M2, **Fluid mechanics** 2013/2014
Friday, December 8th, 2013

Multiscale Hydrodynamic Phenomena

Part I. : 60 minutes, NO documents

1. Quick Questions In few words :
1.1 What is "dominant balance" ?
1.2 What is the dimension of the kinematic viscosity ?
1.3 Write Navier Stokes without dimension
1.4 What is the usual scale for pressure in incompressible NS equation ?
1.5 What is the usual scale for pressure in incompressible NS equation at small Reynolds ?
1.6 What is the natural selfsimilar variable for heat equation ?
1.7 In which one of the 3 decks of Triple Deck is flow separation ?

2. Exercice
2.1 What is the name of the following equation

$$\frac{\partial \eta}{\partial t} + \frac{3}{2}\eta \frac{\partial \eta}{\partial x} + \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3} = 0.$$

2.2 Say in few sentences where does it come from (what are the hypothesis)
2.3 Linearise the equation (and question 2.4 is for this linearized equation)
2.4 Suppose that the integral of η over the domain is conserved. Show that the self similar variable is $\frac{x}{t^{1/3}}$

$$\eta(x,t) = t^{-1/3} f(\frac{x}{t^{1/3}})$$

where f is linked to the Airy function, which solves $Ai''(y) = yAi(y)$

3. Exercice
Let us look at the following ordinary differential equation :

$$(E_\varepsilon) \qquad \varepsilon \frac{d^2 y}{dx^2} + 2\frac{dy}{dx} = 1,$$

valid for $0 \leq x \leq 1$, with boundary conditions $y(0) = 0$ and $y(1) = 1$. Of course ε is a given small parameter. We want to solve this problem with the Matched Asymptotic Expansion method (if you prefer use Multiple Scales or WKB, do it if you have time).
3.1) Why is this problem singular ?
3.2) What is the outer problem obtained from (E_ε) and what is the possible general form of the outer solution ?
3.3) What is the inner problem of (E_ε) and what is the inner solution ?
3.4) Solve the problem at first order (up to power ε^0).
3.5) Suggest the plot of the inner and outer solution.
3.6) What is the exact solution for any ε.

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Part II. : 1h30min all documents.

Flow in a kitchen sink : the circular hydraulic jump

We will try to reobtain all the equations from Higuera's paper. Read first the first page of Higuera and read the Watson paper (parts).

0.1 Draw a sketch of the problem with the variables.

Equations from Watson

1.1 Inviscid theory of Watson, reconstruct equation (1) which is simply the integral mass conservation to obtain (2).

1.2 Equations (8)-(12) are with dimension, write them without dimensions as in (1)-(5) from Higuera

1.3 Follow the developments to obtain f' of Watson.

1.4 Recompute (23) of Watson.

1.5 Put the good axes and ordinates on figure 1 of Watson.

Equations from Higuera

2.1 Comment the S parameter from question 1.2 which is not in (9) from Watson.

2.2 Comment the boundary conditions

2.3 Verify that the H_w and U_w are consistent with Watson.

2.4 Read Higuera up to the end and give a summary of what happens pages 1477-1478

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The radial spread of a liquid jet over a horizontal plane

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When a smooth jet of water falls vertically on to a horizontal plane, it spreads out radially in a thin layer bounded by a circular hydraulic jump, outside which the depth is much greater. The motion in the layer is studied here by means of boundary-layer theory, both for laminar and for turbulent flow, and relations are obtained for the radius of the hydraulic jump. These relations are compared with experimental results. The analogous problems of two-dimensional flow are also treated.

1. Introduction

It is a familiar observation that when a smooth jet of water falls vertically from a tap on to a horizontal plane, such as the bottom of an empty sink, the water spreads out in a thin layer until a sudden increase of depth occurs. This is an hydraulic jump, or standing wave, the stationary counterpart of a tidal bore. The formation of the thin layer and the circular jump was noticed by Rayleigh (1914), who derived the properties of bores and jumps. Rayleigh's analysis refers to flow along a channel of constant breadth, and assumes the speed ahead of the wave to be uniform. In the present case the flow in the thin layer is radial and strongly influenced by viscosity, but the principles of momentum and continuity apply at the jump as in Rayleigh's theory.

