 and 10.5) gives the true values of A: $A = 0.623\,23$ for $x' = 0$ and $A = 0.466\,51$ for $x' = 1$. The errors are quite small considering the large size of ε.

It is just as easy to evaluate integrals of powers of G_{unif} in (10.3.7). For example (see Prob. 10.13),

$$\int_{-\infty}^{\infty} [G_{\text{unif}}(x, x')]^N\, dx = \frac{(2\varepsilon)^{1-N}}{N} [Q(x')]^{-(N+1)/2} + O(\varepsilon^{2-N}), \qquad \varepsilon \to 0+. \tag{10.3.10}$$

To solve the inhomogeneous Schrödinger equation (10.3.1), one need only apply these same integration methods to evaluate the integral in (10.3.4). At points of continuity of $R(x)$, asymptotic analysis of (10.3.4) as $\varepsilon \to 0+$ yields (10.3.2) (see Prob. 10.16).

Example 3 *Uniform approximation to the Schrödinger equation with discontinuous inhomogeneity.* If we choose $R(x)$ in (10.3.1) to be the step function

$$R(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$$

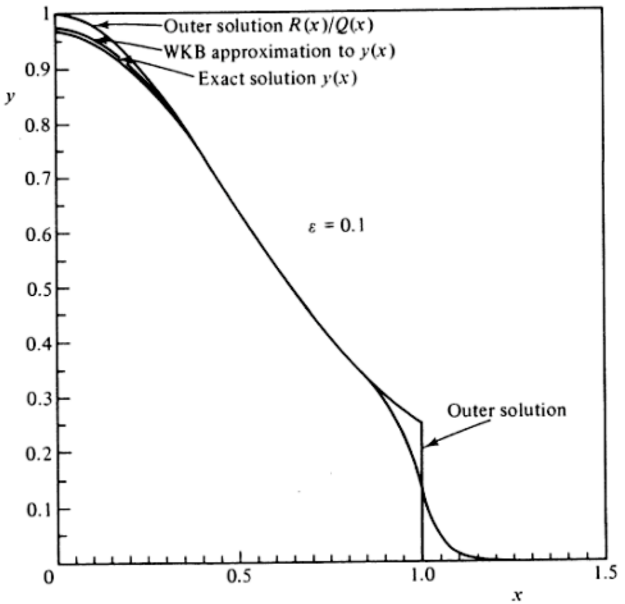

Figure 10.6 A comparison of the exact solution to (10.3.12) for $\varepsilon = 0.1$ with the leading-order WKB approximation in (10.3.11). Also plotted is the outer solution $R(x)/Q(x)$, which cuts off abruptly at $x = 1$. Observe that the WKB approximation is uniform and is especially good near the discontinuity in $R(x)$ at $x = 1$, where it is indistinguishable from the exact solution.

then we obtain $y(x) \sim \int_{-1}^{1} G_{\text{unif}}(x, x')\, dx'$ $(\varepsilon \to 0+)$. This is a nontrivial result whose accuracy we examine numerically. If we take $Q(x) = (1 + x^2)^2$, then we have a uniform approximation to $y(x)$ for all x:

$$y(x) \sim \frac{1}{2\varepsilon\sqrt{x^2 + 1}} \int_{-1}^{1} \frac{dx'}{\sqrt{x'^2 + 1}} \exp\left[-\frac{1}{\varepsilon}\left|x^3/3 + x - x'^3/3 - x'\right|\right], \qquad \varepsilon \to 0+. \tag{10.3.11}$$

Figures 10.6 and 10.7 compare the WKB prediction in (10.3.11) with the exact solution to the inhomogeneous Schrödinger equation

$$\varepsilon^2 y''(x) - (1 + x^2)^2 y(x) + \begin{Bmatrix} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{Bmatrix} = 0, \qquad y(\pm\infty) = 0. \tag{10.3.12}$$

In these figures, we also plot the outer approximation $R(x)/Q(x)$. Observe that this outer approximation is not uniformly valid in the neighborhood of the discontinuities of $R(x)$ at $x = \pm 1$.

A uniform approximation to $y(x)$ can also be derived using boundary-layer theory. In this case the outer solution is given by $R(x)/Q(x)$ and the inner approximation is valid at discontinuities of $R(x)$. (See Prob. 10.16.)

Example 4 *Uniform approximation to the Schrödinger equation with singular inhomogeneity.* If we choose $R(x)$ to be the singular function

$$R(x) = \begin{cases} (1 - x^2)^{-1/2}, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases}$$

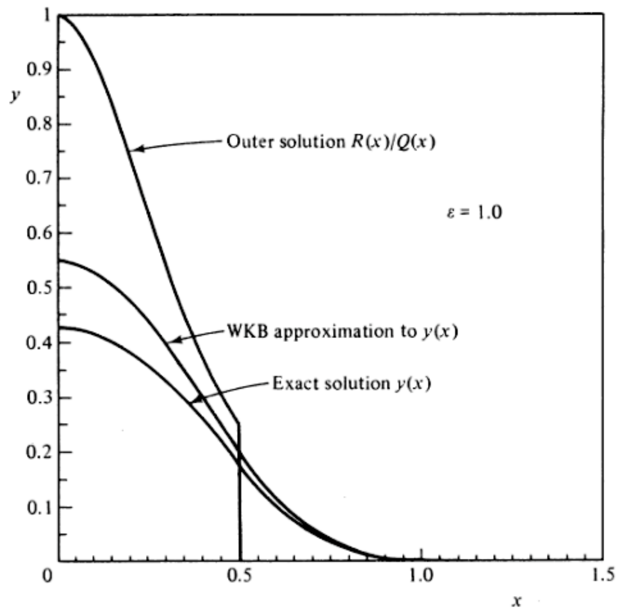

Figure 10.7 Same as in Fig. 10.6 except that $\varepsilon = 1$.

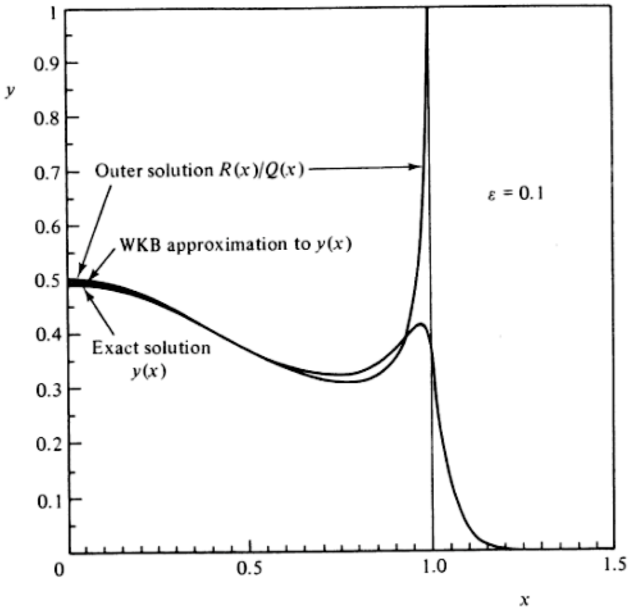

Figure 10.8 Same as in Fig. 10.6 with $\varepsilon = 0.1$ except that $R(x)$ is the singular function $(1 - x^2)^{-1/2}$ $(x < 1)$. The outer solution $R(x)/Q(x)$ becomes singular at $x = 1$, but the WKB approximation to $y(x)$ in (10.3.13) is accurate for all values of x.

and $Q(x) = (1 + x^2)^2$ as in Example 3, then a uniform approximation to $y(x)$ for all x is given by

$$y(x) \sim \frac{1}{2\varepsilon\sqrt{x^2+1}} \int_{-1}^{1} \frac{dx'}{\sqrt{1-x'^4}} \exp\left[-\frac{1}{\varepsilon}\left|x^3/3 + x - x'^3/3 - x'\right|\right], \qquad \varepsilon \to 0+. \qquad (10.3.13)$$

In Figs. 10.8 and 10.9 this WKB prediction is compared with the exact solution $y(x)$ to (10.3.1).

(I) 10.4 MATCHED ASYMPTOTIC APPROXIMATIONS: SOLUTION OF THE ONE-TURNING-POINT PROBLEM

We saw in Sec. 10.2 that the WKB exponential approximation for the Schrödinger equation is not valid in the neighborhood of a turning point. In fact, the physical-optics approximation in (10.1.13) is singular at a turning point. Nevertheless, we will see that there is a general procedure, which is based on the method of matched asymptotic expansions, for constructing a global approximation to the solution of a differential equation having turning points. The approach is very similar to that

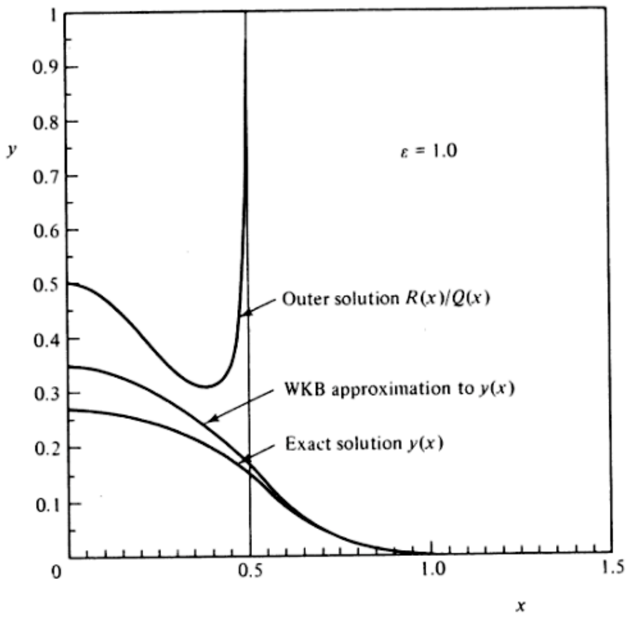

Figure 10.9 Same as in Fig. 10.8 except that $\varepsilon = 1$. Even for this large value of ε, the leading-order WKB approximation in (10.3.13) is a good estimate of $y(x)$.

used in boundary-layer theory. It consists of joining together various WKB approximations which hold in their respective regions of validity.

In this section we begin rather modestly by considering a differential equation which has just *one* turning point. Specifically, we will solve the equation

$$\varepsilon^2 y'' = Q(x)y, \qquad y(+\infty) = 0, \qquad (10.4.1)$$

where $Q(x)$ is a continuous function which passes through zero just once. For simplicity, we take the turning point to lie at the origin: $Q(0) = 0$.

The Simple One-Turning-Point Problem

We begin by analyzing in detail the one-turning-point problem in which $Q(x)$ has a simple (first-order) zero: $Q(x) \sim ax$ $(x \to 0)$. For definiteness, we assume that $Q(x)$ has positive slope at $x = 0$ $(a > 0)$ and that $Q(x)$ is positive when x is positive and negative when x is negative. We also assume that $Q(x) \gg x^{-2}$ as $x \to \pm\infty$. $Q(x) = \sinh x$ and $x + x^3$ satisfy these criteria. In Probs. 10.26 and 10.29 we generalize to the case in which $Q(x)$ has a zero of order α: $Q(x) \sim ax^\alpha$ $(x \to 0)$.

Our analysis of the simple-zero one-turning-point problem proceeds as follows. We divide the x axis into three regions: region I with $x > 0$ and $x \gg \varepsilon^{2/3}$, region II with $|x| \ll 1$, and region III with $x < 0$ and $(-x) \gg \varepsilon^{2/3}$. In regions I and III the physical-optics approximation in (10.1.13) is uniformly valid. The restriction that $Q(x) \gg x^{-2}$ as $|x| \to \infty$ ensures that the physical-optics approximation is valid all the way out to $+\infty$, where we impose the boundary condition. In region II the WKB approximation is not valid because there is a turning point at $x = 0$, but we can solve the approximate differential equation

$$\varepsilon^2 y'' = axy, \tag{10.4.2}$$

valid in the neighborhood of $x = 0$, in terms of Airy functions. We show that regions I and II and regions II and III have an overlap in which both the WKB and Airy function approximations are valid. This enables us to match together asymptotically the solutions in the various regions. From this matching we obtain three formulas which together constitute a global approximation to the solution of (10.4.1). We then combine these three formulas into a single expression which is a uniformly valid approximation to $y(x)$ for all x. The global approximations to $y(x)$ are determined only up to an overall multiplicative constant because we impose only the single boundary condition $y(+\infty) = 0$. Therefore, we consider various methods for normalizing $y(x)$.

The calculation that we have just outlined begins with an analysis of the equation in region I. The physical-optics approximation to $y(x)$ in this region has the form

$$y_{\rm I}(x) = C[Q(x)]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_0^x \sqrt{Q(t)}\, dt\right]. \tag{10.4.3}$$

The boundary condition $y_{\rm I}(+\infty) = 0$ has been used to eliminate the exponentially growing physical-optics solution and is explicitly satisfied by $y_{\rm I}(x)$ in (10.4.3). We have arbitrarily chosen the lower limit of integration in (10.4.3) to lie at the turning point $x = 0$; this choice is not necessary, but it simplifies expressions appearing later in our analysis.

It is essential to determine the region of validity of the approximation (10.4.3). The two criteria for the validity of physical optics that we derived in Sec. 10.2 are $S_0/\varepsilon \gg S_1 \gg \varepsilon S_2$ $(\varepsilon \to 0+)$ and $\varepsilon S_2 \ll 1$ $(\varepsilon \to 0+)$. Because $Q(x)$ is nonzero for $x \neq 0$ and $Q(x) \gg x^{-2}$ as $|x| \to \infty$, we are assured that for x bounded away from the origin the difference between the exact solution $y(x)$ of (10.4.1) and $y_{\rm I}(x)$ is of order ε as $\varepsilon \to 0+$. How small may x be before the physical-optics approximation $y_{\rm I}(x)$ breaks down? When x is small, $Q(x) \sim ax$ so $S_0(x) \sim \pm\frac{2}{3}a^{1/2}x^{3/2}$ $(x \to 0+)$, $S_1(x) \sim -\frac{1}{4}\ln x$ $(x \to 0+)$, $S_2(x) \sim \pm\frac{5}{48}a^{-1/2}x^{-3/2}$ $(x \to 0+)$. Thus, the criteria for validity of the WKB physical-optics approximation are satisfied if

$$x \gg \varepsilon^{2/3}, \qquad \varepsilon \to 0+. \tag{10.4.4}$$

This relation defines the lower boundary of region I.

Next we turn to the analysis of the equation in region II. To solve the approximate differential equation (10.4.2), we make the substitution

$$t = \varepsilon^{-2/3}a^{1/3}x. \tag{10.4.5}$$

In terms of t, the differential equation for $y_{\rm II}$ is $d^2y_{\rm II}/dt^2 = ty_{\rm II}$, which we recognize as the Airy equation. The general solution of this equation is a linear combination of Airy functions:

$$y_{\rm II}(x) = D\,\mathrm{Ai}\,(\varepsilon^{-2/3}a^{1/3}x) + E\,\mathrm{Bi}\,(\varepsilon^{-2/3}a^{1/3}x), \tag{10.4.6}$$

where D and E are constants to be determined by asymptotic matching with $y_{\rm I}(x)$. The approximation $y_{\rm II}(x)$ is valid so long as

$$x \ll 1, \qquad \varepsilon \to 0+, \tag{10.4.7}$$

because it is only when x is small that we may replace $Q(x)$ by ax and thereby obtain (10.4.2) from (10.4.1). The relation in (10.4.7) defines the upper boundary of region II.

Combining (10.4.4) and (10.4.7), we observe that $y_{\rm I}(x)$ in (10.4.3) and $y_{\rm II}(x)$ in (10.4.6) have a common region of validity; namely, $\varepsilon^{2/3} \ll x \ll 1$ $(\varepsilon \to 0+)$. Inside this overlap region $y_{\rm I}(x)$ and $y_{\rm II}(x)$ are both approximate solutions to the same differential equation and therefore they must match asymptotically. However, since $y_{\rm I}$ and $y_{\rm II}$ bear so little resemblance to each other, more analysis is required to demonstrate that they actually match. We must further approximate $y_{\rm I}(x)$ and $y_{\rm II}(x)$ in the overlap region.

First, we consider $y_{\rm I}(x)$. In the overlap region x is small so $Q(x)$ is approximately ax. Therefore, $[Q(x)]^{-1/4} \sim a^{-1/4}x^{-1/4}$ $(x \to 0+)$ and

$$\int_0^x \sqrt{Q(t)}\, dt \sim \tfrac{2}{3}a^{1/2}x^{3/2}, \qquad x \to 0+.$$

Hence,

$$y_{\rm I}(x) \sim Ca^{-1/4}x^{-1/4}e^{-2a^{1/2}x^{3/2}/3\varepsilon}, \qquad x \to 0+. \tag{10.4.8}$$

What is the precise region of validity of (10.4.8)? We already know that the WKB approximation is not valid unless $x \gg \varepsilon^{2/3}$ $(\varepsilon \to 0+)$. However, the upper edge of the region depends on the function $Q(x)$. Suppose, for example, that $Q(x) - ax \sim bx^2$ $(x \to 0)$. Then, a careful estimation gives

$$\begin{aligned}\int \sqrt{Q(t)}\, dt &\sim \int_0^x \sqrt{at + bt^2}\, dt\\ &\sim \int_0^x \sqrt{at}\left(1 + \frac{bt}{2a}\right) dt\\ &\sim \frac{2}{3}a^{1/2}x^{3/2} + \frac{b}{5\sqrt{a}}x^{5/2}, \qquad x \to 0+.\end{aligned}$$

To obtain (10.4.8) it was necessary to assume that x is sufficiently small so that $\exp\,(bx^{5/2}/5\varepsilon\sqrt{a}) \sim 1$ $(\varepsilon \to 0+)$. Hence we arrive at the condition that $x \ll \varepsilon^{2/5}$ $(\varepsilon \to 0+)$. Thus, (10.4.8) is valid in the restricted region $\varepsilon^{2/3} \ll x \ll \varepsilon^{2/5}$ $(\varepsilon \to 0+)$.

Next we consider $y_{\rm II}(x)$. In the overlap region we approximate the Airy functions by their leading asymptotic behaviors for large positive argument. The appropriate formulas are

$$\mathrm{Ai}\,(t) \sim \frac{1}{2\sqrt{\pi}} t^{-1/4} e^{-2t^{3/2}/3}, \qquad t \to +\infty,$$

$$\mathrm{Bi}\,(t) \sim \frac{1}{\sqrt{\pi}} t^{-1/4} e^{2t^{3/2}/3}, \qquad t \to +\infty.$$

These approximations may be used if the arguments of the Airy functions in (10.4.6) are large. Thus,

$$y_{\rm II}(x) \sim \frac{1}{\sqrt{\pi}} a^{-1/12} \varepsilon^{1/6} x^{-1/4} \left(\frac{1}{2} D e^{-2a^{1/2}x^{3/2}/3\varepsilon} + E e^{2a^{1/2}x^{3/2}/3\varepsilon} \right). \qquad (10.4.9)$$

This result is valid if two criteria are satisfied. First, we require that $x \ll 1$ as $\varepsilon \to 0+$, so that the Airy equation (10.4.2) is a good approximation to the differential equation (10.4.1). Second, the use of the asymptotic approximations to the Airy functions requires that $t = \varepsilon^{-2/3} a^{1/3} x$ be large or equivalently that $x \gg \varepsilon^{2/3}$ as $\varepsilon \to 0+$. Thus, the region of validity of (10.4.9) is $\varepsilon^{2/3} \ll x \ll 1$ $(\varepsilon \to 0+)$.

Now observe two things. First, unlike (10.4.3) and (10.4.6), (10.4.8) and (10.4.9) have the same functional form and can therefore be matched. Second, (10.4.8) and (10.4.9) have a common region of validity over which the matching can take place:

$$\varepsilon^{2/3} \ll x \ll \varepsilon^{2/5}, \qquad \varepsilon \to 0+. \qquad (10.4.10)$$

Requiring that (10.4.8) and (10.4.9) match on the overlap region (10.4.10) determines the constants D and E:

$$D = 2\sqrt{\pi}(a\varepsilon)^{-1/6} C, \qquad (10.4.11a)$$

$$E = 0. \qquad (10.4.11b)$$

You may recall that it was emphasized in Chaps. 7 and 9 that asymptotic matching must be performed throughout a region whose extent becomes infinite as the perturbation parameter $\varepsilon \to 0$. At first sight the overlap region (10.4.10) appears to violate this principle. However, the matching variable is not x but rather t as given by (10.4.5). In this variable the matching region is $1 \ll t \ll \varepsilon^{-4/15}$ $(\varepsilon \to 0+)$, which does indeed become infinite as $\varepsilon \to 0+$.

The problem is now half solved. We have completed the asymptotic match between regions I and II. Next we must analyze region III and match to the solution just found in region II.

The physical-optics approximation in region III is a linear combination of two rapidly oscillating WKB expressions:

$$y_{\rm III}(x) = F[-Q(x)]^{-1/4} \exp\left[\frac{i}{\varepsilon}\int_x^0 \sqrt{-Q(t)}\,dt\right] + G[-Q(x)]^{-1/4} \exp\left[-\frac{i}{\varepsilon}\int_x^0 \sqrt{-Q(t)}\,dt\right].$$

We will shortly verify that in order for this expression to match to $y_{\rm II}(x)$ in the overlap of regions II and III, the constants F and G must be chosen so that

$$y_{\rm III}(x) = 2C[-Q(x)]^{-1/4} \sin\left[\frac{1}{\varepsilon}\int_x^0 \sqrt{-Q(t)}\,dt + \frac{\pi}{4}\right]. \qquad (10.4.12)$$

The result in (10.4.12) is established by comparing the asymptotic approximations to $y_{\rm III}(x)$ and to $y_{\rm II}(x)$ in the overlap of regions II and III which is $\varepsilon^{2/3} \ll (-x) \ll \varepsilon^{2/5}$ $(\varepsilon \to 0+)$. In this overlap region we may approximate $y_{\rm III}(x)$ in (10.4.12) by

$$2Ca^{-1/4}(-x)^{-1/4} \sin\left[\frac{2}{3\varepsilon} a^{1/2}(-x)^{3/2} + \frac{\pi}{4}\right].$$

Also, using the formula for the asymptotic behavior of Ai (t) for large negative argument,

$$\mathrm{Ai}\,(t) = \frac{1}{\sqrt{\pi}}(-t)^{-1/4} \sin \phi(t), \qquad \phi(t) \sim \frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}, \qquad t \to -\infty,$$

we may approximate $y_{\rm II}(x)$ in (10.4.6) with $E = 0$ by

$$D\pi^{-1/2} a^{-1/12} \varepsilon^{1/6} (-x)^{-1/4} \sin\left[\frac{2}{3\varepsilon} a^{1/2}(-x)^{3/2} + \frac{\pi}{4}\right].$$

The approximations we have just found for $y_{\rm II}(x)$ and $y_{\rm III}(x)$ in the overlap region match exactly because D and C are related by (10.4.11a). This completes the analysis of regions II and III.

In summary, we have found approximations to $y(x)$ in each of regions I, II, and III. These approximations are:

$$y_{\rm I}(x) = C[Q(x)]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_0^x \sqrt{Q(t)}\,dt\right],$$
$$x > 0,\ x \gg \varepsilon^{2/3},\ \varepsilon \to 0+; \qquad (10.4.13a)$$

$$y_{\rm II}(x) = 2\sqrt{\pi}(a\varepsilon)^{-1/6} C\ \mathrm{Ai}\,(\varepsilon^{-2/3} a^{1/3} x), \quad |x| \ll 1,\ \varepsilon \to 0+; \qquad (10.4.13b)$$

$$y_{\rm III}(x) = 2C[-Q(x)]^{-1/4} \sin\left[\frac{1}{\varepsilon}\int_x^0 \sqrt{-Q(t)}\,dt + \frac{\pi}{4}\right],$$
$$x < 0,\ (-x) \gg \varepsilon^{2/3},\ \varepsilon \to 0+. \qquad (10.4.13c)$$

The first and third of these formulas are sometimes called *connection formulas* because they express the connection between the oscillatory and the exponentially decreasing behavior of $y(x)$ on opposite sides of the turning point. The constant C remains undetermined because we have specified only the one boundary condition $y(+\infty) = 0$. A second boundary condition is needed to determine C. For example, if we require that $y(0) = 1$, then since Ai $(0) = 3^{-2/3}/\Gamma(\frac{2}{3}) \doteq 0.355\,028\,053\,9$, we have

$$C = \tfrac{1}{2}(a\varepsilon)^{1/6}\Gamma(\tfrac{2}{3})3^{2/3}\pi^{-1/2}. \qquad (10.4.14)$$

Observe the global nature of the WKB approximation; we have specified the boundary condition at $x = 0$ and at $x = \infty$ and we can predict the value of $y(-27)$, say, correct to order ε.

Uniform Asymptotic Approximation

In 1935 Langer made the amazing observation that all three formulas in (10.4.13) may be replaced by a *single* formula which is a uniformly valid approximation to $y(x)$ for all x:

$$y_{\text{unif}}(x) = 2\sqrt{\pi}\, C \left(\frac{3}{2\varepsilon} S_0\right)^{1/6} [Q(x)]^{-1/4} \operatorname{Ai}\left[\left(\frac{3}{2\varepsilon} S_0\right)^{2/3}\right], \qquad (10.4.15)$$

where $S_0 = \int_0^x \sqrt{Q(t)}\, dt$. This result is not at all obvious. The best way to explain it is simply to verify it in all three regions. [For a derivation of the Langer formula directly from the differential equation (10.4.1) see Prob. 10.18.]

First, we consider region I, where $x \gg \varepsilon^{2/3}$. Throughout this region $3S_0(x)/2\varepsilon \gg 1$, so we may approximate Ai $[(3S_0/2\varepsilon)^{2/3}]$ by its leading asymptotic behavior:

$$\operatorname{Ai}\left[\left(\frac{3S_0}{2\varepsilon}\right)^{2/3}\right] \sim \left(\frac{3S_0}{2\varepsilon}\right)^{-1/6} \frac{e^{-S_0(x)/\varepsilon}}{2\sqrt{\pi}}, \qquad x \gg \varepsilon^{2/3}.$$

If we substitute this expression into (10.4.15), it reduces to the first formula in (10.4.13).

In region II, where $|x| \ll 1$, the integral $S_0(x)$ may be evaluated approximately by using the first term in the Taylor series for $Q(t)$:

$$\operatorname{Ai}\left[\left(\frac{3}{2\varepsilon} S_0\right)^{2/3}\right] \sim \operatorname{Ai}\,(\varepsilon^{-2/3} a^{1/3} x),$$

$$\left(\frac{3}{2\varepsilon} S_0\right)^{1/6} [Q(x)]^{-1/4} \sim (a\varepsilon)^{-1/6}, \qquad |x| \ll 1,\ \varepsilon \to 0+.$$

Hence (10.4.15) reduces exactly to the second formula of (10.4.13).

In region III, where $(-x) \gg \varepsilon^{2/3}$, one must be very careful about + and − signs (see Prob. 10.19). Now,

$$S_0(x) = \int_0^x \sqrt{Q(t)}\, dt = e^{3\pi i/2} \int_x^0 \sqrt{-Q(t)}\, dt.$$

Thus, $S_0^{2/3}$ is large and *negative*, and (10.4.15) may be simplified by using the asymptotic behavior of Ai for negative argument:

$$\operatorname{Ai}\left[\left(\frac{3}{2\varepsilon} S_0\right)^{2/3}\right] \sim \frac{1}{\sqrt{\pi}} \left[\frac{3}{2\varepsilon} \int_x^0 \sqrt{-Q(t)}\, dt\right]^{-1/6}$$
$$\times \sin\left[\frac{1}{\varepsilon} \int_x^0 \sqrt{-Q(t)}\, dt + \frac{\pi}{4}\right], \qquad \varepsilon \to 0+.$$

Also,

$$Q^{-1/4} = (-Q)^{-1/4} e^{-i\pi/4},$$

$$\left(\frac{3}{2\varepsilon} S_0\right)^{1/6} = e^{i\pi/4} \left[\frac{3}{2\varepsilon} \int_x^0 \sqrt{-Q(t)}\, dt\right]^{1/6}.$$

Thus, (10.4.15) reduces exactly to the third formula of (10.4.13).

Example 1 *Numerical comparison between exact and one-turning-point WKB solutions.* In Figs. 10.10 to 10.13 we compare the exact and uniform one-turning-point solutions in (10.4.15) to $\varepsilon^2 y''(x) = \sinh x(\cosh x)^2 y(x)$ [$y(0) = 1$, $y(+\infty) = 0$] for $\varepsilon = 0.2, 0.3, 0.5$, and 1. Note that for this choice of $Q(x)$, $a = 1$ and $\int_0^x \sqrt{Q(t)}\, dt = \frac{2}{3}(\sinh x)^{3/2}$. The agreement between the exact and the approximate solution is extremely impressive, even when ε is not small.

