Multiscale Hydrodynamic Phenomena: Matched Asymptotic Expansions

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Abstract

We aim to introduce on simple examples the Method of Matched Asymptotic Expansions ("Méthode des Développements Asymptotiques Raccordés"). We first introduce the concept of "singular problem" on a problem depending on a small parameter (the famous "ε"). We will define "singular problems" to be not "regular problems". Problems depending on a small parameter are said regular problems if the solution of the problem does not change so much if the small parameter ε is changed a little bit. We will see that at first on a simple second order equation. Then we will show that in singular problems we have to "change the scales" of the variables to focus on the local solution. The apparition of multiple scale is then natural. We show on examples that we have to rescale the equations in order to obtain the relevant solution of the problem. Hence we will present the "Dominant Balance" (or "distinguished limit") principle which is a principle to avoid over- simplification of a problem. The "matching principle" allows to re connect the simplified problem at the new scale to the previous at normal scale. The Friedrichs problem (an über classical toy model for Navier Stokes) will be discussed, it contains all the principles of the method: rescaling, dominant balance and matching. As the examples are rather simple, they have a simple analytic solution, so that we can check the approximations.

1 Introduction

The game consists to find an approximate solution of a physical problem depending on a small parameter say "epsilon" (the ε, with LaTeX code `$\varepsilon$`, not ϵ `$\epsilon$`). This solution is obtained as an "asymptotic development" in powers (often) of ε.

Often, in fact, the problem comes from a physical problem. In practice we will see in the next chapters the boundary layer problem (Matched Asymptotic Expansion is a technique, first introduced by Prandtl 1904 it allowed decisive progress in aerodynamics during the WWII for the Germans, it was expanded latter on, in the 50' 70' during the cold war). First, one has to make this problem non dimensional in using adequate scales. Using the Π theorem (or Vaschy-Buckingham) may be a good guide to find relevant non dimensional numbers (we will see this in a next chapter). Then, due to small aspect ratios, or small values of the parameters, one obtains a problem with small numbers.

Then, one has to solve the problem with this small parameter, it is not always simple. Sometimes, from the Physics of the phenomena (experiments or computations), one is guided to find where the problem arises. We will see that the small parameter is sometimes a "hidden" small scale.

The methods that we will present are found in several books that the reader should read (Hinch [14] or Bender Orzag [3] or Van Dyke [25] and others, for example Lagerstrom and Casten [19], see the bibliography at the end).

2 Definitions

2.1 Notation o and O

We start by some standard definitions and notations, that everybody knows. Let us introduce the O notation ("big oh" or "large oh"). This is called Landau notation, but Edmund, not Lev). Then, by definition, $f = 0(g)$ for ε tending to 0 means that there exists a neighbourhood of the origin and a constant A so that for any ε in the neighbourhood of the origin: $|f| < A|g|$ or $\underset{\varepsilon\to 0}{(f/g)}$ is a constant.

$$f = o(g), \quad \text{for } \varepsilon \text{ tending to } 0$$

$$\text{means : } \underset{\varepsilon\to 0}{(f/g)} \text{ approaches } 0.$$

One also writes $f \ll g$, which read f is 'much less than' g, if $f = o(g)$,

We introduce the "order", Ord: $f = Ord(g)$, it reads f is of order g, means $f = O(g)$ and $g = O(f)$ or:

$$f \sim g \tag{1}$$

which reads: f is asymptotic to g.

2.2 Series

Let us take any (ordered) sequence $f_0, f_1, f_2, f_3, ...f_n$ of terms (that is, numbers, functions, vectors, tensors...). A series is represented (or denoted) by an expression like (a sum of terms):

$$\sum_{n=0}^{N} f_n.$$

If the number of terms is infinite, a series may be called an infinite series.

2.3 Convergence

When an expansion is convergent, it means that there is a number L, such by definition:

$$\forall\varepsilon,\ \exists N_0, \ \text{ such that } \forall N > N_0, \ \ |\sum_{n=0}^{N} f_n - L| < \varepsilon.$$

L is the sum of the series.

2.4 Asymptotic approximation/ asymptotic expansion

Having a function of a variable x and of a small parameter ε say $f(x,\varepsilon)$. We have the sum for a given integer N:

$$F_N(x,\varepsilon) = \nu_0(\varepsilon)a_0(x) + \nu_1(\varepsilon)a_1(x) + \nu_2(\varepsilon)a_2(x) + ... + \nu_N(\varepsilon)a_N(x)$$

so that this sum is said to to be an "asymptotic approximation" (or "Poincaré Asymptotic Approximation"):

$$\frac{f(x,\varepsilon) - F_N(x,\varepsilon)}{a_N} \to 0 \text{ as } \varepsilon \to 0$$

the remainder is smaller than the last term included once ε is enough small. And we have:

$$f(x,\varepsilon) \sim \sum_{n=0}^{N} a_n(x)\nu_n(\varepsilon)$$

We call "asymptotic sequence" ("séquence asymptotique" or "suite asymptotique") the series of:

$$\nu_0(\varepsilon), \nu_1(\varepsilon), \nu_2(\varepsilon), ...\nu_i(\varepsilon), ...$$

with $\nu_0(\varepsilon) \gg \nu_1(\varepsilon) \gg \nu_2(\varepsilon) \gg ...$ satisfying the order relation:

$$\nu_i(\varepsilon) \gg \nu_{i+1}(\varepsilon), \quad \nu_{i+1}(\varepsilon) = o(\nu_i(\varepsilon)).$$

with the $a_n(x)$ sequence of continuous functions of order one.

In the asymptotic approximation

$$F_N(x,\varepsilon) = \sum_{n=0}^{N} a_n\nu_n(\varepsilon)$$

the coefficients can be evaluated inductively from:

$$a_0(x) = \lim_{\varepsilon\to 0} \frac{f(x,\varepsilon)}{\nu_0(\varepsilon)} \quad \text{then} \quad a_k(x) = \lim_{\varepsilon\to 0} \frac{f(x,\varepsilon) - \sum_{n=0}^{k-1} a_n\nu_n(\varepsilon)}{\nu_k(\varepsilon)}.$$

Remarks and properties

• The general problem deals with $f(x,\varepsilon)$, in the series we have then the $a_i(x)$ functions of x. The $a_i(x)$ functions are of order one. In the first examples that we will present, we will consider $f(\varepsilon)$ (with no x), so at first we consider the a_i as constants. This will be the case up to the section devoted to Matched Asymptotic Expansion §4.
• The approximation is unique for the given sequence.
• A given function may have many asymptotic approximations.
• Many different functions can share the same asymptotic approximation.
• When $\nu_i(\varepsilon) = \varepsilon^i$ this is an asymptotic power series.
• We can call gauge function (*fonction de Jauge* in French) the ν_i.
• When we construct to $N = \infty$, we call it an "asymptotic expansion".

$$F(\varepsilon) = \sum_{n=0}^{\infty} a_n \nu_n(\varepsilon)$$

Do not confuse a convergent series which is:

$$\lim_{\varepsilon \text{fixed}, N\to\infty} F_N(x,\varepsilon)$$

and an asymptotic approximation, or a Poincaré approximation:

$$\lim_{N \text{fixed}, \varepsilon\to 0} F_N(x,\varepsilon)$$

See Hinch [14] page 20, Kevorkian & Cole ([16],[17]) page 1...

• A change of scale $x = \nu(\varepsilon)\bar{x}$ is a special case. This is the rational way to say that x is small, the smallness is reported in the gauge $\nu(\varepsilon)$ and the variable $\bar{x}$ is of order one.

• The development is not unique:

$$\cos(\varepsilon) = 1 - \varepsilon^2/2 + \varepsilon^4/24 + O(\varepsilon^6)$$

$$\cos(\varepsilon) = \sqrt{1-\sin^2(\varepsilon)} = 1 - \sin^2(\varepsilon)/2 - \sin^4(\varepsilon)/8 + O(\sin^6(\varepsilon))$$

• Recently, Mauss introduced "the generalised asymptotic expansion" in which, contrary to what he calls the "regular asymptotic expansion", the a_n depend on ε as well:

$$F(x,\varepsilon) = \sum_{n=0}^{\infty} a_n(x,\varepsilon)\nu_n(\varepsilon)$$

• As noted by Boyd [5] this usual Poincaré definition of asymptoticity, which refers only to powers of ε, is rather misleading. The neologism "superasymptotic" was therefore coined by Berry and Howls [4] and is better...

2.4.1 Divergent series... Carrier's Rule

Do not confuse a convergent series:

$$J_0(\varepsilon) = 1 - \frac{\varepsilon^2}{4} + \frac{\varepsilon^4}{64} - \frac{\varepsilon^6}{2304} + ...$$

of infinite radius of convergence but with slow convergence, and the asymptotic approximation:

$$J_0(1/\varepsilon) = \sqrt{2\varepsilon/\pi}[(1 - \frac{9\varepsilon^2}{128} + \frac{3675\varepsilon^4}{32768}...)\cos(\frac{1}{\varepsilon} - \frac{\pi}{4}) +$$

$$+(\frac{\varepsilon}{8} - \frac{75\varepsilon^2}{1024} - \frac{75\varepsilon^3}{1024} + ...)\sin(\frac{1}{\varepsilon} - \frac{\pi}{4})]$$

which is divergent for all ε no matter how small. Nevertheless a few terms give good accuracy for moderately small ε. This behavior is on the next figure extracted from Van Dyke [25]. In practice, only few terms are calculated (as it is difficult to go from order to order), so that the point of increasing error is never reached....

A more tractable exemple for those non familiar with Bessel function is the function $f(x) = \int_x^\infty \frac{e^{x-t}}{t}dt$ (Jones [15]), repeating integration by parts, one obtains:

$$f(\varepsilon) = \varepsilon - \varepsilon^2 + ... + (-1)^n(n-1)!\varepsilon^n + R_n(\varepsilon), \text{ with } R_n(\varepsilon) = (-1)^n n! \int_{1/\varepsilon}^\infty \frac{e^{(1/\varepsilon)-t}}{t^{n+1}}dt.$$

this term is such $R_n(\varepsilon)| \le n! \int_x^\infty \frac{dt}{t^{n+1}} = (n-1)!\varepsilon^n$. It is straightforward to see take $a_n = (-1)^n(n-1)!$ and $\nu_n(\varepsilon) = \varepsilon^n$ and that the amplitude of the product $(n-1)!\varepsilon^n$ is larger and larger at fixed ε. We see the trade of, if we want precision with more en more terms, ε should be smaller and smaller.

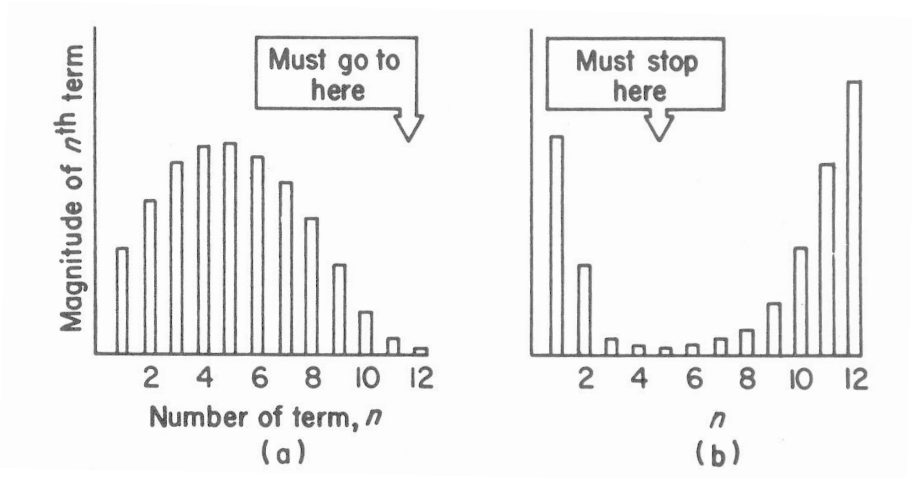


Figure 1: Behavior of terms in series. Left slowly convergent series, Right, divergent asymptotic series (from Van Dyke [25]).

Finally, once we have in mind that those expansion work indeed for $\varepsilon = 1$, so that the Carrier's Rule is *"Divergent series converge faster than convergent series because they don't have to converge."* is a way to say that you can use asymptotic expansion up to $\varepsilon = 1$

2.5 Regular perturbations, singular perturbations

In practice, we will have a problem, say:

$$E_\varepsilon = 0$$

to solve, which depends on a small parameter ε and we look at an asymptotic approximation of this problem $F_N(x, \varepsilon)$.
• The perturbation induced by the parameter ε is regular:

$$F_N(x, \varepsilon) = \sum_{n=0}^{N} a_n(x)\nu_n(\varepsilon)$$

when it is uniformly valid in the domain of definition.

• When the solution can be obtained by simply setting the small parameter to zero:

$$Solution\left[\underset{\varepsilon\to 0}{E_\varepsilon}\right] = \underset{\varepsilon\to 0}{Solution}[E_\varepsilon] \tag{2}$$

We say that the **problem is regular**. In other words, the perturbed problem for small values of ε is not very different from the unperturbed problem for $\varepsilon = 0$.

• We will say that if

$$Solution\left[\underset{\varepsilon\to 0}{E_\varepsilon}\right] \neq \underset{\varepsilon\to 0}{Solution}[E_\varepsilon] \tag{3}$$

the **problem is singular**. Or this is a "singular perturbation problem" in which the limit point $\varepsilon = 0$ differs in an important way from the approach to the limit $\varepsilon \to 0$.

2.5.1 About the examples

At this point, we may quote a famous citation by Niels Henrik Abel 1828 *The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever... Yet for the most part, the results are valid, it is true, but it is a curious thing. I am looking for the reason, a most interesting problem.*

As says Hinch [14]: "*interesting problems are often singular.*"

To solve this kind of problem we will have to change the scales.

First, we will look at some simple regular problem and singular ones, in mechanics, on a polynomia, on an ordinary differential equation and on a Navier Stokes case. In fact all the examples are nearly the same one presented with various point of view. We will adopt a heuristical point of view, with few maths, or low level maths.

3 A first Example

3.1 Mechanical Model

Here we look at what happens to simple second order equations when a small parameter is present. One may imagine that it is a pure abstract problem, but here to visualise, we may imagine that we solve the shooting problem. It means, that for a given velocity v_0, one shoots at angle α a bullet in a gravity field g. We want to hit some target at altitude h.

So first we play with a pen and its spring [you have to be present in the class room to understand].

One wants to know the relation between the longitudinal position of the target for a given h, and all the parameters. Then using this length of scale h, the trajectory is:

$$y/h = \tan(\alpha)x/h - \frac{(x/h)^2}{2(v_0^2/(gh))\cos^2(\alpha)}, \text{ or } \bar{y} = \tan(\alpha)\bar{x} - \frac{\bar{x}^2}{2a\cos^2(\alpha)},$$

with $a = v_0^2/(gh)$ a parameter without dimension, and we want to shoot $\bar{y} = 1$ and find the possible position $\bar{x}$ (maybe two positions, we guess, $\bar{x}_1$ and $\bar{x}_2$).

The first problem that we will see corresponds to what happens if we change a little bit α to $\alpha + \varepsilon$. For example is

$$1 = (-\frac{1}{2a\cos^2(\alpha)} - \frac{\varepsilon\tan(\alpha)}{a\cos^2(\alpha)})\bar{x}^2 + (\tan(\alpha) + \varepsilon(\tan^2(\alpha)+1)\bar{x}$$

Changing a bit ε does not change very much the two roots $\bar{x}_1$ and $\bar{x}_2$. Next the second problem corresponds to what happens if we increase a and say that $\varepsilon = 1/a$:

$$1 = \tan(\alpha)\bar{x} - \varepsilon\frac{\bar{x}^2}{2\cos^2(\alpha)},$$

changing a bit ε does not change very much the first root $\bar{x}_1$ (we are near the canon) but changes a lot $\bar{x}_2$. The second root is rejected far away....

