# Model Equations

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## 1 What is a Model ?

A usual definition of "model" is a representation of a thing typically on a smaller scale than the original. One speaks of "physical model". In hydrodynamics or aerodynamics, it is clear that the flow around the model of a boat or around a model of a plane will give valuable informations on the flow around the real boat or the real plane.

The world comes from italian modello XVIth century, as a representation in small of something build (with the meaning of "modèle réduit" in french, or even "maquette" in french)
wikipedia `https://fr.wiktionary.org/wiki/mod%C3%A8le`

By extension it is used as "a representation of a system. It consists of concepts used to help people know, understand, or simulate a subject the model represents. It is also a set of concepts. In contrast, physical models are physical objects, such as a toy model that may be assembled and made to work like the object it represents."
Wikipedia `https://en.wikipedia.org/wiki/Conceptual_model`.

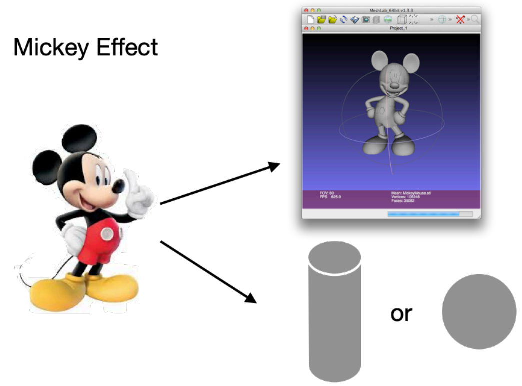


FIGURE 1 – The good model for Mickey is not the openGL one, top, but bottom, the cylinder or the sphere...

### 1.1 The Mickey Effect

A first sensible way to present this notion of model is to invoke Mickey mouse himself.

This is a way to understand the limit of a model : what I call "Mickey's effect". Suppose you want to do a "model" of Mickey Mouse. This is an hard job, as Mickey is 2D, you have to do it in 3D. And there are many details, the thickness of the tail, of the ears, the fact that Mickey has height fingers not ten....

Many people will spend too much time and numerical ressources to do a nice "model" of the mice. But depending of your application, maybe the best model is just a sphere or a cylinder...

It's up to you to decide the good level of precision with the pertinent details of your model for the use you need. This is the interesting job of modelling : extracting the pertinent informations.

Let us look to an other example to fix the ideas.


FIGURE 2 – *Models* of boats : top left, a "real" ship : "L'Étoile Molène", dundee thonier constructed in 1954 (PYL), top right, "L'Étoile du Roy" a full size replica of a Sixth-rate ship (20 canons) form XVIIIth century constructed in 1996 (PYL), bottom left a balsa model sailing on Luxembourg pound circa 1995 (boat and photo PYL), bottom right, model ship to display "La Licorne" in Tintin (1943), D.R. Hergé (a sketch or a drawing is a model it self). Polar curves of an Oceanis 31. A paper boat, and finally" *une coquille de Noix (Marine) (Sens figuré) (Familier) Petit bateau peu susceptible de voguer."*

## 1.2 A floating-model and a model ship in a display case

Hence the question is : **A model for which use ?**

We will make a differentiation between different *models* depending on what we want to observe. Let us look now at a boat (this another visual application of Mickey's effect, but on boats). For example, a model boat that is placed on a chimney (or on a table, a sideboard...) is not a model of boat that sails on a pound. In both cases it is however a model. In the first case one seeks pure representative realism (a nice object), in the other case one wants to make the ship float and move forward (to extrapolate to the real boat). In English / American they also use the word "mockup", or "mock-up" but in this case it is more of a 1 :1 scale prototype, "mok-up" is a deformation of "maquette" pronounced by an englishman.
Wikipedia `https://fr.wiktionary.org/wiki/mockup`

## 1.3 Digital Twins

This is a rather new word (2014) `https://books.google.com/ngrams/graph?content=digital+twins&year_start=1800&year_end=2019&corpus=en-2019&smoothing=3`.

"A digital twin is a virtual representation of a real-world physical system or product that serves as the indistinguishable digital counterpart of it for practical purposes, such as system simulation, integration, testing, monitoring, and maintenance." `https://en.wikipedia.org/wiki/Digital_twin`].