Directional Character of the Connection Formula

There is a subtle feature of the solution (10.4.13) to the one-turning-point problem. You will recall that in our analysis of this problem we started with the

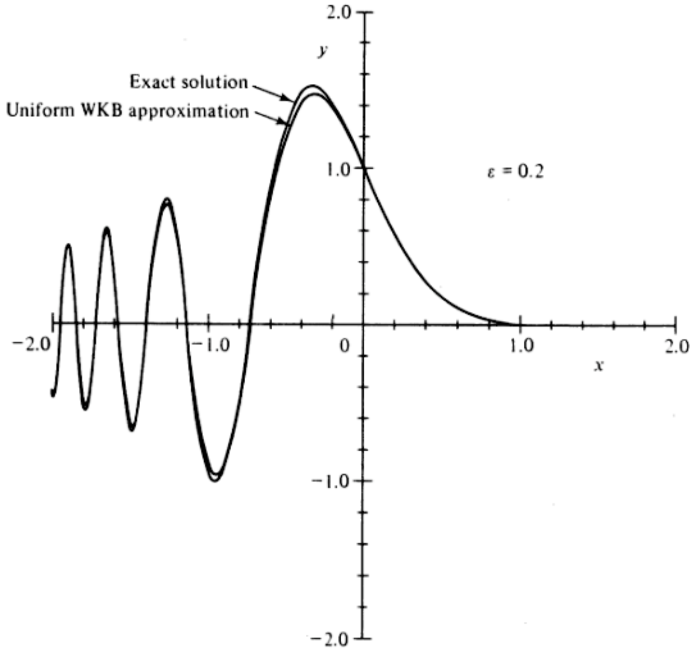

Figure 10.10 A comparison of the exact solution to $\varepsilon^2 y''(x) = \sinh x(\cosh x)^2 y(x)$ [$y(0) = 1$, $y(+\infty) = 0$], with the approximate solution from a one-turning-point WKB analysis. The WKB approximate formulas are given in (10.4.14) and (10.4.15).

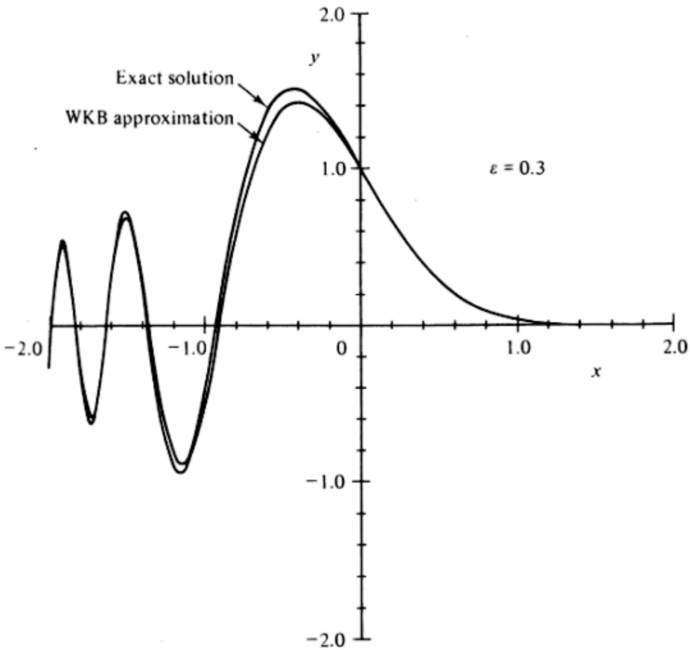

Figure 10.11 Same as in Fig. 10.10 except that $\varepsilon = 0.3$.

boundary condition $y \to 0$ as $x \to +\infty$ in region I and deduced the structure of the solution in regions II and III *in that order*. It is remarkable that the sequence in which this asymptotic analysis is carried out cannot be reversed. To wit, suppose it is given that as $x \to -\infty$ in region III, the solution to $\varepsilon^2 y'' = Q(x)y$ behaves as $2C[-Q(x)]^{-1/4} \sin [\int_x^0 \sqrt{-Q(t)}\, dt/\varepsilon + \frac{1}{4}\pi]$. One is tempted to conclude that the behavior of $y(x)$ in region I is necessarily exponentially decaying: $C[Q(x)]^{-1/4} \exp [-\int_0^x \sqrt{Q(t)}\, dt/\varepsilon]$. But this inference is wrong because the asymptotic matching through the turning-point region is only valid to leading order in ε. We may only conclude that the coefficient of the exponentially growing solution in region I vanishes to leading order in ε. We cannot be sure that the exponentially growing solution in region I is really absent unless the boundary condition $y(+\infty) = 0$ is explicitly imposed.

Apparently, the connection formula for the one-turning-point problem is directional in character. The analysis always proceeds from the region where the solution is exponentially decaying through the turning point and into the oscillatory region. For this reason the descriptive notation

$$\boxed{\begin{aligned} &2[-Q(x)]^{-1/4} \sin \left[\frac{1}{\varepsilon}\int_x^0 \sqrt{-Q(t)}\, dt + \frac{1}{4}\pi\right] \text{ in region III} \\ &\leftarrow [Q(x)]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_0^x \sqrt{Q(t)}\, dt\right] \text{ in region I} \end{aligned}} \tag{10.4.16}$$

is often used to denote the connection formula.

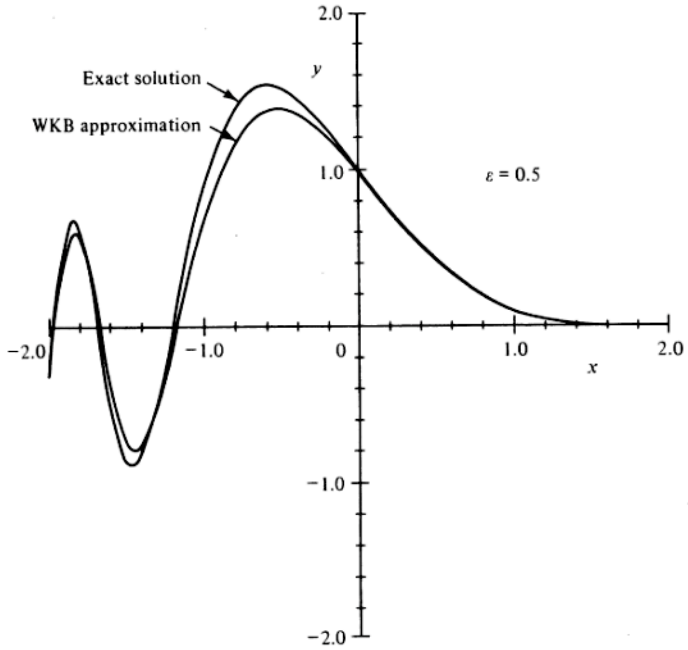

Figure 10.12 Same as in Fig. 10.10 except that $\varepsilon = 0.5$.

Normalization of the One-Turning-Point Solution

The one-turning-point solution (10.4.13) has an arbitrary multiplicative constant C because the Schrödinger equation (10.4.1) and the boundary condition $y(+\infty) = 0$ are homogeneous. In Example 1 we showed how to determine C by

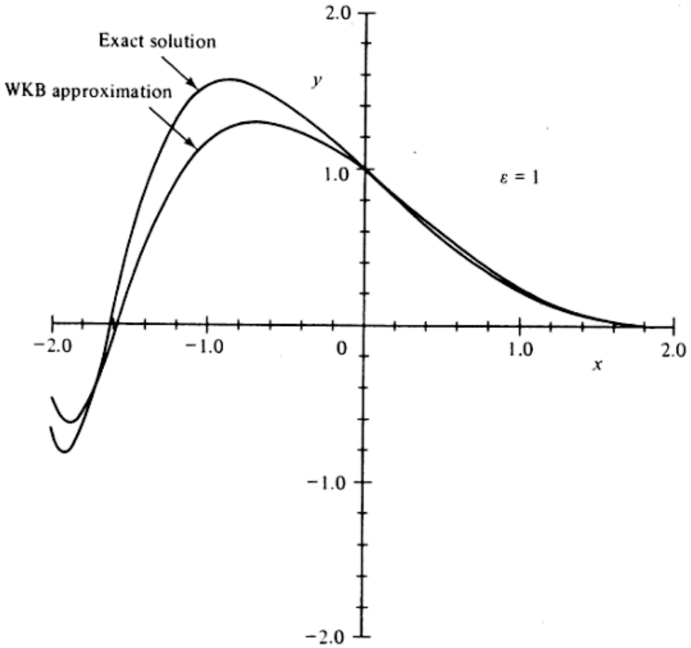

Figure 10.13 Same as in Fig. 10.10 except that $\varepsilon = 1.0$. Even for this large value of ε the agreement between the approximate and exact solutions is impressive.

imposing the additional inhomogeneous boundary condition $y(0) = 1$. Another way to determine C is to require that

$$\int_{-\infty}^{\infty} y(x)\,dx = 1 \tag{10.4.17}$$

or that

$$\int_{-\infty}^{\infty} [y(x)]^2\,dx = 1. \tag{10.4.18}$$

In contrast to the boundary condition $y(0) = 1$ which is imposed at one point, the normalization conditions (10.4.17) and (10.4.18) are global in character. The methods we shall use to evaluate these integrals are especially important because they are a prototype of the techniques for evaluating integrals of functions approximated by matched asymptotic formulas.

To evaluate the integral in (10.4.17), we introduce two arbitrary points A and B where A lies in the overlap of regions II and III and B lies in the overlap of regions I and II. Next we approximate the integral in (10.4.17) as the sum of three

integrals:

$$\int_{-\infty}^{\infty} y\,dx \sim \int_{-\infty}^{A} y_{\text{III}}\,dx + \int_{A}^{B} y_{\text{II}}\,dx + \int_{B}^{\infty} y_{\text{I}}\,dx, \qquad \varepsilon \to 0+, \tag{10.4.19}$$

where y_{I}, y_{II}, y_{III} are given in (10.4.13). Since the points A and B are arbitrary and do not appear in the original integral in (10.4.17), the final answer must be completely independent of the particular choice of A and B. However, the approximations (10.4.13) are only leading-order approximations, so we expect that A and B will disappear from (10.4.19) only to leading order in ε. The cancellation of A and B in the final result is a nontrivial test of the correctness of the asymptotic approximations y_{I}, y_{II}, and y_{III}.

We evaluate $\int_B^\infty y_{\text{I}}\,dx$ using integration by parts:

$$\begin{aligned}
\int_{B}^{\infty} y_{\text{I}}\,dx &= C\int_{B}^{\infty} [Q(x)]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_0^x \sqrt{Q(t)}\,dt\right] dx \\
&= -C\varepsilon\int_{B}^{\infty} [Q(x)]^{-3/4} \frac{d}{dx} \exp\left[-\frac{1}{\varepsilon}\int_0^x \sqrt{Q(t)}\,dt\right] dx \\
&= +C\varepsilon[Q(B)]^{-3/4} \exp\left[-\frac{1}{\varepsilon}\int_0^B \sqrt{Q(t)}\,dt\right] + \mathrm{O}(\varepsilon^2), \qquad \varepsilon \to 0+, \\
&\sim C\varepsilon(aB)^{-3/4} \exp\left(-\frac{2\sqrt{a}}{3\varepsilon} B^{3/2}\right), \qquad \varepsilon \to 0+,
\end{aligned} \tag{10.4.20}$$

where we have retained only the boundary term after integrating by parts because a second integration by parts shows that the remaining integral is $\mathrm{O}(\varepsilon^2)$. The last step in the calculation, where we have replaced Q by the first term in its Taylor series, is valid because B lies in the overlap region (10.4.9).

We evaluate $\int_{-\infty}^{A} y_{\text{III}}\,dx$ similarly. The result is

$$\int_{-\infty}^{A} y_{\text{III}}\,dx \sim 2C\varepsilon(-aA)^{-3/4} \cos\left[\frac{2\sqrt{a}}{3\varepsilon}(-A)^{3/2} + \frac{1}{4}\pi\right], \qquad \varepsilon \to 0+, \tag{10.4.21}$$

provided, of course, that the integral converges at $-\infty$. The integral converges if $Q(x) \to -\infty$ as $x \to -\infty$.

Finally, we evaluate $\int_A^B y_{\text{II}}\,dx$ by expressing it as the sum of three integrals: $\int_A^B y_{\text{II}}\,dx = \int_{-\infty}^{\infty} y_{\text{II}}\,dx - \int_{-\infty}^{A} y_{\text{II}}\,dx - \int_B^\infty y_{\text{II}}\,dx$. The first of the integrals may be done exactly using the identity

$$\int_{-\infty}^{\infty} \mathrm{Ai}\,(t)\,dt = 1. \tag{10.4.22}$$

(See Prob. 10.20 for a derivation of this identity.) The second and third integrals may be evaluated by substituting the asymptotic behaviors of the Airy function for large negative and large positive arguments and then using integration by parts. The final result is (see Prob. 10.21)

$$\int_A^B y_{\text{II}}\,dx \sim 2C\sqrt{\frac{\pi\varepsilon}{a}} - 2C\varepsilon(-aA)^{-3/4}\cos\left[\frac{2\sqrt{a}}{3\varepsilon}(-A)^{3/2} + \frac{1}{4}\pi\right]$$

$$- C\varepsilon(aB)^{-3/4}\exp\left(-\frac{2\sqrt{a}}{3\varepsilon}B^{3/2}\right), \qquad \varepsilon \to 0+. \tag{10.4.23}$$

We combine the results in (10.4.20) and (10.4.21) and (10.4.23) and are pleased to find that all reference to A and B cancels to leading order ε:

$$\int_{-\infty}^{\infty} y\,dx \sim 2C\sqrt{\frac{\pi\varepsilon}{a}}, \qquad \varepsilon \to 0+. \tag{10.4.24}$$

Example 2 *Numerical verification of (10.4.24).* Suppose that we use physical optics to solve the one-turning-point problem

$$\varepsilon^2 y'' = (x + x^3)y, \qquad y(0) = 1,\ y(+\infty) = 0. \tag{10.4.25}$$

For this choice of $Q(x)$, $a = 1$. Equation (10.4.14) implies that the leading-order solution is given by (10.4.13) with $C = \frac{1}{2}\varepsilon^{1/6}\Gamma(\frac{2}{3})3^{2/3}\pi^{-1/2}$. Thus (10.4.24) implies that

$$\begin{aligned} \int_{-\infty}^{\infty} y\,dx &\sim \varepsilon^{2/3}\Gamma(\tfrac{2}{3})3^{2/3}, \qquad \varepsilon \to 0+ \\ &\doteq 2.816\,679\varepsilon^{2/3}, \qquad \varepsilon \to 0+. \end{aligned} \tag{10.4.26}$$

We have solved (10.4.25) numerically and computed $\int_{-\infty}^{\infty} y\,dx$. The results given in Table 10.2 verify the accuracy of this WKB analysis.

To evaluate the integral in (10.4.18), we again introduce two arbitrary points A and B in the overlap regions and express the integral as $\int_{-\infty}^{\infty} y^2\,dx \sim \int_{-\infty}^{A} y_{\text{III}}^2\,dx + \int_A^B y_{\text{II}}^2\,dx + \int_B^{\infty} y_{\text{I}}^2\,dx$ ($\varepsilon \to 0+$). We evaluate $\int_B^{\infty} y_{\text{I}}^2\,dx$ as before using integration by parts. The result is

$$\int_B^{\infty} y_{\text{I}}^2\,dx \sim \frac{C^2\varepsilon}{2aB}\exp\left(-\frac{4\sqrt{a}}{3\varepsilon}B^{3/2}\right), \qquad \varepsilon \to 0+. \tag{10.4.27}$$

To evaluate $\int_{-\infty}^{A} y_{\text{III}}^2\,dx$, we use the identity $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$ and again approximate the resulting integrals using integration by parts (see Prob. 10.22):

$$\int_{-\infty}^{A} y_{\text{III}}^2\,dx \sim 2C^2\int_{-\infty}^{0}\frac{dt}{\sqrt{-Q(t)}} - \frac{4C^2}{\sqrt{a}}\sqrt{-A} - \frac{C^2\varepsilon}{aA}\cos\left[\frac{4}{3\varepsilon}\sqrt{a}(-A)^{3/2}\right],$$

$$\varepsilon \to 0+. \tag{10.4.28}$$

Finally, we evaluate $\int_A^B y_{\text{II}}^2\,dx$ using a nice trick. Ai (t) satisfies the differential equation Ai$''(t) = t$ Ai (t), so

$$\frac{d}{dt}\{t[\text{Ai}(t)]^2 - [\text{Ai}'(t)]^2\} = [\text{Ai}(t)]^2.$$

Table 10.2 Comparison between the exact value of $\int_{-\infty}^{\infty} y(x)\,dx$, where $y(x)$ satisfies $\varepsilon^2 y'' = (x + x^3)y$ [$y(0) = 1$, $y(+\infty) = 0$] and the physical-optics approximation to this integral

$$\int_{-\infty}^{\infty} y(x)\,dx \sim \varepsilon^{2/3}\Gamma(\tfrac{2}{3})3^{2/3}, \qquad \varepsilon \to 0+,$$

as given in (10.4.26)

Observe that as ε gets smaller, the accuracy of the WKB approximation increases

ε	Exact value of $\int_{-\infty}^{\infty} y(x)\,dx$	WKB approximation to $\int_{-\infty}^{\infty} y(x)\,dx$
0.2	0.9751	0.9633
0.1	0.6136	0.6068
0.05	0.3844	0.3823
0.02	0.2079	0.2075
0.01	0.1308	0.1307
0.005	0.0823	0.0824
0.002	0.0447	0.0447

Therefore, since $y_{\text{II}}(x)$ is a constant multiple of Ai (t) with t given by (10.4.5), the integral of $[y_{\text{II}}(x)]^2$ can be evaluated in closed form in terms of Airy functions:

$$\int_A^B [y_{\text{II}}(x)]^2\,dx = 4\pi C^2\varepsilon^{1/3}a^{-2/3}\{t[\text{Ai}(t)]^2 - [\text{Ai}'(t)]^2\}\Big|_{t=A\varepsilon^{-2/3}a^{1/3}}^{t=B\varepsilon^{-2/3}a^{1/3}}. \tag{10.4.29}$$

Naturally, we wish to approximate this expression by replacing Ai (t) and Ai$'(t)$ by their asymptotic expressions. However, we are surprised to find that if we use only the leading asymptotic behaviors of Ai (t) and Ai$'(t)$, then we obtain a vanishing result at the upper endpoint! We have emphasized repeatedly that an asymptotic calculation is wrong if the result is zero. Therefore, we must use a higher-order asymptotic approximation to Ai (t) and Ai$'(t)$. The appropriate formulas are

$$\text{Ai}(t) \sim \frac{1}{2\sqrt{\pi}}t^{-1/4}e^{-2t^{3/2}/3}\left(1 - \frac{5}{48t^{3/2}}\right), \qquad t \to \infty,$$

$$\text{Ai}'(t) = -\frac{1}{2\sqrt{\pi}}t^{1/4}e^{-2t^{3/2}/3}\left(1 + \frac{7}{48t^{3/2}}\right), \qquad t \to \infty.$$

We also use the higher-order asymptotic expansions of Ai (t) and Ai$'$ (t) at the lower endpoint:

$$\mathrm{Ai}\,(-t) \sim \frac{1}{\sqrt{\pi}} t^{-1/4} \left[\sin\left(\frac{2}{3} t^{3/2} + \frac{\pi}{4}\right) - \cos\left(\frac{2}{3} t^{3/2} + \frac{\pi}{4}\right) \frac{5}{48 t^{3/2}} \right], \qquad t \to \infty,$$

$$\mathrm{Ai}'\,(-t) \sim -\frac{1}{\sqrt{\pi}} t^{1/4} \left[\cos\left(\frac{2}{3} t^{3/2} + \frac{\pi}{4}\right) - \sin\left(\frac{2}{3} t^{3/2} + \frac{\pi}{4}\right) \frac{7}{48 t^{3/2}} \right], \qquad t \to \infty.$$

If these formulas are used to approximate the expression in (10.4.29), the result is

$$\int_A^B [y_{\mathrm{II}}(x)]^2\, dx \sim \frac{4C^2}{\sqrt{a}} \sqrt{-A} - \frac{C^2 \varepsilon}{aA} \cos\left[\frac{4}{3\varepsilon} \sqrt{a}\, (-A)^{3/2}\right] - \frac{C^2 \varepsilon}{2aB} \exp\left(-\frac{4\sqrt{a}}{3\varepsilon} B^{3/2}\right), \qquad \varepsilon \to 0+. \qquad (10.4.30)$$

Combining the results (10.4.27), (10.4.28), (10.4.30) gives the final answer

$$\int_{-\infty}^{\infty} [y(x)]^2\, dx \sim 2C^2 \int_{-\infty}^{0} \frac{dt}{\sqrt{-Q(t)}}, \qquad \varepsilon \to 0+. \qquad (10.4.31)$$

Once again, the answer is independent of A and B.

Table 10.3 Comparison between the exact value of $\int_{-\infty}^{\infty} [y(x)]^2\, dx$, where $y(x)$ satisfies $\varepsilon^2 y'' = (x + x^3)y$ $[y(0) = 1,\ y(+\infty) = 0]$ and the physical-optics approximation to this integral

$$\int_{-\infty}^{\infty} [y(x)]^2\, dx \sim \tfrac{1}{4}\varepsilon^{1/3}[\Gamma(\tfrac{2}{3})\Gamma(\tfrac{1}{4})]^2 3^{4/3} \pi^{-3/2}$$

$(\varepsilon \to 0+)$ as given in (10.4.32)

As ε decreases, the accuracy of the WKB approximation increases

ε	Exact value of $\int_{-\infty}^{\infty} [y(x)]^2\, dx$	WKB approximation to $\int_{-\infty}^{\infty} [y(x)]^2\, dx$
0.2	2.9085	2.7382
0.1	2.2308	2.1733
0.05	1.7437	1.7249
0.02	1.2751	1.2710
0.01	1.0101	1.0088
0.005	0.8011	0.8006
0.002	0.5900	0.5899

Example 3 *Numerical verification of (10.4.31).* Consider once again the differential equation in (10.4.25). For this equation, $Q(x) = x + x^3$. Thus,

$$2 \int_{-\infty}^{0} \frac{dt}{\sqrt{-Q(t)}} = \left[\Gamma\left(\frac{1}{4}\right)\right]^2 \pi^{-1/2}.$$

Therefore, using C as determined in Example 2,

$$\int_{-\infty}^{\infty} [y(x)]^2\, dx \sim \tfrac{1}{4}\varepsilon^{1/3}[\Gamma(\tfrac{2}{3})\Gamma(\tfrac{1}{4})]^2 3^{4/3} \pi^{-3/2}, \qquad \varepsilon \to 0+. \qquad (10.4.32)$$

We examine the accuracy of this formula in Table 10.3 by comparing it with the integral of the numerical solution to (10.4.25).

(I) 10.5 TWO-TURNING-POINT PROBLEMS: EIGENVALUE CONDITION

In this section we show how to use the physical-optics approximation to obtain an approximate solution to the homogeneous boundary-value problem

$$\varepsilon^2 y'' = Q(x) y, \qquad y(\pm\infty) = 0, \qquad (10.5.1)$$

where $Q(x)$ has two simple turning points at $x = A$ and $x = B$ with $A < B$. We also assume that $Q > 0$ if $x > B$ or $x < A$, that $Q < 0$ if $A < x < B$, and that $Q(x) \gg x^{-2}$ as $|x| \to \infty$. For most functions $Q(x)$ satisfying these conditions the only solution to (10.5.1) is $y(x) = 0$. This is because the solution to (10.5.1) which decays exponentially as $x \to +\infty$ is, in general, a mixture of growing and decaying solutions as $x \to -\infty$. We will derive an approximate constraint which must be satisfied by $Q(x)$ for the problem (10.5.1) to have nontrivial solutions. To leading order in ε this constraint is

$$\frac{1}{\varepsilon} \int_A^B \sqrt{-Q(t)}\, dt = \left(n + \frac{1}{2}\right)\pi + \mathrm{O}(\varepsilon), \qquad \varepsilon \to 0+, \qquad (10.5.2)$$

where $n = 0, 1, 2, \ldots$ is a nonnegative integer.

The constraint in (10.5.2) is useful if the function Q depends on a parameter E, which we call an eigenvalue. Then (10.5.2) determines the approximate value of E correct to terms of order ε.

The derivation of (10.5.2) is done by asymptotically matching two one-turning-point solutions: the first one-turning-point solution is valid from $+\infty$ through the turning point at B and down to near the turning point at A; the second is valid from $-\infty$ through the turning point at A and up to near the turning point at B. Since these one-turning point solutions overlap in the region between the turning points at A and B, we must require that they match asymptotically. This matching condition translates into the constraint on $Q(x)$ in (10.5.2).

The one-turning-point solution that decays like

$$C_1 [Q(x)]^{-1/4} \exp\left[-\frac{1}{\varepsilon} \int_B^x \sqrt{Q(t)}\, dt\right]$$

as $x \to +\infty$ behaves like

$$2C_1[-Q(x)]^{-1/4} \sin\left[\frac{1}{\varepsilon}\int_x^B \sqrt{-Q(t)}\,dt + \frac{1}{4}\pi\right] \tag{10.5.3}$$

in the region between A and B [so long as the distance between x and A or x and B is much greater than $\varepsilon^{2/3}$ (see Sec. 10.4)]. This is merely a restatement of the connection formula (10.4.16) when the turning point lies at $x = B$ instead of at $x = 0$.

The one-turning-point solution that decays like

$$C_2[Q(x)]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_x^A \sqrt{Q(t)}\,dt\right]$$

as $x \to -\infty$ behaves like

$$2C_2[-Q(x)]^{-1/4} \sin\left[\frac{1}{\varepsilon}\int_A^x \sqrt{-Q(t)}\,dt + \frac{1}{4}\pi\right] \tag{10.5.4}$$

in the region between A and B. This result is derived in Prob. 10.30.

In order that the two physical-optics solutions in (10.5.3) and (10.5.4) match in the region between A and B, we must require that they have the same functional form. Both solutions already have identical factors of $[-Q(x)]^{-1/4}$. However, the arguments of the sine functions are not identical. To achieve the match, we rewrite (10.5.3) as

$$-2C_1[-Q(x)]^{-1/4} \sin\left[\frac{1}{\varepsilon}\int_A^x \sqrt{-Q(t)}\,dt + \frac{\pi}{4} - \left\{\frac{1}{\varepsilon}\int_A^B \sqrt{-Q(t)}\,dt + \frac{\pi}{2}\right\}\right].$$

In order that this expression be functionally identical to that in (10.5.4), it is necessary that the expression in curly brackets be an integral multiple of π. Moreover, since the expression in curly brackets is positive, it follows that we must require $(1/\varepsilon)\int_A^B \sqrt{-Q(t)}\,dt = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$, which is just (10.5.2). To complete the match of (10.5.3) and (10.5.4), we must choose $C_1 = (-1)^n C_2$ where n is defined in (10.5.2). This completes the derivation of (10.5.2).

The above analysis has neglected terms of order ε, namely, the higher terms in the WKB series $(\varepsilon S_2, \varepsilon^2 S_3, \ldots)$. Consequently, the constraint (10.5.2) is only accurate to terms of order ε. In Sec. 10.7 we will derive a more accurate constraint that is valid to all orders in powers of ε by taking into account the presence of higher-order terms in the WKB series.

Linear Eigenvalue Problems

We now examine a special class of eigenvalue problems in which the eigenvalue E appears linearly: $Q(x) = V(x) - E$. In the study of quantum mechanics, if $V(x)$ rises monotonically as $x \to \pm\infty$, the differential equation

$$\varepsilon^2 y'' = [V(x) - E]y(x), \qquad y(\pm\infty) = 0, \tag{10.5.5}$$

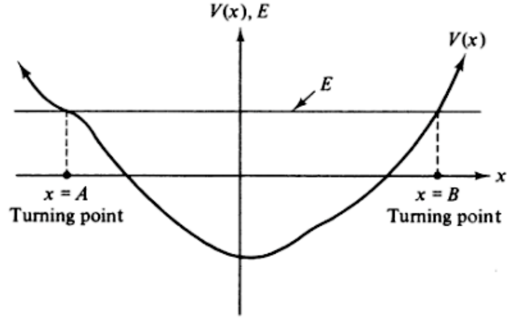

Figure 10.14 Schematic plot of the function $V(x)$ in (10.5.5). Turning points occur when $V(x) = E$. In classical mechanics if we interpret this configuration to represent a particle of energy E in a potential $V(x)$, then the particle is confined to the region between the turning points at A and B where the total energy E is greater than or equal to the potential energy $V(x)$. In classical mechanics the energy of a particle in a potential well is arbitrary so long as $E \geq V_{\min}$. In quantum mechanics E can only have special discrete values which are the eigenvalues of (10.5.5). The energy of such a particle is said to be quantized.

describes a particle of energy E confined to a potential well $V(x)$ (see Fig. 10.14). By (10.5.2) the eigenvalue E of (10.5.5) must satisfy

$$\frac{1}{\varepsilon}\int_A^B \sqrt{E - V(x)}\,dx = \left(n + \frac{1}{2}\right)\pi + O(\varepsilon), \qquad \varepsilon \to 0, \tag{10.5.6}$$

where the turning points A and B are the two solutions to the equation $V(x) - E = 0$.