As playing with coefficients like $\tan(\alpha)$ and $1/\cos^2(\alpha)$ is not fun, we use specific numerical values in the sequel.

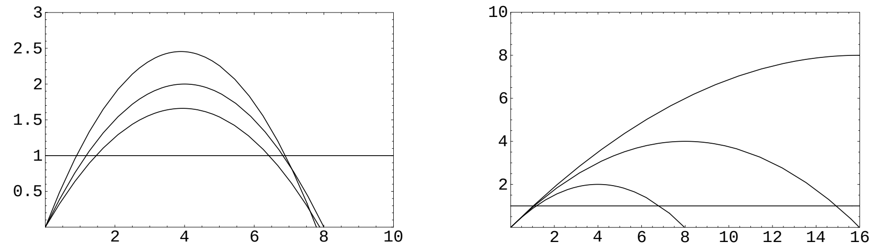


Figure 2: Trajectory $(x(t), y(t))$ of a bullet and level $y = 1$. The bullet starts in $x = 0, y = 0$ with an angle α at velocity v_0 with the horizontal, it crosses the ordinate $y = 1$ in two points. Left, changing slightly the angle of shooting α does not change a lot the abscissae of the intersections with the level 1: the two values change slightly when α changes. Right, increasing just a bit the shooting velocity v_0 does not change so much the position of the first point but changes dramatically the position of the second abscissa of the level 1. So, shooting a bullet at larger and larger velocity gives a singular problem!

3.2 Regular case

3.2.1 The problem

Let us take $\alpha = \pi/4$, and $a = 5$ for the first case, the equation is

$$E_\varepsilon = -\left(\frac{1}{5} + \frac{2}{5}\varepsilon\right)x^2 + (1+2\varepsilon)x - 1$$

we want to solve $E_\varepsilon = 0$.

Unfortunately, the problem is complicated to solve (at the black board) as it implies $\sqrt{5}$...

So as we are lazy and not clever in computations, we will solve the following equation

$$E_\varepsilon = (1-\varepsilon)x^2 + (1+\varepsilon)x - 2,$$

it looks like the previous one, but is less difficult to compute.

3.3 Regular case

3.3.1 Exact solution

We have here a problem depending on a small parameter ε. Let us suppose that we do not know how to solve the problem for $\varepsilon \neq 0$, but we know how to for $\varepsilon = 0$.

$$\text{Let } E_\varepsilon = (1-\varepsilon)x^2 + (1+\varepsilon)x - 2$$

we want to solve $E_\varepsilon = 0$.

OK, the solution is simple! Let us write it. Solution of the full problem are, $\Delta = 1 + \varepsilon + \varepsilon^2 + 8(1-\varepsilon) = (3-\varepsilon)^2$:

$$\{1, \frac{2}{-1+\varepsilon}\}$$

so that after taking the Taylor series:

$$1 \text{ and } -2 - 2\epsilon - 2\epsilon^2 - 2\epsilon^3 + O(\epsilon)^4.$$

But, say that we play the game: we do not know the full solution. Let us construct the solution of $E_\varepsilon = 0$ starting from $\varepsilon = 0$ by Taylor series.

3.3.2 Asymptotic solution

Let us solve $E_\varepsilon = 0$ by asymptotic expansion.

$$E_\varepsilon = (1-\varepsilon)x^2 + (1+\varepsilon)x - 2$$

We start from $E_0 = 0$ which comes from the problem $E_\varepsilon = 0$ in which we set $\varepsilon = 0$:

$$x^2 + x - 2 = 0.$$

We know how to solve the problem ($\sqrt{\Delta} = 3$) which is:

$$x_1 = 1, \text{ and } x_2 = -2$$

Let us go further. We propose an "asymptotic sequence"

$$x_1 = 1 + \epsilon\, x_{11} + \epsilon^2\, x_{12} + ... \text{ and } x_2 = -2 + \epsilon\, x_{21} + \epsilon^2\, x_{22} + ...$$

We put x_1 in the equation, by powers of ε we clearly have $x_{11} = 0$ and $x_{12} = 0$

$$x_1 = 1$$

we do the same with x_2 that we write as:

$$x_2 = -2 + \epsilon\, x_{21} + \epsilon^2\, x_{22} + ...$$

then by substitution in the equation and by identification of the powers of ε: (note $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$)

$$E_\varepsilon = (1-\varepsilon)(4 - 4\epsilon\, x_{21} - 4\epsilon^2\, x_{22} + 2\epsilon^3\, x_{22}x_{21} + \epsilon^2\, {x_{21}}^2 + \epsilon^4\, {x_{22}}^4) +$$

$$+(1+\varepsilon)(-2 + \epsilon\, x_{21} + \epsilon^2\, x_{22}) - 2 + ...$$

so rearrange it

$$E_\varepsilon = 0 + (-6 - 3x_{21})\varepsilon + (5x_{21} + x_{21}^2 - 3x_{22})\varepsilon^2 + O(\varepsilon^3)$$

so that $(-6 - 3x_{21}) = 0$ and $(5x_{21} + x_{21}^2 - 3x_{22}) = 0$ which gives $x_{21} = -2$ and $x_{22} = -2$, we then obtain the expansion of the variable

$$x_2 = -2 - 2\varepsilon - 2\varepsilon^2 + 0(\epsilon^3)$$

We see that we obtain the same solution using the full solution or the asymptotic expansion, of course it is not a proof, it is just an observation.

We see that: "the solution (obtained from the exact solution) in which ε approaches zero" is the same that "the solution obtained from the problem (in which ε tends to zero)"

$$Solution\Big[\underset{\varepsilon\to 0}{E_\varepsilon}\Big] = \underset{\varepsilon\to 0}{Solution}[E_\varepsilon] \qquad (4)$$

We say that **the problem is regular**. In other words, the perturbed problem for small ε is not very different from the unperturbed problem for $\varepsilon = 0$.

3.3.3 Exercise

Our real problem was to solve

$$-\left(\frac{1}{5}+\frac{2}{5}\varepsilon\right)x^2+(1+2\varepsilon)x-1=0$$

Solution of the full problem are, as

$$\Delta = 1/5+12\varepsilon/5+4\varepsilon^2, \text{ hence } \sqrt{\Delta}=\sqrt{(1+12\varepsilon+20\varepsilon^2)/5}$$

$$\left\{x_1=\frac{5\left(-\frac{\sqrt{20\varepsilon^2+12\varepsilon+1}}{\sqrt{5}}+2\varepsilon+1\right)}{2(2\varepsilon+1)}, x_2=\frac{5\left(\frac{\sqrt{20\varepsilon^2+12\varepsilon+1}}{\sqrt{5}}+2\varepsilon+1\right)}{2(2\varepsilon+1)}\right\}$$

taking the Taylor series of the solution of $E_\varepsilon=0$, we compute

$$\sqrt{\Delta}=\frac{1}{\sqrt{5}}\left(1+6\varepsilon-8\varepsilon^2+48\varepsilon^3-320\varepsilon^4+O\left(\varepsilon^5\right)\right)$$

$$\frac{1}{1+2\varepsilon}=1-2\varepsilon+4\varepsilon^2-8\varepsilon^3+16\varepsilon^4+O\left(\varepsilon^5\right)$$

The expansion of the solutions are

$$x_1=\left(\frac{5}{2}-\frac{\sqrt{5}}{2}\right)-2\sqrt{5}\varepsilon+8\sqrt{5}\varepsilon^2-40\sqrt{5}\varepsilon^3+O\left(\varepsilon^4\right)$$

and

$$x_2=\frac{1}{2}\left(5+\sqrt{5}\right)+2\sqrt{5}\varepsilon-8\sqrt{5}\varepsilon^2+40\sqrt{5}\varepsilon^3+O(\varepsilon)^4$$

We see that solving $(1-\varepsilon)x^2+(1+\varepsilon)x-2$ has exactly the same structure and same conceptual difficulties, but a little bit more simple to compute.

Those two examples are "regular perturbation" problems.

3.4 Singular case

3.4.1 Solution

OK, everything OK, no surprise. But there exist pathological cases where we can not swap the limits. Let us see that right now.

The equation of motion was

$$1=\tan(\alpha)\bar{x}-\frac{\bar{x}^2}{2a\cos^2(\alpha)},$$

Let us take $a=1/\varepsilon$ and $\alpha=\pi/4$, so that $\tan\pi/4=1, \cos\pi/4=\sqrt{2}/2$, the equation is

$$1=\bar{x}-\varepsilon\bar{x}^2.$$

Let us loook at the problem $E_\varepsilon=0$ with $E_\varepsilon=1-x+\varepsilon x^2$.

So let us do the same than previously, as it worked well. We start by putting $\varepsilon=0$. We solve $1-x=0$. We find $x=1$. We want to be more precise, we expand $x=1+x_1\varepsilon+x_2\varepsilon^2+...$, we put it into $E_\varepsilon=0$,

$$1-1-x_1\varepsilon-x_2\varepsilon^2+\varepsilon(1+x_1^2\varepsilon^2+x_2^2\varepsilon^4+2x_1x_2\varepsilon^3+2x_1\varepsilon+2x_2\varepsilon^2)+...=0$$

re writing

$$0+\varepsilon(-x_1+1)+\varepsilon^2(-x_2+2x_1)+...=0$$

we identify $x_1=1$ and $x_2=2$, so

$$x=1+\varepsilon+2\varepsilon^2+...$$

If we go on, we are stuck on the same solution and we do not recover the second one. We have lost one solution of the problem. The equation was of order 2, but E_0 is of order one.

In fact, we know that the solutions of $E_\varepsilon=0$ are

$$\frac{1-\sqrt{1-4\varepsilon}}{2\varepsilon} \quad \text{and} \quad \frac{1-\sqrt{1-4\varepsilon}}{2\varepsilon},$$

but the Taylor series we have obtained gives $x=1+\varepsilon+2\varepsilon^2+...$, we note that it corresponds to the expansion of $\frac{1-\sqrt{1-4\varepsilon}}{2\varepsilon}$, but the other root is lost.

Why?

Because simply, the Taylor series of the second is infinite when ε approaches zero:

$$\frac{1-\sqrt{1-4\varepsilon}}{2\varepsilon}=\frac{1}{\varepsilon}-1-\varepsilon-2\varepsilon^2+....$$

We see that there is a problem of scales.

3.4.2 Change of scales

When we have neglected the coefficient εx^2 in E_ε we supposed, without saying it, that it was... negligible!. But, this is not true. The solution is so large that this term becomes not negligible.

To solve the problem, we have to change the scale. We say that we introduce a new "gauge":

$$x = \mu\bar{x}.$$

This means that the order of magnitude of x is not "1", as μ is very large. The idea of the method is that all the reduced variables are of order one ($\bar{x}$ is of order one), the gauge being in front of them.

$E_\varepsilon = 0$ or $\varepsilon x^2 - x + 1$ becomes $E_\varepsilon = \varepsilon\mu^2\bar{x}^2 - \mu\bar{x} + 1$. As we guess that μ is large and that $\bar{x}$ is of order one, the third term "+1" is smaller that the second $\mu\bar{x}$. The first term was previously lost, and we want it to come back, so we choose μ so that:

$$\varepsilon\mu^2\bar{x}^2 \text{ is of same order than } \mu\bar{x}$$

so

$$\varepsilon\mu^2 = \mu \text{ i.e. } \quad \mu = \frac{1}{\varepsilon}.$$

We have introduced here the "Least Degeneracy Principle" or "Dominant Balance", **Principe de Moindre dégénérescence** in French: we retain as much terms as possible in the equation.

This is called as well "Principle of Maximal Balance" (or of "Principle of Maximal Complication" Kruskal in Asymptotlogy 1962).

The principle of "least degeneracy" by Van Dyke is called "significant degeneracy" by Eckhaus.

Another terminology is to say that is is a "distinguished limit". There are many different names for this concept...

The equation is $\bar{x}^2 - \bar{x} + \varepsilon = 0$. We next use the standard method, we put $\varepsilon = 0$, the equation is then in the new variables:

$$\bar{x}^2 - \bar{x} = 0 \text{ with two solutions } \quad \bar{x} = 1 \text{ and } \bar{x} = 0$$

the first one is $x = -1/\varepsilon$, that is perfect, that is the lost solution which is large. But, the second is zero, which may be surprising. It is not, because at the large scale $1/\varepsilon$ it is straightforward that a quantity of order one is zero!

To be sure, let us look at the next order of this null solution:

$$\bar{x} = 0 + \bar{x}_2\varepsilon + \text{...inserted into } \bar{x}^2 - \bar{x} + \varepsilon = 0$$

gives

$$\bar{x}_2^2\varepsilon^2 - \bar{x}_2\varepsilon + \varepsilon = 0 \text{ so } \bar{x}_2 = 1$$

so that the solution in "bar" is $0 + \varepsilon + ...$ so going back to the initial scale:

$$x = 1 + ...$$

which is the solution we found at the very beginning.

We say that if

$$Solution\left[\underset{\varepsilon\to 0}{E_\varepsilon}\right] \neq \underset{\varepsilon\to 0}{Solution}[E_\varepsilon] \tag{5}$$

the **problem is singular**.

To solve this kind of problem we have to change the scales.

4 A simple example of M.A.E., Matched Asymptotic Expansion

Let us consider a model problem (*c.f.* Van Dyke p 79, Hinch p 52, Germain, *etc*), this simple problem has been introduced by **Friedrichs (1942)**. [Note that Kurt Otto Friedrichs (1901 -1982) is co author with Courant (his PhD advisor in Göttingen) and Lewy of "one the partial difference equations of mathematical physics 1928" which introduces CFL condition]. Let us consider the second order linear ODE:

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\ f(1) = 1. \tag{6}$$

ε is a small parameter. We wish to obtain the behaviour of the solution $f(y)$ of problem (6) when the parameter ε approaches to 0. Of course, in this simple case we have the exact solution! We compare this exact solution to the result of the M.A.E. (théorie des développements asymptotiques raccordés, *matched asymptotic expansion*).

4.1 Exact solution

The exact solution of the problem 6 is:

$$f(y) = \frac{1 - e^{-y/\varepsilon}}{2(1 - e^{-1/\varepsilon})} + \frac{y}{2}$$

The smaller ε, the closer the solution of (6) is of the line $(y+1)/2$. The exact solution of (6) allows us to see that there exists two regions. One where

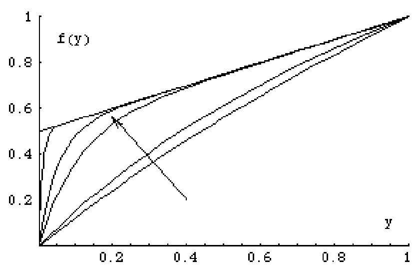


Figure 3: Solution of the Friedrichs problem (6) for $\varepsilon = 1$, $\varepsilon = 0,5$ ε=.1 ε=.05 ε=.01 et ε=0. (the arrow indicates decreasing ε).

y is of order one, where the solution varies with a order one magnitude. A second where y is small of order ε, and where the solution changes abruptly.

In practice the exact solution is not known. We have to develop a strategy to solve the equation. So, we will now present the technique: we have to simplify the problem to emphasize the role of the small ε...