OK, so, please do not say "digital twins", just say "Model".

## 1.4 Never forget reality

A feedback with experiment is important as noted by Madame du Châtelet (1706- 1749) (https://fr.wikipedia.org/wiki/Emilie_du_Chatelet). $\partial$'Alembert was her friend. Voltaire was her boyfriend.

She translated in french the "principia" of Newton. She wrote the first book or textbook in Physics (in fact in point mechanics) "Institutions de Physique 1740" https://gallica.bnf.fr/ark:/12148/bpt6k75646k. She was one of the very first to understand the difference between $mv$ and $mv^2/2$. In this book (written for her son), she insists on the importance of observation and experience before settling theory :

"Souvenez-vous, mon fils, dans toutes vos Études, que l'Expérience est le bâton que la Nature a donné à nous autres aveugles, pour nous conduire dans nos recherches ; nous ne laissons pas avec son secours de faire bien du chemin, mais nous ne pouvons manquer de tomber si nous cessons de nous en servir ; c'est à l'Expérience à nous faire connaître les qualités Physiques, & c'est à notre raison à en faire usage & à en tirer de nouvelles connaissances & de nouvelles lumières".


FIGURE 3 – "Institutions de Physique 1740" , Voltaire et Emilie du Châtelet.

# 2 Model equations

The game is as follows : looking at a phenomena, one can distinguish some pertinent parameters. They are used to make the problem non dimensional. Then some numbers without dimension appear. By dominant balance, some terms are removed as some numbers without dimension are small. The remaining problem is solved with scales different in the various directions. A hierarchy of problems may be constructed, all without dimensions.

Some of the final most classical problems obtained in fluid dynamics are in the next section..... They are combinations of all partial derivatives $\partial_t u$, $\partial_x u$ at various powers $\partial_t^2 u$, $\partial_x^2 u$ $\partial_x^3 u$ up to $\partial_x^4 u$, down to $\int u dx$ and combinations of $u$, $u^2$ up to $u^3$....

# 3 Feynman unwordliness equation

Feynman (Nobel 1965), the total *unwordliness* of the world, the great "law of nature" :

$$U = 0$$

"Equation of the Universe".

# 4 Simple Model Equations

## 4.1 ODE

### 4.1.1 Linear

The most simple model

$$\frac{\partial u}{\partial t} = Lu$$

solution in exponential (if $L$ is a matrix $\mathrm{e}^L = \sum_{k\in\mathbb{N}} \frac{1}{k!} L^k$)

### 4.1.2 Non Linear

The non linear differential equation

$$\frac{\partial u}{\partial t} = Lu^2$$

admits singular solutions... $u(t) = \frac{1}{L(t_0-t)}$, $t_0$ time of the singularity (doomsday)

### 4.1.3 Transcritical bifurcation

Change of type of solution depending on the value of the control parameter $L$

$$\frac{\partial u}{\partial t} = Lu - u^2$$

### 4.1.4 Pitchfork bifurcation

Change of type of solution depending on the value of the control parameter $L$

$$\frac{\partial u}{\partial t} = Lu - u^3$$

### 4.1.5 Verhulst, logistic growth

Self-limiting process when a population becomes too large (logistic map $r, K$ parameters)

$$\frac{dN}{dt} = rN(1 - N/K)$$

### 4.1.6 Lotka-Volterra

Lapin & Renards, predator-prey equations ($\alpha, \beta, \delta\gamma$ parameters) :

$$\begin{cases} \frac{dL(t)}{dt} & = & L(t)\Big(\alpha - \beta R(t)\Big) \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} & = & R(t)\Big(\delta L(t) - \gamma\Big) \end{cases}$$

### 4.1.7 SIR

Covid and deases model : SIR Sain, Infected, Recovered ($r, a$ parameters) :

$$\begin{cases} \frac{dS(t)}{dt} & = & -r\ S\ I \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} & = & rSI - aI, \\ \frac{dR(t)}{dt} & = & a\ I, \end{cases}$$

where $r > 0$ is the infection rate and $a > 0$ the removal rate of infectives.