Since the central layer of fluid is thin, it is natural to apply the ideas of boundary-layer theory in order to discuss the motion. A necessary condition for this approach to be valid is that the Reynolds number of the impinging jet shall be large. The depth is observed to be much greater on the outside of the

2. Inviscid theory

When viscosity is ignored, the motion produced by a round jet falling vertically on to a horizontal plane is one of potential flow with free streamlines. Methods for the solution of problems of this type are described by Birkhoff & Zarantonello (1957). When r, the distance from the axis of the jet, is large compared with a, the radius of the impinging jet, the depth h of the fluid on the plane is small and the motion is almost radial with speed U_0, the speed with which the jet strikes the plane. Hence the volume rate of flow is

$$Q = \text{[redacted]} = \text{[redacted]}, \qquad (1)$$

so that

$$h = a^2/2r. \qquad (2)$$

The condition to be applied at the jump (due originally to Bélanger 1838) is that the thrust of the pressure is equal to the rate at which momentum is destroyed. The depth on the inside of the jump is given by (2) with $r = r_1$, the

(...)

3. Similarity solution of the boundary-layer equations

According to the boundary-layer approximations the flow in the thin layer satisfies the equations

$$\partial(ru)/\partial r+\partial(rw)/\partial z=0, \tag{8}$$

$$u(\partial u/\partial r)+w(\partial u/\partial z)=\nu(\partial^2u/\partial z^2), \tag{9}$$

with the conditions

$$u=w=0 \quad \text{at} \quad z=0, \tag{10}$$

$$\partial u/\partial z=0 \quad \text{at} \quad z=h(r), \tag{11}$$

$$2\pi r\int_0^{h(r)} u\,dz=Q. \tag{12}$$

Here r, z are cylindrical co-ordinates, with z measured vertically upwards from the plate, and u, w are the corresponding velocity components. In equation (9) the gravitational pressure gradient $(-\rho g\,dh/dr)$ has been ignored. Equation (11) asserts that the shearing stress falls to zero at the free surface $z=h(r)$, since the viscosity of air is negligible, and (12) is the condition of constant volume flux.

In this section a similarity solution will be derived by assuming that

$$u=U(r)f(\eta), \tag{13}$$

$$\eta=z/h(r), \tag{14}$$

where $U(r)$ is the speed at the free surface. Then from (10) and (11)

$$f(0)=0,\quad f(1)=1,\quad f'(1)=0, \tag{15}$$

and from (12)

$$Q=2\pi rUh\int_0^1 f(\eta)\,d\eta. \tag{16}$$

Hence rUh is constant, and (8) then leads to

$$w=Uh'\eta f(\eta). \tag{17}$$

The equation of motion (9) now reduces to

$$\nu f''(\eta)=h^2U'f^2(\eta),$$

from which it follows that h^2U' is constant. Also $f''(\eta)\leqslant 0$, since the shearing stress is greatest at the plate, and it is convenient to write

$$h^2U'=-\tfrac{3}{2}c^2\nu, \tag{18}$$

where c is a number. Then $2f''=-3c^2f^2$, which, from (15), may be integrated to $f'^2=c^2(1-f^3)$. Since $f'\geqslant 0$,

$$c\eta=\int_0^f(1-x^3)^{-\frac{1}{2}}dx. \tag{19}$$

The condition $f(1)=1$ now gives

$$c=\int_0^1(1-x^3)^{-\frac{1}{2}}dx=\frac{1}{3}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})}=1{\cdot}402. \tag{20}$$

Then

$$\int_0^1 f(\eta)\,d\eta=c^{-1}\int_0^1 f(1-f^3)^{-\frac{1}{2}}df=\frac{1}{3c}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{6})}=\frac{2\pi}{3\sqrt{3c^2}}. \tag{21}$$

Consequently (16) gives

$$rUh=3\sqrt{3}c^2Q/4\pi^2. \tag{22}$$

The only conditions on $U(r), h(r)$ necessary for the similarity solution are (18) and (22). The general solution of these equations is

$U(r)=$ [redacted] (23)

$$h(r)=\frac{2\pi^2}{3\sqrt{3}}\frac{\nu(r^3+l^3)}{Qr}, \tag{24}$$

where l is an arbitrary constant length.