4.2 Matched Asymptotic Expansion

4.2.1 Guide to MAE, Van Dyke rule

Let us apply the "Matched Asymptotic Expansion" to the Friedrichs problem. To do that we present the principles of the method. They are summarized in one sentence by Van Dyke [25] page 86, who says: *"The guiding principles are that the inner problem shall have the least possible degeneracy, that it must include in the first approximation any essential elements omitted in the first outer solution, and that the inner and outer solutions shall match."*

4.2.2 External problem, Outer Problem (<=> ideal fluid)

We treat the problem as regular perturbation, even though we know it is not. To solve the problem, the most simple is *a priori* to put ε to 0 in (6), it means that we treat the problem as a regular perturbation, and look what happens:

$$\frac{df(y)}{dy} = \frac{1}{2},\ f(0) = 0;\ f(1) = 1,$$

Both B.C. cannot be satisfied in general. If we keep the second boundary condition i $y = 1$, then

$$f(y) = \frac{y+1}{2},\ \ f(1) = 1$$

but condition in 0 is not full fit $f(0) \neq 0$ as $f(0) = 1/2$. Nevertheless, this is a good approximation of (6) except in a small layer near the origin, called "boundary layer". We observe that we have lost the higher derivative term. So that we need only one. We have computed the solution of (6) in which ε approaches 0

$$Solution\Big[\underset{\varepsilon\to 0}{E_\varepsilon}\Big] \neq \underset{\varepsilon\to 0}{Solution}[E_\varepsilon]\,. \tag{7}$$

We see that the problem is "singular".

4.2.3 Inner Problem (<=> Boundary Layer)

We do a rescaling in order to see what happens in the neighborhood of the origin $y = 0$. So we write $y = \delta\tilde{y}$, we call δ the "gauge", the new scale or the boundary layer scale.
We substitute and try to find the gauge which allows to retain a maximal number of terms in the equation. It seems possible to think that the potentially interesting scalings are those which produce a balance between two or more terms in the equations. They are sometimes called "distinguished limits". (6) is now:

$$\varepsilon\frac{d^2\tilde{f}(\tilde{y})}{\delta^2 d\tilde{y}^2} + \frac{d\tilde{f}(\tilde{y})}{\delta d\tilde{y}} = \frac{1}{2}. \quad (8)$$

To satisfy the *least possible degeneracy* or *dominant balance* ("Principe de Moindre Dégénérescence", "Principe de Non Simplification Abusive) we take $\varepsilon = \delta$. The inner problem is $\tilde{f}'' + \tilde{f}' = \varepsilon/2$, so when ε approaches 0:

$$\frac{d^2\tilde{f}}{d\tilde{y}^2} + \frac{d\tilde{f}}{d\tilde{y}} = 0, \quad (9)$$

we take again the solution in $\tilde{y} = 0$ which is always $f\tilde{(0)} = 0$, that is for it that we did all this work. The solution depends now on an up to now indeterminate constant A:

$$\tilde{f}(\tilde{y}) = A(1 - e^{-\tilde{y}}).$$

4.2.4 Asymptotic Matching

The last ingredient is the "Asymptotic Matching" between the developments, it states:

$$\lim_{y\to 0}[f(y)] = \lim_{\tilde{y}\to\infty}[\tilde{f}(\tilde{y})]$$

This is written in Van Dyke's book p 90:

The inner limit of (the outer limit)

= the outer limit of (the inner limit).

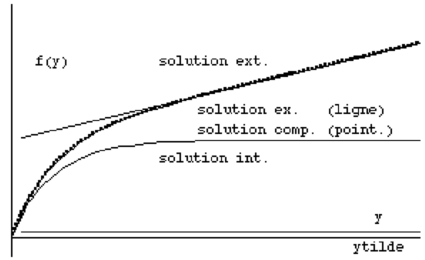


Figure 4: Solution of (6): plot of the exact solution (`ex.`), composite solution (`comp.`) outer solution (`ext.`) as function of y. We superposed the inner solution (`int.`) which evolves with the streched variable $\tilde{y}$

The first is A the second is $1/2$. The internal solution is:

$$\tilde{f} = \frac{1}{2}(1 - e^{-\tilde{y}}).$$

So that the problem is now solved in the two layers, the external and the internal at a different scale.

4.2.5 "Composite expansion" or "uniform approximation"

One problem of the previous solution is that it has two representations in two regions. The ε is supposed vanishingly small. To obtain a practical solution usable in the whole domain and with a given enough small ε, one creates the "composite expansion" or "uniform approximation". The composite solution is written as the sum of the solution in the external layer plus the solution in the internal layer (written with the external variable) minus the common limit:

$$f_{comp}(y) = \frac{y+1}{2} + \frac{1 - e^{-y/\varepsilon}}{2} - \frac{1}{2},$$

This approximation is uniformly valid in the whole domain. For any given enough small ε one has then the solution of the problem.

Fundamental remark:
On this example we see that the full solution:

$$f(y) = \frac{1 - e^{-y/\varepsilon}}{2(1 - e^{-1/\varepsilon})} + \frac{y}{2}, \quad \text{develops for small } \varepsilon \text{ in } f(y) = \frac{1 - e^{-y/\varepsilon}}{2(1-0)} + \frac{y}{2}$$

But we have not finished to deal with small ε. It is a first stage. At this fist stage, we see that we have the expression

$$\frac{1-e^{-y/\varepsilon}}{2}+\frac{y}{2}=\frac{1-e^{-y/\varepsilon}}{2}-\frac{1}{2}+\frac{y}{2}+\frac{1}{2},$$

which is no more that the composite expansion.

We go further in the development: if y is fixed to a peculiar value, then y/ε approaches infinity and $f(y)$ becomes $(y+1)/2$. We have found the external solution. If now we consider that y approaches 0 as well as ε approaches 0, then y/ε may be fixed and is of order one, so that the full solution $f(y)$ becomes

$$f(y)=\frac{1-e^{-y/\varepsilon}}{2(1-0)}+0.$$

We have found again the internal solution.

So, an a simple example, we have seen that the method is consistent. The exact full solution contains the composite expansion. And of course the composite expansion contains the two asymptotic expansions. So the method works well.

4.2.6 Boundary condition: where is the Boundary Layer?

We have just seen that the method works well. But we can have some questions about the position of the singularity. The position of the boundary condition must be discussed. Instead of looking at what happens in $y=0$ and instead of putting $f(1)=1$, let us say $f(0)=0$. What happens next, is the technique still working? Then for the outer problem, we find $f=y/2$ we can not satisfy $f(1)=1$. So we introduce in 1 a boundary layer where we write

$$y=1+\varepsilon\tilde{y}$$

so that we have always $\tilde{f}''+\tilde{f}'=0$ to solve (but at another place). We have always a solution involving exponentials: $\tilde{f}=Ae^{-\tilde{y}}-A+1$. But the matching is $\tilde{y}\to-\infty$ to be matched with $y\to 1^-$. It is impossible as the exponential is not bounded. The boundary layer was in 0, not in 1. So the method works well, it does not introduce spurious boundary layers.

Finding the exact position of the singularity of the equation is difficult for a given abstract E_ε problem. in practice, we have some clues coming from the physical problem which has been modeled by the E_ε problem.

4.2.7 Other scale: what happens if we take a larger/ smaller scale?

We can have some questions about the scales. If we take another scale ε^α, with $\alpha\neq 1$, what happens? Simply we have less terms and we obtain linear solutions that we can not match. Either we recover alway $f'=1/2$ for $\alpha<1$ (we are always too far from the origin). Or we recover $f''=0$ for $\alpha>1$, which is not enough to solve the problem, we are always too close of the origin, and any linear solution is solution. So the method with the "Dominant Balance" works well.

4.3 Back to the Matching and other orders

4.3.1 Other orders

Up to now we implicitly were looking at the first order solution f_0 of the problem. We introduced the matching, and solved the problem. But a better approximation will imply the other terms f_1, f_2 .... and when there are more than one term the asymptotic matching we defined is not enough. Let us do the computations from scratch

- Looking for an expansion as:

$$f=f_0+\nu_1 f_1+\nu_2 f_2+...$$

by substitution in the $\varepsilon f''+f'=1/2$ problem:

$$\varepsilon(f_0''+\nu_1 f_1''+\nu_2 f_2''+...)+(f_0'+\nu_1 f_1'+\nu_2 f_2'+...)=1/2$$

we identify $\nu_1=\varepsilon$, and it is clear that the good choice is $\nu_k=\varepsilon^k$, so that we order:

$$(f_0'-1/2)\varepsilon^0+(f_0''+f_1')\varepsilon^1+(f_1''+f_2')\varepsilon^2+O(\varepsilon^3)=0$$

we have $f_0=(y+1)/2$ (as $f_0(1)=1$) and for the next order the equation is $f_0''+f_1'=0$, as $f_0''=0$ then we have $f_1'=0$ so to $f_1=0$ verifies the boundary conditions.

The expansion is $f = \frac{y+1}{2} + O(\varepsilon^2)$

- Looking now at the inner problem: $\tilde{f}'' + \tilde{f}' = \varepsilon/2$:

$$\tilde{f} = \tilde{f}_0 + \tilde{\nu}_1 \tilde{f}_1 + \tilde{\nu}_2 \tilde{f}_2 + ...$$

by substitution, at order 0 : $\tilde{f}_0'' + \tilde{f}_0' = 0$, giving $\tilde{f}_0 = A(1 - e^{-\tilde{y}})$. Then we have

$$(\tilde{f}_0'' + \tilde{f}_0') + \tilde{\nu}_1(\tilde{f}_1'' + \tilde{f}_1') - \varepsilon/2 + ... = 0$$

gives $\tilde{\nu}_1 = \varepsilon$ as a good choice for the expansion and then $\tilde{f}_1'' + \tilde{f}_1' = 1/2$ with $\tilde{f}_1(0) = 0$. The solution is then: $\tilde{f}_1 = A_1(e^{-\tilde{y}} - 1) + \tilde{y}/2$. The new constant must be obtained by the matching, we guess that putting $\tilde{y}$ to infinity is not working here. So we come back to the "matching".

4.3.2 Back to the matching: intermediate layer

Let us come back to the concept of matching and present another point of view. We forget the paragraph on matching at first order and start again with the solutions with unknown constants.

We have two asymptotic expansions for the solution, one for fixed y and the other for fixed $\tilde{y}$. The outer solution $\frac{y+1}{2} + o(\varepsilon)$ and the inner solution is $A(1 - e^{-\tilde{y}}) + \varepsilon(A_1(e^{-\tilde{y}} - 1) + \tilde{y}/2) + o(\varepsilon)$. We claim that the two expansions are of similar form in an overlap region which has both $\tilde{y}$ large and y small. The matching is the process which consists in forcing the two expansions to be equal in this overlap region.

Say $\hat{y} = y/\eta(\varepsilon) = \tilde{y}\varepsilon/\eta(\varepsilon)$ with $\varepsilon << \eta(\varepsilon) << 1$.

$$f(y) = 1/2 + y/2 + O(\varepsilon^2) = 1/2 + \eta(\varepsilon)\hat{y} + ...$$

$$\tilde{f}(\tilde{y}) = A(1 - e^{-\tilde{y}}) + O(\varepsilon) = A - Ae^{-\eta\hat{y}/\varepsilon} + ...$$

comparing the two solutions we see that $A = 1/2$ as $e^{-\eta\hat{y}/\varepsilon}$ is exponentially small (because $\hat{y} = O(1)$ and $\eta/\varepsilon >> 1$.)

At next order

$$f(y) = 1/2 + y/2 + o(\varepsilon) = 1/2 + \eta(\varepsilon)\hat{y}/2 + ...$$

$$\tilde{f}(\tilde{y}) = A(1 - e^{-\tilde{y}}) + \varepsilon(A_1(e^{-\tilde{y}} - 1) + \tilde{y}/2) + o(\varepsilon)$$

$$= A - \varepsilon A_1 + \eta\hat{y}/2 - Ae^{-\eta\hat{y}/\varepsilon} + \varepsilon A_1 e^{-\eta\hat{y}/\varepsilon} + ...$$

then again $A = 1/2$ and $A_1 = 0$. the overlapping development is:

$$\hat{f} = 1/2 + \eta(\varepsilon)\hat{y}/2$$

The concept of overlapping layer is the more useful to construct the expansion at several orders.

4.3.3 Back to the matching: Van Dyke pq-qp Rule

Matching in an intermediate layer is sometimes difficult. So that Van Dyke has robotized the process in his pq-qp Rule (p220 Note 3, is in fact written $\Delta\delta - \delta\Delta$). Let us introduce the outer development with $p+1$ terms at y fixed and $\varepsilon \to 0$

$$D_p^o f = \sum_0^p \varepsilon^n f_n$$

and the inner development with $q+1$ terms at $\tilde{y}$ fixed and $\varepsilon \to 0$

$$D_q^i f = \sum_0^q \varepsilon^n \tilde{f}_n$$

The Van Dyke pq-qp Rule is then

$$D_p^o D_q^i f = D_q^i D_p^o f.$$

For the left hand side: one takes the inner solution to $q+1$ terms and change $\tilde{y}$ by y/ε. The outer limit of y fixed as $\varepsilon \to 0$ is then taken retaining $p+1$ terms. A similar process is done for the right hand side.

- With $p = q = 0$:

$$\begin{aligned} D_0^o D_0^i f =& D_0^o(A(1 - e^{-\tilde{y}})) \\ =& D_0^o(A(1 - e^{-y/\varepsilon})) \\ =& A \\ D_0^i D_0^o f =& D_0^i(1/2 + y/2)) \\ =& D_0^i(1/2 + \tilde{y}\varepsilon/2)) \\ =& 1/2 \end{aligned} \tag{10}$$

so that $A = 1/2$.

- With $p = q = 1$:

$$\begin{aligned}
D_1^o D_1^i f =& D_1^o(A(1 - e^{-\tilde{y}}) + \varepsilon(A_1(e^{-\tilde{y}} - 1) + \tilde{y}/2)) \\
=& D_1^o(A(1 - e^{-y/\varepsilon}) + \varepsilon A_1(e^{-y/\varepsilon} - 1) + y/2)) \\
=& A + y/2 - \varepsilon A_1 \\
D_1^i D_1^o f =& D_1^i(1/2 + y/2)) \\
=& D_1^i(1/2 + \tilde{y}\varepsilon/2)) \\
=& 1/2 + \tilde{y}\varepsilon/2
\end{aligned} \tag{11}$$

so that $A = 1/2$ and $A_1 = 0$.

4.3.4 Back to the matching: in practice

In practice (and as introduced at first in this curse), often we use the asymptotic matching in the following simple way :

$$\lim_{\tilde{y}\to\infty}[\tilde{f}(\tilde{y})] = \lim_{y\to 0}[f(y)]$$

which is the $p = q = 0$.

The overlapping region is a good approach as well.

5 Simple mechanical examples with small mass (ODE)

5.1 Cole oscillator example:

5.1.1 Physical problem

We come back here to simple mechanical systems. Let us consider another famous example introduced by Cole in his courses on asymptotics see [10]. Let us take a simple oscillator with stifness k, damping β, and small mass m. At initial time the mass is at rest, but one kicks on it so that it gains a momentum P_0. The equations are:

$$m\frac{d^2y}{dt^2}+\beta\frac{dy}{dt}+ky=0,$$

with initial conditions $y(0)=0$ and $m\frac{dy}{dt}(0)=P_0$. This last condition corresponds to an instantaneous release of momentum through a shock.

We look at a solution for a small mass. So first we make the problem non dimensional to quantify the smallness of the mass.

5.1.2 non dimensional problem

Let be Y oscillation size and τ characteristic time, the non dimensional variables are $\bar{y}=y/Y$ and $\bar{t}=t/\tau$. So that

$$m\tau^{-2}k^{-1}\frac{d^2\bar{y}}{d\bar{t}^2}+\beta\tau^{-1}k^{-1}\frac{d\bar{y}}{d\bar{t}}+\bar{y}=0$$

$$\bar{y}(0)=0,\text{ and }mY\tau^{-1}\frac{d\bar{y}}{d\bar{t}}(0)=P_0.$$

initial momentum is given, so Y and τ are such that $mY/\tau=P_0$.

As we guess that the oscillator will be damped a lot, due to his small mass, we think that the spring and dash pot terms will be important, as the mass is small. So we take $\tau=\beta/k$. With this choice, we have $m\tau^{-2}k^{-1}=mk/\beta^2$, so that now it is clear that a small mass problem is a problem with $m\ll\beta^2/k$, with the asymptotic point of view, we define $\varepsilon=mk/\beta^2$, with $\varepsilon\ll 1$.