### 4.1.8 Lorentz attractor

Famous attractor, the simpliest model of meteo ($\sigma, \rho, \beta$ parameters) :

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} & = \sigma\big(y(t) - x(t)\big) \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} & = \rho\,(x(t) - \ z(t)) - y(t) \\ \frac{\mathrm{d}z(t)}{\mathrm{d}t} & = x(t)\,y(t) - \beta\,z(t) \end{cases}$$

## 4.2 Oscillators/ second order equations/ ODE

### 4.2.1 Harmonic Oscillators

the ubiquitous oscillator

$$y'' = -y$$

with given $y(0)$ and $y'(0)$.

### 4.2.2 Damped harmonic oscillator

The same with linear dissipation in velocity

$$\frac{d^2\bar{y}}{d\bar{t}^2} + \varepsilon\frac{d\bar{y}}{d\bar{t}} + \bar{y} = 0$$

$$\bar{y}(0) = 0, \text{ and } \frac{d\bar{y}}{d\bar{t}}(0) = 1.$$

### 4.2.3 Forced harmonic oscillator

The same with a forcing source term

$$\frac{d^2\bar{y}}{d\bar{t}^2} + \varepsilon\frac{d\bar{y}}{d\bar{t}} + \bar{y} = \cos\omega t$$

### 4.2.4 Friedrich problem,

Used for singular perturbation as a simplification of Blasius model

$$\varepsilon y'' + y' = \frac{1}{2}, \quad y(x=0) = 0 \quad y(x=1) = 1,$$

### 4.2.5 Carrier's problem, Carrier-Pearson problem

Introduced by Carrier 1970 as an *ad hoc* equation. See Bender Orszag p 464

$$\varepsilon y'' + y^2 = 1, \quad y(x=\pm 1) = 0 \quad -1 \leq x \leq 1,$$

See Hinch page 108

$$(x + \varepsilon y)y' + y = 1, \quad y(x=1) = 2 \quad 0 \leq x \leq 1,$$

### 4.2.6 Lagerstrom problem

Model problem for the Navier-Stokes equation with small Reynolds numbers around a sphere

$$y'' + 2\frac{y'}{r} + \varepsilon y y' = 0, \quad y(r=0) = 0 \quad y(r=\infty) = 1$$

### 4.2.7 Lagerstrom worse problem

Model problem for the Navier-Stokes equation with small Reynolds numbers around a cylinder

$$y'' + \frac{y'}{r} + \varepsilon y y' = 0, \quad y(r=0) = 0 \quad y(r=\infty) = 1,$$

### 4.2.8 Lagerstrom terrible problem

Another model problem for the Navier-Stokes equation with small Reynolds numbers around a cylinder (unusually difficult, see Hinch p 77)

$$y'' + \frac{y'}{r} + (y')^2 + \varepsilon y y' = 0, \quad y(r=0) = 0 \quad y(r=\infty) = 1,$$

### 4.2.9 Cole oscillator

An harmonic damped oscillator with a small mass, with a given impulse

$$\varepsilon y'' + y' + y = 0, \ \ y(0) = 0, \ \ \varepsilon y'(0) = 1$$

### 4.2.10 Van der Pol Oscillator

An oscillator with a damping depending of the amplitude, so that it is unstable at small amplitude

$$y'' - \varepsilon(1 - y^2)y' + y = 0, \ y(0) = y_0, \ y'(0) = 0$$

comes from electonics with vacuum tubes (non linear in intensity)

### 4.2.11 Canard cycle Oscillator Van der Pol

A relaxation oscillator like Van der Pol with very fast transition (slow fast system)

$$\varepsilon \dot{y} = z - y^3/3 + y$$
$$\dot{z} = a - y$$

### 4.2.12 Rayleigh Oscillator

An oscillator with a damping depending of the velocity $y'$, so that it is unstable at small velocity

$$y'' - \varepsilon(1 - (y')^2)y' + y = 0,\ y(0) = y_0,\ y'(0) = 0$$

### 4.2.13 Duffing Oscillator

An oscillator with a non linear elastic force

$$y'' + y - \varepsilon y^3 = 0,\ y(0) = 0,\ y'(0) = 1.$$

### 4.2.14 Mathieu Oscillator

An oscillator with a coefficient function periodical of time

$$y'' + (1 + \varepsilon \cos(t))y = 0,\ y(0) = 0,\ y'(0) = 1.$$

### 4.2.15 Heat in a fin

Steady heat equation averaged across the section

$$T''(x) = \frac{h}{ka}(T(x) - T_0)$$

The heat equation for a thin body at small Biot number.

