**M2, Fluid mechanics** 2017/2018
Friday, December 1st, 2017

# Multiscale Hydrodynamic Phenomena

Part I. : 90 minutes, NO documents

**1. Quick Questions** In few words :
1.1 What is the usual scale for pressure in incompressible NS equation at small Reynolds ?
1.2 What is the usual scale for pressure in incompressible NS equation at large Reynolds ?
1.3-4-5 ∂'Alembert, Laplace, Heat : give the equation and a physical example of use of this equation.
1.6 What is the Bürgers equation ? Which balance is it ?
1.7 What is the KDV equation ? Which balance is it ?

**2. Exercice**
Let us look at the following ordinary differential equation : $(E_\varepsilon)$ $\quad \dfrac{d^2y}{dt^2} + 2\varepsilon\dfrac{dy}{dt} + y = 0,$ valid for any $t > 0$ with boundary conditions $y(0) = 0$ and $y'(0) = 1$. Of course $\varepsilon$ is a given small parameter.
We want to solve this problem with Multiple Scales.
2.1 Expand up to order $\varepsilon$ : $y = y_0(t) + \varepsilon y_1(t)$, show that there is a problem for long times.
2.2 Introduce two time scales, $t_0 = t$ and $t_1 = \varepsilon t$
2.3 Compute $\partial/\partial t$ and $\partial^2/\partial t^2$
2.4 Solve the problem.
2.5 Suggest the plot of the solution.

**3. Exercice**
Consider the following equation (of course $\varepsilon$ is a given small parameter)

$$(E_\varepsilon) \qquad \varepsilon\frac{d^2u}{dx^2} + \frac{du}{dx} = e^x \text{ with } u(0) = 0 \quad u(1) = e.$$

We want to solve this problem with the Matched Asymptotic Expansion method.
3.1 Why is this problem singular ?
3.2 What is the outer problem and what is the possible general form of the outer solution ?
3.3 What is the inner problem of $(E_\varepsilon)$ and what is the inner solution ?
3.4 Suggest the plot of the inner, outer and composite solution.

**4. Exercice**
Solve with WKB approximation the Airy problem

$$\varepsilon y'' = xy,$$

Hint : show that $S_0 = \pm \int \sqrt{|x|}dx$ and $S_1 \propto ln(|x|)$

**Multiscale Hydrodynamic Phenomena**

**M2, Fluid mechanics** 2017/2018
Friday, December 1st, 2017

Part II. : 1h 15 min all documents. **Triple Deck**

This is a part of "D'Alembert's Paradox" by Keith Stewartson SIAM Review, Vol. 23, No. 3 (1981). We consider the flow on a smooth body, which is may be a flat plate at first approximation. It is of characteristic length $L$ in an incompressible viscous fluid which, at infinite distance, is in uniform motion at velocity $U_\infty$. First we consider and establish the classical Boundary layer equations. Then we will write the triple deck equations (Main, Lower and Upper Decks).

1.1 Write Navier Stokes equations with boundary conditions, with and without dimension.
1.2 Write Euler's problem with associated boundary conditions (discuss if the problem is singular ?).
1.3 According to Stewartson 1981, what is d'Alembert's paradox ?
1.4 Comment figure 1. from Stewartson 1981. What is the analytical solution of incompressible irrotational flows which gives figure 1 top ?
1.5 Write Prandtl problem, with all boundary conditions, to do that, show quickly that dominant balance gives $\delta = L/\sqrt{R}$ for the order of magnitude of the boundary layer thickness. Write scale of velocities and pressure. Use $\varepsilon = R^{-1/8}$ to write the final scales.
1.6 Write with dimensions the wall shear stress $\tau_w$ defined by (1.6) without dimension. Is it small or large compared to variations of pressure ?
We consider the equations of triple deck. We denote $\varepsilon = R^{-1/8}$
2.1 Main deck, we do a perturbation of the boundary layer. Stewartson notations are obscure, show that (2.2) may be obtained as :

$$\tilde{u} = U_B(\tilde{y}) + \varepsilon^n A(\tilde{x})\frac{dU_B(\tilde{y})}{d\tilde{y}} + ..., \;\; \tilde{v} = -\varepsilon^2 \frac{dA}{d\tilde{x}} U_B(\tilde{y}) + ..., \;\; \tilde{p} = \varepsilon^2 \tilde{p}_2(x) + ...$$

if $x = L(1+\varepsilon^3 \tilde{x})$ and $y = L\varepsilon^4 \tilde{y}$, and $(u, v) = U_\infty(\tilde{u}, \tilde{v})$ is substituted in NS equations. Value of $n$ ?
2.2 Write the behavior/ boundary condition for the velocity at the bottom.
2.2 Write the behavior/ boundary condition for the velocity at the top.
3.1 Lower deck in 2D : show that (3.5) in 2D may be obtained with :

$$u/U_\infty = \varepsilon \hat{u}, \;\; v/U_\infty = \varepsilon^3 \hat{v}, \;\; ..., \;\; (p - p_\infty)/(\rho U_\infty^2) = \varepsilon^n \hat{p}(\hat{x}) + ...$$

and $x = L(1+\varepsilon^3 \hat{x})$ and $y = L\varepsilon^5 \tilde{y}$, is substituted in NS equations. Value of $n$ ?
3.2 Lower deck in 3D : What is the scale of the velocity $w$ consistent with 3D effects in eq. (3.5) ?
3.3 Discuss boundary conditions (3.6a)
3.4 Upper deck incompressible 2D, show that (2.4b) may be obtained with :

$$u/U_\infty = 1 + \varepsilon^2 \bar{u}, \;\; v/U_\infty = \varepsilon^2 \bar{v}, \;\; ..., \;\; (p - p_\infty)/(\rho U_\infty^2) = \varepsilon^n \bar{p} + ...$$

and $x = L(1+\varepsilon^3 \bar{x})$ and $y = L\varepsilon^3 \bar{y}$, is substituted in NS equations. Value of $n$ ?
3.5 Upper deck incompressible 3D, discuss (4.6).
4. Write full problem with all boundary conditions in 2D.