We will see that if $V(x) \to \infty$ as $x \to \pm\infty$, then there are an infinite number of solutions E_n to (10.5.6) and that $E_n \to \infty$ as $n \to \infty$. In this case (10.5.6) becomes asymptotically exact as $n \to \infty$ for any *fixed* value of ε and we set $\varepsilon = 1$ as we did in our discussion of the eigenvalue problem in (10.1.27). The accuracy of the WKB approximation increases as $n \to \infty$ because, except at the turning points, $|S_0| = |\int^x \sqrt{V(t) - E}\,dt|$ increases as E increases. Thus, the conditions for the validity of WKB,

$$\frac{1}{\varepsilon} S_0 \gg S_1 \gg \varepsilon S_2 \qquad \text{and} \qquad 1 \gg \varepsilon S_2, \tag{10.5.7}$$

are satisfied either as $\varepsilon \to 0$ with E fixed or as $E \to \infty$ with ε fixed (see Prob. 10.31).

Example 1 *Eigenvalues for $y'' = (|x| - E)y$ [$y(\pm\infty) = 0$].* For this equation the solutions of $V(x) - E = 0$ are $A = -E$ and $B = E$. Thus, the WKB eigenvalue condition becomes $\int_{-E}^{E} \sqrt{E - |x|}\,dx \sim (n + \frac{1}{2})\pi$ $(n \to \infty)$. But $\int_{-E}^{E} \sqrt{E - |x|}\,dx = \frac{4}{3}E^{3/2}$. Thus, for large n,

$$E_n \sim \left[\frac{3\pi}{4}\left(n + \frac{1}{2}\right)\right]^{2/3}, \qquad n \to \infty. \tag{10.5.8}$$

This result may be reproduced by solving the differential equation exactly. When $x > 0$, $|x| = x$ and the differential equation becomes $y'' = (x - E)y$. The exact solution to this equation is a linear combination $\mathrm{Ai}\,(x - E)$ and $\mathrm{Bi}\,(x - E)$. However, only $\mathrm{Ai}\,(x - E)$ vanishes as $x \to +\infty$. Thus,

$$y(x) = c\,\mathrm{Ai}\,(x - E), \qquad x \ge 0, \tag{10.5.9}$$

where c is a constant.

When $x < 0$, $|x| = -x$ and the only solution to the differential equation $y'' = (-x - E)y$ which vanishes as $x \to -\infty$ is

$$y(x) = d\,\mathrm{Ai}\,(-x - E), \qquad x \le 0, \tag{10.5.10}$$

where d is a constant.

The two solutions (10.5.9) and (10.5.10) must be patched at $x = 0$. Demanding that $y(x)$ and $y'(x)$ be continuous at $x = 0$ requires that $c\,\mathrm{Ai}\,(-E) = d\,\mathrm{Ai}\,(-E)$ and that $c\,\mathrm{Ai}'\,(-E) = -d\,\mathrm{Ai}'\,(-E)$. Thus, if $c = -d \neq 0$ then

$$\mathrm{Ai}\,(-E) = 0 \tag{10.5.11}$$

and if $c = d \neq 0$ then

$$\mathrm{Ai}'\,(-E) = 0. \tag{10.5.12}$$

The solutions of the two transcendental equations (10.5.11) and (10.5.12) comprise the complete set of eigenvalues for the differential equation. When E is large, these solutions had better agree with the WKB prediction in (10.5.8)! To check this, we replace $\mathrm{Ai}\,(-E)$ and $\mathrm{Ai}'\,(-E)$ by their leading asymptotic expansions for large negative argument:

$$\mathrm{Ai}\,(-E) \sim \frac{1}{\sqrt{\pi}} E^{-1/4} \sin\left(\frac{2}{3}E^{3/2} + \frac{\pi}{4}\right), \qquad E \to \infty,$$

has zeros whenever

$$\frac{2}{3}E^{3/2} + \frac{\pi}{4} = k\pi, \qquad k = 1, 2, 3, \ldots, \tag{10.5.13}$$

and

$$\mathrm{Ai}'\,(-E) \sim -\frac{1}{\sqrt{\pi}} E^{1/4} \cos\left(\frac{2}{3}E^{3/2} + \frac{\pi}{4}\right)$$

has zeros whenever

$$\frac{2}{3}E^{3/2} + \frac{\pi}{4} = \left(k + \frac{1}{2}\right)\pi, \qquad k = 0, 1, 2, \ldots. \tag{10.5.14}$$

Combining (10.5.13) and (10.5.14) into a single formula gives

$$\frac{2}{3}E^{3/2} + \frac{\pi}{4} = \left(\frac{n}{2} + \frac{1}{2}\right)\pi, \qquad n = 0, 1, 2, \ldots,$$

which is equivalent to the WKB result in (10.5.8) and which is also valid when n is a large positive integer.

Example 2 *Eigenvalues of the parabolic cylinder equation.* We have seen in Example 9 of Sec. 3.8 that the eigenvalues of $(-d^2/dx^2 + x^2/4 - E)y(x) = 0$ $[y(\pm\infty) = 0]$ are exactly $E_n = n + \frac{1}{2}$ $(n = 0, 1, 2, \ldots)$. How well does WKB reproduce this result?

The turning points lie at $A = -2\sqrt{E}$ and $B = 2\sqrt{E}$. Thus, the WKB eigenvalue condition reads

$$\int_{-2\sqrt{E}}^{2\sqrt{E}} \sqrt{E - \tfrac{1}{4}x^2}\,dx \sim (n + \tfrac{1}{2})\pi, \qquad n \to \infty.$$

Upon substituting $x = 2\sqrt{E}\,t$ the above integral becomes $2E \int_{-1}^{1} dt \sqrt{1 - t^2} = E\pi$.

Thus, the WKB prediction is $E_n \sim n + \frac{1}{2}$ $(n \to \infty)$, which is not only valid as $n \to \infty$ but is *exact* for all n.

It is accidental that the leading-order (physical-optics) WKB result is exact. Indeed, the physical-optics approximation to the nth eigenfunction $y_n(x)$ is only approximate. The physical-optics approximation to the eigenvalues is exact because the corrections to E_n that result from a higher-order WKB treatment of the eigenvalue problem all happen to vanish (see Example 1 of Sec. 10.7).

Example 3 *Eigenvalues for $y'' = (x^4 - E)y$.* The turning points are at $A = -E^{1/4}$ and $B = E^{1/4}$. Thus, $\int_{-E^{1/4}}^{E^{1/4}} \sqrt{E - x^4}\,dx \sim (n + \frac{1}{2})\pi$ $(n \to \infty)$ becomes

$$E_n \sim \left[\frac{3\Gamma(\frac{3}{4})(n + \frac{1}{2})\sqrt{\pi}}{\Gamma(\frac{1}{4})}\right]^{4/3}, \qquad n \to \infty \tag{10.5.15}$$

(see Prob. 10.32).

In Table 10.4 we compare the exact eigenvalues with the WKB prediction for the eigenvalues in (10.5.15). Observe that as n increases the accuracy of the WKB prediction increases dramatically.

In Figs. 10.15 to 10.17 we compare the physical-optics approximation to $y(x)$ with the solution to the differential equation obtained by computer. Again, the accuracy increases very rapidly with n.

Table 10.4 Comparison of exact eigenvalues E_n for $y'' = (x^4 - E)y$ $[y(\pm\infty) = 0]$ and the WKB prediction for E_n in (10.5.15): $E_n \sim [3\Gamma(\frac{3}{4})(n + \frac{1}{2})\sqrt{\pi}/\Gamma(\frac{1}{4})]^{4/3}$ $(n \to \infty)$

Observe that the relative error [% relative error = 100 (WKB E_n − exact E_n)/(exact E_n)] decreases rapidly as n increases. [Note that $\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4}) \doteq 0.337\,99$.]

n	Exact E_n	WKB E_n	Relative error, %
0	1.060	0.867	−18.00
2	7.456	7.414	−0.56
4	16.262	16.234	−0.17
6	26.528	26.506	−0.08
8	37.923	37.904	−0.05
10	50.256	50.240	−0.03

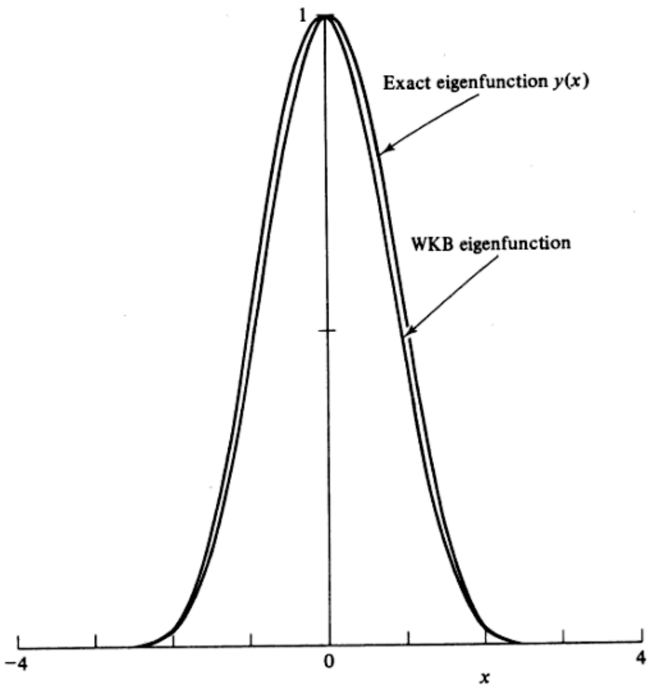

Figure 10.15 Comparison between the exact eigenfunction of $y'' = (x^4 - E)y$ $[y(\pm\infty) = 0]$, with the lowest eigenvalue $E_0(n = 0)$ and the corresponding uniform physical-optics (WKB) approximation (10.4.15) with the WKB approximation (10.5.15) to E_n. Both eigenfunctions are normalized by $y(0) = 1$. The WKB approximation is given by (10.4.15) with $Q(x) = x^4 - E_n$ for $x > 0$ and by $y(x) = y(-x)$ for $x < 0$.

(D) 10.6 TUNNELING

Tunneling is the remarkable quantum-mechanical phenomenon by which a particle passes through a potential barrier that classical mechanics predicts is impenetrable. In this section we use WKB theory to make a quantitative study of tunneling. We begin by introducing the notion of a wave.

Right-Moving and Left-Moving Waves

The phenomenon of tunneling implicitly involves motion. Thus, to describe tunneling we must begin with the time-dependent Schrödinger wave equation

$$\frac{1}{i}\frac{\partial}{\partial t}\psi(x, t) = \left[-\varepsilon^2 \frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x, t). \tag{10.6.1}$$

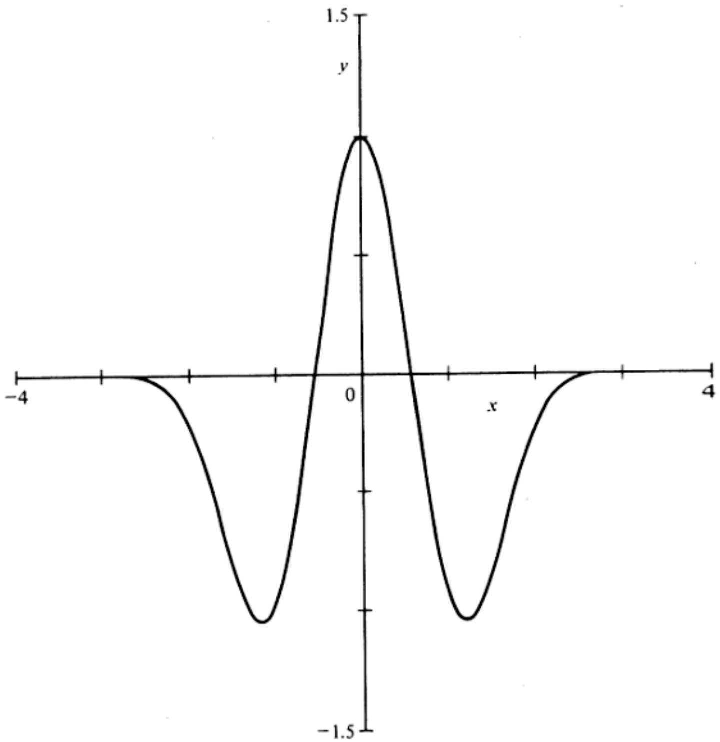

Figure 10.16 Same as Figure 10.15 except for the third lowest eigenvalue $E_2(n = 2)$ of $y'' = (x^4 - E)y$. The exact eigenfunction and the physical-optics approximation to it are not distinguishable on the scale of the plot. See Fig. 10.17 for a plot of the error $y_{\text{WKB}} - y_{\text{exact}}$.

$\psi(x, t)$ is called a wave function. Let us assume that the time dependence of $\psi(x, t)$ is purely oscillatory:

$$\psi(x, t) = y(x)e^{iEt}. \tag{10.6.2}$$

Substituting (10.6.2) into (10.6.1) gives the ordinary differential equation

$$\varepsilon^2 y'' = [V(x) - E]y(x), \tag{10.6.3}$$

which we have already examined using the WKB approximation.

In regions where $E > V(x)$ (classically allowed regions), WKB solutions to (10.6.3) are oscillatory:

$$y_{\text{WKB}}(x) = C_\pm[E - V(x)]^{-1/4} \exp\left[\pm\frac{i}{\varepsilon}\int^x \sqrt{E - V(t)}\, dt\right]. \tag{10.6.4}$$

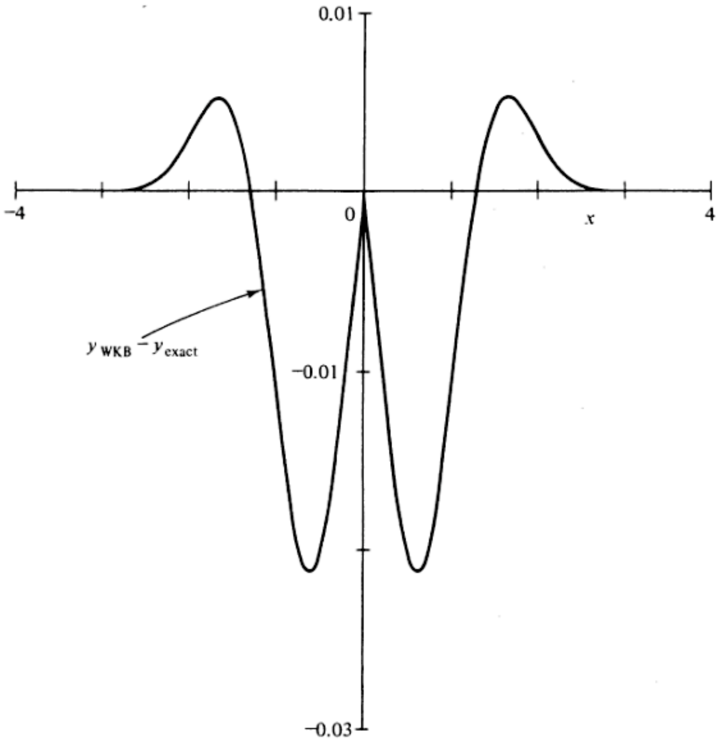

Figure 10.17 A plot of the error $y_{WKB} - y_{exact}$ in the uniform physical-optics (WKB) approximation to the third lowest eigenfunction ($n = 2$) of $y'' = (x^4 - E)y$. Both y_{exact} and y_{WKB} are normalized by $y_{exact}(0) = y_{WKB}(0) = 1$. See Fig. 10.16.

Even though the WKB approximation in (10.6.4) is time independent, we will refer to solutions having positive (negative) phase as left-moving (right-moving) waves. This is because the wave function

$$\psi_{WKB}(x, t) = C_{\pm}[E - V(x)]^{-1/4} \exp i[Et \pm (1/\varepsilon) \textstyle\int^x \sqrt{E - V(t)}\, dt]$$

represents a wave which moves to the left (right) as t increases.

An Exactly Soluble Model of Tunneling

To illustrate the phenomenon of tunneling, we make a very simple choice for the potential V: $V(x) = \delta(x)$. Classically, this delta function potential confers an impulse to a particle which arrives at $x = 0$. If $E < (4\varepsilon^2)^{-1}$, a classical particle

traveling toward $x = 0$ *always* bounces back (reverses its direction) when it reaches $x = 0$. This is called reflection. If $E > (4\varepsilon^2)^{-1}$, a classical particle *always* continues on when it reaches $x = 0$. This is called transmission (see Prob. 10.39).

In quantum mechanics there are well-defined probabilities T and R that a particle will undergo transmission or reflection. We now compute these probabilities exactly. To solve

$$\varepsilon^2 y'' = [\delta(x) - E]y$$

we consider two regions. When $x < 0$, we have $\varepsilon^2 y'' + Ey = 0$, whose general solution is

$$y(x) = a \exp(-ix\sqrt{E}/\varepsilon) + b \exp(+ix\sqrt{E}/\varepsilon), \qquad x < 0.$$

When $x > 0$, we have the same differential equation whose general solution is now

$$y(x) = c \exp(-ix\sqrt{E}/\varepsilon) + d \exp(+ix\sqrt{E}/\varepsilon), \qquad x > 0.$$

To observe tunneling we must choose the boundary conditions properly; we aim a monoenergetic incident beam of particles toward $x = 0$ from the left. We represent this incident beam as a right-moving wave of unit amplitude: $\exp(-ix\sqrt{E}/\varepsilon)$. There will then be a reflected (left-moving) wave for $x < 0$ of amplitude b, $b \exp(+ix\sqrt{E}/\varepsilon)$, and a transmitted (right-moving) wave for $x > 0$ of amplitude c, $c \exp(-ix\sqrt{E}/\varepsilon)$. There is *no* left-moving wave for $x > 0$.

We must patch the two solutions

$$y(x) = \exp(-ix\sqrt{E}/\varepsilon) + b \exp(+ix\sqrt{E}/\varepsilon), \qquad x < 0, \tag{10.6.5}$$

$$y(x) = c \exp(-ix\sqrt{E}/\varepsilon), \qquad x > 0, \tag{10.6.6}$$

at $x = 0$. We require that:

1. $y(x)$ be continuous at $x = 0$ and
2. $\lim_{\eta \to 0+} \varepsilon^2[y'(\eta) - y'(-\eta)] = y(0)$.

Why? (See Sec. 10.3.)

From these two conditions we obtain (see Prob. 10.38)

$$b = \frac{2\varepsilon\sqrt{E}\, i - 1}{4\varepsilon^2 E + 1}, \qquad c = \frac{2\varepsilon\sqrt{E}\,(2\varepsilon\sqrt{E} + i)}{4\varepsilon^2 E + 1}. \tag{10.6.7}$$

We define $R = |b|^2$ as the *reflection coefficient* and $T = |c|^2$ as the *transmission coefficient*. R is the probability that an incident particle of energy E will be reflected and T is the probability that the incident particle will be transmitted. We compute that

$$R = 1/(4\varepsilon^2 E + 1), \qquad T = 4\varepsilon^2 E/(4\varepsilon^2 E + 1).$$

Observe that $T + R = 1$; thus, the total probability that a particle will be reflected or transmitted is 1. Note also that $R = 0$ and $T = 1$ when $E = \infty$ and that $T = 0$ and $R = 1$ when $E = 0$. These are the only values of E for which the classical and quantum-mechanical predictions agree. Classically, $R=1$ and $T=0$ for $E<(4\varepsilon^2)^{-1}$ and $R=0$ and $T=1$ for $E>(4\varepsilon^2)^{-1}$. Thus, quantum mechanics predicts that there is a nonzero probability that a particle will penetrate (tunnel through) a potential barrier, even if its energy is smaller than the minimum energy required by classical mechanics for transmission. [What happens classically and what happens quantum mechanically when $E=(4\varepsilon^2)^{-1}$?]

WKB Description of Tunneling through Potential Barriers

Now let us take $V(x)$ in (10.6.3) to be any continuous function which vanishes as $x\to\pm\infty$ and which rises monotonically to its maximum $V_{\max}$ ($V_{\max}>E$) at $x=x_0$ as x approaches x_0 from either the left or the right side of x_0. For this potential barrier $V(x)$ there are two turning points $x=A$ and $x=B$, $A<B$, at which $V(x)=E$. Thus, there are two classically allowed regions $x<A$ and $x>B$ in which oscillatory solutions occur and a classically forbidden region $A<x<B$ in which there are exponentially growing and decaying solutions.

We will also make the technical assumption that as $x\to\pm\infty$, $V(x)\to 0$ faster than $1/x$. We then have

$$\int_B^x dt\sqrt{E-V(t)}=\sqrt{E}\int_B^x dt+\int_B^x dt[\sqrt{E-V(t)}-\sqrt{E}]$$
$$\sim x\sqrt{E}+I, \qquad x\to+\infty,$$

where $I=\int_B^\infty dt[\sqrt{E-V(t)}-\sqrt{E}]-B\sqrt{E}$ exists and the corrections to this asymptotic relation vanish as $x\to+\infty$ (see Prob. 10.40).

Similarly, we have

$$\int_x^A dt\sqrt{E-V(t)}\sim -x\sqrt{E}+J, \qquad x\to-\infty,$$

where $J=A\sqrt{E}+\int_{-\infty}^A dt[\sqrt{E-V(t)}-\sqrt{E}]$ also exists and the corrections to this asymptotic relation also vanish as $x\to-\infty$. Consequently, as $x\to\pm\infty$, the WKB approximations to $y(x)$ in the classically allowed regions $x>B$ and $x<A$ approach plane waves as $x\to+\infty$ and $x\to-\infty$:

$$Y_{\text{WKB}}(x)=C_\pm[E-V(x)]^{-1/4}\exp\left[\pm\frac{i}{\varepsilon}\int_B^x dt\sqrt{E-V(t)}\right]$$
$$\sim C_\pm E^{-1/4}e^{\pm iI/\varepsilon}\exp(\pm ix\sqrt{E}/\varepsilon), \qquad x\to+\infty, \tag{10.6.8}$$

$$Y_{\text{WKB}}(x)=D_\pm[E-V(x)]^{-1/4}\exp\left[\pm\frac{i}{\varepsilon}\int_x^A dt\sqrt{E-V(t)}\right]$$
$$\sim D_\pm E^{-1/4}e^{\pm iJ/\varepsilon}\exp(\mp ix\sqrt{E}/\varepsilon), \qquad x\to-\infty. \tag{10.6.9}$$

As in the exactly soluble model of tunneling that we discussed above, we must choose an appropriate set of boundary conditions to describe tunneling. We postulate a unit incident right-moving plane wave at $x=-\infty$. This gives rise to a

right-moving transmitted plane wave at $x=+\infty$ and a left-moving reflected wave at $x=-\infty$. We formulate these boundary conditions as asymptotic relations:

$$y(x)\sim\exp(-ix\sqrt{E}/\varepsilon)+b\exp(+ix\sqrt{E}/\varepsilon), \qquad x\to-\infty, \tag{10.6.10}$$

$$y(x)\sim c\exp(-ix\sqrt{E}/\varepsilon), \qquad x\to+\infty. \tag{10.6.11}$$

The relations in (10.6.10) and (10.6.11) are the asymptotic generalizations of the exact equations in (10.6.5) and (10.6.6). The objective is to compute the constants b and c using WKB theory.

The WKB calculation requires the solution of a new kind of one-turning-point problem which reads as follows. Let $Q(0)=0$, $Q(x)>0$ if $x<0$ and $Q(x)<0$ if $x>0$, $Q(x)\sim ax$, $a<0$ $(x\to 0)$. If the WKB approximation to the solution of

$$\varepsilon^2y''(x)=Q(x)y \tag{10.6.12}$$

has negative phase for positive x,

$$y_{\text{WKB}}(x)=[-Q(x)]^{-1/4}\exp\left[-\frac{i}{\varepsilon}\int_0^x\sqrt{-Q(t)}\,dt\right], \tag{10.6.13}$$

how does $y(x)$ behave for negative x?

To solve this problem, we first allow x in (10.6.13) to approach the turning point at $x=0$ and obtain

$$y_{\text{WKB}}(x)\sim(-ax)^{-1/4}\exp\left(-\frac{2i\sqrt{-a}}{3\varepsilon}x^{3/2}\right), \qquad x\to 0+. \tag{10.6.14}$$

We know that near $x=0$ the differential equation (10.6.12) may be approximated by

$$y''=-ty, \qquad t=\varepsilon^{-2/3}(-a)^{1/3}x,$$

whose solution is

$$y(t)=\alpha\operatorname{Ai}(-t)+\beta\operatorname{Bi}(-t). \tag{10.6.15}$$

When t is large and positive, we can replace Ai $(-t)$ and Bi $(-t)$ in (10.6.15) by their asymptotic expansions

$$\operatorname{Ai}(-t)\sim\frac{1}{\sqrt{\pi}}t^{-1/4}\sin\left(\frac{2}{3}t^{3/2}+\frac{\pi}{4}\right), \qquad t\to+\infty,$$

$$\operatorname{Bi}(-t)\sim\frac{1}{\sqrt{\pi}}t^{-1/4}\cos\left(\frac{2}{3}t^{3/2}+\frac{\pi}{4}\right), \qquad t\to+\infty.$$

By comparing the resulting expression with that in (10.6.14), we determine α and β to leading order in the WKB approximation:

$$\beta=\sqrt{\pi}(-\varepsilon a)^{-1/6}e^{i\pi/4},$$
$$\alpha=\sqrt{\pi}(-\varepsilon a)^{-1/6}e^{-i\pi/4}. \tag{10.6.16}$$

The problem is now half solved.

Next, we allow t in (10.6.15) to be large and *negative*. The expansion of Ai $(-t)$ is negligible compared with the expansion of Bi $(-t)$ because it is exponentially small (subdominant). Using (10.6.16) and the expansion

$$\text{Bi}\,(-t) \sim \frac{1}{\sqrt{\pi}}(-t)^{-1/4}e^{2(-t)^{3/2}/3}, \qquad t \to -\infty,$$

we determine that for negative x the WKB approximation to $y(x)$ is given by

$$y_{\text{WKB}}(x) = [Q(x)]^{-1/4} \exp\left[\frac{i\pi}{4} + \frac{1}{\varepsilon}\int_x^0 dt\sqrt{Q(t)}\,dt\right]. \tag{10.6.17}$$

This completes the solution to the one-turning-point problem. Using the notation in (10.4.16) we summarize our result as a connection formula:

$$\boxed{\begin{aligned} &[Q(x)]^{-1/4} \exp\left[\frac{i\pi}{4} + \frac{1}{\varepsilon}\int_x^0 dt\sqrt{Q(t)}\,dt\right], && x < 0 \\ &\leftarrow [-Q(x)]^{-1/4} \exp\left[-\frac{i}{\varepsilon}\int_0^x \sqrt{-Q(t)}\,dt\right], && x > 0. \end{aligned}} \tag{10.6.18}$$

Now we return to the solution of the two-turning-point tunneling problem for the potential $V(x)$. When $x > B$, the WKB approximation to the solution $y(x)$ of (10.6.3) which satisfies the boundary condition in (10.6.3) is [see (10.6.8)]

$$y_{\text{WKB}}(x) = ce^{iI/\varepsilon}\left[\frac{E - V(x)}{E}\right]^{-1/4} \exp\left[-\frac{i}{\varepsilon}\int_B^x \sqrt{E - V(t)}\,dt\right]. \tag{10.6.19}$$

By the connection formula in (10.6.18), this expression asymptotically matches to

$$y_{\text{WKB}}(x) = ce^{iI/\varepsilon}\left[\frac{V(x) - E}{E}\right]^{-1/4} e^{i\pi/4} \exp\left[\frac{1}{\varepsilon}\int_x^B dt\sqrt{V(t) - E}\right], \tag{10.6.20}$$

which is valid when $A < x < B$.