In the actual flow this similarity solution can only be expected to hold when r is sufficiently large for the conditions in the incident jet to have lost their influence. The value of l, however, depends on these conditions, and must be found by consideration of the growth of the boundary layer from the point of impact of the jet. A method for the estimation of l will be described in § 4.

The velocity profile in the similarity solution is given by (19), which can also be expressed by means of Jacobian elliptic functions (Neville 1944) as

$$f(\eta)=\sqrt{3}+1-\frac{2\sqrt{3}}{1+\mathrm{cn}\,\{3^{\frac{1}{4}}c(1-\eta)\}}, \tag{25}$$

where the modulus is sin 75°. Hence, in terms of the elliptic integral $F(\theta)$ with this modulus,

$$\left.\begin{aligned}\nu(r^3+l^3)\,u/Q^2&=(27c^2/8\pi^4)\,(1-\sqrt{3}\tan^2\tfrac{1}{2}\theta),\\ Qrz/\nu(r^3+l^3)&=(2\pi^2/3\sqrt{3})\,\{1-3^{-\frac{1}{4}}c^{-1}F(\theta)\}.\end{aligned}\right\} \tag{26}$$

$\theta=0$ corresponds to the free surface and $\theta=\cos^{-1}(2-\sqrt{3})\simeq 74\tfrac{1}{2}°$ to the surface of the plate. The profile (25) is shown in figure 1.

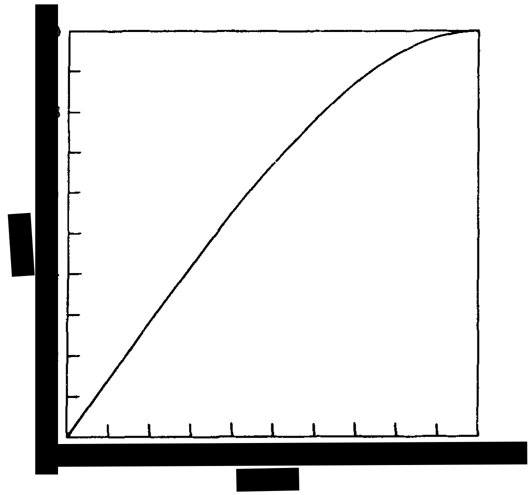

FIGURE 1. The velocity distribution function.

It is of interest to observe that the analogous problems of the wall jet (Glauert 1956, 1958) and the radial free jet (Squire 1955) also yield similarity solutions in which the combination (r^3+l^3) occurs as in equations (23) and (24). In this connexion see Riley (1961, 1962).

The circular hydraulic jump

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An asymptotic order-of-magnitude description is given of the structure of a circular laminar hydraulic jump for large values of the Reynolds and Froude numbers of the flow entering the jump. The results are compared with numerical solutions of the boundary layer equations for the flow in a liquid layer on a horizontal disk and with experimental results existing in the literature. © *1997 American Institute of Physics.* [S1070-6631(97)01405-0]

Circular hydraulic jumps are transitions from supercritical to subcritical flow in radially spreading horizontal liquid layers. Laminar flows displaying this phenomenon occur in cooling systems by impinging liquid jets, and can also be seen in a kitchen sink, while large scale turbulent jumps are common in many hydraulic devices. Strong laminar jumps are fairly long structures,[1–3] despite their relative abruptness in the scale of the whole layer, owing to the need of disposing of a large amount of kinetic energy by viscous dissipation only. The circular jumps exhibit differences with their planar counterparts due to the divergence of the flow in their interior. Thus, treating the jump as a discontinuity, in accordance with the classical theory, Watson[4] found discrepancies with some of his own experimental results and showed how the discrepancies could be decreased by taking into account the length of the jump, while Koloseus and Ahmad[5] derived mass and momentum conservation conditions for finite-length jumps assuming a linear variation of the liquid depth, and their results have been made more precise using correlations valid for turbulent flow. Craik *et al.*[1] observed that the laminar jump contains a long recirculating eddy attached to the wall, as in the inset of Fig. 1, which shortens when the jump is made unstable by increasing the downstream depth and pushing it toward the origin. Laminar jumps with a roller at the surface and with a roller and an eddy (double roller jumps) have been observed by Liu and Lienhard,[2] who also found that the jumps are stabilized by the effect of the surface tension. An asymptotic high Reynolds number analysis of the flow around the upstream end (toe) of a laminar jump in a layer of uniform velocity with a thin viscous sublayer was carried out by Gajjar and Smith[6] using the interacting boundary layer theory, and their analysis was subsequently extended by Bowles and Smith[3] to large Froude number fully developed layers. The rest of the structure of a planar jump in this limit was analyzed in Ref. 7. Here a similar analysis is presented for the laminar circular jump in the absence of surface tension effects.