At this point we can notice that the characteristic time of oscillation is clearly $\sqrt{m/k}$, it is smaller than the relaxation time β/k, so $m/k\ll\beta^2/k^2$ or $\varepsilon\ll 1$.)

The amplitude is obtained from $(mY/\tau)\bar{y}'(0)=P_0$ but with ε this is $(\varepsilon\beta^2/k)Y/(\beta/k)\bar{y}'(0)=P_0$. The amplitude is $Y=P_0/\beta$.

When the mass is small (linked to ε) we obtain the following problem $E_\varepsilon=0$ without dimension, we have removed the bars for simplicity ("Cole" problem [16] p 40):

$$\varepsilon y''+y'+y=0,\;\; y(0)=0,\;\; \varepsilon y'(0)=1.$$

5.1.3 Exact solution

It has an exact solution

$$\frac{1}{\sqrt{1-4\varepsilon}}\left(exp(-(1-\sqrt{1-4\varepsilon}t/(2\varepsilon)-exp(-(1+\sqrt{1-4\varepsilon}t/(2\varepsilon)\right),$$

this exact solution will be useful to compare with the approximate one that we are now building.

5.1.4 Asymptotic solution

But, if we apply the technique, we try to solve $E_\varepsilon=0$ with small ε.

- the outer problem is :

$$y'+y=0.$$

We can verify that the outer solution is $y(t)=Ae^{-t}$, but BC in 0 have disappeared! A is undetermined.

- the inner problem. We look at what happens in 0, by dominant balance the scale is ε. so the inner variable is defined by $t=\varepsilon\tilde{t}$, and the inner problem

$$\tilde{y}''+\tilde{y}'=0,\;\; \tilde{y}(0)=0\;\tilde{y}'(0)=1,$$

the inner solution is

$$\tilde{y}=1-e^{-\tilde{t}}.$$

The matching $\tilde{y}(\infty)=y(0^+)$ gives $A=1$.

• The composite expansion

$$y_{comp}(t) = e^{-t} - e^{-t/\varepsilon}$$

• The next order gives after expansion and matching in an intermediate layer:

$$outer.\ \ y(t) = e^{-t} + \varepsilon(2-t)e^{-t} + ...$$

$$inner.\ \ \tilde{y}(\tilde{t}) = 1 - e^{-\tilde{t}} + \varepsilon[(2-\tilde{t}) - (2+\tilde{t})e^{-\tilde{t}}] + ...$$

• we can expand the exact solution and verify the expansions.

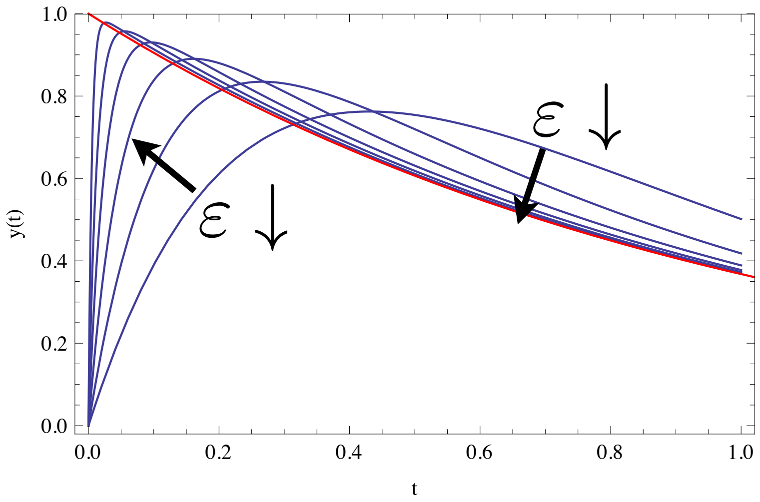


Figure 5: The smaller ε (0.2 0.1 0.05 0.0125 .005) the closer the exact solution is from e^{-t} the outer solution (red).

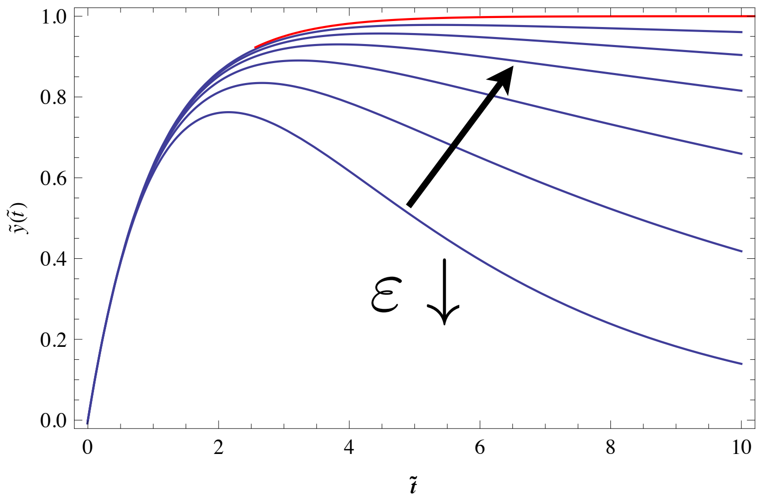


Figure 6: The smaller ε (0.2 0.1 0.05 0.0125 .005) the closer the exact solution is from $1 - e^{-\tilde{t}}$ the inner solution (red).

5.2 Free fall of a small bead

We look at the displacement of a small ball of very small mass in a very viscous flow, in the gravity field. The ball is initially at rest, we look at the position as function of time. This was an exam in 2021.

5.2.1 Questions

2.1 Show that we obtain the following equation, (of course ε is a given small parameter that you have to define with the parameters of the problem and you have to decide the proper orientation of motion)

$$(E_\varepsilon) \quad \varepsilon y''(t) = -y'(t) - 1 \text{ with } y(0) = 0, \quad y'(0) = 0.$$

We want to solve this unsteady problem with the Matched Asymptotic Expansion method.
2.2 Why is (E_ε) problem singular?
2.3 What is the outer problem and what is the possible general form of the outer solution?
2.4 What is the inner problem of (E_ε) and what is the inner solution? (hint: for the inner problem time is small and displacement y is small as well)
2.5 Suggest the plot of the inner and outer solution.
2.6 What is the exact solution of (E_ε) for any ε. Check that we recover inner and outer solution.
2.7 Comments?

5.2.2 Answers

2.1 Newton's law for a mass falling in gravity with viscous friction is

$$m\frac{d^2y}{dt^2} = -mg - 6\pi\mu R\frac{dy}{dt}.$$

We have for sure a competition between free fall mg and viscous drag. A natural velocity is the Stokes velocity $V_s = mg/(6\pi\mu R)$, this is the terminal chute velocity. We define the scales $y = Y\bar{y}$ and $t = \tau\bar{t}$, we have:

$$\frac{V_s}{g}\frac{Y}{\tau^2}\frac{d^2\bar{y}}{d\bar{t}^2} = -V_s - \frac{Y d\bar{y}}{\tau d\bar{t}},$$

hence we take $\frac{Y}{\tau} = V_s$ and we identify $\varepsilon = V_s/(g\tau)$, so that we obtain the following ODE

$$\varepsilon\frac{d^2\bar{y}}{d\bar{t}^2} = -1 - \frac{d\bar{y}}{d\bar{t}}.$$

Boundary condition are same: $\bar{y}(0) = 0$ and $\bar{y}'(0) = 0$. Indeed, the ratio $\frac{V_s}{g\tau}$ is small if velocity scale $g\tau$ of free fall is large compared to the Stokes velocity. Or when the time scale τ compared to the time scale V_s/g is large. Or if the mass is small, or if viscosity is small.....

2.2 Problem singular for small ε, indeed, if we put $\varepsilon = 0$, we have 2 BC, but only one degree of derivation , $\bar{y}_{out}(0) = 0$ and $\bar{y}'_{out}(0) = 0$

$$0 = -1 - \frac{d\bar{y}_{out}}{d\bar{t}}$$

we take $\bar{y}_{out}(0) = 0$ so that $\bar{y}_{out}(t) = -\bar{t}$, the problem is in $\bar{t} = 0$ where $\bar{y}'_{out}(0) = -1 \neq 0$

2.3 So, as we have identified a problem at small time scale, near the origin, we change the scale of time $\bar{t} = \tau_\varepsilon\tilde{t}$ and space $\bar{y} = \nu_\varepsilon\tilde{y}$

$$\varepsilon\frac{\nu_\varepsilon d^2\tilde{y}}{\tau_\varepsilon^2 d\tilde{t}^2} = -1 - \frac{\nu_\varepsilon d\tilde{y}}{\tau_\varepsilon d\tilde{t}}$$

A full dominant balance gives $\nu_\varepsilon = \tau_\varepsilon$ and $\varepsilon\frac{\nu_\varepsilon}{\tau_\varepsilon^2} = 1$ so that $\nu_\varepsilon = \tau_\varepsilon = \varepsilon$.
The problem is in the new small scales :

$$\frac{d^2\tilde{y}}{d\tilde{t}^2} = -1 - \frac{d\tilde{y}}{d\tilde{t}}, \text{ 2 BC:} \quad \tilde{y}(0) = 0, \quad \tilde{y}'(0) = 0$$

It is no more singular, the solution is $\tilde{y} = -\tilde{t} + A + Be^{-\tilde{t}}$ with BC in 0 gives $0 = -0 + A + B$ and $\tilde{y}'(0) = 0$ which give $0 = -1 + 0 - B$ hence :

$$\tilde{y} = 1 - \tilde{t} - e^{-\tilde{t}} \text{ and } \tilde{y}' = -1 + e^{-\tilde{t}}.$$

There is no need to match at this order, matching will appear at next order. Note the matching on velocity is verified

$$\lim_{\tilde{y}\to\infty}\left(\frac{\varepsilon d\tilde{y}}{\varepsilon d\tilde{y}}\right) = \lim_{\bar{t}\to 0}\left(\frac{d\bar{y}}{d\bar{t}}\right).$$

As when $\tilde{y} \to \infty$ then $\tilde{y} \sim -1 + \tilde{t}$ shows that the displacement induced at small time is of order ε. This will be used at next order....

2.4 The full solution of the problem is

$$\bar{y} = \varepsilon - \bar{t} - \varepsilon e^{-\bar{t}/\varepsilon}$$

we see that indeed, for $\varepsilon \to 0$

$$\bar{y} = -\bar{t}$$

as seen for the external solution, we see as well the ε small displacement induced by the small time, $\bar{t} = \varepsilon\tilde{t}$, corresponding to the interanl problem:

$$\varepsilon\tilde{y} = \varepsilon - \varepsilon\tilde{t} - \varepsilon e^{-\tilde{t}}$$

Finally, note that if we take very small time

$$\bar{y} = \varepsilon - \bar{t} - \varepsilon(1 - \bar{t}/\varepsilon + \bar{t}^2/\varepsilon^2/2... = -\bar{t}^2/\varepsilon/2 = -\varepsilon\tilde{t}^2/2 + ...$$

this is the free fall.

```
DSolve[{eps y''[t] == -y'[t] - 1, y[0] == 0, y'[0] == 0}, y[t], t]
Expand[E^(-(t/eps)) (-eps + E^(t/eps) eps - E^(t/eps) t)]

DSolve[{ y''[t] == -y'[t] - 1, y[0] == 0, y'[0] == 0}, y[t], t]
```

6 A very simple example from Fluid Mechanics

6.1 Problem: Steady Poiseuille flow with uniform blowing at one wall and aspiration at the opposite wall.

We will look at the viscous flow of an incompressible fluide. this is an exact solution of Navier- Stokes equations (see Paterson [23] and Kundu as well [18]). This flow is not very realistic but allows a complete instructive resolution. Let us look at a 2D plan flow (with no velocity along z) steady

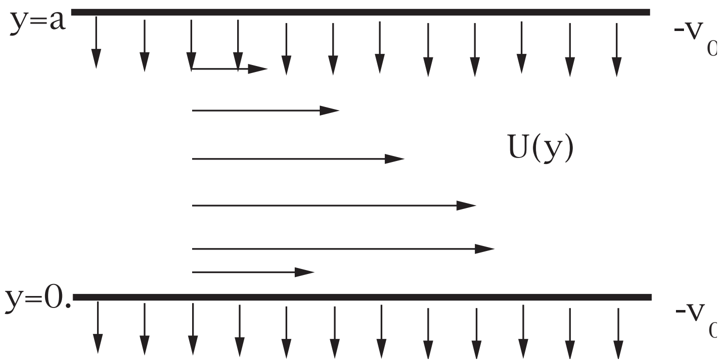


Figure 7: A 2D channel flow from left to right due to an imposed pressure gradient with suction at the lower wall and blowing at the upper.

and incompressible of fluid of contant density ρ and constant viscosītye ν. The fluid flows between two parallel infinite flate plates. One is in $y = 0$, the other in $y = a$. The two plates are porous so that fluid is blow at one wall and sucked at the other. So, v is $-v_0$ at the walls. A pressure gradient (k) along x is imposed to drive the flow. We will see that a new scale (which is not a the width of the channel) appears from the resolution. Of course, the physical problem has itself another scale : the size of the small holes in which the fluid is aspired. This would induce a local analysis out of the scope of this chapter but in tractable with asymptotic analysis.

6.2 Direct resolution:

We look at a solution which is invariant by translation in x:

$$\vec{u}(x, y, z) = U(y)\,\vec{e}_x + V(y)\,\vec{e}_y + 0\,\vec{e}_z.$$

With the boundary conditions and incompressibility we have the transverse velocity:

$$V(y) = -v_0.$$

The Navier Stokes equation then reads:

$$0 = -\frac{\partial p}{\partial y} \quad \text{so} \quad p(x) = -kx + P_0.$$

Which is a simple ODE (12) for $U(y)$:

$$-v_0 \frac{\partial U}{\partial y} = \frac{k}{\rho} + \nu \frac{\partial^2 U}{\partial y^2}. \tag{12}$$

With the no slip boundary conditions we have:

$$U_{exact}(y) = \frac{ka}{\rho v_0}\left(-\frac{y}{a} + \left(\frac{1 - exp(-v_0 y/\nu)}{1 - exp(-v_0 a/\nu)}\right)\right).$$

In fact, for this flow, there are several characteristic velocities that we can use to make the problem non-dimensional: v_0, $(\frac{ka^2}{\rho\nu})$. We can construct two non dimensional numbers

6.3 Non dimensional equation

Say $y = a\bar{y}$ et $u = U_0\bar{u}$, and play the game: we do not know the exact solution. We do not know U_0 up to now. The non dimensional equation is:

$$-(v_0 a/\nu)\frac{\partial \bar{U}}{\partial \bar{y}} = ka^2/(\rho\nu U_0) + \frac{\partial^2 \bar{U}}{\partial \bar{y}^2}. \tag{13}$$

If we take $U_0 = ka^2/\rho\nu$, the driving pressure balances the viscous dissipation. The parameter for aspiration is $(v_0 a/\nu)$ (a Reynolds). The non dimensional equation is:

$$-(v_0 a/\nu)\frac{\partial \bar{U}}{\partial \bar{y}} = 1 + \frac{\partial^2 \bar{U}}{\partial \bar{y}^2} \tag{14}$$

with $\bar{U}(0) = \bar{U}(1) = 0$.

Figure 8: Basic Poiseuille profile, [click on the Image, Acrobat Quick-Time Required] to see the effect of aspiration

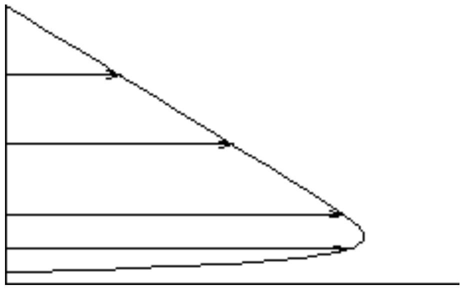


Figure 9: The profile with strong suction at the lower wall and blowing at the upper. [click on the Image, Acrobat QuickTime Required] to see the effect of aspiration.