The expression in (10.6.20) may be rewritten as

$$\begin{aligned} y_{\text{WKB}}(x) = ce^{iI/\varepsilon} &\exp\left[\frac{1}{\varepsilon}\int_A^B dt\sqrt{V(t) - E} + \frac{i\pi}{4}\right] \\ &\times \left[\frac{V(x) - E}{E}\right]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_A^x dt\sqrt{V(t) - E}\right], \end{aligned}$$

which decays exponentially as x increases toward B. We connect to the oscillatory solution which is valid for $x < A$ using the connection formula in (10.4.16) and obtain the WKB approximation

$$\begin{aligned} y_{\text{WKB}}(x) = ce^{iI/\varepsilon} &\exp\left[\frac{1}{\varepsilon}\int_A^B dt\sqrt{V(t) - E}\right]\left[\frac{E - V(x)}{E}\right]^{-1/4} \\ &\times \left\{\exp\left[\frac{i}{\varepsilon}\int_x^A dt\sqrt{E - V(t)}\right] + i\exp\left[-\frac{i}{\varepsilon}\int_x^A dt\sqrt{E - V(t)}\right]\right\}, \end{aligned} \tag{10.6.21}$$

which is valid when $x < A$.

Finally, we let $x \to -\infty$. Recall that in this limit $V(x) \to 0$ faster than $1/x$, so (10.6.21) becomes [see (10.6.9)]

$$\begin{aligned} y_{\text{WKB}}(x) \sim ce^{iI/\varepsilon} &\exp\left[\frac{1}{\varepsilon}\int_A^B dt\sqrt{V(t) - E}\right] \\ &\times [e^{iJ/\varepsilon}\exp(-i\sqrt{E}\,x/\varepsilon) + ie^{-iJ/\varepsilon}\exp(i\sqrt{E}\,x/\varepsilon)], \\ &\qquad x \to -\infty. \end{aligned} \tag{10.6.22}$$

Comparing this formula with that in (10.6.8) gives expressions for the constants b and c:

$$b = ie^{-2iJ/\varepsilon},$$

$$c = \exp\left[-\frac{1}{\varepsilon}\int_A^B dt\sqrt{V(t) - E}\right]e^{-i(I+J)/\varepsilon}.$$

Thus, the reflection coefficient is

$$R = |b|^2 \sim 1, \qquad \varepsilon \to 0+, \tag{10.6.23}$$

and the transmission coefficient is

$$T = |c|^2 \sim \exp\left[-\frac{2}{\varepsilon}\int_A^B dt\sqrt{V(t) - E}\right], \qquad \varepsilon \to 0+. \tag{10.6.24}$$

We observe that only an exponentially small portion of the incident wave is transmitted (tunnels through the potential barrier). Notice that the leading-order WKB prediction for R and T in (10.6.23) and (10.6.24) appears to violate the constraint that $R + T = 1$, which is always exactly satisfied. Indeed, R is not **1**, but differs from 1 by an exponentially small (subdominant) quantity. However, the principles of asymptotics require that we *always* disregard subdominant corrections. (Of course, we do not replace T by 0 because it is not small compared with 0!)

Scattering off the Peak of a Potential Barrier

The reflection and transmission coefficients in (10.6.23) and (10.6.24) are good asymptotic approximations only if $E < V_{\max}$. What happens when $E = V_{\max}$? To answer this question, we consider the simple model problem for which $V(x) = e^{-x^2/4}$, $E = 1$. Now $V_{\max} = E = 1$.

A classical particle of energy $E = 1$ moving under the influence of this potential slows down as it approaches the origin. Classically, we cannot define a reflection or transmission coefficient because the particle never actually reaches the origin! The quantum-mechanical result is much more interesting, as we will now see.

The differential equation

$$\left(-\varepsilon^2\frac{d^2}{dx^2} + e^{-x^2/4} - 1\right)y(x) = 0 \tag{10.6.25}$$

is quite different from that in (10.6.3) where $V_{max} > E$ because here there is just *one* real turning point which lies at $x = 0$.

To solve this equation using asymptotic matching, we divide the x axis into three regions: region I, in which $x > 0$; region II, the immediate neighborhood of $x = 0$; and region III, in which $x < 0$. For a precise asymptotic determination of the boundaries of these regions in terms of the small parameter ε, see Prob. 10.41.

We begin our analysis in region II. Since $|x|$ is small there, we may replace $e^{-x^2/4} - 1$ by $-x^2/4$, the first term in its Taylor expansion. We thereby replace (10.6.25) by the simpler differential equation

$$\left(-\varepsilon^2 \frac{d^2}{dx^2} - x^2/4\right) y_{\text{II}}(x) = 0. \tag{10.6.26}$$

Note that $x = 0$ is a *second-order* turning point.

The differential equation (10.6.26) is closely related to the parabolic cylinder equation $(-d^2/dt^2 + \frac{1}{4}t^2 - \nu - \frac{1}{2})z(t) = 0$, whose general solution is $z(t) = \alpha D_\nu(t) + \beta D_\nu(-t)$ when $\nu \neq 0, 1, 2, \ldots$ (see Example 8 of Sec. 3.8). The general solution to (10.6.26) is

$$y_{\text{II}}(x) = \alpha D_{-1/2}(e^{-i\pi/4}x/\sqrt{\varepsilon}) + \beta D_{-1/2}(-e^{-i\pi/4}x/\sqrt{\varepsilon}). \tag{10.6.27}$$

Now we examine the behavior of $y(x)$ in (10.6.27) as $x/\sqrt{\varepsilon} \to \pm\infty$. We use the formulas for the asymptotic behavior of $D_\nu(t)$ as $|t| \to \infty$ in the complex plane [see (3.8.22) and (3.8.24)]:

$$D_\nu(t) \sim t^\nu e^{-t^2/4}, \qquad t \to \infty;\ |\arg t| < \frac{3\pi}{4},$$

$$D_\nu(t) \sim t^\nu e^{-t^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu} t^{-\nu-1} e^{t^2/4}, \qquad t \to \infty;\ \frac{\pi}{4} < \arg t < \frac{5\pi}{4}.$$

Note that $\arg(e^{-i\pi/4}\varepsilon^{-1/2}x) = -\pi/4$ and $\arg(-e^{-i\pi/4}\varepsilon^{-1/2}x) = 3\pi/4$ when $x > 0$. Thus,

$$y_{\text{II}}(x) \sim \varepsilon^{1/4} x^{-1/2}[(\alpha e^{i\pi/8} + \beta e^{-3i\pi/8})e^{ix^2/4\varepsilon} + \beta\sqrt{2}\, e^{i\pi/8} e^{-ix^2/4\varepsilon}],$$
$$x/\sqrt{\varepsilon} \to +\infty. \tag{10.6.28}$$

A similar expansion exists when x is large and negative:

$$y_{\text{II}}(x) \sim \varepsilon^{1/4}(-x)^{-1/2}[(\alpha e^{-3i\pi/8} + \beta e^{i\pi/8})e^{ix^2/4\varepsilon} + \alpha\sqrt{2}\, e^{i\pi/8} e^{-ix^2/4\varepsilon}],$$
$$x/\sqrt{\varepsilon} \to -\infty. \tag{10.6.29}$$

Now we return to the differential equation (10.6.25) and examine it in region I where $x > 0$ and in region III where $x < 0$. In region I the WKB solution to (10.6.25) is

$$y_{\text{I}} = A(1 - e^{-x^2/4})^{-1/4} \exp\left(-\frac{i}{\varepsilon}\int_0^x dt\sqrt{1 - e^{-t^2/4}}\right), \tag{10.6.30}$$

where we have included only a negative phase solution to represent only a right-going transmitted wave for positive x. Note that

$$y_{\text{I}} \sim Ae^{-iI/\varepsilon}e^{-ix/\varepsilon}, \qquad x \to +\infty,$$

where $I = \int_0^\infty dt(\sqrt{1 - e^{-t^2/4}} - 1)$. We therefore impose the outgoing wave boundary condition at $+\infty$ in (10.6.11) by requiring that

$$c = Ae^{-iI/\varepsilon}. \tag{10.6.31}$$

Next we examine (10.6.30) as $x \to 0+$. When $x \to 0$, we have $1 - e^{-x^2/4} \sim x^2/4$. Thus, performing the integral in (10.6.30) we have

$$y_{\text{I}} \sim A(2/x)^{1/2}e^{-ix^2/4\varepsilon}, \qquad x \to 0+.$$

This expression must match asymptotically with that in (10.6.28). Thus, we require that

$$\varepsilon^{1/4}\beta e^{i\pi/8} = A, \qquad \alpha e^{i\pi/8} + \beta e^{-3i\pi/8} = 0. \tag{10.6.32}$$

Finally, we write down the WKB solution to (10.6.25) in region III:

$$y_{\text{III}} = (1 - e^{-x^2/4})^{-1/4}$$
$$\times \left[B \exp\left(\frac{i}{\varepsilon}\int_x^0 dt\sqrt{1 - e^{-t^2/4}}\right) + C \exp\left(-\frac{i}{\varepsilon}\int_x^0 dt\sqrt{1 - e^{-t^2/4}}\right)\right]. \tag{10.6.33}$$

If we allow $x \to -\infty$, we have $y_{\text{III}} \sim Be^{iI/\varepsilon}e^{-ix/\varepsilon} + Ce^{-iI/\varepsilon}e^{ix/\varepsilon}$. We impose the boundary condition in (10.6.10) by requiring that

$$Be^{iI/\varepsilon} = 1, \qquad Ce^{-iI/\varepsilon} = b. \tag{10.6.34}$$

Next we match y_{III} to y_{II}. In the limit $x \to 0-$, we have

$$y_{\text{III}} \sim (-2/x)^{1/2}(Be^{ix^2/4\varepsilon} + Ce^{-ix^2/4\varepsilon}), \qquad x \to 0-.$$

Comparing this expression with that in (10.6.29) gives

$$B\sqrt{2} = \varepsilon^{1/4}(\alpha e^{-3i\pi/8} + \beta e^{i\pi/8}), \qquad C = \alpha e^{i\pi/8}\varepsilon^{1/4}. \tag{10.6.35}$$

Now, we combine the algebraic equations in (10.6.31), (10.6.32), (10.6.34), and (10.6.35). We obtain the following expressions for b and c:

$$b = \frac{i}{\sqrt{2}}e^{-2iI/\varepsilon}, \qquad c = \frac{1}{\sqrt{2}}e^{-2iI/\varepsilon}.$$

Hence, the reflection and transmission coefficients are $R = |b|^2 = \frac{1}{2}$ and $T = c^2 = \frac{1}{2}$. We obtain the elegant result that half of the incident wave is reflected and half is transmitted!

In Probs. 10.43 to 10.46 we examine other aspects of the phenomenon of tunneling.

(D) 10.7 BRIEF DISCUSSION OF HIGHER-ORDER WKB APPROXIMATIONS

In this section we show how to perform a WKB approximation beyond the leading-order approximation of physical optics. We begin by constructing a higher-order solution to the one-turning-point problem discussed in Sec. 10.4. Then we use this result to obtain an eigenvalue condition which is a higher-order version of (10.5.6).

Second-Order Solution to the One-Turning-Point Problem

We follow closely the notation of Sec. 10.4. We are given the differential equation

$$\varepsilon^2 y''(x) = Q(x)y(x), \qquad y(+\infty) = 0, \tag{10.7.1}$$

where $Q(0) = 0$, $Q(x) > 0$ for $x > 0$, and $Q(x) < 0$ for $x < 0$. We assume, as we did in Sec. 10.4, that $Q(x) \gg x^{-2}$ as $|x| \to \infty$, so that the WKB approximation is valid for all x away from the turning point at $x = 0$. We also assume that

$$Q(x) = ax + bx^2 + O(x^3), \qquad x \to 0. \tag{10.7.2}$$

First, we examine region I ($x > 0$), in which the WKB approximation is valid. For a second-order calculation we must retain one term beyond the physical-optics approximation:

$$y_{\mathrm{I}}(x) = Ce^{S_0/\varepsilon + S_1 + \varepsilon S_2}, \tag{10.7.3a}$$

where

$$S_0(x) = -\int_0^x \sqrt{Q(t)}\, dt, \tag{10.7.3b}$$

$$S_1(x) = -\frac{1}{4}\ln Q(x), \tag{10.7.3c}$$

and, integrating (10.1.4) for $S_2(x)$ once by parts, we obtain

$$S_2(x) = -\frac{5}{48}\frac{Q'(x)}{Q^{3/2}(x)} - \int_\mu^x \frac{Q''(t)}{48Q^{3/2}(t)}\, dt. \tag{10.7.3d}$$

Observe that in the expression for $S_0(x)$ we have integrated from the turning point at $x = 0$. However, in the expression for $S_2(x)$ we must integrate from $\mu > 0$ to x because the integral is divergent at $x = 0$. We will treat μ as a small fixed positive number like ε^2, for example.

Next, we consider region II, the turning-point region. When $|x|$ is small, then we may replace $Q(x)$ in (10.7.1) by the first *two* terms in its Taylor expansion and obtain

$$\varepsilon^2 y_{\mathrm{II}}''(x) = (ax + bx^2)y_{\mathrm{II}}(x), \tag{10.7.4}$$

where $y_{\mathrm{II}}(x)$, as in Sec. 10.4, stands for the approximation to $y(x)$ in the turning-point region.

It is clear that by making a linear transformation of the form $x = \alpha t + \beta$ the constants α and β can be chosen so that (10.7.4) becomes a parabolic cylinder equation. However, this trick is worthless for a third-order WKB calculation. (Why?) We prefer to use a more general approach which is equally useful in all orders. We substitute $x = \varepsilon^{2/3}a^{-1/3}t$, as in (10.4.5). This converts (10.7.4) to an approximate Airy equation

$$\frac{d^2 y_{\mathrm{II}}}{dt^2} = (t + \varepsilon^{2/3}a^{-4/3}bt^2)y_{\mathrm{II}}, \tag{10.7.5}$$

which has a small correction of order $\varepsilon^{2/3}$. We can represent the approximate solution to this equation in terms of an Airy function whose argument also has small corrections of order $\varepsilon^{2/3}$:

$$y(t) \sim D(1 + \alpha_1\varepsilon^{2/3}t + \alpha_2\varepsilon^{4/3}t^2 + \cdots) \times \mathrm{Ai}\,(t + \beta_1\varepsilon^{2/3}t^2 + \beta_2\varepsilon^{4/3}t^3 + \cdots), \qquad \varepsilon \to 0+. \tag{10.7.6}$$

Substituting (10.7.6) into (10.7.5) determines the values of the constants $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$. But this is a second-order calculation, so we retain only the α_1 and β_1 terms:

$$y_{\mathrm{II}}(x) \sim D\left(1 - \frac{bx}{5a}\right)\mathrm{Ai}\left[a^{1/3}\varepsilon^{-2/3}\left(x + \frac{bx^2}{5a}\right)\right]. \tag{10.7.7}$$

If we were doing a third-order calculation, we would replace $Q(x)$ by $ax + bx^2 + cx^3$ and then compute and retain the α_1, α_2, β_1, and β_2 terms in (10.7.6).

Now we must demonstrate that $y_{\mathrm{I}}(x)$ and $y_{\mathrm{II}}(x)$ match in the overlap of regions I and II and in doing so we must find the relation between the constants C and D. To perform the asymptotic match, we replace both $y_{\mathrm{I}}(x)$ in (10.7.3a) and $y_{\mathrm{II}}(x)$ in (10.7.7) by simpler functions.

First, we examine $y_{\mathrm{I}}(x)$ for small x. We make the following approximations:

$$Q^{-1/4}(x) \sim x^{-1/4}a^{-1/4}\left(1 - \frac{bx}{4a}\right), \qquad x \to 0+,$$

$$\int_0^x \sqrt{Q(t)}\, dt \sim \frac{2}{3}\sqrt{a}\, x^{3/2} + \frac{b}{5\sqrt{a}}x^{5/2}, \qquad x \to 0+,$$

$$\frac{5}{48}\frac{Q'(x)}{Q^{3/2}(x)} \sim \frac{5}{48\sqrt{a}}x^{-3/2}, \qquad x \to 0+,$$

$$\frac{1}{48}\int_\mu^x \frac{Q''(t)}{Q^{3/2}(t)}\, dt \sim \frac{1}{12}ba^{-3/2}(\mu^{-1/2} - x^{-1/2}), \qquad x \to 0+.$$

Substituting these formulas into y_1 in (10.7.3) gives

$$y_1(x) \sim Ca^{-1/4}x^{-1/4}\left(1-\frac{bx}{4a}\right)$$
$$\times \exp\left(-\frac{2}{3\varepsilon}a^{1/2}x^{3/2}-\frac{b}{5\varepsilon}a^{-1/2}x^{5/2}-\frac{5\varepsilon}{48}a^{-1/2}x^{-3/2}-\frac{b\varepsilon a^{-3/2}}{12\sqrt{\mu}}\right),$$
$$x, \varepsilon \to 0+. \qquad (10.7.8)$$

Here we are not bothering to specify the precise size of the matching region (see Prob. 10.47). We have discarded a term of the form $\varepsilon x^{-1/2}$ in (10.7.8) because it is small in the limit $\varepsilon \to 0+$; it must be included, however, in a third-order match.

Finally, we approximate $y_{\rm II}(x)$ in (10.7.7) by expanding the Airy function for large positive argument. We take *two* terms in the expansion of Ai (t):

$$\text{Ai}\,(t) \sim \frac{1}{2\sqrt{\pi}}t^{-1/4}e^{-2t^{3/2}/3}\left(1-\tfrac{5}{48}t^{-3/2}\right), \qquad t \to +\infty.$$

Thus, (10.7.7) gives

$$y_{\rm II}(x) \sim D\left(1-\frac{bx}{4a}\right)\frac{1}{2\sqrt{\pi}}\varepsilon^{1/6}a^{-1/12}x^{-1/4}\left(1-\frac{5\varepsilon}{48}a^{-1/2}x^{-3/2}\right)$$
$$\times \exp\left(-\frac{2}{3\varepsilon}a^{1/2}x^{3/2}-\frac{b}{5\varepsilon}a^{-1/2}x^{5/2}\right), \qquad x, \varepsilon \to 0+. \qquad (10.7.9)$$

Observe that (10.7.8) and (10.7.9) match perfectly! What is more, we obtain the condition

$$C = \frac{D}{2\sqrt{\pi}}a^{1/6}\varepsilon^{1/6}\exp\left(\frac{b\varepsilon a^{-3/2}}{12\sqrt{\mu}}\right). \qquad (10.7.10)$$

The one-turning-point problem is now half done.

The next step is to write down the oscillatory WKB solution in region III and match it to $y_{\rm II}(x)$. In Prob. 10.48 you are asked to verify that

$$y_{\rm III}(x) = \frac{D}{\sqrt{\pi}}\varepsilon^{1/6}a^{1/6}[-Q(x)]^{-1/4}$$
$$\times \sin\left[\frac{1}{\varepsilon}\int_x^0 \sqrt{-Q(t)}\,dt - \frac{5\varepsilon Q'(x)}{48[-Q(x)]^{3/2}} + \frac{\varepsilon}{48}\int_x^{-\mu}\frac{Q''(t)}{[-Q(t)]^{3/2}}\,dt - \frac{b\varepsilon a^{-3/2}}{12\sqrt{\mu}} + \frac{\pi}{4}\right].$$
$$(10.7.11)$$

Equations (10.7.3), (10.7.7), and (10.7.11) constitute a complete second-order solution to the one-turning-point problem.

Second-Order Solution to the Two-Turning-Point Eigenvalue Problem

If we follow Sec. 10.5 and combine two second-order one-turning-point solutions, we obtain the second-order generalization of (10.5.2) (see Prob. 10.49):

$$\frac{1}{\varepsilon}\int_A^B \sqrt{-Q(t)}\,dt + \frac{\varepsilon}{48}\int_{A+\mu}^{B-\mu}\frac{Q''(t)}{[-Q(t)]^{3/2}}\,dt - \frac{b_A\varepsilon a_A^{-3/2}}{12\sqrt{\mu}} - \frac{b_B\varepsilon a_B^{-3/2}}{12\sqrt{\mu}}$$
$$= \left(n+\frac{1}{2}\right)\pi + O(\varepsilon^2), \qquad \varepsilon \to 0+, \qquad (10.7.12)$$

where a_A, b_A and a_B, b_B are the expansion coefficients of $Q(x)$ in the neighborhood of A and B. If after evaluating the integrals in (10.7.2) one allows μ to tend to $0+$, one obtains a finite answer independent of μ (see Probs. 10.50 and 10.51).

If $Q(x)$ is an analytic function of x, then (10.7.12) can be replaced by a much simpler contour integral

$$\frac{1}{2i}\oint_C\left[\frac{1}{\varepsilon}S_0'(z)+\varepsilon S_2'(z)\right]dz = \left(n+\frac{1}{2}\right)\pi + O(\varepsilon^2), \qquad \varepsilon \to 0+, \qquad (10.7.13)$$

where the contour C encircles the two turning points, which are connected by a branch cut on the real-z axis, and S_0 and S_2 are given in (10.7.3). Although $S_2'(z)$ is infinite at the two turning points, the integral in (10.7.13) is finite because the contour encircles the turning points without passing through them.

Complete Perturbative Solution to the Two-Turning-Point Eigenvalue Problem

Dunham discovered a lovely generalization of (10.7.13) to all orders in perturbation theory:

$$\frac{1}{2i}\oint_C \frac{1}{\varepsilon}\sum_{k=0}^{\infty}\varepsilon^k S_k'(z)\,dz \sim n\pi, \qquad \varepsilon \to 0+. \qquad (10.7.14)$$

Let us see how this formula reduces to (10.7.13). Recall that

$$S_1'(z) = -\frac{1}{4}\frac{d}{dz}\ln\,[Q(z)].$$

Thus,

$$\frac{1}{2i}\oint S_1'(z)\,dz = -\frac{1}{8i}\ln\,Q(z)\Bigg|_{\substack{\text{evaluated once}\\ \text{around the contour } C}} = -\frac{1}{8i}(4\pi i) = -\frac{\pi}{2}.$$

Note that evaluating $Q(z)$ once around the contour C gives $4\pi i$ because the contour encircles the two simple zeros of $Q(z)$ at the turning points A and B. This accounts for the $\pi/2$ in (10.7.13).

It is a fact that (see Prob. 10.52) $S'_{2k+1}(z)$ $(k = 1, 2, 3, \ldots)$ is a total derivative. For example [see (10.1.15)],

$$S'_3(z) = \frac{d}{dz}\left\{\frac{5[Q'(z)]^2}{64[Q(z)]^3} - \frac{Q''(z)}{16[Q(z)]^2}\right\}.$$

This becomes a single-valued function once the turning points are joined by a branch cut. Therefore, evaluating this expression once around the closed contour C gives 0. This allows us to simplify (10.7.14) to

$$\frac{1}{2i}\oint_C \frac{1}{\varepsilon}\sum_{k=0}^{\infty} \varepsilon^{2k} S'_{2k}(z)\, dz = \left(n + \frac{1}{2}\right)\pi, \qquad \varepsilon \to 0+. \tag{10.7.15}$$

Observe that only *even* orders in WKB perturbation theory contribute to a calculation of the eigenvalues.

Example 1 *Eigenvalues of the parabolic cylinder equation.* In Example 2 of Sec. 10.5 we found that the eigenvalues E of the parabolic cylinder equation $(-d^2/dx^2 + x^2/4 - E)y(x) = 0$ $[y(\pm\infty) = 0]$ are given exactly by leading-order WKB: $E = n + \frac{1}{2}$ $(n = 0, 1, 2, \ldots)$. The explanation for this surprising result is simply that all terms in (10.7.15) after the first happen to vanish upon explicit calculation (see Prob. 10.53).

Example 2 *Eigenvalues for* $y'' = (x^4 - E)y$. After much effort we have managed to evaluate the integrals and thus to calculate the first seven terms in the WKB series in (10.7.15). The series takes the form of a power series in inverse fractional powers of E:

$$E^{3/4}\sum_{n=0}^{\infty} A_{2n} E^{-3n/2} \sim \left(n + \frac{1}{2}\right)\pi, \qquad n \to \infty, \tag{10.7.16}$$

where we have set $\varepsilon = 1$ and

$$A_0 = \frac{1}{3} R\sqrt{\pi} \doteq 1.748,$$

$$A_2 = -\frac{1}{4}\frac{\sqrt{\pi}}{R} \doteq -0.1498,$$

$$A_4 = \frac{11}{3 \cdot 2^9} R\sqrt{\pi} \doteq 0.03756,$$

$$A_6 = \frac{7 \cdot 11 \cdot 61}{3 \cdot 5 \cdot 2^{11}}\frac{\sqrt{\pi}}{R} \doteq 0.09160,$$

$$A_8 = -\frac{5 \cdot 13 \cdot 17 \cdot 353}{7 \cdot 2^{19}} R\sqrt{\pi} \doteq -0.5574,$$

$$A_{10} = -\frac{11 \cdot 11 \cdot 19 \cdot 23 \cdot 1{,}009}{3 \cdot 2^{21}}\frac{\sqrt{\pi}}{R} \doteq -5.080,$$

$$A_{12} = \frac{5 \cdot 17 \cdot 29 \cdot 49{,}707{,}277}{3 \cdot 11 \cdot 2^{28}} R\sqrt{\pi} \doteq 72.54,$$

in which $R = \Gamma(\frac{1}{4})/\Gamma(\frac{3}{4}) \doteq 2.958\,675\,119\,188\,638\,892\,310\,821\,4$.

We have not been able to discover a simple formula for the terms in this series, but the series certainly looks like a typical asymptotic series. Like the Stirling series for the gamma function,

Table 10.5 Comparison of the exact eigenvalues of the x^4 potential with the 0, 2, 4, 6, 8, 10, and 12th-order WKB predictions from (10.7.16)

Observe how rapidly the maximal accuracy increases as E_n increases

E_0(exact) $\doteq$ 1.060 362 090 484 182 899 65	E_6(exact) $\doteq$ 26.528 471 183 682 518 191 81
$(\text{WKB})_0$ 0.87	$(\text{WKB})_0$ 26.506 335 511
$(\text{WKB})_2$ 0.98 (1 part in 10)	$(\text{WKB})_2$ 26.528 512 552
$(\text{WKB})_4$ 0.95	$(\text{WKB})_4$ 26.528 471 873
$(\text{WKB})_6$ 0.78	$(\text{WKB})_6$ 26.528 471 147
$(\text{WKB})_8$ 1.13	$(\text{WKB})_8$ 26.528 471 179
$(\text{WKB})_{10}$ 1.40	$(\text{WKB})_{10}$ 26.528 471 182 (7 parts in 10^{11})
$(\text{WKB})_{12}$ 1.64	$(\text{WKB})_{12}$ 26.528 471 181
E_2(exact) $\doteq$ 7.455 697 937 986 738 392 16	E_8(exact) $\doteq$ 37.923 001 027 033 985 146 52
$(\text{WKB})_0$ 7.414 0	$(\text{WKB})_0$ 37.904 471 845 068
$(\text{WKB})_2$ 7.455 8 (1 part in 10^5)	$(\text{WKB})_2$ 37.923 021 140 528
$(\text{WKB})_4$ 7.455 3	$(\text{WKB})_4$ 37.923 001 229 358
$(\text{WKB})_6$ 7.455 2	$(\text{WKB})_6$ 37.923 001 021 414
$(\text{WKB})_8$ 7.455 2	$(\text{WKB})_8$ 37.923 001 026 832
$(\text{WKB})_{10}$ 7.455 2	$(\text{WKB})_{10}$ 37.923 001 027 043 (7 parts in 10^{14})
$(\text{WKB})_{12}$ 7.455 2	$(\text{WKB})_{12}$ 37.923 001 027 030
E_4(exact) $\doteq$ 16.261 826 018 850 225 937 89	E_{10}(exact) $\doteq$ 50.256 254 516 682 919 039 74
$(\text{WKB})_0$ 16.233 614 7	$(\text{WKB})_0$ 50.240 152 319 172 36
$(\text{WKB})_2$ 16.261 936 7	$(\text{WKB})_2$ 50.256 265 932 002 07
$(\text{WKB})_4$ 16.261 828 6 (5 parts in 10^8)	$(\text{WKB})_4$ 50.256 254 592 948 49
$(\text{WKB})_6$ 16.261 824 5	$(\text{WKB})_6$ 50.256 254 515 324 64
$(\text{WKB})_8$ 16.261 824 9	$(\text{WKB})_8$ 50.256 254 516 650 43
$(\text{WKB})_{10}$ 16.261 825 0	$(\text{WKB})_{10}$ 50.256 254 516 684 34
$(\text{WKB})_{12}$ 16.261 825 0	$(\text{WKB})_{12}$ 50.256 254 516 682 99 (1 part in 10^{15})

the coefficients get smaller for a while but eventually appear to grow without bound. We would therefore expect that, for any given value of n, successive approximations to the nth eigenvalue E_n, obtained by solving (10.7.16) with the series truncated after more and more terms, should improve to some maximal accuracy and then become worse. Moreover, since E_n increases with n, more terms in the series should be required to reach maximal accuracy as n increases and the accuracy should also increase with n. This is precisely what happens (see Table 10.5). The rate at which the accuracy increases is particularly impressive. The error of the most accurate WKB approximations are indicated in parentheses.

