M2, **Fluid mechanics** 2014/2015
Friday, December 5th, 2014

Multiscale Hydrodynamic Phenomena

Part I. : 90 minutes, NO documents

1. Quick Questions In few words :
1.1 What is "dominant balance" ?
1.2 What is the dimension of the kinematic viscosity ?
1.3 Write Navier Stokes without dimension
1.4 What is the usual scale for pressure in incompressible NS equation ?
1.5 What is the usual scale for pressure in incompressible NS equation at small Reynolds ?
1.6 Which problem exhibits logarithms ?
1.7 What is the Bürgers equation ?
1.8 What is the KDV equation ?
1.9 What is the natural selfsimilar variable for heat equation ?
1.10 In which one of the 3 decks of Triple Deck is flow separation ?

2. Exercice
2.1 What is the name of the following equation (of course ε is a given small parameter)

$$(E_\varepsilon) \qquad u\frac{\partial u}{\partial x} = -\varepsilon\frac{\partial^2 u}{\partial x^2}. \text{ with } u(-\infty) = -1, \quad u(\infty) = 1.$$

We want to solve this problem with the Matched Asymptotic Expansion method.
2.2 Say in few sentences where does it come from 2.3 Why is this problem singular ?
2.4 What is the outer problem and what is the possible general form of the outer solution ?
2.5 What is the inner problem of (E_ε) and what is the inner solution ?
2.6 Solve the problem at first order (up to power ε^0).
2.7 Suggest the plot of the inner and outer solution.
2.8 What is the exact solution for any ε.
Hint : $tanh'(x) = 1 - tanh^2(x)$

3. Exercice
Let us look at the following ordinary differential equation : $(E_\varepsilon) \qquad \frac{d^2y}{dt^2} + \varepsilon\frac{dy}{dt} + y = 0,$ valid for any $t > 0$ with boundary conditions $y(0) = 0$ and $y'(0) = 1$. Of course ε is a given small parameter.
We want to solve this problem with Multiple Scales.
3.1 Expand up to order ε : $y = y_0(t) + \varepsilon y_1(t)$, show that there is a problem for long times.
3.2 Introduce two time scales, $t_0 = t$ and $t_1 = \varepsilon t$
3.3 Compute $\partial/\partial t$ and $\partial^2/\partial t^2$
3.4 Solve the problem.
3.5 Suggest the plot of the solution.
3.6 What is the exact solution for any ε, compare.

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Part II. : 1h 15 min all documents.

Compressible flow in a hypersonic reentry problem

We will settle the ideal fluid boundary layer decomposition for compressible flows. Read Bush for more informations

1.1 The Navier Stokes equations in steady 2D are :

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{xy} \text{ and } \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\tau_{yx} + \frac{\partial}{\partial y}\tau_{yy}$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xx}u + \tau_{xy}v + k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\tau_{yx}u + \tau_{yy}v + k\frac{\partial T}{\partial y})$$

where $H = h + \frac{1}{2}(u^2 + v^2)$ is total enthalpy $h = c_p T$ is enthalpy ;
the viscous stress tensor is $(\lambda = -2/3\mu)$:
$\tau_{xx} = \mu(2\frac{\partial u}{\partial x} - \frac{2}{3}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}))$, $\tau_{yy} = \mu(2\frac{\partial v}{\partial y} - \frac{2}{3}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}))$, $\tau_{xy} = \tau_{yx} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$,
the ideal gas law $p = r\rho T$, or $p/p_\infty = (\rho/\rho_\infty)(T/T_\infty)$, and heat capacity $c_p = \gamma c_v = \gamma r/(\gamma - 1)$
Write them without dimension using for any variable $q = q_\infty \bar{q}$ (see Bush paper).
Define Re_L, the Mach number $M^2 = U_\infty^2/(\gamma p_\infty/\rho_\infty)$
1.2 What are the equations for $1/Re_L = 0$?