6.4 Light suction/blowing

We verify in this section that $Solution\left[\underset{\varepsilon\to 0}{E_\varepsilon}\right] = \underset{\varepsilon\to 0}{Solution}[E_\varepsilon]$. The E_ε problem is:

$$\bar{U}'' + \varepsilon\bar{U}' + 1 = 0, \text{ and } \bar{U}(0) = \bar{U}(1) = 0$$

For the $Solution\left[\underset{\varepsilon\to 0}{E_\varepsilon}\right]$, we easily verify that the Poiseuille with light suction problem is regular: the solution for $\varepsilon = 0$ is :

$$\frac{\bar{y}}{2} - \frac{\bar{y}^2}{2}$$

the next order is defined by (cf figure 6.2): $\bar{U} = \frac{\bar{y}}{2} - \frac{\bar{y}^2}{2} + \varepsilon\bar{u}_1 + ...$ if we put this in the ε problem $\bar{U}'' + \varepsilon\bar{U}' + 1 = 0$, the equation for $\bar{u}_1$ is

$$\bar{u}_1'' + 1/2 - \bar{y} = 0 \text{ and } \bar{u}_1(0) = \bar{u}_1(1) = 0$$

so that the pertubation is:

$$\bar{u}_1 = \frac{\bar{y}^3}{6} - \frac{\bar{y}^2}{4} + \frac{\bar{y}}{12}$$

etc at the next order.

Now we turn to $\underset{\varepsilon\to 0}{Solution}[E_\varepsilon]$. So solution of E_ε is :

$$\bar{U}_{exact} = -\frac{-e^\varepsilon\bar{y} + \bar{y} + e^\varepsilon - e^{\varepsilon-\varepsilon\bar{y}}}{\varepsilon - \varepsilon e^\varepsilon}$$

we verify that everything is OK:

$$\bar{U}_{exact} = -\frac{-e^\varepsilon\bar{y} + \bar{y} + e^\varepsilon - e^{\varepsilon-\varepsilon\bar{y}}}{\varepsilon - \varepsilon e^\varepsilon} = (\frac{\bar{y}}{2} - \frac{\bar{y}^2}{2}) + (\frac{\bar{y}^3}{6} - \frac{\bar{y}^2}{4} + \frac{\bar{y}}{12})\varepsilon + ...$$

So, we obtain the Poiseuille flow plus the previous small perturbation $\bar{u}_1$. The problem is regular.

6.5 Strong suction/blowing

Now we take $U_0 = ka/\rho/v_0$, with this choice, it correspond to a balance between aspiration and pressure gradient. The new parameter is again the suction Reynolds $(v_0 a/\nu)$,but this time $\varepsilon = \nu/(av_0)$. The E_ε problem is

$$\varepsilon\bar{U}'' + \bar{U}' + 1 = 0; \text{ with } \bar{U}(0) = \bar{U}(1) = 0.$$

6.5.1 case $\varepsilon = 0$

Let put $\varepsilon = 0$, so:

$$\bar{U} = 1 - \bar{y}.$$

If we take the boundary in $\bar{y} = 1$, there is a problem in $\bar{y} = 0$!!! The velocity slips: $\bar{U}(0) = 1$, it should be zero. We call this solution the "outer solution". The problem is clearly singular as near the wall, there is a problem?

6.5.2 Boundary layer "inner solution"

To full fill the boundary condition, we have to change the scale. In fact, near the wall, the velocity changes very quickly from 1 to 0. This is so fast that the $\varepsilon\bar{U}''$ term is no more negligible.

Let us define $\tilde{U} = \bar{U}$ and $\bar{y} = \delta\tilde{y}$, with $\delta << 1$. By substitution, with these new scales:

$$\varepsilon\frac{1}{\delta^2}\frac{d^2\tilde{U}}{d\tilde{y}^2} + \frac{1}{\delta}\frac{d\tilde{U}}{d\tilde{y}} + 1 = 0$$

In this new desxcription, $\frac{d\tilde{U}}{d\tilde{y}}$ is of order one, the velocity changes slowly. But $\frac{d\bar{U}}{d\bar{y}}$ is very large, the order of magnitude is $\frac{1}{\delta}$. Looking at the equation, we guess that the third term (1) is smaller than the second. The first one contains a large parameter ($\frac{1}{\delta^2}$), but is multiplied by a very small parameter ε.

We have here to introduce the "Dominant Balance Principle" (*Principe de Moindre dégénérescence*, in french) : we want to simplify the problem but to retain in the problem as much term as possible. So we simplified previously the second order derivative, this was not a good choice of simplification. As we have lost a boundary condition, we keep it and we take

$$\varepsilon\frac{1}{\delta^2} = \frac{1}{\delta} \quad \text{which is} \quad \delta = \varepsilon$$

The equation is then

$$\frac{d^2\tilde{U}}{d\tilde{y}^2} + \frac{d\tilde{U}}{d\tilde{y}} = 0$$

The solution is $\tilde{U} = A(1 - e^{\tilde{y}})$ which satifies the lost $\tilde{U}(0) = 0$. That is due to this condition that we did such an hard job. But, we have now an indeterminate constant A.

6.6 Matching

To identify this indeterminate constant, let us introduce the last ingredient, the asymptotic matching:

$$\lim_{\bar{y}\to 0} \bar{U}(\bar{y}) = \lim_{\tilde{y}\to\infty} \tilde{U}(\tilde{y})$$

this tells that we have now two asymptotic expansions for the solution, one for $\bar{y}$ of order one, and the other for $\tilde{y}$ of order one. These two expansions should be somewhere in a similar form. This is a kind of overlap region which has $\bar{y}$ small and $\tilde{y}$ large. Forcing the two expansion to be equal in this limits of $\bar{y}$ small and $\tilde{y}$ large gives the unknown A.

In practice, it gives $\bar{U}(0) = 1$which should be equal to $\tilde{U}(\tilde{y})$ at infinity, i.e. $\tilde{U}(\infty) = A$. So $A = 1$, the velocity in the inner region is then:

$$\tilde{U} = (1 - e^{-\tilde{y}}).$$

So we have expressed the solution after slpitting the the domain in two regions. An outer one, where the scale is the most evident one, and a second the inner one, where we had to process a change of scale to focus on the boundary. Both region over lap.

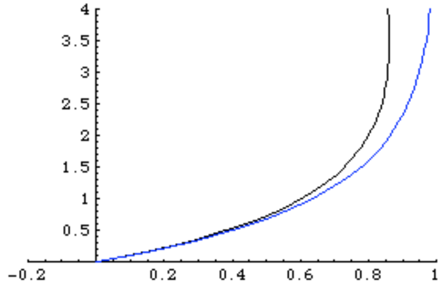


Figure 10: The exact solution profile seen in the boundary Layer scales and the asymptotic solution, for increasing suction [Adobe / QuickTime]

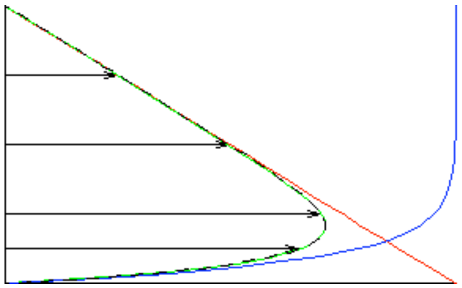


Figure 11: Composite and exact solutions seen in the 0(1) scales for increasing suction [Adobe / QuickTime].

6.7 Composite Expansion

But, as we have just seen, both solutions, the outer and inner ones, are valid for different scales and different layers:

$$1 \geq \bar{y} > 0 \quad \text{we have} \quad \bar{U} = 1 - \bar{y}$$

$$\infty > \tilde{y} \geq 0 \quad \text{we have} \quad \tilde{U} = (1 - e^{-\tilde{y}})$$

We can put all this together if we define a composite expansion:

$$\bar{U}_{composite} = \bar{U}(\bar{y}) + \tilde{U}(\bar{y}/\varepsilon) - \bar{U}(0)$$

which gives

$$\bar{U}_{composite} = 1 - \bar{y} + (1 - e^{-\bar{y}/\varepsilon}) - 1$$

On figure 11 we present whith a dashed curve the exact solution and in green the comosite one. They are very close.

6.8 Next Orders

it is simple to see that for this example, $\varepsilon\bar{U}'' + \bar{U}' + 1 = 0$ for the outer problem, we develop

$$\bar{U} = \bar{U}_0 + \varepsilon\bar{U}_1 + O(\varepsilon^2)$$

and by substitution, we have to solve $\bar{U}_0' + 1 = 0$ and $\bar{U}_0'' + \bar{U}_1' = 0$ so that $\bar{U}_0 = 1 - \bar{y}$ and $\bar{U}_1 = 0$.

For the inner problem: $\varepsilon\tilde{U}'' + \tilde{U}' + \varepsilon\tilde{U} = 0$ we developp

$$\tilde{U} = \tilde{U}_0 + \varepsilon\tilde{U}_1 + O(\varepsilon^2)$$

we have to solve $\tilde{U}_0'' + \tilde{U}_0' = 0$ and next $\tilde{U}_1'' + \tilde{U}_1' + 1 = 0$ so that $\tilde{U}_0 = A(1 - exp(-\tilde{y}))$ and $\tilde{U}_1 = A_1(exp(-\tilde{y}) - 1) - \tilde{y}$

6.8.1 Intermediate layer

Matching in the intermediate layer $\hat{y} = \bar{y}/\varepsilon$ gives

$$\bar{U} = 1 - \eta\hat{y} + 0\varepsilon$$

$$\tilde{U} = A - 0 - \eta\hat{y} - A_1\varepsilon$$

so that $A = 1$ and $A_1 = 0$.

6.8.2 pq-qp

Again the Van Dyke rule will give the same result.

$$D_p^o D_q^i \bar{U} = D_q^i D_p^o \bar{U}.$$

For the left hand side: one takes the inner solution to $q + 1$ terms and change $\tilde{y}$ by y/ε. The outer limit of y fixed as $\varepsilon \to 0$ is then taken retaining $p + 1$ terms. A similar process is done for the right hand side.

With $p = q = 1$:

$$\begin{aligned}
D_1^o D_1^i \bar{U} =& D_1^o(A(1 - e^{-\tilde{y}}) + \varepsilon(A_1(e^{-\tilde{y}} - 1) - \tilde{y})) \\
=& D_1^o(A(1 - e^{-\bar{y}/\varepsilon}) + \varepsilon A_1(e^{-\bar{y}/\varepsilon} - 1) - \bar{y})) \\
=& A - \varepsilon\bar{y} - \varepsilon A_1 \\
D_1^i D_1^o U =& D_1^i(1 - \bar{y})) \\
=& D_1^i(1 - \bar{y}\varepsilon)) \\
=& 1
\end{aligned} \tag{15}$$

so that $A = 1$ and $A_1 = 0$.

6.9 For the "Saint Thomas"

"Saint Thomas" (in french) is a guy who believes only what he sees (in english "Doubting Thomas" a skeptic guy). Let us take teh exact solution of the ODE :

$$\bar{U}_{exact} = -\bar{y} + \frac{(1 - e^{-\bar{y}/\varepsilon})}{(1 - e^{-1/\varepsilon})}$$

when $\bar{y}$ is fixed and when ε approaches 0, we have:

$$\bar{U}_{exact} \simeq -\bar{y} + \frac{(1 - 0)}{(1 - 0)} = 1 - \bar{y}$$

We obtain the external solution.

If now we fix $\bar{y}/\varepsilon$and ε approaches 0, we find:

$$\bar{U}_{exact} \simeq -\varepsilon(\bar{y}/\varepsilon) + \frac{(1 - e^{\bar{y}/\varepsilon})}{(1 - 0)} \simeq (1 - e^{-\bar{y}/\varepsilon})$$

We obtain the internal solution.

Much more interesting, let us keep $\bar{y}/\varepsilon$ and do not expand it, do not touch to $\bar{y}$, and let ε approaches 0, hence:

$$\bar{U}_{exact} \simeq -\bar{y} + \frac{(1 - e^{-\bar{y}/\varepsilon})}{(1 - 0)} \simeq (1 - e^{-\bar{y}/\varepsilon}) = (1 - \bar{y}) + (1 - e^{-\bar{y}/\varepsilon}) - 1$$

We obtain exactly the composite expansion.

6.10 using *Gerris* and freefem++

6.10.1 The problem

For illustration of the utility of the knowledge of existence of singular problems, we will solve the problem (12) with a "solver". A "solver" means a numerical code, "codes" are commercial or free softwares solving the equations of Physics, and here more specifically equations of Mechanics. We will use *Gerris*, a free Navier Stokes solver, and **freefem++** a free PDE solver.

We adimensionalise the problem with the physical parameters so a, and the pressure gradient (k, the pressure gradient is a source term in the equation) will be taken to unity in the final non dimensional problem so in (12) $U = u_0\bar{u}$ and $y = a\bar{y}$:

$$-(\frac{v_0}{u_0})\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{ka}{\rho u_0^2} + \frac{\nu}{u_0 a}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{16}$$

with u_0 so that $\frac{ka}{\rho u_0^2} = 1$, and so, if $v_a = \frac{v_0}{u_0}$:

$$-v_a\frac{\partial \bar{u}}{\partial \bar{y}} = 1 + \frac{1}{Re}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{17}$$

The exact solution:

$$\bar{u}(\bar{y}) = \frac{1}{v_a}(-\bar{y} + (\frac{1 - exp(-v_a Re\bar{y})}{1 - exp(-v_a Re)}))$$

The full problem depends on x, so we use a periodic domain with $\partial_x = 0$ at the borders. so, in fact the problem that we solve is something like:

$$\frac{\partial \bar{u}}{\partial \bar{t}} - v_a\frac{\partial \bar{u}}{\partial \bar{y}} = 1 + \frac{1}{Re}(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}) \tag{18}$$

in a square with homogenous Neumann conditions right and left, and we wait for a steady solution.

Here are the scripts for (12) with first *Gerris*, and then freefem++.