PROBLEMS FOR CHAPTER 10

Section 10.1

(I) **10.1** Derive equations (10.1.14), (10.1.15), (10.1.16), and (10.1.17).

(E) **10.2** Show that, for the Schrödinger equation $\varepsilon^2 y'' = Q(x)y$, if $S'_2 = 0$ then $S'_n = 0$ for $n \ge 2$. Deduce that the most general function $Q(x)$ for which the equation $\varepsilon^2 y'' = Q(x)y$ has the physical-optics approximation as its exact solution is $Q(x) = (c_1 x + c_2)^{-4}$.

(E) **10.3** Estimate how small ε must be for the approximation in (10.1.9) to be accurate to a relative error of ≤ 1 percent when $x \ge 1$.

(I) **10.4** Using the asymptotic methods of Chap. 6, evaluate the integral $\int_0^1 [y(x)]^n\,dx$ to leading order in powers of ε, where $y(x)$ is given in (10.1.19).

(I) **10.5** Use WKB theory to obtain the solution to $\varepsilon y'' + a(x)y' + b(x)y = 0$ $[a(x) > 0, y(0) = A, y(1) = B]$ correct to order ε.

(I) **10.6** Use second-order WKB theory to derive a formula which is more accurate than (10.3.31) for the nth eigenvalue of the Sturm-Liouville problem in (10.1.27). Let $Q(x) = (x + \pi)^4$ and compare your formula with the values of E_n in Table 10.1.

(I) **10.7.** (*a*) Show that the eigenvalues of (10.1.27) are nondegenerate. That is, show that all eigenfunctions having the same eigenvalue E are proportional to each other.

(*b*) Show that distinct eigenfunctions are orthogonal in the sense that the integral $\int_0^\pi Q(x)y_n(x)y_m(x) = 0$ when $n \neq m$.

(*c*) The Sturm-Liouville eigenfunctions $y_n(x)$ for the boundary-value problem (10.1.27) form a complete orthonormal set and may therefore be used to expand functions on the interval $(0, \pi)$ into Fourier series. The Fourier expansion of the function $f(x)$ has the form $\sum_{n=0}^{\infty} a_n y_n(x)$. Show that the Fourier coefficients are given by $a_n = \int_0^\pi dx\, f(x)y_n(x)Q(x)$.

(*d*) Let $f(x)$ be continuous and $f(0) \neq 0$. Find the leading behavior of a_n as $n \to \infty$ from (10.1.33). Is the resulting series differentiable?

Section 10.2

(E) **10.8** Consider the equations $\varepsilon^2 y''(x) = (\sin x)y$, $\varepsilon^2 y''(x) = (\sin x^2)y$, $\varepsilon^2 y''(x) = [1 + (\sin x)^2]y$. For which fixed values of x is the WKB physical-optics approximation a good approximation to $y(x)$ as $\varepsilon \to 0$. Is WKB accurate as $x \to \infty$.

(E) **10.9** For the following equations estimate how small ε must be for $\exp\,[(1/\varepsilon)S_0(x) + S_1(x) + \varepsilon S_2(x)]$ to be accurate to a relative error of less than 0.1 percent for all $x \geq 0$: $\varepsilon^2 y'' = Q(x)y$; (*a*) $Q(x) = \cosh x$, (*b*) $Q(x) = 1 + x^2$, (*c*) $Q(x) = 1 + x^4$.

(I) **10.10** (*a*) Verify that (10.2.8) reproduces the asymptotic formulas in (10.2.6).

(*b*) Show that the leading behavior of solutions to $\varepsilon^2 y'' = x^{-2}(\ln x)^2 y$ as $x \to +\infty$ is given by (10.2.6) with $(\ln x)^{-3/8}$ and $(\ln x)^{-5/8}$ replaced by $(\ln x)^{-1/2+\varepsilon/8}$ and $(\ln x)^{-1/2-\varepsilon/8}$, respectively. Use these results to demonstrate that physical optics is valid as $\varepsilon \to 0$ with $x > 0$ fixed, but that the higher-order approximation (10.2.8) must be used to find the leading behavior of $y(x)$ as $x \to +\infty$ with $\varepsilon > 0$ fixed.

(I) **10.11** (*a*) Derive the physical-optics approximation to the solutions of $\varepsilon^n y^{(n)} = Q(x)y$. Show that $S_0(x)$ and $S_1(x)$ are given correctly by (10.2.10) and (10.2.11).

(*b*) Show that if $Q(x)$ is sufficiently smooth and that $|x^n Q(x)| \to \infty$ as $x \to +\infty$, then physical optics gives the correct leading behavior of solutions to (10.2.9) as $x \to +\infty$ with ε fixed. This justifies the result stated after (3.4.28).

Section 10.3

(E) **10.12** Show that for (10.3.1) to have a solution $y(x)$ for which $y(\pm\infty) = 0$ it is necessary that $R(x) \ll Q(x)$ as $x \to \pm\infty$.

(I) **10.13** Verify (10.3.9) and (10.3.10).

(D) **10.14** Solve the Green's function equation $\varepsilon^2\,\partial^2 G/\partial x^2 - Q(x)G = -\delta(x - x')$ to one order beyond physical optics. That is, include S_0, S_1, and S_2 in the WKB series. Evaluate $\int_{-\infty}^{\infty} G(x, x')\,dx$ and $\int_{-\infty}^{\infty} [G(x, x')]^2\,dx$ correct to order ε.

(I) **10.15** Prove that $\int_{-\infty}^{\infty} [G_{\text{unif}}(x, x')]^3 (x - x')^2\,dx' = [Q(x)]^3/54 + O(\varepsilon)$ $(\varepsilon \to 0+)$, where G_{unif} is given in (10.3.7).

(I) **10.16** Show how to use the notions of boundary-layer theory to derive the approximate solution (10.3.11) to the Schrödinger equation (10.3.12). Show that boundary layers (localized regions of rapid change) occur at $x = \pm 1$. Find the thickness of the boundary layers. Match inner and outer solutions to derive (10.3.11).

(I) **10.17** Consider the fourth-order Green's function equation $\varepsilon^4 d^4y/dx^4 + Q(x)y = \delta(x)$ $[Q(x) > 0, y(\pm\infty) = 0]$. Use physical optics to derive a uniform asymptotic approximation to $y(x)$ for all x. What is $y(0)$? Evaluate $\int_{-\infty}^{\infty} y(x)\,dx$.

Section 10.4

(D) **10.18** Deduce Langer's uniform approximation (10.4.15) to the solution of the Schrödinger equation $\varepsilon^2 y'' = Q(x)y$ directly from the differential equation.

Clue: First introduce a new dependent variable which is some function $f(x)$ times the old dependent variable $y(x)$. Then show that $f(x)$ may be chosen such that for *all* x there is a new independent variable for which the equation may be approximated by the Airy equation.

(I) **10.19** Prove that Langer's formula in (10.4.15) reduces to (10.4.13*c*) in region III.

(I) **10.20** From the integral representation for Ai (x) in Prob. 6.75 prove (10.4.22). Specifically, show that $\int_0^\infty \text{Ai}\,(x)\,dx = \frac{1}{3}$, $\int_{-\infty}^0 \text{Ai}\,(x)\,dx = \frac{2}{3}$.

(I) **10.21** Verify (10.4.23).

(I) **10.22** Verify (10.4.28).

(I) **10.23** Solve $\varepsilon^2 y''(x) = Q(x)y(x)$, where $Q(x)$ is even, $Q(x)$ vanishes just once at $x = 0$, and $Q(x) \sim a|x|$ near $x = 0$ $(a > 0)$. Find that solution y which vanishes as $x \to \infty$.

(D) **10.24** Solve the one-turning-point Green's function equation $y''(x) - Q(x)y(x) = -\delta(x - x')$, where $Q(x)$ has a simple zero at $x = 0$, $Q(x) > 0$ if $x > 0$, $Q(x) < 0$ if $x < 0$, and $Q(x) \sim ax$ $(x \to 0)$. $y(x)$ is required to satisfy $y(\infty) = 0$ and $y(0) = 1$. Compute $y(x')$ and $\int_{-\infty}^{\infty} y(x)\,dx$.

(E) **10.25** Use WKB theory to approximate the Bessel function $J_\nu(x)$ for ν, $x > 0$.

Clue: The Bessel equation is $y'' + y'/x + (1 - \nu^2/x^2)y = 0$. Let $x = e^t$, remember that $J_\nu(0) = 0$ when $\nu > 0$, and use Langer's formula (10.4.15).

(D) **10.26** (*a*) Consider the one-turning-point problem $\varepsilon^2 y''(x) = Q(x)y(x)$, where $Q(x)$ vanishes just once at $x = 0$ and $Q(x) \sim a^2x^2$ as $x \to 0$, $a > 0$. Find a complete physical optics WKB approximation to $y(x)$ for that solution which approaches 0 as $x \to +\infty$ and is normalized by $y(0) = 1$. *Note:* Your answer should consist of three formulas valid in each of three regions. Combine your three formulas, à la Langer, to obtain a *single* formula which is a uniformly valid approximation to $y(x)$ for $-\infty < x < \infty$.

Clue: The final answer is

$$y_{\text{unif}}(x) = \left[\frac{4aS(x)}{\pi^2 Q(x)}\right]^{1/4} \Gamma\left(\frac{3}{4}\right) D_{-1/2}[2\varepsilon^{-1/2}S^{1/2}(x)],$$

where $S(x) = \int_0^x \sqrt{Q(t)}\,dt$.

(*b*) Show that to leading order in powers of ε, $\int_0^\infty x[y(x)]^2\,dx \sim [\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})]^2 2\varepsilon/a$ $(\varepsilon \to 0+)$.

(TI) **10.27** Suppose we attempt to derive the one-turning-point connection formula using WKB in the complex plane. Let $Q(0) = 0$ and write down the WKB approximation

$$y_{\text{WKB}}(z) = C[Q(z)]^{-1/4} \exp\left[-\frac{1}{\varepsilon}\int_0^z \sqrt{Q(t)}\,dt\right],$$

which is valid when z is real and positive. Then analytically continue this expression to negative z along a path which does not pass through $z = 0$ and which goes around the turning point at $z = 0$ in a counterclockwise sense in the upper half plane. We fail to derive (10.4.16). Next take a path which goes around the turning point in a clockwise sense in the lower half plane. Again we fail to derive (10.4.16). Explain the breakdown of these analytic continuations of the WKB solution in terms of the Stokes phenomenon.

(I) **10.28** Use WKB analysis to show that uniform leading-order approximations to the solutions of the differential equations of Prob. 9.33 as $\varepsilon \to 0+$ are

(*a*) $y_{\text{unif},0}(x) = 2A(x + 1) + (B - 3A)e^{(6x-3)/(8\varepsilon)}$;

(*b*) $y_{\text{unif},0}(x) = \dfrac{3A - B}{4}(1 - x) + \dfrac{9B - 3A}{8}e^{(6x-3)/(16\varepsilon)} + \dfrac{3B - A}{8}e^{-(6x+3)/(16\varepsilon)}$.

Explain why there is a boundary layer only at $x = \frac{1}{2}$ in (*a*) while there are boundary layers near both $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ in (*b*). Observe that the approximations $y_{\text{unif},0}(x)$ are *not* exponentially small when $|x| < \frac{1}{2}$ even though these differential equations correspond to case IV of Sec. 9.6. Higher-order corrections do *not* force the solution to be exponentially small within $|x| < \frac{1}{2}$.

Clue: Observe that the outer solutions $1 + x$ and $1 - x$ to the problems are also exact solutions to the differential equations. You need only find a linearly independent solution which must grow exponentially fast away from the internal layer at $x = 0$.

(D) **10.29** Derive the connection formula for a turning point which is a $(2n + 1)$-fold zero, going from a classically forbidden to a classically allowed region. Assume that $y(x) \to 0$ as $x \to \infty$ in the classically forbidden region.

Section 10.5

(I) **10.30** Derive (10.5.4).

(I) **10.31** Show that the eigenvalue condition in (10.5.6) is a valid asymptotic relation as $\varepsilon \to 0+$ with E fixed or as $E \to +\infty$ with ε fixed.

(E) **10.32** Verify (10.5.15).

(E) **10.33** Show that the physical-optics approximation to the eigenvalues E of the equation $-y'' + (x^{2K} - E)y = 0$ $[y(\pm\infty) = 0]$, with $K = 1, 2, 3, 4, \ldots$, is

$$E_n \sim \left\{\frac{(n + \frac{1}{2})\sqrt{\pi}\,\Gamma[(3 + 1/K)/2]}{\Gamma[(2 + 1/K)/2]}\right\}^{2K/(K+1)}, \qquad n \to \infty.$$

(I) **10.34** Show that the WKB physical-optics formula for the eigenvalues of the equation $-\varepsilon^2 y'' + [V(x) - E]y = 0$ $[y(0) = 0,\ y(+\infty) = 0]$, where $V(0) = 0$, $V(+\infty) = +\infty$, $V(x)$ rises monotonically as x increases from 0, is $(1/\varepsilon)\int_0^{x_0} \sqrt{E - V(x)}\,dx = (n - \frac{1}{4})\pi + O(\varepsilon)$ $(\varepsilon \to 0+)$, where $E - V(x_0) = 0$ and $n = 1, 2, 3, \ldots$.

(E) **10.35** The gravitational potential rises linearly with x. Use the result in Prob. 10.34 to find the eigenvalues in a gravitational potential well.

(I) **10.36** Consider the eigenvalue problem $y'' + E(\cos x)y = 0$ $[y(\pm\pi) = 0]$. Find an approximation to E which is valid for large values of E.

(D) **10.37** Find a physical-optics approximation to the eigenvalues E of $d^4y/dx^4 = [E - V(x)]y(x)$ $[y(\pm\infty) = 0]$, where $V(\pm\infty) = \infty$. Check your result by using it to find the eigenvalues of $d^4y/dx^4 = (E - x^2)y(x)$ $[y(\pm\infty) = 0]$, which can be solved using a Fourier transform.

Clue: It will help if you solve Prob. 6.83 first.

Section 10.6

(E) **10.38** Verify (10.6.7).

(I) **10.39** Prove that a classical particle incident on a delta-function potential $V(x) = \delta(x)$ bounces back if $E < (4\varepsilon^2)^{-1}$ and continues on if $E > (4\varepsilon^2)^{-1}$.

(E) **10.40** Show that the corrections to the asymptotic relation $\int_B^x dt\sqrt{E - V(t)} \sim x\sqrt{E} + I$ $(x \to +\infty)$, where $I = \int_B^\infty dt[\sqrt{E - V(t)} - \sqrt{E}] - B\sqrt{E}$, vanish as $x \to +\infty$ if $V(x) \to 0$ faster that $1/x$.

(I) **10.41** Find precise asymptotic estimates of the boundaries of regions I, II, and III for (10.6.25).

(I) **10.42** If $e^{-x^2/4}$ in (10.6.25) were replaced by $1/(1 + x^2)$, how would the leading-order WKB predictions for R and T change?

(D) **10.43** Suppose $e^{-x^2/4}$ in (10.6.25) were replaced by $e^{-x^2/4}/2$ or by $1/(2 + 2x^2)$. In classical mechanics all incident particles penetrate the potential bump at $x = 0$. However, in quantum mechanics, there is an exponentially small reflection coefficient R. Find a physical-optics approximation to R.

Clue: It is necessary to find the connection formula for a turning point in the complex plane. See the discussion of this problem in Ref. 18.

(D) **10.44** In this problem we investigate the quantum-mechanical phenomenon of resonance.

(*a*) Let $V(x)$ in (10.6.3) be two delta functions at $x = 0$ and $x = 1$: $V(x) = \delta(x) + \delta(x - 1)$. Solve (10.6.3) for this $V(x)$ exactly and show that there is an infinite number of discrete energies for which $V(x)$ becomes transparent ($T = 1$, $R = 0$). Compute these energies.

(*b*) Let $V(x)$ in (10.6.3) be $x^2 - x^4$. Use WKB theory to find a physical-optics approximation for the resonant energies E_n.

(D) **10.45** Suppose we wish to calculate the eigenvalues of a double potential well separated by a potential hill (like the letter W). Then there are two cases to consider:

(*a*) If each of the wells has a different shape, then to a good approximation the eigenvalue spectrum (for those eigenvalues below the peak of the hill) is the union of the spectra of each of the two wells separately. Corrections are exponentially small (subdominant). Explain why.

(*b*) Suppose the two wells are identical. For this problem take $V(x) = x^4 - x^2$. Now the eigenvalue spectrum consists of almost degenerate pairs of eigenvalues. Use WKB theory to calculate the splitting between pairs of eigenvalues.

Clue: For the above $V(x)$ assume that eigenfunctions are either even or odd functions of x.

(D) **10.46** In this problem we investigate the quantum-mechanical phenomenon of radioactive decay. Radioactive decay, the tunneling of a wavefunction out of a confined region, is clearly a time-dependent phenomenon. Therefore we must return to the time-dependent Schrödinger equation (10.6.1). We represent the time dependence as in (10.6.2). The decay constant is the imaginary part of E, which must be positive. Establish the following results:

(*a*) Define a probability density $\rho(x, t)$ and a probability current $j(x, t)$ by $\rho(x, t) = \psi^*\psi/\varepsilon$, $j(x, t) = i\varepsilon[\psi^*(\partial\psi/\partial x) - \psi(\partial\psi^*/\partial x)]$. Show that as a consequence of (10.6.1), j and ρ satisfy a local conservation law: $\partial j/\partial x + \partial\rho/\partial t = 0$.

(*b*) Consider a potential V which looks like an upside-down letter W. For example, take $V(x) = x^2 - x^4$. Show that in general

$$\text{Im } E = \frac{j(x_2, t) - j(x_1, t)}{2\int_{x_1}^{x_2} \rho(x, t)\,dx},$$

where x_1 and x_2 are points to the left and right of and outside the potential well. What is the connection between the sign of Im E and the direction of flow of probability current? Does Im E vary with time?

(*c*) Impose outgoing wave boundary conditions and find a physical-optics approximation to Im E. What happens to Im E if we impose incoming wave boundary conditions? Explain.

Section 10.7

(I) **10.47** What is the size of the region in which $y_I(x)$ in (10.7.8) and $y_{II}(x)$ in (10.7.9) match asymptotically?

(I) **10.48** Verify (10.7.11).

(D) **10.49** Derive (10.7.12) by combining two second-order one-turning-point WKB approximations.

(I) **10.50** Evaluate (10.7.12) for $Q(x) = x^2/4 - E$.

Clue: Show that $A = 2\sqrt{E}$ and $B = -2\sqrt{E}$; $a_A = a_B = \sqrt{E}$; $b_A = b_B = \frac{1}{4}$. Then show that for small μ, the left side of (10.7.12) reduces to $E\pi - \sqrt{\mu}\,E^{-3/4}/64$. Thus, in the limit $\mu \to 0+$, we recover the physical-optics result $E \sim n + \frac{1}{2}$ $(n \to \infty)$.

(I) **10.51** Evaluate (10.7.12) for $Q(x) = x^4 - E$. Show that in the limit $\mu \to 0+$, the left side of (10.7.12) becomes

$$\frac{E^{3/4}\Gamma(\frac{1}{4})\sqrt{\pi}}{3\Gamma(\frac{3}{4})} - \frac{E^{-3/4}\Gamma(\frac{3}{4})\sqrt{\pi}}{4\Gamma(\frac{1}{4})}.$$

Conclude that

$$E \sim \left[3\left(n + \frac{1}{2}\right)\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)\Big/\Gamma\left(\frac{1}{4}\right)\right]^{4/3}\left[1 + \frac{1}{81(n + \frac{1}{2})^2\pi}\right]$$

as $n \to \infty$. Can you understand why the numerical results in Table 10.5 improve so dramatically with increasing n?

(TD) **10.52** Formulate a proof that S'_{2k+1} $(k = 1, 2, 3, \ldots)$ is a total derivative. Check your result by explicitly calculating S'_3, S'_5, and S'_7 for arbitrary Q.

(D) **10.53** Calculate explicitly the first three terms in the series (10.7.15) for $Q(x) = \frac{1}{4}x^2 - E$ and show that the only nonvanishing term is the first. Explain why *every* term vanishes except the first.

CHAPTER

ELEVEN

MULTIPLE-SCALE ANALYSIS

And here—ah, now, this really is something a little recherché.

—Sherlock Holmes, *The Musgrave Ritual*
Sir Arthur Conan Doyle

(E) 11.1 RESONANCE AND SECULAR BEHAVIOR

Multiple-scale analysis is a very general collection of perturbation techniques that embodies the ideas of both boundary-layer theory and WKB theory. Multiple-scale analysis is particularly useful for constructing uniformly valid approximations to solutions of perturbation problems.

In this section we show how nonuniformity can appear in a regular perturbation expansion as a result of resonant interactions between consecutive orders of perturbation theory. To illustrate, we examine a simple perturbation problem, show how resonances occur and lead to a nonuniformly valid perturbation expansion, and finally show how to interpret and eliminate these nonuniformities. The formal development of multiple-scale analysis is postponed to Sec. 11.2.

Resonance

The phenomenon of resonance is nicely illustrated by the differential equation

$$\frac{d^2}{dt^2}y(t) + y(t) = \cos(\omega t). \tag{11.1.1}$$

This equation represents a harmonic oscillator of natural frequency 1 which is driven by a periodic external force of frequency ω. The general solution to this equation for $|\omega| \neq 1$ has the form

$$y(t) = A\cos t + B\sin t + \frac{\cos(\omega t)}{1-\omega^2}, \qquad |\omega| \neq 1. \tag{11.1.2}$$

Observe that for all $|\omega| \neq 1$ the solution remains bounded for all t. If $|\omega|$ is close to 1, the amplitude of oscillation becomes large because the system absorbs large amounts of energy from the external force. Nevertheless, the amplitude of the system is still bounded when $|\omega| \neq 1$ because the system is oscillating out of phase with the driving force.

The solution in (11.1.2) is incorrect when $|\omega| = 1$. The correct solution has an amplitude which grows with t:

$$y(t) = A\cos t + B\sin t + \tfrac{1}{2}t\sin t, \qquad |\omega| = 1. \tag{11.1.3}$$

The amplitude of oscillation of this solution is unbounded as $t \to \infty$ because the oscillator continually absorbs energy from the periodic external force. This system is in *resonance* with the external force.

The term $\frac{1}{2}t\sin t$, whose amplitude grows with t, is said to be a *secular* term. The secular term $\frac{1}{2}t\sin t$ has appeared because the inhomogeneity $\cos t$ in (11.1.1) with $|\omega| = 1$ is itself a solution of the homogeneous equation associated with (11.1.1): $d^2y/dt^2 + y = 0$. In general, secular terms always appear whenever the inhomogeneous term is itself a solution of the associated homogeneous *constant-coefficient* differential equation. A secular term always grows more rapidly than the corresponding solution of the homogeneous equation by at least a factor of t.

Example 1 *Appearance of secular terms.*

(*a*) The solution to the differential equation $d^2y/dt^2 - y = e^{-t}$ has a secular term because e^{-t} satisfies the associated homogeneous equation. The general solution is $y(t) = Ae^{-t} + Be^t - \frac{1}{2}te^{-t}$. The particular solution $-\frac{1}{2}te^{-t}$ is secular relative to the homogeneous solution Ae^{-t}; we must regard the term $-\frac{1}{2}te^{-t}$ as secular even though it is negligible compared with the homogeneous solution Be^t as $t \to \infty$.

(*b*) The solution to the differential equation $d^2y/dt^2 - 2dy/dt + y = e^t$ has a secular term because e^t satisfies the associated homogeneous equation. The general solution is $y(t) = Ae^t + Bte^t + \frac{1}{2}t^2e^t$. In this case, the particular solution $\frac{1}{2}t^2e^t$ is secular with respect to all solutions of the associated homogeneous equation.

Nonuniformity of Regular Perturbation Expansions

The appearance of secular terms signals the nonuniform validity of perturbation expansions for large t. The nonlinear oscillator equation (Duffing's equation)

$$\frac{d^2y}{dt^2} + y + \varepsilon y^3 = 0, \qquad y(0) = 1,\ y'(0) = 0, \tag{11.1.4}$$

provides a good illustration of what we mean by nonuniformity. A perturbative solution of this equation is obtained by expanding $y(t)$ as a power series in ε:

$$y(t) = \sum_{n=0}^{\infty} \varepsilon^n y_n(t), \tag{11.1.5}$$

where $y_0(0) = 1$, $y_0'(0) = 0$, $y_n(0) = y_n'(0) = 0$ $(n \geq 1)$. Substituting (11.1.5) into the differential equation (11.1.4) and equating coefficients of like powers of ε gives a sequence of linear differential equations of which all but the first are inhomogeneous:

$$y_0'' + y_0 = 0, \tag{11.1.6a}$$

$$y_1'' + y_1 = -y_0^3, \tag{11.1.6b}$$

and so on.

The solution to (11.1.6*a*) which satisfies $y_0(0) = 1$, $y_0'(0) = 0$ is

$$y_0(t) = \cos t.$$

To solve (11.1.6b) we invoke the trigonometric identity $\cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t$ to rewrite the inhomogeneous term. The formulas in (11.1.2)–(11.1.3) then provide the general solution to (11.1.6b):

$$y_1(t) = A\cos t + B\sin t + \tfrac{1}{32}\cos 3t - \tfrac{3}{8}t\sin t;$$

the particular solution satisfying $y_1(0) = y_1'(0) = 0$ is

$$y_1(t) = \tfrac{1}{32}\cos 3t - \tfrac{1}{32}\cos t - \tfrac{3}{8}t\sin t.$$

We observe that $y_1(t)$ contains a secular term. This secularity necessarily occurs because $\cos^3 t$ contains a component, $\frac{3}{4}\cos t$, whose frequency equals the natural frequency of the unperturbed oscillator.

In summary, the first-order perturbative solution to (11.1.4) is

$$y(t) = \cos t + \varepsilon[\tfrac{1}{32}\cos 3t - \tfrac{1}{32}\cos t - \tfrac{3}{8}t\sin t] + O(\varepsilon^2), \qquad \varepsilon \to 0+. \tag{11.1.7}$$

We emphasize that the term $O(\varepsilon^2)$ in the above expression means that for *fixed t* the error between $y(t)$ and $y_0(t) + \varepsilon y_1(t)$ is at most of order ε^2 as $\varepsilon \to 0+$. The nonuniformity of this result surfaces if we consider large values of t—specifically, values of t of order $1/\varepsilon$ or larger as $\varepsilon \to 0+$. For such large values of t, the secular term in $y_1(t)$ suggests that the amplitude of oscillation grows with t. However, as we will now show, the exact solution $y(t)$ remains bounded for all t.

Boundedness of the Solution to (11.1.4)

To show that the solution to (11.1.4) is bounded for all t, we construct an integral of the differential equation. Multiplying (11.1.4) by the integrating factor dy/dt converts each term in the differential equation to an exact derivative:

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}y^2 + \frac{1}{4}\varepsilon y^4\right] = 0.$$

Thus,

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}y^2 + \frac{1}{4}\varepsilon y^4 = C, \tag{11.1.8}$$

where C is a constant. Since $y(0) = 1$ and $y'(0) = 0$, $C = \frac{1}{2} + \frac{1}{4}\varepsilon$. When $\varepsilon > 0$, the integral in (11.1.8) shows that $\frac{1}{2}y^2 \le C$ for all t. Therefore, $|y(t)|$ is bounded for all t by $\sqrt{1 + \varepsilon/2}$.

The argument just given is frequently used in applied mathematics to prove boundedness of solutions to both ordinary and partial differential equations. The integral in (11.1.8) is called an *energy* integral. Equation (11.1.8) may be interpreted graphically as a closed bounded orbit in the phase plane whose axes are labeled by y and dy/dt (see Fig. 11.1).