2. Ideal Fluid problem

It can easily be shown (from compressible inviscid shock wave theory) that the compressible inviscid flow along a wedge of angle θ is such that the pressure is

$$\frac{p}{p_\infty} = 1 + \frac{\gamma(\gamma+1)}{4}K^2 + \gamma K^2 \sqrt{\left(\frac{(\gamma+1)}{4}\right)^2 + \frac{1}{K^2}}$$

where $K = M\theta$ and $\gamma = c_p/c_v$, for air $\gamma = 1.4$
2.1 expand this relation at small angle θ when $M > 1$, but $M = O(1)$.
2.2 Show that this is the same than the solution of the ∂'Alembert equation for compressible perturbations of a uniform flow (no demonstration).
2.3 What is the incompressible relation linking perturbation of pressure to perturbation of velocity in ideal fluids (no demonstration) ?
2.4 There is another limit for the pressure-angle equation when $M \gg 1$ and $1 \gg \theta$ with $M\theta \gg 1$, show that the pressure is now proportional to the square of the angle. This is the hypersonic strong interaction case.

3. Boundary layer problem
3.1 Show from 1. that the boundary layer problem reads (in Bush Pr is σ.) :

$$\frac{\partial \tilde{\rho}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{\rho}\tilde{v}}{\partial \tilde{y}} = 0, \qquad \tilde{\rho}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{\rho}\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{d\bar{p}_e}{d\bar{x}} + \frac{\partial}{\partial \tilde{y}}(\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{y}}),$$

$$\tilde{\rho}\tilde{u}\frac{\partial \tilde{h}}{\partial \tilde{x}} + \tilde{\rho}\tilde{v}\frac{\partial \tilde{h}}{\partial \tilde{y}} = \tilde{u}\frac{d\bar{p}_e}{d\bar{x}} + \tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{y}}\frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\partial}{\partial \tilde{y}}(\frac{\tilde{\mu}}{Pr}\frac{\partial \tilde{T}}{\partial \tilde{y}})$$

3.2 Notice that due to a subtility of hypersonic flow the velocity at the edge is $\bar{u}_e = 1$ (the flow is very very fast) and that the temperature in the boundary layer will be of order M^2. Boundary conditions ?
3.3 As the viscosity is supposed to be proportional to the temperature, show that $\tilde{\mu}/\tilde{\rho} = C\tilde{T}^2/\tilde{p}$, the C is a constant of proportionality (in Bush paper μ varies with T^ω, here $\omega = 1$).
3.3 Looking at a self similar solution ($\tilde{x} = X^*\hat{x}$, $\tilde{y} = Y^*\hat{y}$ *etc*), show that the invariance of $\tilde{\rho}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}$ versus $\frac{\partial}{\partial \tilde{y}}(\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{y}})$ gives $Y^{*2}/X^{*2} = 1/X^*/P^*$. And show next that $V^* = Y^*/X^*$. Verify that invariance $\tilde{\rho}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}$ versus $\frac{d\bar{p}_e}{d\bar{x}}$ is consistent, *idem* check the heat equation. Do not try to write the final coupled ODEs problem.

4. Interactive Boundary Layer problem (hypersonic strong interaction)
From 2.4 we have the pressure in the ideal fluid p proportional to a certain power of the angle θ of the body in the ideal fluid description. In hypersonic strong interaction, we assimilate this θ to the blowing velocity coming from the boundary layer.
4.1 What do you think of this ? Does it ring a bell ?
4.2 From this relation between pressure and velocity show that the final self similar variable is $\eta = y/x^{3/4}$.
4.3 Identify the interaction parameter defined by Bush.
4.4 Conclusion : the boundary layer is in $x^{3/4}$, what is the final scale ? Find the power of pressure, the power of the angle of the shock wave.

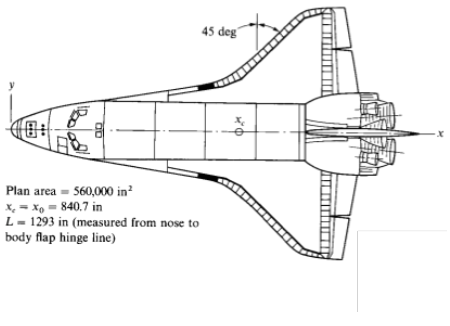

FIGURE 1 – The space shuttle has been designed with lot of experiments and asymptotic theory (and some simulations)

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Hypersonic strong-interaction similarity solutions for flow past a flat plate

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The hypersonic strong-interaction regime for the flow of a viscous, heat-conducting compressible fluid past a flat plate is analysed using the Navier–Stokes equations as a basis. It is assumed that the fluid is a perfect gas having constant specific heats, a constant Prandtl number σ, whose numerical value is of order one, and a viscosity coefficient varying as a power, ω, of the absolute temperature. Limiting forms of solutions are studied as the free-stream Mach number M, the free-stream Reynolds number based on the plate length R_L, and the interaction parameter $\chi = \{(\gamma M^2)^{2+\omega}/R_L\}^{\frac{1}{4}}$, go to infinity.