Consider the liquid layer formed on a horizontal disk of radius R by the radial spread of an impinging vertical liquid jet of radius a. In the asymptotic limit of large Reynolds numbers, $\mathrm{Re}=Q/2\pi\nu R\gg 1$, where Q is the flux of the jet and ν is the kinematic viscosity of the liquid, the boundary layer approximation can be used to describe the flow in the layer at distances from the center large compared with a. In the absence of gravity this flow would become self-similar[4] after a distance from the center of order $R(a/R)^{2/3}\,\mathrm{Re}^{1/3}$, which will be neglected here assuming that $a\ll R/\mathrm{Re}^{1/2}$. In addition, the balance of convection and viscous forces in the momentum equation ($u_c^2/R=\nu u_c/h_c^2$, where u_c and h_c are the characteristic liquid velocity and depth of the layer) and the mass conservation condition ($u_c h_c R=Q/2\pi$) yield $u_c=(Q/2\pi)^2/\nu R^3$ and $h_c=\nu R^2/(Q/2\pi)$. Scaling the horizontal and vertical distances with R and h_c, and the corresponding components of the velocity with u_c and $u_c h_c/R$, the non-dimensional continuity and momentum equations determining the velocity of the fluid and the depth of the layer under the action of the gravity are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial r}+v\frac{\partial u}{\partial y}=-S\,\frac{dh}{dr}+\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

with the boundary conditions

$$u=v=0 \quad \text{at} \quad y=0, \tag{3}$$

$$\partial u/\partial y=0 \quad \text{at} \quad y=h(r), \tag{4}$$

$$h=h_0 \quad \text{at} \quad r=1, \tag{5}$$

$$u\to U_w(y/r^2)/r^3, \quad h\to H_w r^2 \quad \text{for} \quad r\to 0, \tag{6}$$

$$r\int_0^h u\,dy=1, \tag{7}$$

where $S=g\nu^3R^8/(Q/2\pi)^5$ is the inverse of an overall Froude number measuring the effect of the acceleration of the gravity g and h_0 is the non-dimensional depth of the liquid at the edge of the disk, which is taken as data here (though in some practical cases the outflow might be over the edge of the disk or a weir and ought to be treated as a separate problem). Use has been made of the hydrostatic pressure balance in the vertical direction across the layer to write (2) and the density and viscosity of the air above the layer have been neglected, as well as the surface tension. The boundary conditions (6) state that the flow in the layer takes the self-similar Watson's form[4] near the center [with $U_w''+3U_w^2=0$, $U_w(0)=0$, $U_w'(H_w)=0$ and $\int_0^{H_w}U_w\,d(y/r^2)=1$, giving in particular $H_w\approx 0.6046$ and $\phi_{M_W}=\int_0^{H_w}U_w^2\,d(y/r^2)\approx 2.079$]. These boundary conditions can be justified noticing that the local effect of the gravity

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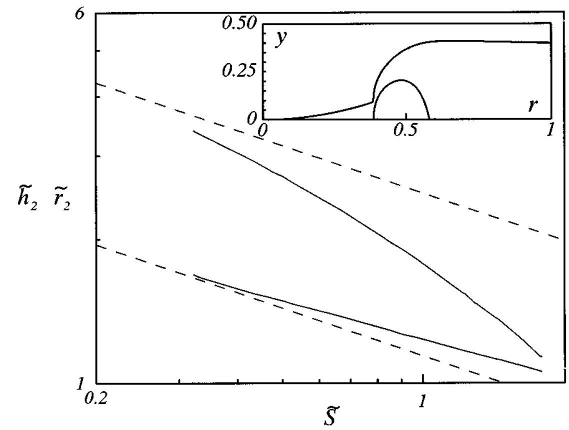