Perturbative Construction of a Bounded Solution to (11.1.4)

We have arrived at an apparent paradox; we have shown that the exact solution $y(t)$ to (11.1.4) is bounded for all t but that the first-order perturbative solution in (11.1.7) is secular (grows with t for large t). The resolution of this paradox lies in

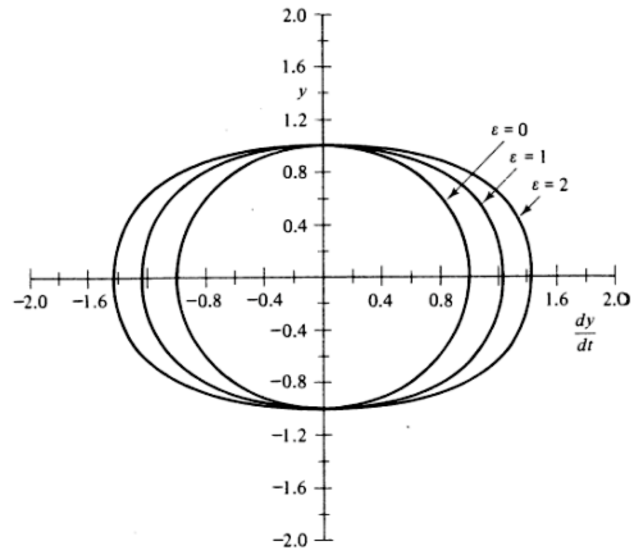

Figure 11.1 A phase-plane plot (y versus dy/dt) of solutions to Duffing's equation $d^2y/dt^2 + y + \varepsilon y^3 = 0$ [$y(0) = 1$, $y'(0) = 0$] for $\varepsilon = 0$, 1, and 2. The orbits shown are constant-energy curves [see (11.1.8)] which satisfy $(dy/dt)^2 + y^2 + \varepsilon y^4/2 = 1 + \varepsilon/2$.

the summation of the perturbation series (11.1.5). We know that the problem (11.1.4) is a regular perturbation problem as $\varepsilon \to 0+$ for fixed t (see Sec. 7.2). Therefore, the series (11.1.5) converges to the solution $y(t)$ for each t. We conclude that although order by order each term in the perturbation expansion may be secular, the secularity must disappear when the series is summed.

To illustrate how summing a perturbation series can eliminate secularity, consider the perturbation series

$$1 - \varepsilon t + \tfrac{1}{2}\varepsilon^2 t^2 - \tfrac{1}{6}\varepsilon^3 t^3 + \cdots + \varepsilon^n t^n[(-1)^n/n!] + \cdots, \qquad \varepsilon \to 0+.$$

Each term in this series is secular when t is of order $1/\varepsilon$ or larger. Nevertheless, the sum of the series $e^{-\varepsilon t}$ is bounded for all positive t!

We will now examine the more complicated perturbation series (11.1.5) and show that the sum of the most secular terms in each order in perturbation theory is actually not secular. We will show, using an inductive argument, that the most secular term in $y_n(t)$ has the form

$$A_n t^n e^{it} + A_n^* t^n e^{-it}, \tag{11.1.9}$$

where * denotes complex conjugation. There are less secular terms in $y_n(t)$ which grow like t^k ($k < n$), but we ignore such terms for now.

The final result of our calculations will be

$$A_n = \frac{1}{2}\frac{1}{n!}\left(\frac{3i}{8}\right)^n. \tag{11.1.10}$$

Using this formula for A_n we see that the sum of the most secular terms in the perturbation series (11.1.5) is a cosine function:

$$\sum_{n=0}^{\infty} \frac{1}{2}\varepsilon^n t^n \left[\frac{1}{n!}\left(\frac{3i}{8}\right)^n e^{it} + \frac{1}{n!}\left(-\frac{3i}{8}\right)^n e^{-it}\right] = \cos\left[t\left(1+\frac{3}{8}\varepsilon\right)\right]. \quad (11.1.11)$$

Observe that this expression is not secular; it remains bounded for all t.

The expression (11.1.11) is a much better approximation to the exact solution $y(t)$ than $y_0(t) = \cos t$ because it is a good approximation to $y(t)$ for $0 \le t = O(1/\varepsilon)$. The difference between $y(t)$ and $\cos t$ is small so long as $0 \le t \ll 1/\varepsilon$ ($\varepsilon \to 0+$), while $\cos[t(1+\frac{3}{8}\varepsilon)]$ is an accurate approximation to $y(t)$ over a much larger range of t. These assertions are explained as follows. In order that $y_0(t)$ be a good approximation to $y(t)$, it is necessary that $\varepsilon^n y_n(t) \ll y_0(t)$ ($\varepsilon \to 0+$) for all $n \ge 1$; this is true if $0 \le t \ll 1/\varepsilon$. On the other hand, the terms that we ignored in deriving (11.1.11) all have the form

$$\varepsilon[A\varepsilon^k(\varepsilon t)^l e^{imt} + A^*\varepsilon^k(\varepsilon t)^l e^{-imt}],$$

where k, l, m are nonnegative integers. Therefore, when $t = O(1/\varepsilon)$, each of these ignored terms is in fact negligible compared to at least one of the secular terms included in (11.1.11). We accept without proof the nontrivial result that the sum of all these small terms is still small. The higher-order terms are analyzed in Probs. 11.5 to 11.7.

We interpret the formula in (11.1.11) to mean that the cubic anharmonic term in (11.1.4) causes a shift in the frequency of the harmonic oscillator $y'' + y = 0$ from 1 to $1 + \frac{3}{8}\varepsilon$. This small frequency shift causes a phase shift which becomes noticeable when t is of order $1/\varepsilon$ (see Figs. 11.2 to 11.4 in Sec. 11.2).

Inductive Derivation of (11.1.10)

Comparing the first-order perturbation theory result in (11.1.7) with (11.1.9) verifies that the coefficient of the most secular terms in zeroth and first order are given correctly by (11.1.10). To establish (11.1.10) for all n, we proceed inductively. The $(n+2)$th equation in the sequence of equations (11.1.6) determines $y_{n+1}(t)$:

$$y''_{n+1} + y_{n+1} = -I_{n+1}, \quad (11.1.12)$$

where the inhomogeneity I_{n+1} is the coefficient of ε^n in the expansion of $[\sum_{j=0}^{\infty} \varepsilon^j y_j(t)]^3$. Thus,

$$I_{n+1} = \sum_{j+k+l=n} y_j y_k y_l. \quad (11.1.13)$$

The most secular term in $y_{n+1}(t)$ is generated by the most secular terms in $y_j(t)$ for $0 \le j \le n$ (see Prob. 11.2). If we assume that (11.1.10) is valid for $A_0, A_1, A_2, \ldots, A_n$, then the coefficient of $t^n e^{it}$ in I_{n+1} is given by

$$\frac{1}{8}\left(\frac{3}{8}\right)^n \sum_{j+k+l=n} \frac{i^{j+k-l} + i^{j+l-k} + i^{k+l-j}}{j!\,k!\,l!} = \frac{1}{8}\left(\frac{3i}{8}\right)^n \sum_{j+k+l=n} \frac{(-1)^l + (-1)^k + (-1)^j}{j!\,k!\,l!}.$$

The sum in the above expression is just three times the coefficient of x^n in the Taylor expansion of $e^x e^x e^{-x}$ (see Prob. 11.3); therefore, it has the value $3/n!$. Thus, the terms in I_{n+1} which generate the most secular terms in $y_{n+1}(t)$ are

$$\tfrac{3}{8}(\tfrac{3}{8}t)^n[i^n e^{it} + (-i)^n e^{-it}]/n!.$$

Substituting these terms into the right side of (11.1.12) and solving for $y_{n+1}(t)$ gives

$$y_{n+1}(t) = (\tfrac{3}{8}t)^{n+1}[i^{n+1}e^{it} + (-i)^{n+1}e^{-it}]/(n+1)! + \text{less secular terms.}$$

By induction, we conclude that since (11.1.10) is true for $n = 0$, it remains true for all n.

(E) 11.2 MULTIPLE-SCALE ANALYSIS

In Sec. 11.1 we showed how to eliminate the most secular contributions to perturbation theory by simply summing them to all orders in powers of ε. The method we used works well but requires a lengthy calculation which can be avoided by using the methods of multiple-scale analysis that are introduced in this section.

Once again, we consider the nonlinear oscillator problem in (11.1.4):

$$\frac{d^2y}{dt^2} + y + \varepsilon y^3 = 0, \qquad y(0) = 1,\ y'(0) = 0. \quad (11.2.1)$$

The principal result of the last section is that when t is of order $1/\varepsilon$, perturbation theory in powers of ε is invalid. Secular terms appear in all orders (except zeroth order) and violate the boundedness of the solution $y(t)$.

A shortcut for eliminating the most secular terms to all orders begins by introducing a new variable $\tau = \varepsilon t$. τ defines a long time scale because τ is not negligible when t is of order $1/\varepsilon$ or larger. Even though the exact solution $y(t)$ is a function of t alone, multiple-scale analysis seeks solutions which are functions of both variables t and τ treated as *independent* variables. We emphasize that expressing y as a function of two variables is an artifice to remove secular effects; the actual solution has t and τ related by $\tau = \varepsilon t$ so that t and τ are ultimately not independent.

The formal procedure consists of assuming a perturbation expansion of the form

$$y(t) = Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \cdots. \quad (11.2.2)$$

We use the chain rule for partial differentiation to compute derivatives of $y(t)$:

$$\frac{dy}{dt} = \left(\frac{\partial Y_0}{\partial t} + \frac{\partial Y_0}{\partial \tau}\frac{d\tau}{dt}\right) + \varepsilon\left(\frac{\partial Y_1}{\partial t} + \frac{\partial Y_1}{\partial \tau}\frac{d\tau}{dt}\right) + \cdots.$$

However, since $\tau = \varepsilon t$, $d\tau/dt = \varepsilon$. Thus,

$$\frac{dy}{dt} = \frac{\partial Y_0}{\partial t} + \varepsilon\left(\frac{\partial Y_0}{\partial \tau} + \frac{\partial Y_1}{\partial t}\right) + O(\varepsilon^2). \quad (11.2.3)$$

Also, differentiating with respect to t again gives

$$\frac{d^2y}{dt^2} = \frac{\partial^2 Y_0}{\partial t^2} + \varepsilon\left(2\frac{\partial^2 Y_0}{\partial\tau\,\partial t} + \frac{\partial^2 Y_1}{\partial t^2}\right) + \mathrm{O}(\varepsilon^2). \tag{11.2.4}$$

Substituting (11.2.4) into (11.2.1) and collecting powers of ε gives

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{11.2.5}$$

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -Y_0^3 - 2\frac{\partial^2 Y_0}{\partial\tau\,\partial t}. \tag{11.2.6}$$

The most general real solution to (11.2.5) is

$$Y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}, \tag{11.2.7}$$

where $A(\tau)$ is an arbitrary complex function of τ.

$A(\tau)$ will be determined by the condition that secular terms do *not* appear in the solution to (11.2.6). From (11.2.7), the right side of (11.2.6) is

$$e^{it}\left[-3A^2A^* - 2i\frac{dA}{d\tau}\right] + e^{-it}\left[-3A(A^*)^2 + 2i\frac{dA^*}{d\tau}\right] - e^{3it}A^3 - e^{-3it}(A^*)^3.$$

Note that e^{it} and e^{-it} are solutions of the homogeneous equation $\partial^2 Y_1/\partial t^2 + Y_1 = 0$. Therefore, if the coefficients of e^{it} and e^{-it} on the right side of (11.2.6) are nonzero, then the solution $Y_1(t, \tau)$ will be secular in t. To preclude the appearance of secularity, we require that the as yet arbitrary function $A(\tau)$ satisfy

$$-3A^2A^* - 2i\frac{dA}{d\tau} = 0, \tag{11.2.8}$$

$$-3A(A^*)^2 + 2i\frac{dA^*}{d\tau} = 0. \tag{11.2.9}$$

These two complex equations do not overdetermine $A(\tau)$ because they are redundant; one is the complex conjugate of the other. If (11.2.8) and (11.2.9) are satisfied, no secularity appears in (11.2.2), at least through terms of order ε.

To solve (11.2.8) for $A(\tau)$, we represent $A(\tau)$ in polar coordinate form:

$$A(\tau) = R(\tau)e^{i\theta(\tau)}, \tag{11.2.10}$$

where R and θ are real. Substituting into (11.2.8) and equating real and imaginary parts gives

$$\frac{dR}{d\tau} = 0, \tag{11.2.11a}$$

$$\frac{d\theta}{d\tau} = \frac{3}{2}R^2. \tag{11.2.11b}$$

Therefore,

$$A(\tau) = R(0)e^{i\theta(0) + 3iR^2(0)\tau/2} \tag{11.2.12}$$

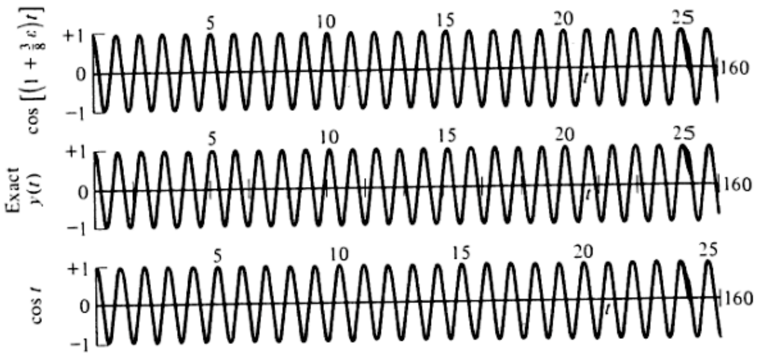

Figure 11.2 The exact solution $y(t)$ to Duffing's equation $d^2y/dt^2 + y + \varepsilon y^3 = 0$ [$y(0) = 1$, $y'(0) = 0$] for $\varepsilon = 0.1$ (middle graph) compared with perturbative approximations to $y(t)$ (upper and lower graphs). The lower graph is a plot of cos t, the first term in the regular perturbation series for $y(t)$, and the upper graph is a plot of cos $[(1 + 3\varepsilon/8)t]$, the leading-order approximation to $y(t)$ obtained from multiple-scale methods. Both approximations, cos t and cos $[(1 + 3\varepsilon/8)t]$, are correct up to additive terms of order ε, but cos t is not valid for large values of t; when $t = 160$, cos t is a full cycle out of phase with $y(t)$. The multiple-scale approximation closely approximates $y(t)$, even for large values of t.

and the zeroth-order solution (11.2.7) is

$$Y_0(t, \tau) = 2R(0)\cos\left[\theta(0) + \tfrac{3}{2}R^2(0)\tau + t\right]. \tag{11.2.13}$$

The initial conditions $y(0) = 1$, $y'(0) = 0$ determine $R(0)$ and $\theta(0)$. The condition $y(0) = 1$ becomes $Y_0(0, 0) = 1$, $Y_1(0, 0) = 0, \ldots$. From (11.2.3), $y'(0) = 0$ becomes $(\partial Y_0/\partial t)(0, 0) = 0$, $(\partial Y_1/\partial t)(0, 0) = -(\partial Y_0/\partial\tau)(0, 0), \ldots$. In order to satisfy these conditions, we must choose $R(0) = \frac{1}{2}$ and $\theta(0) = 0$. Therefore, the zeroth-order solution is $Y_0(t, \tau) = \cos\left[t + \frac{3}{8}\tau\right]$. Finally, since $\tau = \varepsilon t$,

$$y(t) = \cos\left[t(1 + \tfrac{3}{8}\varepsilon)\right] + \mathrm{O}(\varepsilon), \qquad \varepsilon \to 0+, \ \varepsilon t = \mathrm{O}(1), \tag{11.2.14}$$

and we have reproduced (11.1.11). In Figs. 11.2 to 11.4 we compare the exact solution to (11.2.1) with the approximation in (11.2.14).

A higher-order treatment of (11.2.1) is not completely straightforward. When more than two time scales are employed, there is so much freedom in the perturbation series representation that ambiguities can result (see Probs. 11.5 to 11.7).

(I) 11.3 EXAMPLES OF MULTIPLE-SCALE ANALYSIS

In this section we illustrate the formal multiple-scale technique that was developed in Sec. 11.2 by showing how to solve four elementary examples. The third and fourth of these examples are especially interesting because they show

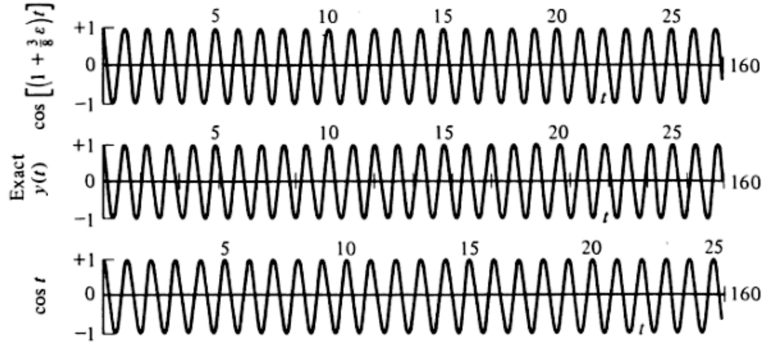

Figure 11.3 Same as in Fig. 11.2 but with $\varepsilon = 0.2$. Note that $\cos t$ is two cycles out of phase with $y(t)$ when $t = 160$.

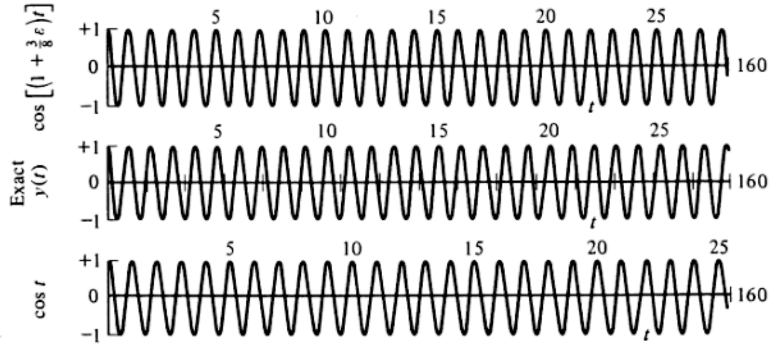

Figure 11.4 Same as in Fig. 11.2 but with $\varepsilon = 0.3$. Note that $\cos t$ is three cycles out of phase with $y(t)$ when $t = 160$.

how multiple-scale analysis can reproduce the results of boundary-layer and WKB analysis.

Example 1 *Multiple-scale analysis of a damped oscillator.* Let us consider an harmonic oscillator with a cubic damping term:

$$y'' + y + \varepsilon(y')^3 = 0, \qquad y(0) = 1,\ y'(0) = 0. \tag{11.3.1}$$

If $\varepsilon > 0$, the solution $y(t)$ must decay to 0 as $t \to \infty$. To prove this assertion, we multiply (11.3.1) by y' and construct an energy integral similar to that in (11.1.8):

$$\frac{d}{dt}\left[\frac{1}{2}(y')^2 + \frac{1}{2}y^2\right] = -\varepsilon(y')^4 \le 0. \tag{11.3.2}$$

This result shows that the energy $\frac{1}{2}(y')^2 + \frac{1}{2}y^2$ is a decreasing function of t unless $y'(t) = 0$ for all t. In Prob. 11.8 it is shown that the energy must decay to 0 as $t \to \infty$ and therefore that $y(t) \to 0$ as $t \to \infty$. [By contrast, when $\varepsilon < 0$, the energy argument just given shows that (11.3.1) represents a negatively damped system (like a self-propelled lawnmower that uses grass for fuel or a rocket with vacuum-cleaner drive that uses space dust for fuel) whose solutions grow explosively with t.]

Multiple-scale analysis may be used to study the behavior of $y(t)$ for large t. We begin by assuming a perturbation expansion for $y(t)$ in (11.3.1) of the form

$$y(t) \sim Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \cdots, \qquad \varepsilon \to 0+,$$

where $\tau = \varepsilon t$. Using (11.2.3) and (11.2.4) and equating coefficients of ε^0 and ε^1 gives two equations which correspond with (11.2.5) and (11.2.6):

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{11.3.3}$$

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t\,\partial \tau} - \left(\frac{\partial Y_0}{\partial t}\right)^3. \tag{11.3.4}$$

The most general real solution to (11.3.3) is

$$Y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}. \tag{11.3.5}$$

Substituting this solution into the right side of (11.3.4) gives

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -e^{it}\left[2i\frac{dA}{d\tau} + 3iA^2A^*\right] - e^{-it}\left[-2i\frac{dA^*}{d\tau} - 3i(A^*)^2A\right]$$
$$+ ie^{3it}A^3 - ie^{-3it}(A^*)^3. \tag{11.3.6}$$

Since the solutions to the homogeneous equation (11.3.3) are $e^{\pm it}$, the solution to (11.3.6) is secular unless the expressions in the square brackets vanish; in order that Y_1 not be secular, we require that $A(\tau)$ satisfy the equations

$$2i\frac{dA}{d\tau} + 3iA^2A^* = 0, \tag{11.3.7a}$$

$$-2i\frac{dA^*}{d\tau} - 3i(A^*)^2A = 0. \tag{11.3.7b}$$

To solve (11.3.7) we set $A(\tau) = R(\tau)e^{i\theta(\tau)}$, where $R(\tau)$ and $\theta(\tau)$ are real. Substituting this expression into (11.3.7) gives equations for $R(\tau)$ and $\theta(\tau)$:

$$\frac{dR}{d\tau} = -\tfrac{3}{2}R^3, \qquad \frac{d\theta}{d\tau} = 0.$$

Therefore,

$$R(\tau) = \frac{R(0)}{\sqrt{3\tau R^2(0) + 1}}, \tag{11.3.8a}$$

$$\theta(\tau) = \theta(0). \tag{11.3.8b}$$

$R(0)$ and $\theta(0)$ are determined by the initial conditions $y(0) = 1$, $y'(0) = 0$. These conditions imply that $Y_0(0, 0) = 1$, $(\partial Y_0/\partial t)(0, 0) = 0$, whence $R(0) = \frac{1}{2}$, $\theta(0) = 0$. Thus, to leading order in ε,

$$y(t) \sim \frac{\cos t}{\sqrt{1 + 3\varepsilon t/4}}, \qquad \varepsilon \to 0+,\ \varepsilon t = O(1). \tag{11.3.9}$$

This result implies that when $\varepsilon > 0$ the solution decays like $t^{-1/2}$ for large t, and that when $\varepsilon < 0$ the solution becomes infinite at a finite value of t approximately equal to $-4/3\varepsilon$. Moreover, this solution does not exhibit any phase shift (or frequency shift) to leading order in ε. These qualitative conclusions are verified numerically in Figs. 11.5 to 11.7.

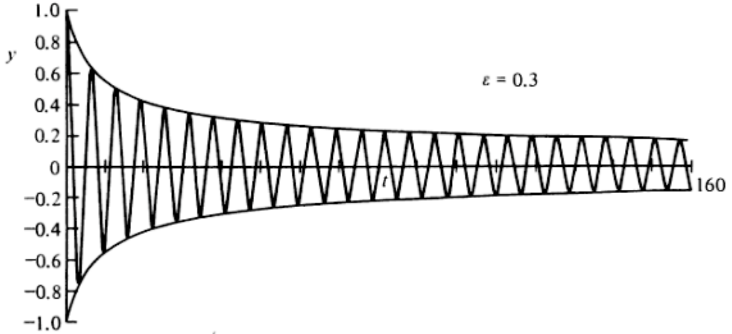

Figure 11.5 A plot of the exact solution to $y'' + y + \varepsilon(y')^3 = 0$ [$y(0) = 1$, $y'(0) = 0$] for $\varepsilon = 0.3$ [see (11.3.1)] together with a plot of the envelope $(1 + 3\varepsilon t/4)^{-1/2}$ of the leading-order multiple-scale approximation to $y(t)$ in (11.3.9). We have not plotted the full multiple-scale approximation to $y(t)$ because it is indistinguishable from the exact solution to within the thickness of the curve.

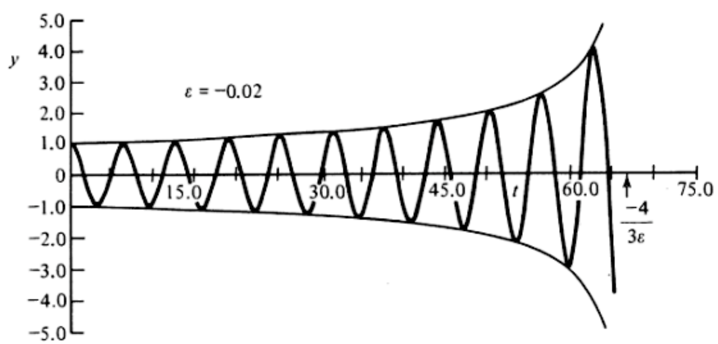

Figure 11.6 Same as in Fig. 11.5 except that $\varepsilon = -0.02$. Observe that the exact solution $y(t)$ and the multiple-scale approximation to it differ noticeably only when t is near the explosive singularity at $t = -4/3\varepsilon = 66\frac{2}{3}$.

Example 2 *Approach to a limit cycle.* The equation

$$y'' + y = \varepsilon[y' - \tfrac{1}{3}(y')^3], \qquad y(0) = 0,\ y'(0) = 2a, \tag{11.3.10}$$

known as the Rayleigh oscillator, is interesting because the solution approaches a limit cycle in the phase plane (see Sec. 4.4 and Example 3 of Sec. 9.7). Multiple-scale analysis determines the shape of this limit cycle and the rate of approach of $y(t)$ to the limit cycle.

As in Example 1, we assume a perturbation expansion for $y(t)$ in (11.3.10) of the form $y(t) \sim Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \cdots$ $(\varepsilon \to 0+)$, where $\tau = \varepsilon t$. Next we substitute (11.2.3) and (11.2.4)

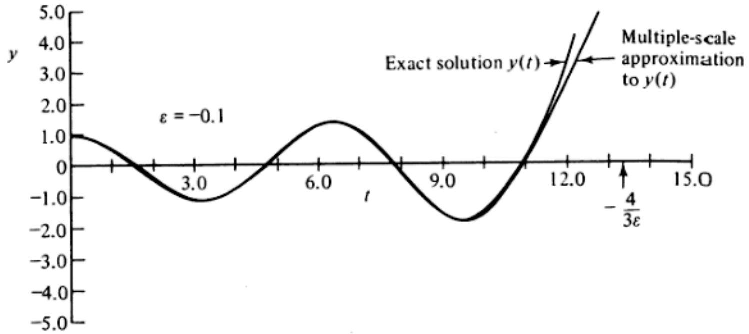

Figure 11.7 A comparison of the multiple-scale approximation and the exact solution to $y'' + y + \varepsilon(y')^3 = 0$ [$y(0) = 1$, $y'(0) = 0$] for $\varepsilon = -0.1$. The approximation to $y(t)$ is extremely accurate except near the singularity at $t = -4/3\varepsilon = 13\frac{1}{3}$.

into (11.3.10) and equate coefficients of ε^0 and ε^1:

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{11.3.11}$$

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t\, \partial \tau} + \frac{\partial Y_0}{\partial t} - \frac{1}{3}\left(\frac{\partial Y_0}{\partial t}\right)^3. \tag{11.3.12}$$

The solution to (11.3.11) is again

$$Y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}.$$

We substitute this expression into (11.3.12) and observe that secular terms in $Y_1(t, \tau)$ will arise unless the coefficients of $e^{\pm it}$ on the right side of (11.3.12) vanish. Thus, the conditions for the absence of secular behavior are

$$-2i\frac{dA}{d\tau} + iA - iA^2A^* = 0, \tag{11.3.13a}$$

$$2i\frac{dA^*}{d\tau} - iA^* + i(A^*)^2A = 0. \tag{11.3.13b}$$

To solve (11.3.13) we again set $A(\tau) = R(\tau)e^{i\theta(\tau)}$, where R and θ are real. The equations for R and θ are

$$2\frac{dR}{d\tau} = R - R^3, \tag{11.3.14a}$$

$$\frac{d\theta}{d\tau} = 0. \tag{11.3.14b}$$

The solutions are

$$R(\tau) = R(0)[e^{-\tau} + R^2(0)(1 - e^{-\tau})]^{-1/2}, \tag{11.3.15a}$$

$$\theta(\tau) = \theta(0). \tag{11.3.15b}$$

The initial conditions $y(0) = 0$, $y'(0) = 2a$ require that $R(0) = a$, $\theta(0) = -\frac{1}{2}\pi$. Thus, to leading order in ε, the solution to (11.3.10) is

$$y(t) \sim \frac{2a \sin t}{\sqrt{e^{-\tau} + a^2(1 - e^{-\tau})}}, \qquad \varepsilon \to 0+, \ \tau = \varepsilon t = O(1). \tag{11.3.16}$$

Observe that for all values of a, this approximate solution smoothly approaches the limit cycle $y(t) = 2 \sin t$ as $t \to \infty$. This limit cycle is represented as a circle of radius 2 in the phase plane of y and y'. If $a < 1$, the solution spirals outward to the limit cycle, and if $a > 1$, the solution spirals inward. A comparison of these asymptotic results and the numerical solution to (11.3.10) is given in Figs. 11.8 to 11.10.

Example 3 *Recovery of the WKB physical-optics approximation.* Let us consider the oscillator

$$y''(t) + \omega^2(\varepsilon t)y(t) = 0. \tag{11.3.17}$$

Note that the frequency $\omega(\varepsilon t)$ is a slowly varying function of time t.