### 4.2.16 Schrödinger, eigen state

A particule in a potential $V$ of a given energy $E$

$$-\frac{\hbar^2}{2m}\Psi'' + V(x)\Psi = E\Psi$$

## 4.3 PDE

### 4.3.1 Diffusion

Diffusion equation

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}$$

### 4.3.2 Reaction Diffusion

Diffusion equation with source term

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + f(c)$$

### 4.3.3 Advection linéaire

Transport in at constant velocity,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

### 4.3.4 Advection

A given $u$ transports another field $c$

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = 0$$

### 4.3.5 Advection Diffusion

Mix of previous ones

$$\frac{\partial c}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 c}{\partial x^2} + f(c)$$

### 4.3.6 Bürgers

Advection with diffusion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

### 4.3.7 Inviscid Bügers

Advection in its own velocity field

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

### 4.3.8 Ginzburg Landau

Remember Landau or Stuart Landau (Pitchfork bifurcation)

$$\frac{\partial A}{\partial t} = \sigma A - \lambda A|A|^2$$

Ginzburg Landau is :

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial x^2} + \varepsilon A - g|A^2|A$$

### 4.3.9 Kuramoto-Sivashinsky

A model for the diffusive-thermal instabilities in a laminar flame front, used for front propagation

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} + \frac{\partial^2 A}{\partial x^2} + \frac{\partial^4 A}{\partial x^4} = 0$$

### 4.3.10 Korteveg de Vries

Solitary wave equation

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} + \frac{\partial^3 A}{\partial x^3} = 0$$

### 4.3.11 Black Scholes

A heat equation with a negative coefficient, price of an option in financial market

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

#### 4.3.12 LWR

Traffic flow : the Lighthill-Whitham-Richards model, in Whitham book

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u(\rho)) = 0 \ \ u(\rho) = 1 - \rho$$

`http://www.clawpack.org/riemann_book/html/Traffic_flow.html`

#### 4.3.13 KPZ

Equation of surface growth

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} h + \lambda(\frac{\partial}{\partial x} h)^2$$

classic model for the evolution of the profile of a growing interface is the Kardar-Parisi-Zhang (KPZ) equation (Parisi Nobel 2021), here without the stochastic source. nor constant feeding, nor an extra slope effect. `http://basilisk.fr/sandbox/M1EMN/BASIC/kpz.c`

#### 4.3.14 Swift-Hohenberg

Model for pattern-forming behaviour (see Manneville)

$$\frac{\partial u}{\partial t} = ru - (1 + \frac{\partial^2}{\partial x^2})^2 u$$

#### 4.3.15 Cahn Hilliard

Model for phase separation

$$\frac{\partial f}{\partial t} = f(1-f) + \frac{\partial^2 f}{\partial x^2}$$

#### 4.3.16 Benney

Model of falling liquid films (see Kalliadasis Ruyer-Quil Scheid & Velarde)

$$\frac{\partial h}{\partial t} + h^2 \frac{\partial h}{\partial x} + \frac{\partial}{\partial x}\left((Ah^6 - Bh^3)\frac{\partial h}{\partial x} + Ch^3 \frac{\partial^3 h}{\partial x^3}\right) = 0$$

#### 4.3.17 First problem of Huppert, viscous dam

Spreading of a thin viscous column of fluids on a horizontal plate

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}(h^3 \frac{\partial h}{\partial x}) = 0$$

#### 4.3.18 Second problem of Huppert, collapse on a slope

Spreading of a thin viscous column of fluids over a slope

$$\frac{\partial h}{\partial t} + h^2 \frac{\partial h}{\partial x} = 0$$

#### 4.3.19 Flow in aquifere : Barenblatt

Non linear diffusion in an aquifer (Barenblatt's self similar problem of second type) in 2D ans axi

$$\frac{\partial h}{\partial t} - \frac{\partial^2}{\partial x^2} h^2 = 0$$

$$\frac{\partial h}{\partial t} - \frac{1}{r} r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} h^2 = 0$$

#### 4.3.20 Keller Segel model

Chimiotaxy of microorganisms

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}(k_1 \frac{\partial}{\partial x} n - k_2 \frac{\partial}{\partial x} c),$$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2}{\partial x^2} c + f(c, n),$$

$n$ density of organisms $c$ concentration of the chimioattractant.