Through the use of asymptotic expansions and matching, it is shown that, for $(1-\omega) \to 0$, three distinct layers for which similarity exists make up the region between the shock wave and the plate. The behaviour of the flow in these three layers is analysed.

1. Introduction

The purpose of this paper is to enlarge upon the existing theory for the hypersonic strong-interaction problem for viscous, compressible flow past a flat plate (cf. e.g. Stewartson 1964).

According to this strong-interaction theory, the flow field is divided into three regions: (*a*) a region extending from the upstream side of a Rankine–Hugoniot shock wave outwards, where there is a uniform, high-speed flow; (*b*) a high temperature, low density, viscous region extending outward from the plate part of the way to the shock, across which the pressure change is small so that the flow in this region can be analysed by boundary-layer theory; and (*c*) an inviscid region, between the clearly defined outer edge of the above viscous boundary layer and the downstream side of the Rankine–Hugoniot shock wave, for which the hypersonic small-disturbance theory holds.

The Stewartson solution of this flow problem for $\omega = 1$ consists of a similar solution for the flow in the inviscid shock layer that is joined to the similar solution for the flow in the viscous boundary layer. Stewartson states, however, that to make this joining, a slight change in the boundary conditions at the outer edge of the viscous layer has to be made. It is this need for modification in these boundary conditions that drew the author's attention to the strong-interaction problem. A standard method for overcoming such a difficulty is the introduction of a layer intermediate to the shock and boundary layers. For $\omega = 1$, unfortunately, the intermediate layer is not the answer to the problem. For $(1-\omega) > 0$,

however, it is found that, with an intermediate layer, strong-interaction similarity solutions, which do not require modification of the boundary conditions at the outer edge of the boundary layer, can be obtained. These similarity solutions for $(1-\omega) > 0$ are discussed in the following sections.

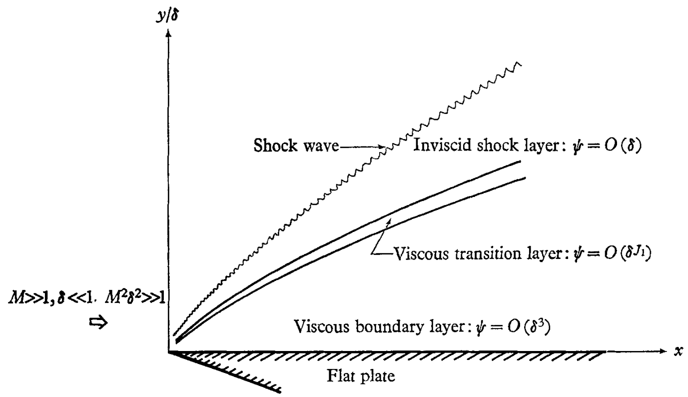

FIGURE 1. Schematic diagram of hypersonic strong interaction for flow past a flat plate.

2. The equations of motion

Consider the (two-dimensional) flow of a viscous, compressible gas past a semi-infinite flat plate. Let $x_1 = Lx$ and $y_1 = Ly$ represent the Cartesian co-ordinates parallel and normal to the flat plate, respectively, with the origin of this co-ordinate system at the leading edge of the plate. The length L is chosen so that x is of order unity in the region where the strong-interaction theory is valid. The velocity components in the x_1- and y_1-directions are $u_1 = u_\infty u$, and $v_1 = u_\infty v$, and the pressure, temperature, and density are $p_1 = p_\infty p$, $T_1 = T_\infty T$, and $\rho_1 = \rho_\infty \rho$, where u_∞, p_∞, T_∞, and ρ_∞ are the velocity in the x_1-direction, pressure, temperature, and density in the undisturbed region upstream of the flat plate.