It is easy to solve (11.3.17) using the WKB approximation. We simply introduce the new variable $\tau = \varepsilon t$ to convert (11.3.17) to standard WKB form:

$$\varepsilon^2 \frac{d^2y}{d\tau^2} + \omega^2(\tau)y = 0. \tag{11.3.18}$$

The physical-optics approximation to (11.3.18) [see (10.1.13)] is then

$$y(t) = [\omega(\tau)]^{-1/2} \exp\left[\pm i\varepsilon^{-1} \int^{\tau} \omega(s)\, ds\right]. \tag{11.3.19}$$

Now, let us rederive (11.3.19) using multiple-scale theory. The procedure requires a bit of subtlety. Suppose we naively assume that there is a linear relation $\tau = \varepsilon t$ between the appropriate long and short time scales. Then, letting $y(t) = Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \cdots$, we obtain

$$\frac{\partial^2 Y_0}{\partial t^2} + \omega^2(\tau)Y_0 = 0, \tag{11.3.20}$$

$$\frac{\partial^2 Y_1}{\partial t^2} + \omega^2(\tau)Y_1 = -2\frac{\partial^2 Y_0}{\partial t\, \partial \tau}. \tag{11.3.21}$$

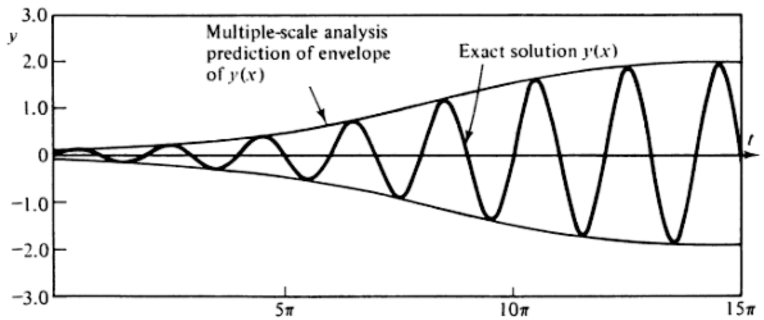

Figure 11.8 Approach to the limit cycle of the Rayleigh oscillator $y'' + y = \varepsilon[y' - \frac{1}{3}(y')^3]$ [$y(0) = 0$, $y'(0) = 2a$] [see (11.3.10)], where we have taken $\varepsilon = 0.2$ and $a = 0.05$. The oscillatory curve is the numerical solution to the differential equation; the envelope is the prediction of multiple-scale analysis [see (11.3.16)]. The two curves agree to better than their thicknesses.

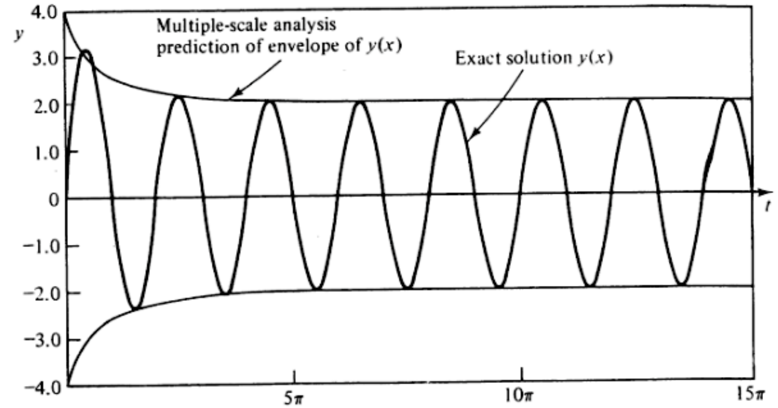

Figure 11.9 Approach to the limit cycle of the Rayleigh oscillator (11.3.10) (see Fig. 11.8). Here, $\varepsilon = 0.2$ and $a = 2.0$. Except for a small discrepancy at $t = \pi/2$ the exact and approximate solutions have nearly perfect agreement.

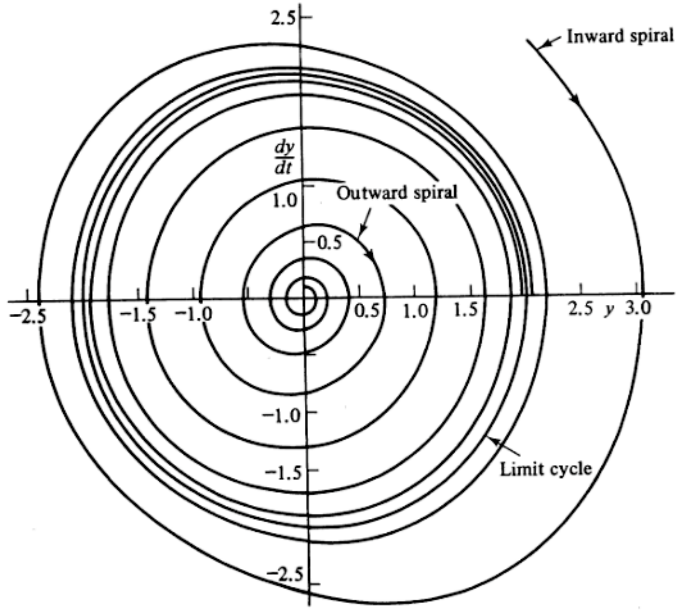

Figure 11.10 A phase-plane plot (y versus dy/dt) of three solutions to the Rayleigh oscillator (11.3.10) with $\varepsilon = 0.2$. Shown are the limit cycle solution which is approximately a circle of radius 2, the solution on Fig. 11.8 (spiraling outward toward the limit cycle), and the solution on Fig. 11.9 (spiraling inward toward the limit cycle).

The solution to (11.3.20) is $y_0 = A(\tau)e^{i\omega(\tau)t} + A^*(\tau)e^{-i\omega(\tau)t}$. Substituting this expression in the right side of (11.3.21) gives

$$\frac{\partial^2 Y_1}{\partial t^2} + \omega^2(\tau)Y_1 = -2ie^{i\omega(\tau)t}\left[\frac{d}{d\tau}(A\omega) + itA\omega\frac{d\omega}{d\tau}\right]$$

$$+ 2ie^{-i\omega(\tau)t}\left[\frac{d}{d\tau}(A^*\omega) - itA^*\omega\frac{d\omega}{d\tau}\right]. \qquad (11.3.22)$$

The presence of the variable t in the square brackets implies that we cannot eliminate secularity without setting $A(\tau) \equiv 0$ (see Prob. 11.9).

This failure illustrates a crucial feature of multiple-scale perturbation methods. If the long-scale variable τ is linearly proportional to the short scale t ($\tau = \varepsilon t$), then multiple-scale methods will fail unless the frequency of the unperturbed oscillator is a constant; it must not vary even on the τ scale. Therefore, before we can apply multiple-scale methods to the oscillator (11.3.17), we must find a transformation which converts (11.3.17) to a fixed-frequency oscillator with a small perturbation term:

$$y'' + y + \varepsilon(\text{some function of } y) = 0. \qquad (11.3.23)$$

With this in mind, we introduce a new time variable T:

$$T = f(t). \qquad (11.3.24)$$

We will try to choose $f(t)$ to convert (11.3.17) to the form in (11.3.23). From (11.3.24) we have $d/dt = f'(t)\, d/dT$, $d^2/dt^2 = f''(t)\, d/dT + [f'(t)]^2\, d^2/dT^2$. Thus, (11.3.17) becomes

$$\frac{d^2}{dT^2}y + \frac{f''(t)}{[f'(t)]^2}\frac{d}{dT}y + \frac{\omega^2(\varepsilon t)}{[f'(t)]^2}y = 0.$$

We achieve the form in (11.3.23) if we choose $f'(t) = \omega(\varepsilon t)$. Thus,

$$T = f(t) = \int^t \omega(\varepsilon x)\, dx = \frac{1}{\varepsilon}\int^\tau \omega(s)\, ds. \qquad (11.3.25)$$

In terms of T the differential equation now reads

$$\frac{d^2y}{dT^2} + y + \varepsilon\frac{\omega'(\tau)}{\omega^2(\tau)}\frac{d}{dT}y = 0. \qquad (11.3.26)$$

This equation may be solved using multiple-scale methods. We expand

$$y = Y_0(T, \tau) + \varepsilon Y_1(T, \tau) + \cdots. \qquad (11.3.27)$$

Using the relation $d\tau/dT = \varepsilon\, dt/dT = \varepsilon/f'(t) = \varepsilon/\omega(\tau)$, we substitute (11.3.27) into (11.3.26) and obtain, as usual, a sequence of partial differential equations:

$$\frac{\partial^2 Y_0}{\partial T^2} + Y_0 = 0, \qquad (11.3.28)$$

$$\frac{\partial^2 Y_1}{\partial T^2} + Y_1 = -\frac{\omega'(\tau)}{\omega^2(\tau)}\frac{\partial Y_0}{\partial T} - \frac{2}{\omega}\frac{\partial^2 Y_0}{\partial T\, \partial\tau}. \qquad (11.3.29)$$

Substituting the solution

$$Y_0 = A(\tau)e^{iT} + A^*(\tau)e^{-iT} \qquad (11.3.30)$$

of (11.3.28) into the right side of (11.3.29) gives

$$\frac{\partial^2 Y_1}{\partial T^2} + Y_1 = -ie^{iT}\left[\frac{2}{\omega}\frac{dA}{d\tau} + \frac{\omega'(\tau)}{\omega^2(\tau)}A\right] + ie^{-iT}\left[\frac{2}{\omega}\frac{dA^*}{d\tau} + \frac{\omega'(\tau)}{\omega^2(\tau)}A^*\right].$$

To eliminate secularity we must require that the expressions in the square brackets vanish for all τ:

$$2\frac{dA}{d\tau} = -\frac{\omega'(\tau)}{\omega(\tau)}A,$$

$$2\frac{dA^*}{d\tau} = -\frac{\omega'(\tau)}{\omega(\tau)}A^*.$$

The solution for $A(\tau)$, apart from a multiplicative constant, is $1/\sqrt{\omega(\tau)}$. Inserting this solution into (11.3.30) gives

$$Y_0 = \frac{1}{\sqrt{\omega(\tau)}}e^{\pm iT},$$

and using the expression for T in (11.3.25) gives

$$Y_0 = \frac{1}{\sqrt{\omega(\tau)}}\exp\left[\pm\frac{i}{\varepsilon}\int^\tau \omega(s)\, ds\right].$$

We have reproduced the WKB result in (11.3.19).

Example 4 *Solution of a boundary-layer problem by multiple-scale perturbation theory.* Consider the elementary boundary-layer problem

$$\varepsilon y'' + ay' + by = 0, \qquad y(0) = A,\ y(1) = B,\ a > 0, \qquad (11.3.31)$$

where a and b are constants. We know (see Fig. 9.4) that the solution to this problem has a boundary layer of thickness ε at $x = 0$ and is slowly varying in the range $\varepsilon \ll x \le 1$ ($\varepsilon \to 0+$). Thus, there are two natural scales for this problem, a short scale t which describes the inner solution in the boundary layer and a long scale $x = \varepsilon t$ which describes the outer solution. Note that (11.3.31) is written in terms of the long scale. If we wish to use multiple-scale theory we must rewrite (11.3.31) in terms of the short scale t in order to eliminate secularity on the long scale:

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + \varepsilon by = 0. \qquad (11.3.32)$$

Assuming that $y(t)$ in (11.3.32) has a perturbation expansion of the form $y(t) = Y_0(t, x) + \varepsilon Y_1(t, x) + \cdots$, we obtain the following sequence of equations:

$$\varepsilon^0\colon \frac{\partial^2 Y_0}{\partial t^2} + a\frac{\partial Y_0}{\partial t} = 0; \qquad (11.3.33)$$

$$\varepsilon^1\colon \frac{\partial^2 Y_1}{\partial t^2} + a\frac{\partial Y_1}{\partial t} = -2\frac{\partial^2 Y_0}{\partial t\, \partial x} - a\frac{\partial Y_0}{\partial x} - bY_0. \qquad (11.3.34)$$

The solution to (11.3.33) has the form

$$Y_0(t, x) = A_1(x) + A_2(x)e^{-at}. \qquad (11.3.35)$$

Substituting (11.3.35) into (11.3.34) gives

$$\frac{\partial^2 Y_1}{\partial t^2} + a\frac{\partial Y_1}{\partial t} = -[aA_1'(x) + bA_1(x)] + [aA_2'(x) - bA_2(x)]e^{-at}.$$

The right side of this equation is a solution to the homogeneous equation in (11.3.33) and therefore gives rise to secular terms. To eliminate the secular term that grows like t (we know from our study of boundary-layer theory that no such term is present in leading order), we set

$$aA_1'(x) + bA_1(x) = 0.$$

Thus,
$$A_1(x) = C_1 e^{-bx/a},$$
where C_1 is a constant.

Note that if $aA_2'(x) - bA_2(x) \neq 0$, then there will be a secular term of the form te^{-t} which does not occur in leading-order boundary-layer theory. It is not necessary to eliminate this secular term because it decays exponentially with increasing t.

To leading order in ε we now have
$$y(t) = C_1 e^{-bx/a} + A_2(x)e^{-at} + O(\varepsilon), \qquad \varepsilon \to 0+. \tag{11.3.36}$$
Recall that $t = x/\varepsilon$. Therefore, for all $x > 0$, it is valid to replace $A_2(x)e^{-ax/\varepsilon}$ by $A_2(0)e^{-ax/\varepsilon} + O(\varepsilon)$ $(\varepsilon \to 0+)$. Setting $A_2(0) = C_2$, (11.3.36) becomes $y(t) = C_1 e^{-bx/a} + C_2 e^{-ax/\varepsilon} + O(\varepsilon)$ $(\varepsilon \to 0+)$.

Finally, we impose the boundary conditions at $x = 0$ and $x = 1$ and obtain
$$y(t) = Be^{b/a}e^{-bx/a} + (A - Be^{b/a})e^{-ax/\varepsilon} + O(\varepsilon), \qquad \varepsilon \to 0+,$$
which agrees with the uniform leading-order boundary-layer solution in (9.1.13).

If a and b in (11.3.31) vary with x, it is necessary to perform a transformation of variable like that in Example 3 before one can use multiple-scale perturbation theory (see Prob. 11.12).

(I) 11.4 THE MATHIEU EQUATION AND STABILITY

The Mathieu equation
$$\frac{d^2y}{dt^2} + (a + 2\varepsilon \cos t)y = 0, \tag{11.4.1}$$
in which a and ε are parameters, is an example of a differential equation whose coefficients are periodic. The general theory of linear periodic differential equations, which is known as Floquet theory, predicts that there may be solutions to (11.4.1) for some values of a and ε which are unstable (grow exponentially with increasing t). As a particularly nice application of multiple-scale perturbation theory (which is valid when ε is small) we find the boundaries between the regions in the (a, ε) plane for which all solutions to the Mathieu equation are stable (remain bounded for all t) and the regions in which there are unstable solutions.

Elementary Floquet Theory

We consider here just the case of second-order linear ordinary differential equations having 2π-periodic coefficient functions. We will make use of two facts. First, since the coefficients are 2π-periodic, we know that if $y(t)$ is any solution of such an equation, so is $y(t + 2\pi)$. Second, since the equation is linear and second order, any solution $y(t)$ may be represented as a linear combination of two linearly independent solutions $y_1(t)$ and $y_2(t)$:
$$y(t) = Ay_1(t) + By_2(t). \tag{11.4.2}$$

Since the coefficients of the differential equation are 2π-periodic, $y_1(t + 2\pi)$ and $y_2(t + 2\pi)$ are also solutions, so they may be represented as linear combinations of $y_1(t)$ and $y_2(t)$:
$$y_1(t + 2\pi) = \alpha y_1(t) + \beta y_2(t), \qquad y_2(t + 2\pi) = \gamma y_1(t) + \delta y_2(t).$$
Thus, for $y(t)$ in (11.4.2) we have
$$\begin{aligned} y(t + 2\pi) &= (A\alpha + B\gamma)y_1(t) + (A\beta + B\delta)y_2(t) \\ &= A'y_1(t) + B'y_2(t). \end{aligned} \tag{11.4.3}$$
The relation between the coefficients A and B and A' and B' in (11.4.3) involves matrix multiplication:
$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{11.4.4}$$

Now let us choose (A, B) to be an eigenvector of the 2×2 matrix in (11.4.4). If the corresponding eigenvalue is λ, then $A' = \lambda A$ and $B' = \lambda B$ and
$$y(t + 2\pi) = \lambda y(t). \tag{11.4.5}$$
Thus, if we introduce $\mu = (\ln \lambda)/2\pi$ so that $\lambda = e^{2\pi\mu}$, then we see that for all t, $y(t)$ takes the form
$$y(t) = e^{\mu t}\phi(t), \tag{11.4.6}$$
where $\phi(t)$ is a 2π-periodic function: $\phi(t + 2\pi) = \phi(t)$. We say that $y(t)$ in (11.4.6) is an *unstable* solution if $\text{Re}\,\mu > 0$ because $y(t)$ grows exponentially with t. We say that $y(t)$ is a *stable* solution if $\text{Re}\,\mu \le 0$.

Stability Boundaries of the Mathieu Equation

The Mathieu equation (11.4.1) is special because it is even under the reflection $t \to -t$. Thus, if $y(t)$ is a solution of the Mathieu equation, so is $y(-t)$. Therefore, for *both* solutions $e^{\mu t}\phi(t)$ and $e^{-\mu t}\phi(-t)$ of the Mathieu equation to be stable, we must have $\text{Re}\,\mu = 0$.

There are well-defined regions of the (a, ε) plane for which all solutions of the Mathieu equation are stable. In Fig. 11.11 we indicate those regions (white) where all solutions are stable and those regions (cross hatched) for which there is an unstable solution. The boundaries between regions of stability and instability are called *stability boundaries*. Our main objective in this section is to find approximate expressions for the stability boundaries which are valid as $\varepsilon \to 0$.

CASE I *Perturbative investigation of stable solutions.* Here we assume that a is positive and that $a \neq n^2/4$ $(n = 0, 1, 2, 3, \ldots)$. We will show that $y(t)$ is stable for sufficiently small ε. We assume a regular perturbation expansion for $y(t)$:
$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots. \tag{11.4.7}$$

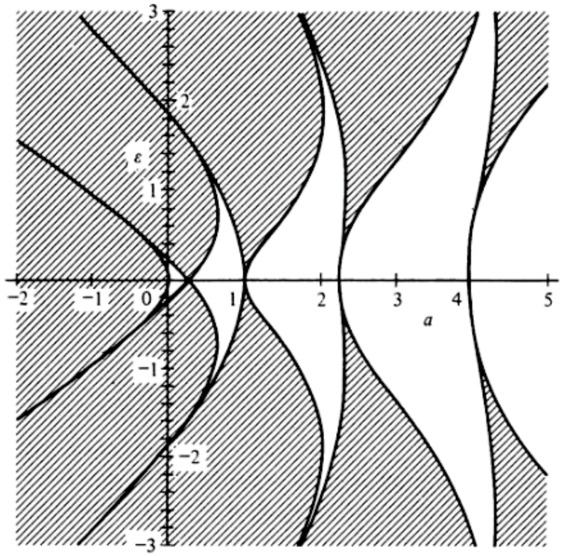

Figure 11.11 A plot of the stability boundaries of solutions to the Mathieu equation (11.4.1). In the white regions of the (a, ε) plane all solutions of the Mathieu equation are stable, while in the cross-hatched regions there is an unstable solution. When $\varepsilon = 0$, the cross-hatched regions meet the a axis at $a = n^2/4$ $(n = 0, 1, 2, \ldots)$.

Substituting $y(t)$ in (11.4.7) into the Mathieu equation (11.4.1) and comparing powers of ε gives a sequence of equations:

$$\varepsilon^0: \frac{d^2 y_0}{dt^2} + a y_0 = 0, \tag{11.4.8}$$

$$\varepsilon^1: \frac{d^2 y_1}{dt^2} + a y_1 = -2y_0 \cos t, \tag{11.4.9}$$

$$\varepsilon^2: \frac{d^2 y_2}{dt^2} + a y_2 = -2y_1 \cos t, \tag{11.4.10}$$

and so on.

The solution to (11.4.8) is secular (grows linearly with time) only if $a = 0$. Since we have assumed that $a > 0$, we have

$$y_0(t) = A \exp(i\sqrt{a}\,t) + \text{c.c.},$$

where c.c. stands for the complex conjugate of the exhibited terms. Substituting this result into (11.4.9), we have

$$\frac{d^2 y_1}{dt^2} + a y_1(t) = -A_0 \exp[i(\sqrt{a} + 1)t] - A_0 \exp[i(\sqrt{a} - 1)t] + \text{c.c.}$$

Now, secular terms appear only if $\sqrt{a} \pm 1 = \pm\sqrt{a}$. But this can only happen if $a = \frac{1}{4}$.

In subsequent orders of perturbation theory the solution will be secular only if $a = 1, 9/4, 4, \ldots$ (see Prob. 11.18). But by assumption $a \neq n^2/4$. Therefore, there is no secularity. After solving to all orders in perturbation theory, we will have

$$y(t) = \exp(i\sqrt{a}\,t)\phi(t) + \exp(-i\sqrt{a}\,t)\phi^*(t),$$

where $\phi(t) = \sum \varepsilon^n A_n e^{int}$, which is a *periodic* series in t. We conclude that since (11.4.7) is a regular perturbation series, the series for $\phi(t)$ converges for sufficiently small ε to a periodic function. Thus, all solutions $y(t)$ are stable for $a > 0$, $a \neq n^2/4$ and sufficiently small ε. Figure 11.11 shows that this prediction is correct.

Case II *Perturbative investigation of unstable solutions for a near* $\frac{1}{4}$. To investigate the behavior of solutions near $a = \frac{1}{4}$ and ε near 0, we treat a as a power series in ε:

$$a = \tfrac{1}{4} + a_1\varepsilon + a_2\varepsilon^2 + \cdots.$$

Thus, the Mathieu equation becomes

$$\frac{d^2 y}{dt^2} + \left[\frac{1}{4} + (a_1 + 2\cos t)\varepsilon + \cdots\right] y = 0. \tag{11.4.11}$$

We will look for nongrowing solutions and we specifically hope to find the stability boundary (the edge of the shaded region on Fig. 11.11).

We already know that a naive perturbation expansion will yield secular terms. Thus, we will use a multiple-scale expansion:

$$y(t) = Y_0(t, \tau) + \varepsilon Y_1(t, \tau) + \cdots, \tag{11.4.12}$$

where $\tau = \varepsilon t$. Substituting (11.4.12) into (11.4.11), recalling (11.2.4), and comparing like powers of ε gives

$$\varepsilon^0: \frac{\partial^2 Y_0}{\partial t^2} + \frac{Y_0}{4} = 0, \tag{11.4.13}$$

$$\varepsilon^1: \frac{\partial^2 Y_1}{\partial t^2} + \frac{Y_1}{4} = -(a_1 + 2\cos t)Y_0 - 2\frac{\partial^2 Y_0}{\partial t\,\partial\tau}. \tag{11.4.14}$$

The solution to (11.4.13) is $Y_0 = A(\tau)e^{it/2} + A^*(\tau)e^{-it/2}$. Substituting this result into (11.4.14) gives

$$\partial^2 Y_1/\partial t^2 + Y_1/4 = -[a_1 A(\tau) + iA'(\tau) + A^*(\tau)]e^{it/2} - A(\tau)e^{3it/2} + \text{c.c.}$$

To eliminate the terms which cause Y_1 to exhibit secular behavior, we take

$$iA'(\tau) = -a_1 A(\tau) - A^*(\tau), \qquad -iA^{*\prime}(\tau) = -a_1 A^*(\tau) - A(\tau).$$

This system becomes simpler if we decompose $A(\tau)$ into its real and imaginary parts:

$$A(\tau) = B(\tau) + iC(\tau).$$

The equations for $B(\tau)$ and $C(\tau)$ are

$$B'(\tau) = (-a_1 + 1)C(\tau), \qquad C'(\tau) = (a_1 + 1)B(\tau).$$

Thus, the equation for $B(\tau)$ is

$$B''(\tau) = (1 - a_1^2)B(\tau),$$

and $B(\tau)$ has solutions of the form

$$B(\tau) = K \exp\left(\pm\sqrt{1 - a_1^2}\,\tau\right), \tag{11.4.15}$$

where K is a constant.

Instability (solutions growing exponentially with τ) occurs if $\sqrt{1 - a_1^2}$ is real. Thus, $|a_1| < 1$ gives unstable solutions and $|a_1| > 1$ gives stable solutions. We conclude that near $\varepsilon = 0$, the stability boundary is the pair of straight lines

$$a = \tfrac{1}{4} \pm \varepsilon + O(\varepsilon^2), \qquad \varepsilon \to 0, \tag{11.4.16}$$

which intersect the a axis at 45° angles. This conclusion is verified in Fig. 11.11.

Higher-Order Corrections to the Stability Boundary near $a = \frac{1}{4}$

We now set $a_1 = 1$ and pursue our analysis to higher order to determine the location of the stability boundary more precisely than in (11.4.16). This analysis is particularly interesting because when $a_1 = 1$ there is apparently a new time scale for the problem. To see why, suppose we set $a_1 = 1 + a_2\varepsilon$ in (11.4.15). Then $B(\tau)$ becomes approximately $K \exp(\pm\sqrt{-2a_2 \varepsilon}\,\tau) = K \exp(\sqrt{-2a_2}\,\varepsilon^{3/2}t)$, which suggests that we must introduce a new time scale $\sigma = \varepsilon^{3/2}t$.

We therefore substitute

$$a = \tfrac{1}{4} + \varepsilon + a_2\varepsilon^2 \tag{11.4.17}$$

into the Mathieu equation (11.4.11), set $\sigma = \varepsilon^{3/2}t$, and expand

$$y = Y_0(t, \sigma) + \varepsilon^{1/2}Y_1(t, \sigma) + \varepsilon Y_2(t, \sigma) + \varepsilon^{3/2}Y_3(t, \sigma) + \varepsilon^2 Y_4(t, \sigma) + \cdots. \tag{11.4.18}$$

We have expanded y in powers of $\varepsilon^{1/2}$ rather than ε as in (11.4.12) because σ will inject powers of $\sqrt{\varepsilon}$ into the perturbation series. Note that it is necessary to go to fourth order in powers of $\sqrt{\varepsilon}$ to determine a_2!

Next, we substitute

$$\frac{d^2y}{dt^2} = \frac{\partial^2 Y_0}{\partial t^2} + \varepsilon^{1/2}\frac{\partial^2 Y_1}{\partial t^2} + \varepsilon\frac{\partial^2 Y_2}{\partial t^2} + \varepsilon^{3/2}\left(\frac{\partial^2 Y_3}{\partial t^2} + 2\frac{\partial^2 Y_0}{\partial t\,\partial\sigma}\right) + \varepsilon^2\left(\frac{\partial^2 Y_4}{\partial t^2} + 2\frac{\partial^2 Y_1}{\partial t\,\partial\sigma}\right) + \cdots$$

into the Mathieu equation (11.4.11) and equate powers of $\varepsilon^{1/2}$:

$$\begin{aligned}
&\varepsilon^0: \frac{\partial^2 Y_0}{\partial t^2} + \frac{1}{4}Y_0 = 0, \qquad \text{so } Y_0 = A_0(\sigma)e^{it/2} + \text{c.c.};\\
&\varepsilon^{1/2}: \frac{\partial^2 Y_1}{\partial t^2} + \frac{1}{4}Y_1 = 0, \qquad \text{so } Y_1 = A_1(\sigma)e^{it/2} + \text{c.c.};\\
&\varepsilon^1: \frac{\partial^2 Y_2}{\partial t^2} + \frac{1}{4}Y_2 = -(1 + 2\cos t)Y_0\\
&\qquad = -A_0 e^{it/2} - A_0 e^{3it/2} - A_0^* e^{it/2} + \text{c.c.}
\end{aligned} \tag{11.4.19}$$

To remove the secularity on this level we take $A_0 = -A_0^*$, so that $A_0(\sigma) = iB(\sigma)$ with B real. Now we can solve for Y_2:

$$Y_2(t, \sigma) = A_2(\sigma)e^{it/2} + \tfrac{1}{2}A_0(\sigma)e^{3it/2} + \text{c.c.} \tag{11.4.20}$$

Equating coefficients of $\varepsilon^{3/2}$ gives

$$\begin{aligned}
\varepsilon^{3/2}: \frac{\partial^2 Y_3}{\partial t^2} + \frac{1}{4}Y_3 &= -2\frac{\partial^2 Y_0}{\partial t\,\partial\sigma} - (1 + 2\cos t)Y_1\\
&= -i\frac{dA_0}{d\sigma}e^{it/2} - A_1 e^{it/2} - A_1^* e^{it/2} - A_1 e^{3it/2} + \text{c.c.}
\end{aligned}$$

Eliminating secularity on this level gives $i(dA_0/d\sigma) = -A_1 - A_1^* = -dB/d\sigma$, so

$$\frac{dB}{d\sigma} = A_1 + A_1^*.$$

Finally, using (11.4.19) and (11.4.20), we equate coefficients of ε^2:

$$\begin{aligned}
\varepsilon^2: \frac{\partial^2 Y_4}{\partial t^2} + \frac{1}{4}Y_4 = &-i\frac{dA_1}{d\sigma}e^{it/2} - A_2 e^{it/2} - A_2^* e^{it/2} - A_2 e^{3it/2} - \frac{1}{2}A_0 e^{3it/2}\\
&- \frac{1}{2}A_0 e^{5it/2} - \frac{1}{2}A_0 e^{it/2} - a_2 A_0 e^{it/2} + \text{c.c.}
\end{aligned}$$

Setting all terms which can give rise to secularity equal to zero gives

$$-i\frac{dA_1}{d\sigma} = A_2 + A_2^* + \tfrac{1}{2}iB + a_2 iB, \qquad i\frac{dA_1^*}{d\sigma} = A_2^* + A_2 - \tfrac{1}{2}iB - a_2 iB.$$

From these equations we have

$$-i\frac{d(A_1 + A_1^*)}{d\sigma} = iB + 2a_2 iB.$$

Letting $A_1 + A_1^* = C$, C real, gives

$$\frac{dB}{d\sigma} = C, \qquad -\frac{dC}{d\sigma} = (1 + 2a_2)B.$$

Finally, eliminating C gives

$$\frac{d^2B}{d\sigma^2} = (2a_2 + 1)B,$$

whose solution is

$$B(\sigma) = (\text{constant}) \exp\left(\pm\sigma\sqrt{2a_2 + 1}\right).$$

We conclude that we have stability when $a_2 < -\frac{1}{2}$ and instability when $a_2 > -\frac{1}{2}$. The higher-order stability boundary is thus given by

$$a(\varepsilon) = \tfrac{1}{4} + \varepsilon - \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3), \qquad \varepsilon \to 0. \tag{11.4.21}$$

For further analysis of the Mathieu equation see Prob. 11.19.