#### 4.3.21 Non Linear Schrödinger

$$i \frac{\partial A}{\partial t} = -\frac{\partial^2 A}{\partial x^2} + |A^2| A$$

### 4.3.22 Schrödinger

$$i\hbar\frac{\partial}{\partial t}\Psi = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x))\Psi$$

type

$$iu_x + u_{yy}$$

do not confuse $u_{xx} - u_{yy} = 0$ or $u_{xx} + u_{yy} = 0$ or $u_{xx} = xu_{yy}$

### 4.3.23 Laplacian

$$u_{xx} + u_{yy} = 0$$

### 4.3.24 $\partial$'Alembert

$$u_{xx} - u_{yy} = 0$$

### 4.3.25 Euler Tricomi

$$u_{xx} = xu_{yy}$$

### 4.3.26 Poisson

$$u_{xx} + u_{yy} + f = 0$$

### 4.3.27 Helmholtz

$$u_{xx} + u_{yy} + k^2 u = 0$$

### 4.3.28 Wave

$$u_{tt} - u_{xx} = 0$$
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2}u = 0$$

### 4.3.29 Klein Gordon

$$u_{tt} - u_{xx} + u = 0$$

### 4.3.30 Sine Gordon

$$u_{tt} - u_{xx} + \sin u = 0$$

### 4.3.31 Wave

$$\left\{\begin{array}{ll} \frac{\partial u}{\partial t} & = -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial t} & = -\frac{\partial u}{\partial x} \end{array}\right.$$

### 4.3.32 Telegrapher's equations

$$\frac{\partial^2}{\partial x^2}V = LC\frac{\partial^2}{\partial t^2}V + (RC + GL)\frac{\partial}{\partial t}V + GRV$$

## 4.4 Mechanics

### 4.4.1 Navier Stokes

$$\underline{\nabla} \cdot \underline{u} = 0, \quad \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla u} = -\rho^{-1}\underline{\nabla} p + \nu \underline{\nabla}^2 \underline{u}. \tag{1}$$

### 4.4.2 Blasius Boundary layer Eq

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \text{ and } u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}$$

### 4.4.3 Shallow Water

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0, \text{ and } \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(\frac{Q^2}{h} + g\frac{h^2}{2}) = -ghZ' - \tau$$

### 4.4.4 Ground Water Flow, Barenblatt

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} h^2$$

### 4.4.5 Viscous collapse, Huppert

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} h^4$$

### 4.4.6 Flow in elastic pipes

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} = 0. \tag{2}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(\frac{Q^2}{S}) = -S\rho^{-1}\frac{\partial p}{\partial x} - 2\pi R\tau. \tag{3}$$

### 4.4.7 Waves ∂'Alembert

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

### 4.4.8 Waves : KdV

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

### 4.4.9 Waves : Benjamin Bona Mahony

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x \partial x \partial t}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial t \partial x^2} = 0$$

at first order same than KdV as $\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

but more stable

### 4.4.10 Waves : Boussinesq

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} & = 0 \\ \frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2)}{\partial x} & = -gh\frac{\partial h}{\partial x} + \frac{h_0^3}{3}\frac{\partial^3 u}{\partial x^2 \partial t} \end{cases}$$