FIG. 1. Maximum depth of the layer (upper curve) and position of the maximum (lower curve) versus $\tilde{S}$ from the numerical solution of (1)–(7) for different values of S and h_0. The dashed lines have slope $\tilde{S}^{-1/3}$. Inset: liquid surface and recirculating eddy for $S=1000$ and $h_0=0.4$.

evaluated with Watson's solution is $(-S\,dh/dr)/\mathbf{v}\cdot\nabla u = (2H_W/3U_W^2)(Sr^8)$, which rapidly tends to zero for $r\to 0$.

Very strong hydraulic jumps are characterized by large local Froude numbers at their upstream sides, where the oncoming flow is therefore little affected by the gravity and the self-similar Watson's solution applies to a good approximation. Calling r_J the location of the toe of the jump, this condition amounts to $\tilde{S}=Sr_J^8 \ll 1$, and rescaled variables appropriate to describe the inner structure of the jump are $\tilde{r}=r/r_J$, $(\tilde{y},\tilde{h})=(y,h)/r_J^2$, $\tilde{u}=r_J^3 u$ and $\tilde{v}=r_J^2 v$. Equations (1)–(4) do not change form when rewritten in terms of these variables, except that S changes to $\tilde{S}$. A short interaction region exists around the toe of the jump[3] where $\tilde{r}-1=O(\tilde{S}^3)$ and the flow is locally planar and not affected by the gravity-induced pressure except in a low velocity viscous sublayer where $\tilde{y}=O(\tilde{S})$. The solution in this region was analyzed by Gajjar and Smith[6] and Bowles and Smith,[3] who showed that the flow in the viscous sublayer separates under the action of the adverse self-induced pressure gradient, leading to an effectively inviscid recirculation region of thickness $\tilde{S}[(\tilde{r}-1)/\tilde{S}^3]^m$ for $(\tilde{r}-1)/\tilde{S}^3 \gg 1$, with $m=\frac{2}{3}\times(\sqrt{7}-2)\approx 0.4305$, where the velocity of the fluid is of order $\tilde{u}_{ri}=\tilde{S}[(\tilde{r}-1)/\tilde{S}^3]^{2/3-m}$. This region is bounded above by a viscous shear layer of thickness $(\tilde{r}-1)^{1/3}$ at the base of the separated stream, and below by a viscous boundary layer of thickness $\tilde{S}[(\tilde{r}-1)/\tilde{S}^3]^{3m/(3m+2)}$ close up to the solid surface, which is the only region affected by the gravity-induced pressure when $(\tilde{r}-1)/\tilde{S}^3 \gg 1$.

The viscous shear layer grows to cover the whole separated flow when $(\tilde{r}-1)=O(1)$ and becomes a self-similar radially spreading jet for $\tilde{r}\gg 1$ and $\eta=(\tilde{y}-\tilde{h})/\tilde{r}=O(1)<0$, riding over a region of slower recirculating flow. The stream function in the jet is $\tilde{\psi}=\tilde{r}f(\eta)$, with $f(\eta)=f_{-\infty}\tanh(f_{-\infty}\eta/4)$ and $f_{-\infty}=-(6\phi_{M_W})^{1/3}\approx -2.319$, from the condition of conservation of the momentum flux of the oncoming Watson's flow. The velocity in the jet decays as $1/\tilde{r}$ and the effect of the gravity finally comes into play when $\tilde{S}\tilde{h}\tilde{r}^2=O(1)$. Assuming that the thickness of the jet, $(\tilde{y}-\tilde{h})=O(\tilde{r})$, is by then of the order of the depth of the whole layer (as seems appropriate for the liquid to recirculate smoothly; see Ref. 7), this estimation yields