6.10.2 The scripts

script:

```
################################################################
# 29/09/10  Poiseuille aspire par PYL, sauver dans "aspois0.gfs"
# lancer avec:    gerris2D -DRe=50 -DVa=0.1 aspois0.gfs | gfsview2D
# valeur du Reynolds et de l'aspiration passes en parametres
# definition de 1 boite avec 1 connection
# met le coin gauche en 0,0
1 1 GfsSimulation GfsBox GfsGEdge{x = 0.5 y = 0.5 } {
  SourceViscosity {} 1./Re
# precision 2**(-4.) = 1/16=0.06   5-> 32  0.03
  Refine 4
# temps initial 0
  Init {} { U = 0 }
# on impose un gradient de pression
  Source {} U 1.
# AdaptGradient { istep = 1 } { cmax = .1 maxlevel = 5 } U
#  GfsAdaptVorticity { istep = 1 } { maxlevel = 5 cmax = 1e-1 }
# sortie tous les 20 pas de calculs du temps en cours
  OutputTime { istep = 20 } stderr
# valeurs qui vont sortir pour entrer dans gfsview
# tous les 20 pas de calcul
  OutputSimulation { istep = 20 } stdout
  OutputSimulation { istep = 20 } SIM/sim-%g.txt {format = text}
  EventScript { istep = 20 } { cp SIM/sim-$GfsTime.txt sim.data}
# arret lorsque la variation de U devient "petite"
  EventStop { istep = 10 } U 1.e-4 DU
}

#conditions aux limites
GfsBox {
# en haut vitesse nulle
          top = GfsBoundary {
            GfsBcDirichlet U 0
            GfsBcDirichlet V -Va
          }
# en bas vitesse nulle
        bottom = GfsBoundary {
            GfsBcDirichlet  U 0
```

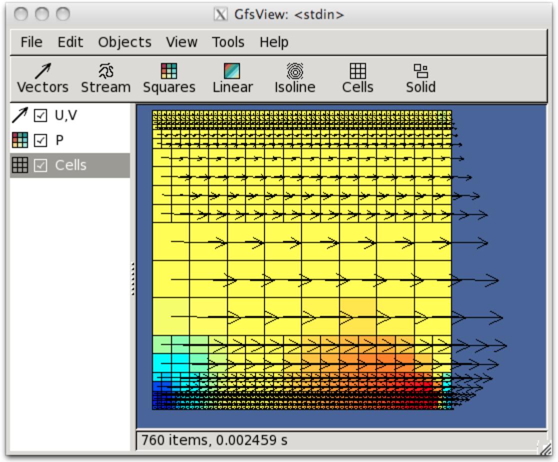
```
            GfsBcDirichlet V -Va
          }
}
# branchement périodique
1 1 right
# fin de fichier
######################################################################
```

file `run.sh` to run the *Gerris* script:

```
#!/bin/bash
mkdir SIM
for zeVa in 0.01 0.1 1 ; do
gerris2D -DRe=50 -DVa=$zeVa aspois0.gfs | gfsview2D v.gfv
cp sim.data sim$zeVa.data
cat <<EOUF | gnuplot
set xlabel 'y'
Va=$zeVa
Re=50
set title "Re=50 Va=$zeVa"
p[0:][0:]"< awk '{if((\$1>0.4)&&(\$1<0.51)){print \$2,\$6}}' sim.data" w p,\
(-x+(1-exp(-Va*x*Re))/(1-exp(-Va*Re)))/Va t'solution exacte'
EOUF
done;
cat <<EOUF | gnuplot
  set term post eps enhanced
   set output "prof_aspi.eps"
   set ylabel "@^{\261}u(@^{\261}y) "
   set xlabel '@^{\261}y'
Re=50
u(x,Va)=(-x+(1-exp(-Va*x*Re))/(1-exp(-Va*Re)))/Va
set title "Re=50 Va"
p[0:][0:]"< awk '{if((\$1>0.4)&&(\$1<0.51)){print \$2,\$6}}' sim0.01.data" t'Va=0.01' w p,\
u(x,0.01) t'solution exacte',\
"< awk '{if((\$1>0.4)&&(\$1<0.51)){print \$2,\$6}}' sim0.1.data"t'Va=0.1' w p,\
u(x,0.1) t'solution exacte',\
"< awk '{if((\$1>0.4)&&(\$1<0.51)){print \$2,\$6}}' sim1.data"t'Va=1' w p,\
u(x,1) t'solution exacte'
EOUF
```

to plot:

```
p[0:]"< awk '{if(($1>0.4)&&($1<0.51)){print $2,$6}}' sim.data"t'gerris' w lp,\
   (-x+(1-exp(-Va*x*Re))/(1-exp(-Va*Re)))/Va t'solution exacte'
```



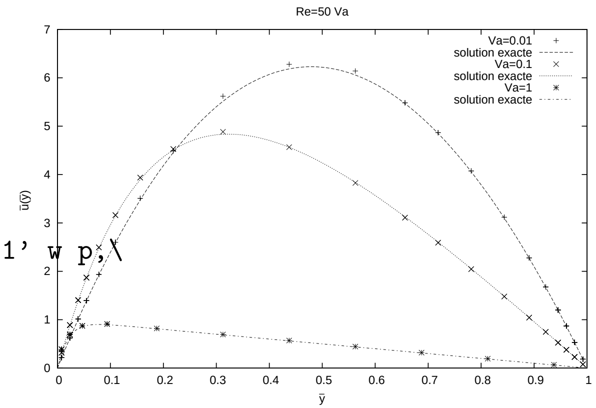


Figure 12: A 2D channel flow with suction and blowing. We see that as we imposed the pressure gradient to be one, and measure in this NS simulation the velocities with the Poiseuille scale, the velocity is smaller and smaller. This shows how powerful is MAE, as it gives the right scale for the velocity.

The same may be done with `freefem++`

```
// resolution aspiration
// PYL  sept 10
// ./FreeFem++-CoCoa aspiration.edp
exec("echo aspiration ");
verbosity=-1;
real s0=clock();
real h0=1;   //hauteur domaine
real L0=1.;   //longueur
int n=25;    //nbre de points

// definition des cotes Maillage
border b(t=0,1) { x= t*L0;    y = 0      ; };
border d(t=0,1) { x= L0;      y = h0 * t ; };
border h(t=1,0) { x= L0*(t);  y = h0     ; };
border g(t=1,0) { x= 0;       y = h0 * t ; };
// maillage
mesh Th= buildmesh(b(n)+d(n)+h(n)+g(n));
//espace EF
fespace Vh(Th,P2);
Vh U,UT;

real dVa = 0.05;
real Va=0.0000000001;
real Re=50;
U=1;

problem Aspi (U,UT) =
    int2d(Th)(
            -1./Re*(dx(U)*dx(UT) + dy(U)*dy(UT))  )
     + int2d(Th)( Va*dy(U)*UT)
     + int2d(Th) (   UT)
 + on(b,U=0)
 + on(h,U=0) ;

 while((Va<20))
 {
 Aspi;
 cout << "Va = "<< Va <<" ----------------   +++++++++++++++" << end
 plot(Th,cmm="U  Re=50 Va="+Va,U,fill=1,wait=1);
{ ofstream gnu("NplotU.gp");
real x,y,Uinf;
gnu << "# vals x  T" << endl;
for (int i=0;i<=4*n;i++)
   {y =(i*h0)/n/4;
    x=L0/2;
    Uinf=    (-y+(1-exp(-Va*y*Re))/(1-exp(-Va*Re)))/Va ;
   gnu<< y   << " " << U(x,y) << " " << Uinf << endl; }
   }
 Va=Va+dVa;
  }
cout << "CPU " << clock()-s0 << "s " << endl;
```

to plot:

```
p[0:]'NplotU.gp' u 2:1 w l,''u 3:1 w l
```

6.10.3 Discussion

With both codes, we compare well the numerical solution and the asymptotic one.

But, we see that the case $Va = 1$ and $Re = 50$ is not so simple to be computed, though the values are moderate. We need an enough large number of points to do this. Let us discuss the number of points that we need to "have enough points in the boundary layer".

• Fixed mesh size:
With our choice of parameters, the order of magnitude of the boundary layer is $1/(Va * Re)$. The solution in the boundary layer is written with $(1 - exp(-Va * y * Re)$. Let us guess the physical "size" of the boundary layer: as $exp(-4.) \simeq 0.02$ is enough small, we guess that a physical layer of size about $4/(Va * Re)$ is involved here. Hence, if we want to put 4 points in the boundary layer to describe it, we guess that the smaller size of the

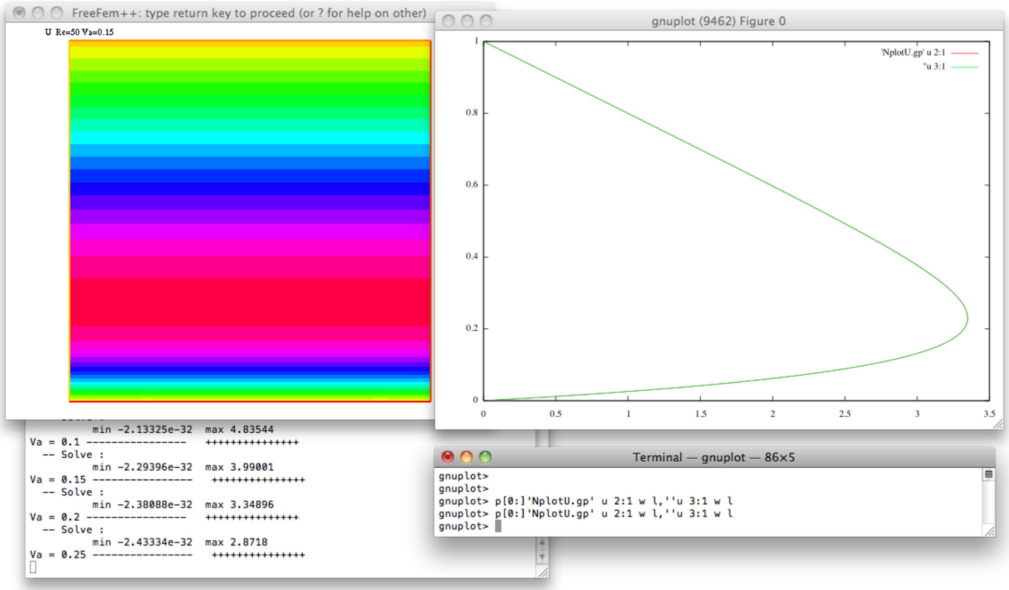


Figure 13: A 2D channel flow with suction and blowing. Increasing the suction shifts the velocity to the wall, it changes at a small scale. Beware of the mesh size!

mesh should be $\Delta x = \Delta y = 1/(VaRe)$. The language *Gerris* proposes the `Refine N` directive to monitor the mesh size. It means (*Gerris* language) that $\Delta x = 2^{-N}$, so we have to adjust the number of points so that

$$N = \log_2(VaRe),$$

so for $Re = 50$, we take `Refine 6` for a correct description, 5 is not enough. Then we see, that the larger Va or Re, the larger the number of points N, for $Re = 1000$ $N = 10$... hence increasing Re increases drastically the computational time and the storage space in memory. But that is the price to catch the solution.

• Adapting the mesh size:
The solver allows mesh refinement, `freefem++` as well. For example, with *Gerris* we can use either `AdaptFunction AdaptGradient AdaptVorticity` to focus on region where the field changes quickly. So, it is important to adjust the number of points in order to have few points where variations of velocity are not large and enough points where the velocity changes quickly.

We change the lines in the script:

```
Refine 4
AdaptGradient { istep = 1} { cmax = .25 maxlevel = 18 } U
```

It means that there are 2^4 points at least, but that every change of value $cmax = 0.25$ of U, we refine. This refinement is done up to the maximal level of say 18. This is an *a priori* large value that we do not want to obtain, that is a kind of limit of precision. To have a correct computation, we should not pass this value.

We will now show that there is a relation between N and Re so that the number of points obtained is "an automatic asymptotic change of scale"

The solution in the boundary layer has a large derivative at the wall $\partial u/\partial y|_0 = -Re$. The numerical derivative in 0 is evaluated as : $u(\Delta y) - u(0))/\Delta y$ hence as with our choice $u(\Delta y) - u(0)) = cmax$ then

$$\Delta y = cmax/Re$$

which is as $\Delta y = 2^{-N}$, we therefore obtain the following equation which gives the number of points necessary as a function of the Reynolds number (and the criteria):

$$N = log_2 Re - log_2(cmax)$$

for $cmax = 0.25$, we have $N = log_2 Re + 2$ for $cmax = 0.5$, we have $N = log_2 Re + 1$, *etc.* This is a way to reobtain automatically the asymptotics of the problem.

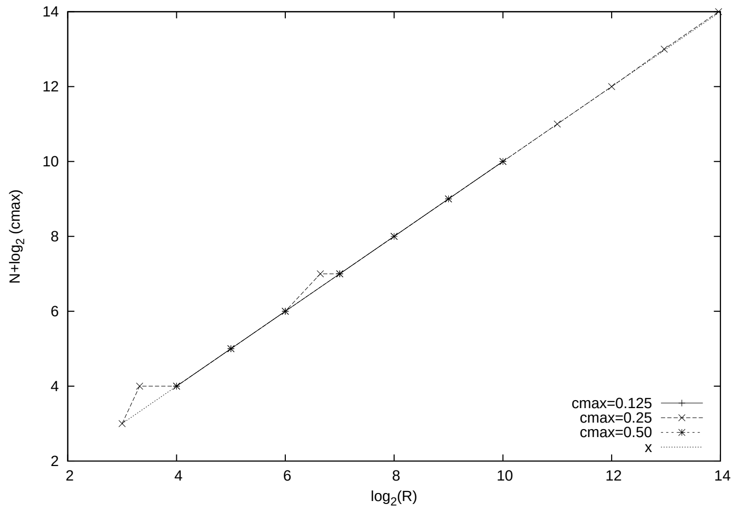


Figure 14: In order to satisfy the criteria of variation of value $cmax$, the mesh refinement N is function of the Reynolds number Re according to the asymptotics of the problem. The relation is $N + log_2(cmax) = log_2 Re$. We verify it here for three values of $cmax$.

• Conclusion

What have here a very important feature. Thanks to adaptative mesh refinement, the computation automatically puts more points at the right place: the boundary layer. This allows to do multiscale computations with enough points disposed at the "good" place. The number of points behaves according to the asymptotics, the number of points obtained is "an automatic asymptotic change of scale".

This interaction between the numerics and the asymptotics is very important to understand the flow and put enough points to describe it with enough precision.

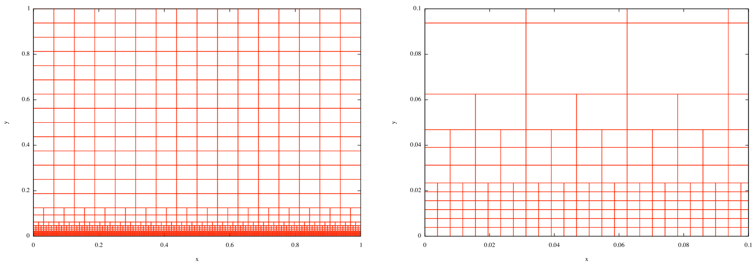


Figure 15: An example of mesh, starting from 2^{-4} to 2^{-8} at the wall, right a zoom near the wall.

This is a new tool which will have an increasing use.

7 Example with a boundary layer in the center of the domain

In this section we have an example of matched asymptotic expansions with a boundary layer either at the right wall, the left or in the flow far from the walls!

This was an exam.

7.1 Questions

Let us look at (see Kevorkian [16] and [17])

$$\varepsilon y''(x) - y(x)y'(x) + y(x) = 0. \tag{19}$$

with the following boundary conditions $y(-1) = 0$ and $y(1) = 0$. The trivial zero solution is excluded $y(x) = 0$.

1. 1.) Solve the outer problem (with $\varepsilon = 0$). Write with "bars" the solution :$y(x) = \bar{y}(\bar{x})$. Show that the problem is singular.

   2.) plot the various *a priori* solution for $\bar{y}$. If there is a boundary layer in $\bar{x} = 1$, what is the value of $\bar{y}$ in $\bar{x} = 1^-$ ($\bar{x} < 1$)

2. Inner problem. Let us first place a boundary layer in $x = 1$.

   1.) So we change the scale $x = 1 + \mu(\varepsilon)\tilde{x}$. Justify this notation: what is the order of magnitude of $\tilde{x}$, what is $\mu(\varepsilon)$? We write $y(x) = \tilde{y}(\tilde{x})$, why? Waht is the order of magnitude of $\tilde{y}$,

   2.) Put this in (19) and say ε goes to 0. Find μ. Obtain the inner problem.

   3.) Integrate once and obtain:

   $$-\tilde{y}'(\tilde{x}) + \frac{(\tilde{y}(\tilde{x}))^2}{2} = 2K^2. \tag{20}$$

   What do you think of K? Show that $\tilde{y}(\tilde{x}) = 2Kth(K\tilde{x}) = 2K\frac{e^{Kx}-e^{-Kx}}{e^{Kx}+e^{-Kx}}$ is solution of (20).

   4.) Find K by asymptotic matching. Plot the inner and outer solutions. Write the composite approximation.

   5.) Observe by symmetry that the behavior in $x = -1$ awith inner solution $\bar{y} = -1 + \bar{x}$ is the same than the one in $x = 1$ we have computed.