#### 4.4.11 Waves : Serre Green Naghdi

the Serre-Green-Naghdi non linear weakly dispersive equations are :

$$\begin{cases} \frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{h}\bar{u}) & = 0 \\ \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}, & = -\frac{\partial \bar{\eta}}{\partial \bar{x}} + \frac{\delta^2}{3\bar{h}}\frac{\partial}{\partial \bar{x}}\left(\bar{h}^2(\frac{\partial^2 \bar{u}}{\partial \bar{t}\partial \bar{x}} + \bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - (\frac{\partial \bar{u}}{\partial \bar{x}})^2)\right) + O(\delta)^4. \end{cases}$$

#### 4.4.12 Waves : Benjamin Ono

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mathcal{H}\frac{\partial^2 u}{\partial x^2} = 0$$

with the Hilbert transform

$$\mathcal{H}f = \frac{1}{\pi}vp\int \frac{f(x)}{x-\xi}d\xi$$

$$\mathcal{H}f = \frac{1}{\pi}vp(\frac{1}{x} * f) = TF^{-1}(-isign(k)TF(f))$$

#### 4.4.13 Waves : in a horn, Webster Lokshin

$$\frac{\partial^2 p}{c_0^2\partial t^2} + \alpha\frac{\partial^{3/2} p}{\partial t^{3/2}} - \frac{1}{R(x)^2}\frac{\partial}{\partial x}(R(x)^2 p) = 0$$

#### 4.4.14 Waves in falling films

Benney

#### 4.4.15 Heat

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

http://basilisk.fr/sandbox/M1EMN/BASIC/heat.c
http://basilisk.fr/sandbox/M1EMN/BASIC/heat_imp.c

#### 4.4.16 heat in a fin

$$\rho c_p \frac{\partial}{\partial t}T(x,t) = k\frac{\partial^2}{\partial x^2}T(x,t) - 2hT(x,t)$$

#### 4.4.17 Beam equations.

$$\rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2}{\partial x^2}w\right) = 0$$

#### 4.4.18 Beam equations with loading and longitudinal tension.

$$\rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2}{\partial x^2}w\right) - \frac{\partial}{\partial x}T\frac{\partial}{\partial x}w = P(x)$$

### 4.5 Weak form

$$\nabla^2 u + f = 0$$

$$\int_\Omega \nabla u \cdot \nabla v \, d\tau = \int_\Omega fv \, d\tau$$

### 4.6 Conservative form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0$$

## 4.7 Dérivée fractionnaire :

$$\frac{d^0}{dx^0} f = f(x)$$

$$\frac{d^{-1}}{dx^{-1}} f = \int_0^x f(s) ds$$

$$\frac{d^{-2}}{dx^{-2}} f = \int_0^x f(s)(x-s) ds$$

$$\frac{d^{-3}}{dx^{-3}} f = \int_0^x \frac{f(s)(x-s)^2}{(2!)} ds$$

$$\frac{d^{-q}}{dx^{-q}} f = \int_0^x \frac{f(s)(x-s)^{q-1}}{(q-1)!)} ) ds$$

if $q \in \mathbb{N},\ \Gamma(q) = (q-1)!$.

$$\frac{d^{-q}}{dx^{-q}} f = \int_0^x \frac{f(s)(x-s)^{q-1}}{\Gamma(q)} ) ds$$

$$\frac{d^{1/2}}{dx^{1/2}} f = \sqrt{\frac{1}{\pi}} \left( \int_0^x \frac{\frac{d}{ds} f(s)}{\sqrt{t-s}} ds \right)$$

# 5 Special functions

## 5.1 exponential, log, cos and sin

OK for every body as well as the description with complex numbers.

## 5.2 Gamma function

by definition $\Gamma(z) = \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt$.
it easy to show $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1) = 1$,
hence $\forall\, n \in \mathbb{N},\ \Gamma(n+1) = n!$.
Other values $\Gamma(1/2) = \sqrt{\pi}$ (integral of gaussian)
$n\,! \simeq \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$ for $n \in \mathbb{N}$ ,

## 5.3 Bessel

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

or $y'' + y'/x + (1 - \alpha^2/x^2)y = 0$ with $\alpha$ real or integer $\alpha = n$

for $\alpha = 0$, $J_0''(x) + J_0'(x)/x + J(x) = 0$.
Bessel function $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + ...$
$J_0$ approximation for large $x$ $J_0 \sim \sqrt{2/(\pi x)}\cos(x - \pi/4)$

for $n$ integer,
first kind : $J_n(x) = \Sigma_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$,
second kind : $Y_\alpha(x) = \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi}$
for $\alpha = n \in \mathbb{N}$, $Y_n(x) = \lim_{\alpha\to n} Y_\alpha(x)$.