$$(\tilde{h},\tilde{r})=O\left(\frac{1}{\tilde{S}^{1/3}}\right) \quad \text{and} \quad \tilde{u}=O(\tilde{S}^{1/3}) \tag{8}$$

in the bulk of the bubble. Notice that the effect of the radial divergence is important to the flow in the bubble because $\tilde{r}\gg 1$; the length of the jump is large compared with the distance from its toe to the center of the disk. A consequence of this divergence is that the order of $\tilde{h}$ in (8) for $\tilde{S}\ll 1$ is not as large as the downstream depth $(2\phi_{M_W}/\tilde{S})^{1/2}\approx 2.039/\tilde{S}^{1/2}$ predicted by the classical theory, which treats the jump as a discontinuity. Notice also that the asymptotic solution for $(\tilde{r}-1)/\tilde{S}^3 \gg 1$ mentioned in the previous paragraph involves reverse flow velocities of order $\tilde{u}_{ri}$ which become $O(\tilde{S}^{1/3})$ for $(\tilde{r}-1)$ of order $l=\tilde{S}^{(4-9m)/(2-3m)}\ll 1$. For $l\ll(\tilde{r}-1)\lesssim 1$ the velocity of the recirculating fluid remains $O(\tilde{S}^{1/3})$ and the depth of the bubble grows as $(\tilde{r}-1)^{2/3}/\tilde{S}^{1/3}$ (from the balance of the recirculating flux and the flux ingested by the shear layer on top of the bubble). The effect of the gravity on this recirculating fluid, measured by $\tilde{S}\tilde{h}/\tilde{u}_{ri}^2=O[(\tilde{r}-1)^{2/3}]$, is negligible for $(\tilde{r}-1)\ll 1$, while the depth of the layer stops growing appreciably for larger values of $\tilde{r}$.

Most of the flux in the separated jet, of order $\tilde{S}^{-1/3}\gg 1$, recirculates in the rear part of the bubble, leaving a flow with velocity of order $\tilde{S}^{2/3}$ downstream of the jump (from the condition of conservation of the mass flux $\tilde{r}\int_0^{\tilde{h}}\tilde{u}\,d\tilde{y}=1$ with $\tilde{h}$ and $\tilde{r}$ of order $\tilde{S}^{-1/3}$). In this region $\tilde{\mathbf{v}}\cdot\nabla\tilde{u}/(\tilde{S}\,d\tilde{h}/d\tilde{r})=O(\tilde{S}^{2/3})\ll 1$ and the balance of hydrostatic pressure and viscous forces gives a parabolic velocity profile and $\tilde{h}=\{\tilde{h}_0^4-(12/\tilde{S})\ln(r_J\tilde{r})\}^{1/4}$, where $\tilde{h}_0=h_0/r_J^2$. An equation determining r_J for given values of S and h_0 is obtained particularizing this expression immediately downstream of the jump, where $\tilde{h}=\tilde{h}_2=\alpha/\tilde{S}^{1/3}$ and $\tilde{r}=\tilde{r}_2=\beta/\tilde{S}^{1/3}$, say, α and β being order unity constants that could be determined from a detailed analysis of the inner structure of the jump. Since $\tilde{S}\ll 1$ for a strong jump, the resulting equation can be simplified to $\tilde{h}_2\approx\tilde{h}_0$, giving

$$\frac{r_J}{h_0^{1/2}}\approx\frac{\alpha^{3/2}}{(Sh_0^4)^{1/2}} \tag{9}$$

and therefore $\tilde{S}=[\alpha^4/(Sh_0^4)]^3$, so the condition $\tilde{S}\ll 1$ amounts to $Sh_0^4\gg 1$ in the present horizontal disk configuration, and the depth of the liquid varies very little downstream of the jump. It may be seen that the radius of the disk is irrelevant insofar as $Sh_0^5\ll 1$, while if this condition is violated the length of the jump becomes comparable to or larger than the radius of the disk. However, the experiments[1,2] show that the jump always becomes unstable before reaching that state.