3. Case of a "shock" in $x = 0$.

   1.) Show that $\bar{y} = 1 + \bar{x}$ for $-1 \le \bar{x} < 0$ and $\bar{y} = -1 + \bar{x}$ for $0 < \bar{x} \le 1$ is a outer solution of the problem.

   2.) Show that in $x = 0$ we may follow 2-1) 2-2) and 2-3).What is the new value of K, plot the solutions.

7.2 Answers

The external solution is $\bar{y}'(x) = 1$ any solution like $\bar{y}(x) = x + C$ is allowed. It is possible to have a boundary layer at the right, at the left, at the right and the left...

We may imagine:

- $\bar{y}(x) = x - 1$ a boundary layer in $x = -1$
- $\bar{y}(x) = x$ a boundary layer in $x = -1$ and another in $x = 1$.
- $\bar{y}(x) = x + 1$ a boundary layer in $x = 1$.
- $\bar{y}(x) = x + 1$ for $x < 0$, a shock in $x = 0$ and $\bar{y}(x) = x - 1$ for $x > 0$.

Say that in any point $x = x_0$ the external solution is y_0 from one side and 0 to the order (boundary layer case). Or the solution is y_0 at one side and $-y_0$ at the other (shock case). Near $x = x_0 + \mu\tilde{x}$, we have $\mu = \varepsilon$ so that $\tilde{y}''(\tilde{x}) - \tilde{y}(\tilde{x})\tilde{y}'(\tilde{x}) = 0$. by integration $-\tilde{y}'(\tilde{x}) + \frac{(\tilde{y}(\tilde{x}))^2}{2} = 2K^2$, with $y_0 = 2K$ by matching.

This problem allows even several shocks.

Note that this is similar to the Bürgers equation $u_t + uu_x = u_{xx}$. Note that the thickness of a shock with NS equations is ν/c_0 which is more or less the free mean path. Navier Stokes equation are not valid at this scale.

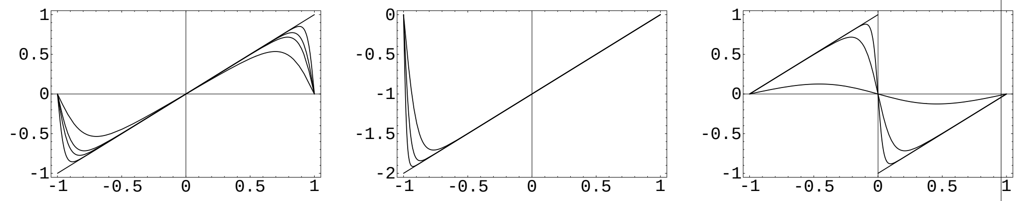


Figure 16: Solutions of (19) for various values of ε. Note that the boundary layer may be any where: either at one boundary, left or right, either on the two boundary, or in the center of the domain (this is a kind of shock)...

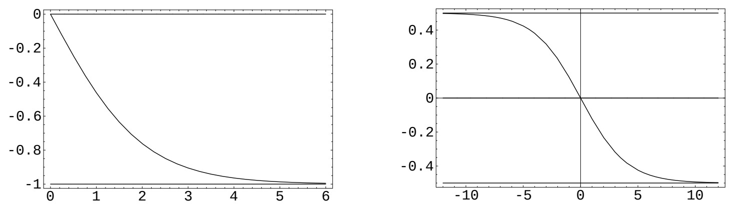


Figure 17: Solutions of the inner problem, left a boundary layer, right a shock.

8 Case with logarithms

8.1 A remark on the large / small scale

The problems are not always defined at scale one, and solved locally at a small scale. The reverse may happen, for some reason the problem is defined in some scales of order one, and one has to look at a larger scale to solve it. Let us take a very simple example.

The following problem needs a "zoom in" in $y = 0$ to be solved:

$$\varepsilon u''(y) - u = 0 \text{ with } f(0) = 1, f(\infty) = 0$$

outer solution for variable y is $u(y) = 0$. This solution does not fit the boundary condition in $y = 0$. The inner solution with $\tilde{y} = y/\sqrt{\varepsilon}$ is $\tilde{u} = Ae^{-\tilde{y}} + Be^{\tilde{y}}$, with $A = 1$ by boundary condition in 0 to reobtain 1 the lost B.C. and $B = 0$ by matching.

The following problem needs a "zoom out" from $y = O(1)$ to be solved:

$$u''(y) - \varepsilon u = 0 \text{ with } f(0) = 1, f(\infty) = 0$$

of solution $u(y) = A + By$ clearly $A = 1$, one can admit that if $B \neq 0$, then $u(\infty)$ is unbounded, so that $B = 0$ is a good candidate. Then $u(y) = 1$ for $y = O(1)$ and using a large variable $Y = \sqrt{\varepsilon}y$, then $U''(Y) - U(Y) = 0$ so that $U(Y) = Ae^{-Y} + Be^{Y}$, with $A = 1, B = 0$ by matching and boundary condition.

The following example seems to be the same than this one, but there is a new difficulty.

8.2 The Lagerstrom 1960 problem

We present here a case which involves logarithms and is more difficult. It is a classical example, it is reminiscent of the Stokes problems around a sphere or around a cylinder. We will see this in the chapter small Reynolds the Stokes paradox for a flow around a cylinder. But here it is a bit more simple. This problem may be interpreted as heat equation in a sphere (see Kevorkian and Cole [16] page 89 or [17] page 101 or Hinch [14] p 67):

$$f_{rr} + \frac{2f_r}{r} + \varepsilon f f_r = 0$$

with conditions $f = 0$ in $r = 1$, and $f \to 1$ as $r \to \infty$.

8.3 Near approximation

Let us try a simple expansion as usual:

$$f = f_0(r) + \varepsilon f_1(r) + ...$$

at order ε^0 we find $log(f_0') = -2log(f_0)$ so that $f_0 = A - 1/r$ with the BC:

$$f_0 = 1 - \frac{1}{r},$$

at order ε^1 we have

$$f_1'' + \frac{2f_1'}{r} = -f_0 f_0'$$

with $f_1(0) = 0$ and $f_1 \to 0$ at infinity. We rewrite in a compact form the derivative

$$\frac{1}{r^2}(r^2 f_1')' = -\frac{1}{r^2} + \frac{1}{r^3}$$

so that the solution is (noting $\int r^{-2}\ln(r)dr = -\ln(r)/r - 1/r$ by integrating by parts), and as $f_1(1) = 0$

$$f_1 = -\ln(r) - \frac{\ln r}{r} + A_1(1 - \frac{1}{r}).$$

The expansion is then, up to now:

$$f = f_0(r) + \varepsilon(-\ln(r) - \frac{\ln r}{r} + A_1(1 - \frac{1}{r})) + ...$$

Let us look at the boundary condition, we must have that f approaches one at infinity. There is a problem as the condition at infinity cannot be satisfied due to the logarithm: far from 1, $\ln(r)$ is large.

Far from the origin

$$f = 1 - \varepsilon\ln(r) + ...$$

so writing $\rho = \varepsilon r$, this new variable is a variable which describes what happens far from the origin. So that, in this layer as $r = \rho/\varepsilon$,,

$$f = 1 - \varepsilon\ln(\rho/\varepsilon) + ...$$

or

$$f = 1 - \varepsilon\ln(1/\varepsilon) - \varepsilon\ln(\rho) + ...$$

This term in $\varepsilon\ln(1/\varepsilon)$ gives us the idea to insert a new term before ε^1:

$$f = f_0 + \varepsilon\ln(\frac{1}{\varepsilon})f_{ln} + \varepsilon f_1 + ...$$

or

$$f = (1 - \frac{1}{r}) + \varepsilon\ln(\frac{1}{\varepsilon})f_{ln} + \varepsilon f_1 + ...$$

The expected development was ε^0, ε, ε^2, ε^3.... The fact that a new unexpected term arises $\varepsilon ln(\frac{1}{\varepsilon})$ and interplays in the sequence was called "Switchback" by S. Kaplun (Low Re number flow J Math & Mec V6 No 5 1957). "in trying to find terms of a certain order one is forced to reconsider lower order terms." (in "fluid mechanics and singular perturbations collection of paper by Kaplun editor Lagerstrom)

After substitution, we obtain for f_{ln} the same equation than f_0 so

$$f_{ln} = A_{ln}(1 - \frac{1}{r}).$$

8.4 far approximation

We have just seen that

$$f = 1 - \varepsilon\ln(1/\varepsilon)(1 - A_{ln}) - \varepsilon\ln(\rho) + ...$$

in a layer far from the origin, where $\rho = \varepsilon r$, so the equation becomes $f_{\rho\rho} + \frac{2f_\rho}{\rho} + ff_\rho = 0$ we are at large distances from the origin, so let search for:

$$f = 1 + \varepsilon\ln(\frac{1}{\varepsilon})g_1(\rho) + \varepsilon g_2(\rho) + ...$$

Both g_1 and g_2 satisfy the same equation

$$g'' + (2/\rho + 1)g' = 0 \quad i.e. \quad (\rho e^\rho g')' = 0.$$

the solution with $g = 0$ at infinity is

$$g_i = B_i \int_\rho^\infty \frac{e^{-t}}{t^2}dt$$

where $\int_\rho^\infty \frac{e^{-t}}{t^2}dt$ is linked to the Incomplete gamma function:

$$\Gamma(a, x) = \int_0^\infty t^{a-1}e^{-t}dt$$

it may integrated by parts: $\int_\rho^\infty \frac{e^{-t}}{t^2}dt = [\frac{e^{-t}}{t^2}]_\rho^\infty + \int_\rho^\infty \frac{e^{-t}}{t}dt.$
The integral $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt$ is called exponential integral (Bender Orzag [3] p 252), it is solution of

$$\frac{dE_1(x)}{dx} = -\frac{e^{-x}}{x} = -\frac{1}{x} + 1 - \frac{x}{2} + ...$$

so that

$$E_1(x) = C - \ln(x) + x - \frac{x^2}{4} + ...$$

Bender Orzag [3] p307 or Abramowitz and Stegun [1] 5.1.11, the constant is $C = \gamma$

$$\gamma = lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln(n)) \simeq 0.5772$$

so that it is classical that:

$$\gamma = lim_{x\to 0^+}(\int_x^\infty \frac{e^{-t}}{t}dt + \ln x).$$

proof
It seems that a way to prove it, is to start from the fact that the definition of Γ from Euler and Weierstrass are:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}dt = e^{-\gamma x}\prod_{n=1}^\infty e^{x/n}\left(1 + x/n\right)^{-1}$$

and so ([8]):

$$\Gamma'(1) = \int_0^\infty Log(t)e^{-t}dt = -\gamma$$

and integrating par parts:

$$\gamma = F(x) - Log(x) - R(x)$$

with

$$F(x) = \int_0^x \frac{1 - e^{-t}}{t}dt = \sum_{n=1}^\infty \frac{(-1)^{n-1}x^n}{nn!} \quad \text{and} \quad R(x) = \int_x^\infty \frac{e^{-t}}{t}dt.$$

QED

So going back to the integral

$$\int_\rho^\infty \frac{e^{-t}}{t^2}dt = -\frac{e^{-\rho}}{\rho^2} + \int_\rho^\infty \frac{e^{-t}}{t}dt$$

we then obtain after developing $e^{-\rho}/\rho = 1/\rho - 1 + \rho/2 + ...$

$$\int_\rho^\infty \frac{e^{-t}}{t^2}dt = (1/\rho - 1 + \rho/2 + ...) + (\gamma - \ln(\rho) + \rho - \frac{1}{\rho} + 1 - \frac{\rho^2}{4} + ...)$$

so the final integral is

$$\int_\rho^\infty \frac{e^{-t}}{t^2}dt \sim \frac{1}{\rho} + \ln(\rho) + \gamma - 1 - \frac{\rho}{2} + o(\rho)$$

Finally we say that

$$f = 1 + \varepsilon\ln(\frac{1}{\varepsilon})g_1(\rho) + \varepsilon g_2(\rho) + ...$$

and near the origin

$$f = 1 + \varepsilon\ln(\frac{1}{\varepsilon})B_1[\frac{1}{\rho} + \ln(\rho) + \gamma - 1 - \frac{\rho}{2}] + \varepsilon B_2[\frac{1}{\rho} + \ln(\rho) + \gamma - 1 - \frac{\rho}{2}] + ...$$

8.5 Matching

At this point, we may guess that from the behavior in $r \gg 1$

$$f = 1 - \varepsilon\ln(1/\varepsilon)(1 - A_{ln}) - \varepsilon\ln(\rho) + ...$$

and the previous expression

$$f = 1 + \varepsilon\ln(\frac{1}{\varepsilon})B_1[\frac{1}{\rho} + \ln(\rho) + \gamma - 1 - \frac{\rho}{2}] + \varepsilon B_2[\frac{1}{\rho} + \ln(\rho) + \gamma - 1 - \frac{\rho}{2}] + ...$$

at first glance we see $-\varepsilon\ln(\rho)$ and $\varepsilon B_2\ln(\rho)$, so $B_2 = -1$ would be good. As εB_2 terms seem to have no counter parts in $-(1 - A_{ln})$, so maybe $(1 - A_{ln}) = 0$ and $B_2 = 0$.

To be more precise, we introduce (again see Hinch [14]) the intermediate variable

$$\hat{r} = \varepsilon^\alpha r = \rho\varepsilon^{\alpha-1}.$$

After substitution of the intermediate variable and matching term to term in the overlap layer the r solution

$$\begin{aligned} &= (1 - \frac{\varepsilon^\alpha}{\hat{r}}) + \\ &+ \varepsilon\ln(1/\varepsilon)A_{ln}(1 - \frac{\varepsilon^\alpha}{\hat{r}}) + \\ &+ \varepsilon[-\alpha\ln(1/\varepsilon) - \ln(\hat{r}) + A_1 - \alpha\ln((1/\varepsilon)\frac{\varepsilon^\alpha}{\hat{r}})) - \varepsilon^\alpha\frac{\ln(\hat{r}) + A_1}{\hat{r}}] + ... \end{aligned}$$

and the ρ description

$$\begin{aligned} &= 1 + \\ &+ \varepsilon\ln(1/\varepsilon)B_1[\frac{\varepsilon^{\alpha-1}}{\hat{r}} + (\alpha - 1)\ln(1/\varepsilon) + \ln\hat{r} + \gamma - 1 + ...] + \\ &+ \varepsilon B_2[\frac{\varepsilon^{\alpha-1}}{\hat{r}} + (\alpha - 1)\ln(1/\varepsilon) + \ln\hat{r} + \gamma - 1 + ...] + ... \end{aligned}$$

Looking at each order :
at ε^0 we have $1 = 1$
at $\varepsilon^\alpha ln(1/\varepsilon)$ we have $0 = B_1/\hat{r}$
at ε^α we have $-1/\hat{r} = B_2/\hat{r}$
at $\varepsilon\ln(1/\varepsilon)$ we have $A_{ln} - \alpha = B_2(\alpha - 1)$
at ε we have $-\ln(\hat{r}) + A_1 = B_2(\ln(\hat{r}) + \gamma - 1$
so we find $B_1 = 0$, $B_2 = -1$, $A_{ln} = 1$ and $A_1 = 1 - \gamma$.

The solution for r fixed:

$$f = (1 - \frac{1}{r}) + \varepsilon\ln(\frac{1}{\varepsilon})(1 - \frac{1}{r}) + \varepsilon[-\ln(r) - \frac{\ln r}{r} + (1 - \gamma)(1 - \frac{1}{r})] + ...$$

while for ρ fixed

$$f = 1 + 0\varepsilon\ln(\frac{1}{\varepsilon}) - \varepsilon\int_\rho^\infty \frac{e^{-t}}{t^2}dt + ...$$

It is to be noticed that the pq-qp rule does not work in this case due to the term $\varepsilon\ln(r)$ which changes of order (see Hinch [14] for full discussion).