PROBLEMS FOR CHAPTER 11

Section 11.1

(I) **11.1** The pendulum of a grandfather clock swings to a maximum angle of 5° from the vertical. How many seconds does the clock gain or lose each day if the clock is adjusted to keep perfect time when the angular swing is 2° from the vertical?

(E) **11.2** Show that the most secular term in y_{n+1} in (11.1.5) arises from the most secular term in y_n.

(E) **11.3** Show that the coefficient of x^n in the Taylor expansion of $e^x e^x e^{-x}$ is $\sum_{j+k+l=n} (-1)^l/(j!\,k!\,l!)$.

Section 11.2

(E) **11.4** There is an alternative to the method discussed in Sec. 11.2 to eliminate secular terms. In the *method of averaging* we consider the integral $I = \int_0^{2\pi} Y_0(\partial^2 Y_1/\partial t^2 + Y_1)\,dt$, taken over the short time-scale period of oscillation of (11.2.1). Y_0 and Y_1 are defined in (11.2.2). Throughout this integration the long time scale τ remains fixed, and thus Y_0 and Y_1 should be periodic in t.

(*a*) Show that if Y_0 and Y_1 are periodic in t and (11.2.2) is uniformly valid, then $I = 0$.

(*b*) Use (11.2.6) and the requirement that $I = 0$ to derive (11.2.8) and (11.2.9).

(I) **11.5** We know that the solution to Duffing's equation. (11.2.1) $d^2y/dt^2 + y + \varepsilon y^3 = 0$ [$y(0) = 1$, $y'(0) = 0$] has the form $y = \cos(\omega t) + O(\varepsilon)$ ($\varepsilon \to 0+$), where $\omega^2 - 1 = a\varepsilon + b\varepsilon^2 + \cdots$ represents the frequency shift caused by the εy^3 term. Let us rewrite Duffing's equation as $d^2y/dt^2 + \omega^2 y + \varepsilon y^3 - a\varepsilon y - b\varepsilon^2 y \cdots = 0$.

(*a*) Adjust a so that no secularity appears to first order in the perturbation expansion of y. (Treat ω^2 as a parameter in this calculation.)

(*b*) Adjust b so that no secularity appears to second order in the perturbation expansion of y.

(*c*) From your determination of a and b, compute ω_1 and ω_2 in the expansion of ω: $\omega = 1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots$. Show that $\omega_1 = \frac{3}{8}$ and $\omega_2 = -\frac{21}{256}$.

(E) **11.6** Rederive the result in Prob. 11.5 by integrating the separable differential equation in (11.1.8) over one period T: $\int dy/\sqrt{1 - y^2 + \varepsilon(1 - y^4)/2} = \int dt = T$. The relation between the frequency ω and the period T is $\omega T = 2\pi$. Show that $\omega = \pi/2I$, where $I = \int_0^{\pi/2} d\theta/\sqrt{1 + \varepsilon(1 + \sin^2\theta)/2}$, and expand I as a power series in ε.

(D) **11.7** Perform a multiple-scale analysis of the Duffing equation (11.2.1) to second order in ε. That is, take three terms in the expansion (11.2.2): $y(t) = Y_0(t, \tau, \sigma) + \varepsilon Y_1(t, \tau, \sigma) + \varepsilon^2 Y_2(t, \tau, \sigma) + \cdots$, where $\tau = \varepsilon t$, $\sigma = \varepsilon^2 t$.

(*a*) Derive the partial differential equations satisfied by Y_0, Y_1, Y_2.

(*b*) Show that first-order multiple-scale analysis gives $Y_0 = A(\tau, \sigma)e^{it} + \text{c.c.}$, where $A(\tau, \sigma) = R(\sigma)e^{i[3R^2(\sigma)\tau/2 + \theta(\sigma)]}$. Also show that $Y_1 = \frac{1}{8}R^3(\sigma)e^{3i[t + 3R^2(\sigma)\tau/2 + \theta(\sigma)]} + B(\tau, \sigma)e^{it} + \text{c.c.}$

(*c*) Show that second-order multiple-scale analysis does *not* determine $R(\sigma)$ and $\theta(\sigma)$ uniquely. However, if it is assumed that B is a constant, then it is possible to reproduce the second-order results cited in Prob. 11.5 and rederived in Prob. 11.6. Can this assumption be weakened?

(*d*) Show that third-order multiple-scale analysis does not remove the ambiguities encountered in second order.

Section 11.3

(E) **11.8** Use the energy integral (11.3.2) to show that all solutions to (11.3.1) decay to 0 as $t \to +\infty$.

(E) **11.9** Show that demanding that secular terms on the right side of (11.3.22) vanish leads to the conclusion that $A(\tau) = 0$.

(I) **11.10** Consider the nonlinear oscillator $d^2y/dt^2 + \omega^2(\varepsilon t)y + \varepsilon y^3 = 0$ [$y(0) = 1$, $y'(0) = 0$]. Use multiple-scale perturbation theory to find an approximation to $y(t)$ which is valid on the εt time scale.

(I) **11.11** The Van der Pol equation is given by $d^2y/dt^2 + y - \varepsilon(1 - y^2)\,dy/dt = 0$. For arbitrary initial conditions the solution to this equation approaches a limit cycle. Find the approach to this limit cycle using multiple-scale perturbation theory.

(D) **11.12** Consider the boundary-layer problem $\varepsilon y''(x) + a(x)y'(x) + b(x)y = 0$ [$y(0) = A$, $y(1) = B$, $a(x) > 0$]. Show that naive multiple-scale perturbation theory, in which the short time scale is $t = x/\varepsilon$ and the long time scale is x, breaks down. Find a suitable transformation for which the method of multiple scales does work and reproduce the result in (9.1.13).

(D) **11.13** Before solving this problem read Example 4 of Sec. 7.2. Now consider an oscillator governed by the equation $\ddot{y} + y - \varepsilon t y = 0$. Assume that $y(0) = 1$ and that $\dot{y}(0)$ has been chosen so that $y \to 0$ as $t \to \infty$.

(*a*) Find the leading asymptotic behavior of $y(t)$ for large positive values of t.

(*b*) Using regular perturbation theory, obtain an approximation to y valid to first order in powers of ε. How large may t be before secular behavior appears.

(*c*) Use multiple-scale theory to eliminate this lowest-order secular behavior in y.

Clue: To do this you must consider three time scales: t, $T_1 = \sqrt{\varepsilon}\,t$, and $T_2 = \varepsilon t$. Then consider the differential equation $\ddot{y} + y - \sqrt{\varepsilon}\,T_1 y = 0$ and try a perturbation series of the form $y(t) = Y_0(t, T_1, T_2) + \varepsilon Y_1(t, T_1, T_2) + \cdots$.

Show that for times $t = O(\varepsilon^{-1/2})$ the effect of the perturbation term is to make the frequency of the oscillator time dependent. Find the frequency of the oscillator to first order in powers of ε.

(*d*) Find the exact solution to the differential equation and use it to verify the result of part (*c*).

(I) **11.14** Use multiple-scale perturbation theory to find a leading-order approximation to $\ddot{y} + y + \varepsilon\dot{y}y^2 = 0$ [$y(0) = 1$, $\dot{y}(0) = 0$, $\varepsilon > 0$].

(I) **11.15** Consider the following perturbation problem: $d^2u/dt^2 + u = \varepsilon u^2$ ($\varepsilon \to 0$) with initial conditions $u(0) = 2$, $u'(0) = 0$.

(*a*) In what order of ε does a secularity first appear in the regular perturbation solution for $u(t)$? Find $u(t)$ to that order in ε in this expansion.

(*b*) Introduce a suitable long time scale to eliminate the secularity found in part (*a*).

(*c*) Find the zeroth-order solution valid on the long time scale. Find the amplitude change and frequency shift.

(I) **11.16** Cheng and Wu considered the following simple differential equation that illustrates the limitations of WKB and multiple-scale perturbation theories: $y'' + e^{-\varepsilon x}y = 0$ $[y(0) = 0,\ y'(0) = 1]$.

(*a*) Show that the exact solution is

$$y(x) = \frac{\pi}{\varepsilon}\left[Y_0\left(\frac{2}{\varepsilon}\right)J_0\left(\frac{2}{\varepsilon}e^{-\varepsilon x/2}\right) - J_0\left(\frac{2}{\varepsilon}\right)Y_0\left(\frac{2}{\varepsilon}e^{-\varepsilon x/2}\right)\right].$$

(*b*) Show that leading-order WKB and multiple-scale (MS) analysis give the approximation

$$y_{\text{WKB}}(x) = y_{\text{MS}}(x) = e^{\varepsilon x/4}\sin\left[\frac{2}{\varepsilon}(1 - e^{-\varepsilon x/2})\right].$$

(*c*) Show that the error in y_{WKB} and y_{MS} is small only if $\varepsilon e^{\varepsilon x/2} \ll 1$ $(\varepsilon \to 0+)$.

(*d*) Show that the reason for the breakdown of the approximations as $x \to +\infty$ is that there is a turning point of the differential equation at ∞. Argue that when $x = O[(1/\varepsilon)\ln(1/\varepsilon)]$ the character of the solution changes. How does it change?

Section 11.4

(TI) **11.17** Prove the following result in Floquet theory. If the eigenvalues of the 2×2 matrix in (11.4.4) are degenerate, then there are solutions $y_1(t)$ and $y_2(t)$ with the properties that $y_1(t) = e^{\mu t}\phi(t)$, $y_2(t) = [t\phi(t) + \psi(t)]e^{\mu t}$, where $\phi(t)$ and $\psi(t)$ are 2π-periodic.

(I) **11.18** Show that the perturbation expansion (11.4.7) has secular terms if and only if $a = n^2/4$ $(n = 0, 1, 2, \ldots)$. When $a = n^2/4$, secularity appears in nth-order perturbation theory.

(D) **11.19** Find the next term in the expansion (11.4.21). Note that when $a_2 = -\frac{1}{2}$, the appropriate long time scale is $\varepsilon^2 t$.

(I) **11.20** It might seem that $A_0 = -A_0^*$ in (11.4.19) gives only one solution to the Mathieu equation (11.4.1). Can you find a second linearly independent solution in the context of multiple-scale analysis?

(D) **11.21** (*a*) Consider a pendulum of length L whose pivot point is oscillating up and down a distance l with frequency ω according to $l\cos(\omega t)$. Show that if the pendulum undergoes a small angular displacement θ, then $\theta(\omega t)$ satisfies the Mathieu equation (11.4.1) in which $s = \omega t$, $2\varepsilon = l/L$, and $a = \pm g/(\omega^2 L)$ if the pendulum is hanging downward (upward).

(*b*) Explain the physical meaning of the instabilities for a near $n^2/4$ when the pendulum is hanging downward. (See Fig. 11.11.) This is a parametric amplifier.

(*c*) Show from Fig. 11.11 that for certain ranges of ω the pendulum undergoes stable oscillation when it is hanging upward. Build a gadget to demonstrate this phenomenon.

APPENDIX

USEFUL FORMULAS

The world is full of obvious things which nobody by any chance ever observes.

—Sherlock Holmes, *The Hound of the Baskervilles*
Sir Arthur Conan Doyle

AIRY FUNCTIONS

1. Differential equation:

$$y'' = xy.$$

Solutions are linear combinations of Ai (x) and Bi (x).

2. Taylor series:

$$\text{Ai}(x) = 3^{-2/3}\sum_{n=0}^{\infty}\frac{x^{3n}}{9^n n!\,\Gamma(n+\frac{2}{3})} - 3^{-4/3}\sum_{n=0}^{\infty}\frac{x^{3n+1}}{9^n n!\,\Gamma(n+\frac{4}{3})},$$

$$\text{Bi}(x) = 3^{-1/6}\sum_{n=0}^{\infty}\frac{x^{3n}}{9^n n!\,\Gamma(n+\frac{2}{3})} + 3^{-5/6}\sum_{n=0}^{\infty}\frac{x^{3n+1}}{9^n n!\,\Gamma(n+\frac{4}{3})},$$

$$\text{Ai}(0) = \text{Bi}(0)/\sqrt{3} = 3^{-2/3}/\Gamma(\tfrac{2}{3}) \doteq 0.355\,028,$$

$$\text{Ai}'(0) = -\text{Bi}'(0)/\sqrt{3} = -3^{-1/3}/\Gamma(\tfrac{1}{3}) \doteq -0.258\,819.$$

3. Functional relations:

$$\text{Ai}(z) + \omega\,\text{Ai}(\omega z) + \omega^2\,\text{Ai}(\omega^2 z) = 0,$$

$$\text{Bi}(z) = i\omega\,\text{Ai}(\omega z) - i\omega^2\,\text{Ai}(\omega^2 z),$$

where $\omega = e^{-2i\pi/3}$.

4. Relation to Bessel functions:

$$\text{Ai}(z) = \pi^{-1}\sqrt{z/3}\,K_{1/3}(2z^{3/2}/3),$$

$$\text{Bi}(z) = \sqrt{z/3}\,[I_{-1/3}(2z^{3/2}/3) + I_{1/3}(2z^{3/2}/3)].$$

5. Asymptotic expansions:

$$\mathrm{Ai}\,(z) \sim \tfrac{1}{2}\pi^{-1/2} z^{-1/4} e^{-2z^{3/2}/3} \sum_{n=0}^{\infty} (-1)^n c_n z^{-3n/2}, \quad z \to \infty;\ |\arg z| < \pi,$$

$$\mathrm{Bi}\,(z) \sim \pi^{-1/2} z^{-1/4} e^{2z^{3/2}/3} \sum_{n=0}^{\infty} c_n z^{-3n/2}, \qquad z \to \infty;\ |\arg z| < \tfrac{1}{3}\pi,$$

$$\mathrm{Ai}\,(z) = w_1(z) \sin\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right] - w_2(z) \cos\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right],$$

$$\mathrm{Bi}\,(z) = w_2(z) \sin\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right] + w_1(z) \cos\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right],$$

$$w_1(z) \sim \pi^{-1/2}(-z)^{-1/4} \sum_{n=0}^{\infty} c_{2n} z^{-3n}, \qquad z \to \infty;\ \frac{\pi}{3} < \arg z < \frac{5\pi}{3},$$

$$w_2(z) \sim \pi^{-1/2}(-z)^{-7/4} \sum_{n=0}^{\infty} c_{2n+1} z^{-3n}, \qquad z \to \infty;\ \frac{\pi}{3} < \arg z < \frac{5\pi}{3},$$

$$c_n = \frac{(2n+1)(2n+3)\cdots(6n-1)}{144^n n!}$$

$$= \frac{1}{2\pi}\left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}, \qquad c_0 = 1.$$

6. Integral representations:

$$\mathrm{Ai}\,(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{1}{3}t^3 + xt\right) dt,$$

$$\mathrm{Bi}\,(x) = \frac{1}{\pi}\int_0^{\infty} \left[e^{-t^3/3 + xt} + \sin\left(\frac{1}{3}t^3 + xt\right)\right] dt.$$

MODIFIED BESSEL FUNCTIONS

1. Differential equation:

$$x^2 y'' + xy' - (x^2 + v^2)y = 0.$$

Solutions are linear combinations of $I_v(x)$ and $K_v(x)$.

2. Frobenius series:

$$I_v(x) = \left(\frac{1}{2}x\right)^v \sum_{n=0}^{\infty} \frac{(\frac{1}{4}x^2)^n}{n!\,\Gamma(n+v+1)},$$

$$K_v(x) = \pi\frac{I_{-v}(x) - I_v(x)}{2\sin v\pi}, \text{ if } v \text{ is nonintegral,}$$

$$K_n(x) = (-1)^{n+1}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] I_n(x) + \frac{1}{2}\left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(-\frac{x^2}{4}\right)^k$$

$$+ (-1)^n\left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \frac{1}{2(k+1)}\right.$$

$$\left. + \frac{1}{2(k+2)} + \cdots + \frac{1}{2(n+k)}\right] \frac{(\frac{1}{4}x^2)^k}{k!\,(n+k)!},$$

where the sum $\sum_{k=0}^{n-1}$ is absent when $n = 0$.

3. Functional relations:

$$I_v(z) = \pm\frac{i}{\pi}K_v(ze^{\pm i\pi}) \mp \frac{ie^{\mp iv\pi}}{\pi}K_v(z),$$

$$K_v(z) = 2\cos(\pi v)K_v(ze^{\pm i\pi}) - K_v(ze^{\pm 2i\pi}).$$

4. Asymptotic expansions:

$$K_v(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{n=0}^{\infty} c_n z^{-n}, \qquad z \to \infty;\ |\arg z| < \frac{3}{2}\pi,$$

$$I_v(z) \sim (2\pi z)^{-1/2} e^{z} \sum_{n=0}^{\infty} (-1)^n c_n z^{-n} + i(2\pi z)^{-1/2} e^{-z + v\pi i} \sum_{n=0}^{\infty} c_n z^{-n},$$

$$z \to \infty;\ -\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi,$$

$$c_n = \frac{(4v^2 - 1^2)(4v^2 - 3^2)(4v^2 - 5^2)\cdots(4v^2 - (2n-1)^2)}{8^n n!}, \qquad c_0 = 1.$$

5. Integral representations:

$$I_v(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos t}\cos(vt)\,dt - \frac{\sin(v\pi)}{\pi}\int_0^{\infty} e^{-z\cosh t - vt}\,dt, \quad |\arg z| < \frac{1}{2}\pi,$$

$$K_v(z) = \int_0^{\infty} e^{-z\cosh t}\cosh(vt)\,dt, \qquad |\arg z| < \frac{1}{2}\pi.$$

6. Difference equations [$y_v(x)$ is either $I_v(x)$ or $K_v(x)$]:

$$y_{v-1}(x) - y_{v+1}(x) = \frac{2v}{x}y_v(x),$$

$$2y_v'(x) = y_{v-1}(x) + y_{v+1}(x),$$

$$I_0'(x) = I_1(x),$$

$$K_0'(x) = -K_1(x).$$

7. Generating function:

$$e^{z\cos t} = I_0(z) + 2\sum_{k=1}^{\infty} I_k(z)\cos(kt).$$

BESSEL FUNCTIONS

1. Differential equation:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

Solutions are linear combinations of $J_\nu(x)$ and $Y_\nu(x)$.

2. Frobenius series:

$$J_\nu(x) = (\tfrac{1}{2}x)^\nu \sum_{n=0}^{\infty} \frac{(-\frac{1}{4}x^2)^n}{n!\,\Gamma(n+\nu+1)},$$

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, \text{ if } \nu \text{ is nonintegral,}$$

$$Y_n(x) = \frac{2}{\pi}[\ln(\tfrac{1}{2}x) + \gamma]J_n(x) - \frac{1}{\pi}(\tfrac{1}{2}x)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(\tfrac{1}{4}x^2)^k$$

$$-\frac{1}{\pi}(\tfrac{1}{2}x)^n \sum_{k=0}^{\infty}\left[1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{k} + \frac{1}{2(k+1)} + \frac{1}{2(k+2)} + \cdots + \frac{1}{2(n+k)}\right]\frac{(-\frac{1}{4}x^2)^k}{k!\,(n+k)!},$$

where the sum $\sum_{k=0}^{n-1}$ is absent if $n = 0$.

3. Functional relations:

$$J_\nu(z) = e^{i\nu\pi/2} I_\nu(ze^{-i\pi/2}), \qquad -\frac{\pi}{2} < \arg z \le \pi,$$

$$Y_\nu(z) = ie^{i\nu\pi/2} I_\nu(ze^{-i\pi/2}) - \frac{2}{\pi}e^{-i\nu\pi/2}K_\nu(ze^{-i\pi/2}), \qquad -\frac{\pi}{2} < \arg z \le \pi.$$

4. Asymptotic expansions:

$$J_\nu(z) = w_1(z)\left(\frac{2}{\pi z}\right)^{1/2}\cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) - w_2(z)\left(\frac{2}{\pi z}\right)^{1/2}\sin(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi),$$

$$Y_\nu(z) = w_2(z)\left(\frac{2}{\pi z}\right)^{1/2}\cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + w_1(z)\left(\frac{2}{\pi z}\right)^{1/2}\sin(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi),$$

$$w_1(z) \sim \sum_{n=0}^{\infty}(-1)^n c_{2n} z^{-2n}, \qquad z \to \infty;\ |\arg z| < \pi,$$

$$w_2(z) \sim \sum_{n=0}^{\infty}(-1)^n c_{2n+1} z^{-2n-1}, \qquad z \to \infty;\ |\arg z| < \pi,$$

$$c_n = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2n-1)^2)}{8^n n!}, \qquad c_0 = 1.$$

5. Integral representations:

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin t - \nu t)\,dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\sinh t - \nu t}\,dt, \qquad |\arg z| < \tfrac{1}{2}\pi,$$

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin t - \nu t)\,dt - \frac{1}{\pi}\int_0^\infty [e^{\nu t} + e^{-\nu t}\cos(\nu\pi)]e^{-z\sinh t}\,dt,$$

$$|\arg z| < \tfrac{1}{2}\pi.$$

6. Difference equations [$y_\nu(x)$ is either $J_\nu(x)$ or $Y_\nu(x)$]:

$$y_{\nu-1}(x) + y_{\nu+1}(x) = \frac{2\nu}{x}y_\nu(x),$$

$$2y_\nu'(x) = y_{\nu-1}(x) - y_{\nu+1}(x),$$

$$J_0'(x) = -J_1(x),$$

$$Y_0'(x) = -Y_1(x).$$

7. Generating function:

$$e^{z(t-1/t)/2} = \sum_{k=-\infty}^{\infty} t^k J_k(z).$$

8. Other differential equations:

(*a*)
$$y'' + a^2x^{k-2}y = 0,$$
$$y = \sqrt{x}[\alpha J_{1/k}(2ax^{k/2}/k) + \beta Y_{1/k}(2ax^{k/2}/k)].$$

(*b*)
$$\frac{d^{2n}y}{dx^{2n}} = (-a^2)^n x^{-n} y,$$
$$y = x^{n/2}[\alpha J_n(2a\omega x^{1/2}) + \beta Y_n(2a\omega x^{1/2})],$$

where $\omega^n = 1$.

PARABOLIC CYLINDER FUNCTIONS

1. Differential equation:

$$y'' + (\nu + \tfrac{1}{2} - \tfrac{1}{4}x^2)y = 0.$$

Solutions are $D_\nu(\pm x)$ and $D_{-\nu-1}(\pm ix)$. Only two of these functions are linearly independent.

2. Taylor series:

$$D_\nu(x) = \frac{\pi^{1/2}2^{\nu/2}}{\Gamma(\frac{1}{2} - \frac{1}{2}\nu)}\sum_{n=0}^{\infty}\frac{a_{2n}x^{2n}}{(2n)!} - \frac{\pi^{1/2}2^{(\nu+1)/2}}{\Gamma(-\frac{1}{2}\nu)}\sum_{n=0}^{\infty}\frac{a_{2n+1}x^{2n+1}}{(2n+1)!},$$

where $a_0 = a_1 = 1$ and $a_{n+2} = -(\nu + \frac{1}{2})a_n + \frac{1}{4}n(n-1)a_{n-2}$.

$$D_\nu(0) = \pi^{1/2}2^{\nu/2}/\Gamma(\tfrac{1}{2} - \tfrac{1}{2}\nu).$$
$$D_\nu'(0) = -\pi^{1/2}2^{(\nu+1)/2}/\Gamma(-\tfrac{1}{2}\nu).$$

3. Functional relation:

$$D_\nu(z) = e^{i\nu\pi}D_\nu(-z) + \frac{(2\pi)^{1/2}}{\Gamma(-\nu)}e^{i(\nu+1)\pi/2}D_{-\nu-1}(-iz).$$

4. Asymptotic expansions:

$$D_\nu(z) \sim z^\nu e^{-z^2/4}\sum_{n=0}^{\infty}(-1)^n c_n z^{-2n}, \qquad z \to \infty;\ |\arg z| < \tfrac{3}{4}\pi,$$

$$D_\nu(z) \sim z^\nu e^{-z^2/4}\sum_{n=0}^{\infty}(-1)^n c_n z^{-2n} - \frac{(2\pi)^{1/2}}{\Gamma(-\nu)}e^{i\pi\nu}z^{-\nu-1}e^{z^2/4}\sum_{n=0}^{\infty}d_n z^{-2n},$$

$$z \to \infty;\ \frac{1}{4}\pi < \arg z < \frac{5}{4}\pi,$$

$$c_n = \frac{\nu(\nu-1)\cdots(\nu-2n+1)}{2^n n!}, \qquad c_0 = 1,$$

$$d_n = \frac{(\nu+1)(\nu+2)\cdots(\nu+2n)}{2^n n!}, \qquad d_0 = 1.$$

5. Integral representation:

$$D_\nu(x) = \sqrt{\frac{2}{\pi}}\,e^{x^2/4}\int_0^\infty e^{-t^2/2}t^\nu \cos(xt - \nu\pi/2)\,dt, \qquad \operatorname{Re}\nu > -1.$$

6. Difference equations:

$$xD_\nu(x) = D_{\nu+1}(x) + (\nu + \tfrac{1}{2})D_{\nu-1}(x),$$
$$D_\nu'(x) = -\tfrac{1}{2}xD_\nu(x) + (\nu + \tfrac{1}{2})D_{\nu-1}(x).$$

7. Relation to Hermite polynomials:

$$D_n(x) = \mathrm{He}_n(x)e^{-x^2/4}.$$

GAMMA AND DIGAMMA (PSI) FUNCTIONS

1. Integral representation:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad \operatorname{Re} z > 0.$$

2. Difference equation:

$$\Gamma(x+1) = x\Gamma(x).$$

3. Special values:

$$\Gamma(0) = 1,\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \qquad \Gamma(n+1) = n!.$$

4. Stirling's asymptotic formula:

$$\Gamma(z) \sim (z/e)^z\sqrt{2\pi/z}\left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \cdots\right],$$

$$z \to \infty;\ |\arg z| < \pi.$$

5. Other formulas:

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z),$$
$$\Gamma(2z) = \tfrac{1}{2}\pi^{-1/2}4^z\Gamma(z)\Gamma(z + \tfrac{1}{2}),$$
$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \Gamma(x)\Gamma(y)/\Gamma(x+y), \qquad \operatorname{Re} x > 0,\ \operatorname{Re} y > 0.$$

6. Psi function:

$$\psi(z) = \Gamma'(z)/\Gamma(z).$$

7. Difference equation:

$$\psi(z+1) = \psi(z) + \frac{1}{z}.$$

8. Special values:

$$\psi(1) = -\gamma, \qquad \psi(n+1) = -\gamma + \sum_{k=1}^{n} 1/k,$$

where $\gamma \doteqdot 0.5772$ is Euler's constant.

9. Taylor series:

$$\psi(1+z) = -\gamma - \sum_{n=2}^{\infty}\zeta(n)(-z)^{n-1},$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann zeta function.

10. Asymptotic expansion:

$$\psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots, \qquad z \to \infty;\ |\arg z| < \pi.$$

EXPONENTIAL INTEGRALS

1. Integral representation:

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n}\,dt, \qquad \operatorname{Re} z > 0.$$