Abramowitz and Stegun. 1965 Handbook of Mathematical Functions. `http://people.math.sfu.ca/~cbm/aands/page_358.htm`

(`https://fr.wikipedia.org/wiki/Fonction_de_Bessel`).

## 5.4 Modified Bessel

Modified Bessel Equation of order $n$.

$$z^2 \frac{d^2 K_n(z)}{dz^2} + z\frac{dK_n(z)}{dz} - \left(z^2 + n^2\right) K_n(z) = 0,$$

from Abramowitz and Stegun, 1965, The general expression is that $I_n(z)$ and $K_n(z)$ are the two linearly independent solutions to the modified Bessel's equation :
and there is a recursive relation

$$\frac{dK_n(z)}{dz} = \frac{nK_n(z)}{z} - K_{n+1}(z).$$

We have for $z \ll 1$ the expansion $K_0(z) = -\gamma - \ln(z/2) + ...$ where $\gamma = 0.577216...$ (the Euler constant $\gamma$).
And for $z \gg 1$

$$I_0(z) = \frac{1}{\sqrt{2\pi z}}\exp(+z)(1+O(z^{-1})), \text{ and } K_0(z) = \frac{\sqrt{\pi}}{\sqrt{2z}}\exp(-z)(1+O(z^{-1}))$$

The expansions for $K_1$ for $z \ll 1$ and for $z \gg 1$ are respectively

$$K_1 = \frac{1}{z} + \frac{z}{2}\ln\frac{z}{2} + ...resp.\ K_1 = \sqrt{\frac{\pi}{2z}}\exp(-z) + ....$$

The expansions for $K_n$ for $z \ll 1$ and for $n \geq 1$ is

$$K_n = 2^n (n-1)! z^{-n} + ...$$

### 5.4.1 Airy

$$\frac{d^2 y}{dx^2} - xy = 0$$

The Queen of all the functions...

solution of $y''(x) = xy(x)$ with $y(\infty) = 0$ and $\int_{-\infty}^{\infty} Ai(x)dx = 1$ is $y = Ai(x)$ Airy's function.

There is a second solution $y = Bi(x)$ which goes to infinity
Integral definition

$$Ai(z) = \frac{1}{\pi}\int_0^\infty cos(sz + \frac{sz^3}{3})ds$$

asymptotics by stationary phase, steepest descent Hinch page 34, Erdély page 41

$$\text{for } z<0 \; Ai(z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4}\sin(\frac{2}{3}|z|^{3/2} + \pi/4)$$

$$\text{for } z>0 \quad \text{we have } Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}}$$

for $|arg(z)| < \pi/3$

$$Bi(z) \sim \frac{1}{\sqrt{\pi}\, z^{1/4}} \exp\left(\frac{2}{3} z^{3/2}\right).$$

$$Ai(x) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} - \frac{x}{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)} + \frac{x^3}{6\; 3^{2/3}\Gamma\left(\frac{2}{3}\right)} - \frac{x^4}{12\left(\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)\right)} + O\left(x^5\right)$$

$$Ai(x) = \frac{1 + \frac{x^3}{2*3} + \frac{x^{(2*3)}}{2*3*5*6} + \frac{x^{3*3}}{2*3*5*6*8*9} + ... + \frac{a^+_{k-1}}{(3k-1)3k}x^{3k} + ...}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} -$$

$$-\frac{x + \frac{x^4}{3*4} + \frac{x^7}{3*4*6*7} + \frac{x^{10}}{3*4*6*7*9*10} + ... + \frac{a^-_{k-1}}{3k(3k+1)}x^{3k+1} + ...}{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)}$$