8.6 Remark

This problem (introduced by Lagerstrom in a seminar in 1960) is reminiscent to the flow round a a sphere. The problem

$$f_{rr} + \frac{f_r}{r} + \varepsilon f f_r = 0$$

is more reminiscent to the flow around a cylinder at small Reynolds, the so called Stokes Oseen problem (see François [10] among others). A log arises from the very beginning. This problem is "a worse problem" according to Hinch [14].

He proposes even a "terrible problem" : $f_{rr} + \frac{f_r}{r} + f_r^2 + \varepsilon f f_r = 0$, which looks like in Stokes flow, near the cylinder, it seems to be impossible to match with the imposed free-stream velocity as log terms arise. A $Re/ln(Re)$ term arises. Far from the cylnder, the velocity is the imposed velocity plus an expansion in powers of $1/\ln(Re)$. The Stokes paradox around a cylinder is in the chapter small Reynolds,

9 Exercices

9.1 Polynomia

For each of the following cases, say if the E_ε is singular of regular. Find the solution with the change of scale, compare to the exact solution.
• Is $x^2 - x + \varepsilon = 0$ singular or regular?
• Is $\varepsilon x^2 - x + 1 = 0$ singular or regular?
• Is $\varepsilon x^3 - x + 1 = 0$ singular or regular?
hint: recover that the solutions of the last one are:
$x_1 = 1 + \varepsilon^{1/2} + 3\varepsilon + ...$, $x_2 = \varepsilon^{-1/2} - 1/2 + ...$ and $x_3 = -\varepsilon^{-1/2} - 1/2 + ...$

9.2 ODE

For each of the following cases, slove the external problem show that the problem is singular. Change of scale, solve the inner problem, match the layers.
• Look at the problem

$$\varepsilon f' = -f$$

with $f(0) = 1$.

• Look at the problem

$$\varepsilon \frac{d^2 f}{dy^2} + y\frac{df}{dy} - yf = 0$$

with $f(0) = 0$ and $f(1) = e$.

Find that the scale is $\sqrt{\varepsilon}$ and that outer and inner solution are:

$$f_0(\bar{y}) = e^{\bar{y}} \quad \text{and} \quad \tilde{f}_0(\tilde{y}) = \sqrt{2/\pi}\int_0^{\tilde{y}} e^{-\tilde{y}^2} d\tilde{y}$$

• Solve

$$0 = 1 - u + \varepsilon u''$$

with $u(0) = u(1) = 0$. Show that we have two boundary layers.

10 Conclusion

We have demonstrated nothing, but we have seen what is a singular and a regular problem. In the framework of Matched Asymptotic Developments "Méthode des Développements Asymptotiques Raccordés", we have seen that the first thing to do when we deal with a problem with small parameter ε is to solve the external problem when the parameter ε is set to 0. Then we do a rescaling. This rescaling uses the "Least Degenerascy Principle", or "Dominant Balance" ("Principe de Moindre Dégénérescence"). It means that we try to recover the maximum of terms in the equation. In fact, the terms that we neglected were not so small in some regions. Of course, we try to reobtain the terms previously neglected. We solve then the internal problem. It is indeterminate. The indetermination is resolved by the "Asymptotic Matching" ("Raccord Asymptotique"), the top of the layer matches with the bottom of the other. Or in other words, there exist an overlapping intermediate region where the two developments are the same. This is called "matching".

Again, we have demonstrated nothing (see Guiraud [13] for an introduction to "non standard analysis" which is maybe the most sound basis for MAE), but we have check that the method works on simple cases.

Some times we have logarithms, this makes the solution more complicated. The Friedrichs problem looks like high Reynolds number and Boundary Layer, the Lagerstrom problem is for Stokes flow.

We will use this method intensively in the case of aerodynamics (from Blasius solution to Triple Deck theory).

Finally, the result of those expansions is that it works even for $\varepsilon = 1$, so that the Carrier's Rule is verified : "Divergent series converge faster than convergent series because they don't have to converge."

We have seen what are Matched Asymptotic Expansions. In the following books we find those classical examples with different points of view. Most of them are in the "Bibliothèque de Mécanique" Tower 55-65, 4th floor.
• Chapter 1 and 2 of Kevorkian & Cole (81)
• Chapter 1 and 4 of Nayfeh (73)

- Chapter I, (II) and III from Claude François (81)
- Chapter IX5. from Paul Germain (86)
- Chapter 5 Milton Van Dyke
- Chapter 7 and 9 Bender Orzag
- Chapter IX p 149 Paterson
- Chapter 1 of A. Ruban
- Chapter 1,2 and 5 of John Hinch

Of course there are other classical Books from settlers of this theory Cole, Kaplun, Lagerstorm, Eckhaus... Look at the following bibliography (after the annex), at some wiki links and partial google books.

11 Annex: Restricted Three-Body system

11.1 Hill equations

In fact most of the techniques were developed for use in celestial mechanics (for instance see Hinch [14] p92). Here, as an example among a lot of other, we consider here the motion of three gravitational masses (earth, moon and a satellite) in the limit when one of the masses is smaller than the other two....

$$m(\frac{d^2\overrightarrow{r}}{dt^2}+2\overrightarrow{\omega}\times\frac{d\overrightarrow{r}}{dt}+\overrightarrow{\omega}\times(\overrightarrow{\omega}\times\overrightarrow{r}))=-\mathcal{G}m_1\frac{(\overrightarrow{r}-\overrightarrow{r}_1)}{|\overrightarrow{r}-\overrightarrow{r}_1|^3}-\mathcal{G}m_2\frac{(\overrightarrow{r}-\overrightarrow{r}_2)}{|\overrightarrow{r}-\overrightarrow{r}_2|^3}$$

$$\omega^2=\mathcal{G}(m+m_1+m_2)/(|\overrightarrow{r}_1-\overrightarrow{r}_2|)^3$$

First the equations are written in a non dimensional form as:

$$\frac{d^2x}{dt^2}=-(1-\mu)\frac{x-\xi_1}{((x-\xi_1)^2+(y-\eta_1)^2)^{3/2}}-\mu\frac{x-\xi_2}{((x-\xi_2))^2+(y-\eta_2)^2)^{3/2}}.$$

$$\frac{d^2y}{dt^2}=-(1-\mu)\frac{y-\eta_1}{((x-\xi_1)^2+(y-\eta_1)^2)^{3/2}}-\mu\frac{y-\eta_2}{((x-\xi_2))^2+(y-\eta_2)^2)^{3/2}}.$$

where μ is the reduced mass. The equations are then written in a rotating frame centered to one of the large bodies.

After some algebra and using $\tau\bar{t}=t$ and $\delta\bar{x}=x$ $\delta\bar{y}=y$ the good scales are $\delta=\mu^{1/3}$ and $\tau=1$:

$$\frac{d^2x}{dt^2}=-\frac{x}{(x^2+y^2)^{3/2}}+2\frac{dy}{dt}+3x,$$

$$\frac{d^2y}{dt^2}=-\frac{y}{(x^2+y^2)^{3/2}}-2\frac{dx}{dt},$$

which are known as Hill equations, see [16] page 179.

11.2 Rendez Vous Spatial/ Space Rendez Vous

Very near one planet the equations reduce to the Clohessy - Wiltshire equations (`http://www.lmm.jussieu.fr/%7Elagree/SIEF/GRAVITATION/RDV/rdvtheo/rdvtheo.html`):

$$x''=2\omega y'+3\omega^2x \text{ and } y'=-2\omega x'$$

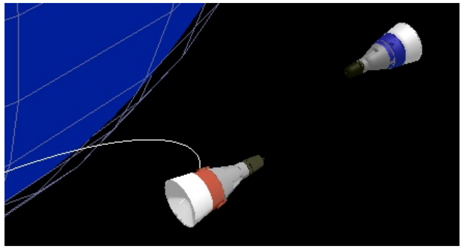


Figure 18: The Space Rendez-Vous is a problem of asymptotic expansions

with $\omega=(GM/R^3)^{1/2}$. The general solution is obtained after derivation: $x'''=-4\omega x'+3\omega^2x'$ so $(x')''=\omega^2(x')$ which gives $x=B_1\cos(\omega t)+B_2\sin(\omega t)+B_3$
Then we obtain $y=C_1+tC_2+2B_2\cos(\omega t)-2B_1\sin(\omega t)$ Next, putting as initial conditions: $x(0)=x_0, y(0)=0, x'(0)=u_0, y'(0)=v_0$, we have:

$$x(t)=(2v_0/\omega+4x_0)-(2v_0/\omega+3x_0)\cos(\omega t)+u_0/\omega\sin(\omega t)$$

$$y(t)=(-2u_0/\omega+y_0)-(3v_0+6\omega x_0)t+2u_0/\omega\cos(\omega t)+(4v_0/\omega+6x_0)\sin(\omega t)$$

Some examples from left to right
$x_0=y_0=0, u_0=-1, v_0=0$ we go to the left, Coriolis makes turn to the right... ellipsis
$x_0=y_0=0, u_0=0, v_0=1$ forward, and then backward!!
$x_0=y_0=0, u_0=0v_0=-1$ backward and then forward!!

To play, use the "flash" in this pdf, or download it on `http://www.lmm.jussieu.fr/%7Elagree/SIEF/GRAVITATION/RDV/rdv.html`.

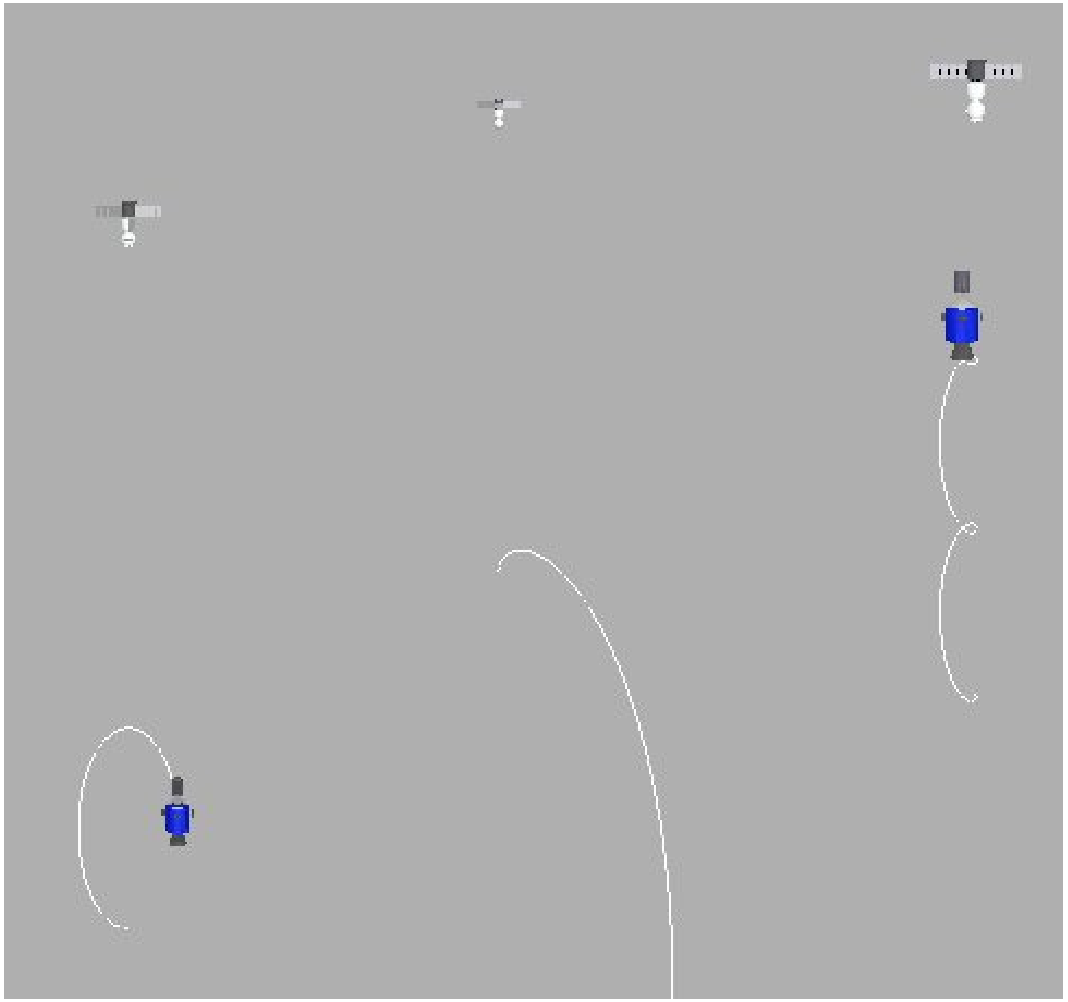


Figure 19: Click to launch a "flash" game: "Space Rendez Vous by PYL", [Acrobat Reader Required]; use arrow to move the Apollo vessel and join the Soyouz crew. The Hewlett Packard HP-65 was used by the astronauts during the historical rendez vous in July 15th 1975. This was the end of the cold war.

Figure 20: Asymptotics in Paris: the ε is everywhere! [update : Oups, recently this famous Café of place Jussieu changed his marvellous name to a stupide one]

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Kruskal 63

[27] Some similar and more complete courses on the web (in Great Britain mostly, as this kind of course is widespread in Applied Math departments):
http://www.math.ucdavis.edu/∼hunter/notes/asy.pdf
http://homepage.ntlworld.com/harper7/asymptoticmethods.pdf
http://www.maths.manchester.ac.uk/∼gajjar/MATH44011/notes/

up to date December 3, 2024

The up to date web page of this text is:
http://www.lmm.jussieu.fr/∼lagree/COURS/M2MHP/MAE.pdf

The web page of the french version of text is:
http://www.lmm.jussieu.fr/∼lagree/COURS/LOME/DAR.pdf

The web page of the course is:
http://www.lmm.jussieu.fr/∼lagree/COURS/M2MHP

The scope of this course is to present the classical "perturbation methods" of fluid mechanics. This first curse introduces the necessity of adimentionalisation to extract small parameters. Those small parameters may have a tremendous influence on the solution of the problem. They oblige to introduce new variables with new scales, those scales were not evident *a priori*. The Van Dyke "dominant balance" will guide us to find those scales. Having simplified just enough the equations, they are solved in the new pertinent variables. After, the two solutions at two different scales are joined by the "matching" process which tells that there exists some intermediate scale in which both developments are the same. Here, we presented the "Matched Asymptotic Expansion" (sometimes MAX: Matched Asymptotic eXpansion) ("méthode des développements asymptotiques raccordés") which will be useful to construct the boundary layer theory for vanishingly small viscous flow over a flat plate.

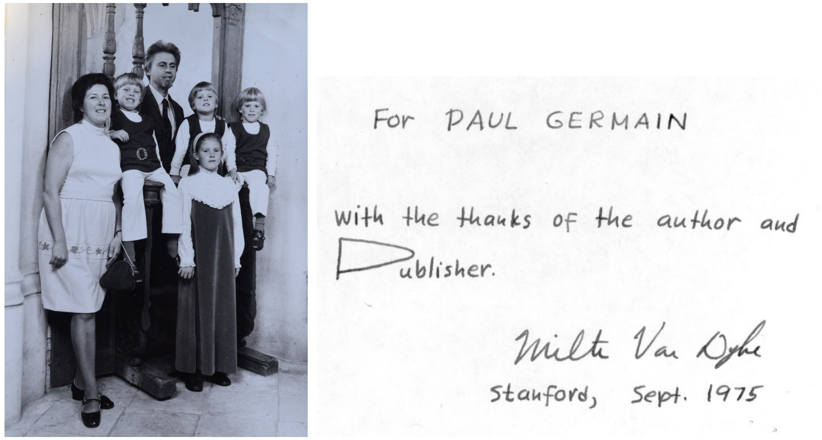

Figure 21: Milton Van Dyke and family at wedding of the son of Paul Germain (collection MH Germain), Van Dyke dedicace to Paul Germain in the book "Perturbation methods in Fluid Mechanics (1975)"