The gas is assumed to be a perfect one ($p = \rho T$), having (i) constant specific heats, c_{v_1} and c_{p_1}, with $\gamma = (c_{p_1}/c_{v_1})$, such that $(\gamma - 1) = O(1)$, (ii) a constant Prandtl number of order unity ($\sigma = \text{const.} = O(1)$), and (iii) its 'normal' viscosity coefficient proportional to a power, ω, of the absolute temperature ($\mu_1 = \mu_\infty \mu = \mu_\infty T^\omega$, with $\frac{1}{2} \leqslant \omega < 1$, as will be shown to be required in the succeeding analysis), while its 'bulk' viscosity coefficient is taken to be zero, although such an assumption is not necessary.

The von Mises forms of the Navier–Stokes equations for the flow of such a gas are

$$\frac{\partial}{\partial \psi}\left(\frac{v}{u}\right) - \frac{\partial}{\partial \xi}\left(\frac{1}{\rho u}\right) = 0, \qquad (2.01)$$

Part I

Error, it was originally

$$(E_\varepsilon) \qquad u\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}. \text{ with } u(-\infty) = -1, \quad u(\infty) = 1.$$

in fact, nobody really detected this error (it was such as ε is négative !).

The external problem is

$$(E_0) \qquad u\frac{\partial u}{\partial x} = 0$$

so the solution is $u(x < 0) = -1$ and $u(x > 0) = 1$, solution is discontinuous in $x = 0$, and this gives the limits for the matching $u(0^-) = -1$ and $u(0^+) = +1$

By dominant balance we find the scale of the solution is with $x = \varepsilon\tilde{x}$ so that

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} = -\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}. \text{ with } \tilde{u}(-\infty) = u(0^-), \quad \tilde{u}(\infty) = u(0^+)$$

first integral

$$\frac{\partial}{\partial \tilde{x}}(\frac{\tilde{u}^2}{2} - \frac{1}{2}) = -\frac{\partial \tilde{u}}{\partial \tilde{x}} + 0$$

$$\frac{d\tilde{u}}{1-\tilde{u}^2} = d\tilde{x}/2 \text{ so that } \frac{d\tilde{u}}{1-\tilde{u}} + \frac{d\tilde{u}}{1+\tilde{u}} = d\tilde{x}/2 \text{ hence } -Log(1-\tilde{u}) + Log(1+\tilde{u}) = \tilde{x}/2$$

so $\tilde{u} = tanh(\tilde{x}/2)$

Part II

Clearly the expansion at small θ gives

$$\frac{p}{p_\infty} = 1 + \frac{\gamma(\gamma+1)}{4}K^2 + \gamma K^2\sqrt{\frac{1}{K^2} + ...} = 1 + \gamma K + ...$$

so pressure

$$\frac{p}{p_\infty} = 1 + \gamma M\theta + ...$$

the Ackeret formula valid for small angles

$$\frac{p}{p_\infty} = 1 + \gamma\frac{M^2}{\sqrt{M^2-1}}\theta + ...$$

at small θ and for M enough large (but $M\theta << 1$) gives the same

we have $\mu/\mu_\infty = CT/T_\infty$ (linear behavior of the viscosity) but as $p/p_\infty = (\rho T)/(\rho_\infty T_\infty)$ then

$$\frac{\mu}{\rho} = \frac{\mu_\infty}{\rho}C\frac{T}{T_\infty} = C\frac{\mu_\infty}{\rho_\infty}\frac{p_\infty}{p}(\frac{T}{T_\infty})^2$$

orders of magnitude $\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}$ is $1/X^*$ as the velocity is $U^* = 1$, versus $\frac{\partial}{\tilde{\rho}\partial \tilde{y}}(\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{y}})$ which is $(1/P*)(1/Y^{*2})$ (using the $\frac{\mu}{\rho}$ order of magnitude)

This gives $Y^{*2}/X^{*2} = 1/X^*/P^*$. And we show next that $V^* = Y^*/X^*$.

hence

$$Y^* = X^{*3/4}$$

We have a boundary layer ideal fluid interaction, the result is a shock wave in $x^{3/4}$ and a boundary layer in $x^{3/4}$. Both interact strongly : pressure increases the boundary layer, which in turn develops the shock wave and pressure change, looping back. This is called viscous strong interaction.