In order to check the asymptotic results (8), the problem (1)–(7) was numerically solved for different values of S and h_0. The numerical solutions display jumps with a single eddy

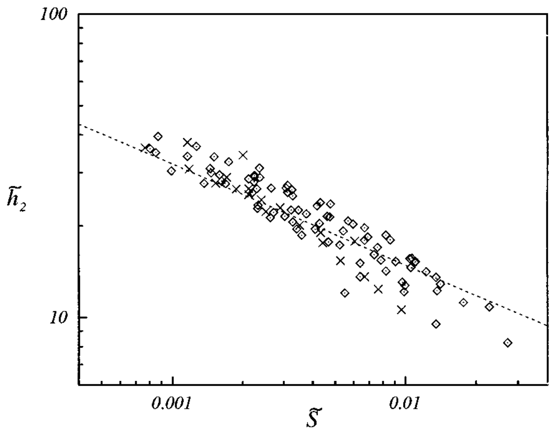

FIG. 2. Experimental results of Ref. 2. Diamonds: normal water surface tension. Crosses: reduced surface tension. The dashed line is $\tilde{h}_2 = 3.2/\tilde{S}^{1/3}$.

on the solid surface; see inset of Fig. 1 for an example. [The results of Ref. 7 suggest that surface rollers are a feature of laminar hydraulic jumps associated with cross-stream pressure variations due to the curvature of the streamlines, which is a finite Re effect not included in (1)–(7).] The resulting $\tilde{h}_2$ and $\tilde{r}_2$, defined as the maximum depth of the layer and its position on the disk, respectively, are plotted in Fig. 1 versus $\tilde{S} = Sr_J^8$, where r_J is taken as the position of the separation point. The curves of Fig. 1 contain data obtained with $S = 20$, 30, 200, 500, 1000, and 2000 and with $h_0 = 0.1$ to 1.2, confirming the expected dependence of the scaled (tilde) variables on the combination Sh_0^4 only. For small values of $\tilde{S}$ the results agree with (8), and the approximate values $\alpha \approx 2.5$ and $\beta \approx 1.14$ can be obtained fitting straight lines of slope $\tilde{S}^{-1/3}$ to the logarithmic plots.

In Fig. 2 the experimental results of Liu and Lienhard[2] are represented in a $\tilde{S}$–$\tilde{h}_2$ graph, using the assumption that Watson's solution holds up to the toe of the jump to compute $\tilde{S}$ and $\tilde{h}_2$ from the data given in Ref. 2. The connection of these experiments with the present analysis is only partial because, as was pointed out by Liu and Lienhard, the effect of the surface tension is important in the conditions of their experiments and, in addition, cross-stream pressure variations are probably also important,[7] while both effects have been left out of the analysis. Despite these important differences, the experimental results, which correspond to values of $\tilde{S}$ much smaller than the numerical solutions presented before, still show reasonable agreement with the estimate $\tilde{h}_2 = \alpha/\tilde{S}^{1/3}$, though with $\alpha \approx 3.2$, a value somewhat higher than the one found before.

Summarizing, an asymptotic structure has been proposed of the strong circular laminar hydraulic jump consisting of an interaction region around the toe containing the separation point, already analyzed in Refs. 3 and 6, and a recirculation region of length much larger than the standoff distance from the toe to the center of the spreading layer. Downstream of the interaction region the depth of the liquid layer is first proportional to $(\tilde{r}-1)^m$, with $m \approx 0.4305$, and then becomes proportional to $(\tilde{r}-1)^{2/3}$ at a distance from the separation point estimated in the paragraph below (8). This depth is of the order of the depth of the liquid downstream of the jump for $(\tilde{r}-1) = O(1)$, at which point the fast separated flow in the upper part of the layer is a jet much thinner than the region of slow recirculating flow in the lower part, and it is not yet affected by the gravity-induced pressure force. Further downstream the thickness of this jet grows proportionally to $\tilde{r}$ without increasing very much the total depth of the liquid, until its thickness and velocity become comparable to the corresponding magnitudes of the recirculating flow, for $\tilde{r} = O(1/\tilde{S}^{1/3})$, and the adverse pressure gradient due to the gravity leads to the closure of the recirculation bubble. The asymptotic results (8) obtained from this analysis are in fair agreement with numerical solutions of the equations governing the flow in the layer and with experimental results existing in the literature.

ACKNOWLEDGMENT

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