### 5.4.2 Lambert function

The Lambert function is defined in implicit way

$$z = W(z)e^{W(z)}$$

approximation as $e^W = z/W$ so $W = \ln(z) - \ln(W)$ hence $W = \ln(z) - \ln(\ln(z) - \ln(W))$ as $W \ll z$ we have $W \simeq \ln(z) - \ln(\ln(z))$, a better approximation of $W$ is :

$$W = \ln(z) - \ln(\ln(z)) + \ln(\ln(z))/\ln(z) + ...$$

### 5.4.3 Exponential integral

Solution of

$$zw'''(z) + 2w''(z) - zw'(z) = 0$$

is (Bender Orzag p 252) $w = A + BE_1(z) + CE_1(-z)$ where

$$E_1(z) = \int_z^\infty e^{-t}/tdt$$

note that $Ei(z) = -\int_{-z}^\infty e^{-t}/tdt$ and $E_1(z) = -Ei(-z)$ where $Ei$ is the exponential integral defined in the software Mathematica.

The integral $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt$ is called exponential integral (Bender Orzag p 252), by definition solution of

$$\frac{dE_1(x)}{dx} = -\frac{e^{-x}}{x} = -\frac{1}{x} + 1 - \frac{x}{2} + ...$$

so that

$$E_1(x) = C - ln(x) + x - \frac{x^2}{4} + ...$$

Bender Orzag [?] p307 or Abramowitz and Stegun [3] 5.1.11, the constant is $C = \gamma$

$$\gamma = lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - ln(n)) \simeq 0.5772$$

Euler-Mascheroni constant

$$\gamma \simeq 0.577215664901532860606512090082402431042159...$$

so that it is classical that :

$$\gamma = lim_{x\to 0^+}(\int_x^\infty \frac{e^{-t}}{t}dt + lnx).$$

**proof**
It seems that a way to prove it, is to start from the fact that the definition of $\Gamma$ from Euler and Weierstrass are :

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}dt = e^{-\gamma x}\prod_{n=1}^\infty e^{x/n}\left(1 + x/n\right)^{-1}$$

and so ([?]) :

$$\Gamma'(1) = \int_0^\infty Log(t)e^{-t}dt = -\gamma$$

and integrating par parts :

$$\gamma = F(x) - Log(x) - R(x)$$

with

$$F(x) = \int_0^x \frac{1-e^{-t}}{t}dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{nn!} \quad \text{and} \quad R(x) = \int_x^\infty \frac{e^{-t}}{t}dt.$$

QED

### 5.5 Hypergeometric functions

The Gaussian or ordinary hypergeometric function ${}_2F_1(a,b,c,x)$ is by definition :

$${}_2F_1(a,b,c,x) = 1 + \frac{abx}{c} + \frac{a(a+1)b(b+1)x^2}{2c(c+1)} +$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)x^3}{6c(c+1)(c+2)} + O\left(x^4\right)$$

where ${}_2F_1(a,b,c,z)$ is solution of the homogenous second order differential equation :

$$z(1-z)\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + [c-(a+b+1)z]\frac{\mathrm{d}y}{\mathrm{d}z} - aby = 0.$$

### 5.6 Kummer's functions

${}_1F_1(a,b,z)$ is solution of the homogenous second order differential equation :

$$z\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + [b-z]\frac{\mathrm{d}y}{\mathrm{d}z} - ay = 0.$$

relation to hypergeometric functions

$${}_1F_1(a,b,z) = \lim_{b\to\infty} {}_2F_1(a,b,c,z/b)$$

### 5.7 ParabolicCylinder"

"ParabolicCylinder" function : $y(x) = D_\nu(x)$ satisfies the Weber differential equation . . $\frac{d^2y(x)}{dx^2} + (\nu + \frac{1}{2} - \frac{x^2}{4})y(x) = 0$

$$D_a(z) = \frac{1}{\sqrt{\pi}}2^{a/2}e^{-\frac{z^2}{4}}\left(\cos\left(\frac{\pi a}{2}\right)\Gamma\left(\frac{a+1}{2}\right){}_1F_1\left(-\frac{a}{2};\frac{1}{2};\frac{z^2}{2}\right) + \sqrt{2}z\sin\left(\frac{\pi a}{2}\right.\right.$$

### 5.8 Meijer G-function

Meijer G-function is a very general function intended to include most of the known special functions as particular cases.

## Références


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