extra questions

- unsteady triple Deck scale of time ?
- Show that $(\hat{u}, \hat{v}, \hat{p}) = (\hat{y}, 0, 0)$ is a base flow. interpretation ?
- Show that $\hat{u}, = \hat{y} - af'(\hat{y})e^{i(k\hat{x}-\omega\hat{t})}$, with $\hat{p} = ae^{i(k\hat{x}-\omega\hat{t})}$ is a linearized possible solution with $f''$ solving an Airy equation $Ai''(\eta) = \eta Ai(\eta)$. Deduce the dispersion relation $F(\omega, k) = 0$ for linear waves in the triple deck.

SIAM REVIEW
Vol. 23, No. 3, July 1981


0036-1445/81/2303-0003$01.00/0

# D'ALEMBERT'S PARADOX*

KEITH STEWARTSON†

**Abstract.** Since classical inviscid theory leads to the patently absurd conclusion that the resistance experienced by a rigid body moving through a fluid with uniform velocity is zero, great efforts have been made during the last hundred or so years to propose alternate theories and to explain how a vanishingly small frictional force in the fluid can nevertheless have a significant effect on the flow properties. The methods used are a combination of experimental observation, computation often on a very large scale, and analysis of the structure of the asymptotic form of the solution as the friction tends to zero. This three-pronged attack has achieved considerable success, especially during the last ten years, so that now the paradox may be regarded as largely resolved. The lecture will review these achievements in subsonic and supersonic flow, for bluff bodies, for trailing-edge flows and for internal flows. Most of the work has been on steady two-dimensional problems but the special difficulties in unsteady and three-dimensional flow will also be touched on.

**1. Introduction.** Conjectures are a vital part of the fabric of pure mathematics. Not only have they engaged the lasting attention of many mathematicians in all periods of history, but their resolution has often led to a deeper appreciation of the discipline and to a broadening of its power, to say nothing of suggesting new problems. Some conjectures, still open, are both deceptively simple and quite old: one of my own favorites is Goldbach's conjecture, first stated in a letter to Euler (1742), that every even number is the sum of two primes.

Theoretical fluid mechanics also has its conjectures which have long tantalized and disconcerted scientists. Some of the most persistent have a history of over two hundred years, as they arise from the famous paradox discovered by d'Alembert in 1752. This paradox, like Goldbach's conjecture, is extremely easy to state, namely that there is no drag on a finite body at rest in an infinite, incompressible, inviscid fluid otherwise in uniform motion. That such a result is to be expected can be seen from the following argument. If $p$ is the pressure in the fluid, $\rho$ the density and $\mathbf{q}$ the local velocity, then, in virtue of Bernoulli's theorem,

$$p+\tfrac{1}{2}\rho\mathbf{q}^2=\text{const}, \tag{1.1}$$

since conditions are uniform at infinite distances from the body. Hence, the drag, i.e., the component of the force on the body in the direction $\mathbf{i}$ of the main stream, is the same if $\mathbf{i}$ is replaced by $-\mathbf{i}$. Thus, if the drag is positive for one flow it will become negative if the direction of fluid motion at infinity is reversed, contradicting the second law of thermodynamics. It is easy to imagine the consternation which such a conclusion immediately caused because it is so entirely at variance with one's everyday experience. Indeed the reduction of drag on moving bodies is a central problem for practical fluid dynamicists such as ship designers and aeronautical engineers. The paradox haunted theoretical workers throughout the nineteenth century and beyond, and only gradually were they able to come to terms with it and learn that, nevertheless, theory has much to contribute to the understanding of the way bodies move through fluids.

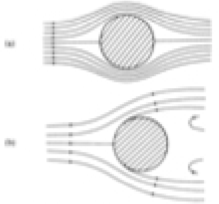


FIG. 1. Two of the candidates for the steady solution of the Navier-Stokes equations for flow past a circular cylinder at $R=\infty$. (a) attached potential flow. (b) Kirchhoff free-streamline flow.

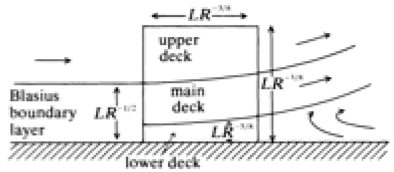


FIG. 2. *Sketch of the triple-deck region near separation on a flat plate.*

$$\tau(x)=\left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{1.6}$$

found by an application of Prandtl's transposition theorem (1905). Suppose the basic boundary layer has the profile $U_B(y)$ in the interaction region and we base $L$ on the distance of the origin of the triple-deck from the nose or leading edge. Then, in the main deck,

$$u^*=U_B(y)+A(x)U_B'(y), \qquad v^*=-A'(x)U_B(y)R^{-1/2}+\cdots \tag{2.2}$$

from (1.4), giving us an automatic match between the $x$-components of velocity in the main and lower decks. As $y\to\infty$ on the main-deck scale

$$u^*\to U_B(\infty), \qquad v^*\to -R^{-1/2}U_B(\infty)A'(x), \tag{2.3}$$

Although the lower-deck equations are essentially the same for supersonic or subsonic external flows, the same is not true for the upper deck, which assumes a hyperbolic form in supersonic flow ($M_\infty>1$ where $M_\infty$ is the Mach number of the inviscid flow just outside the triple deck) and elliptic in subsonic flow. In each case a simple expression may be written down connecting $p$ and $A'$. After applying appropriate but algebraically complicated scales (see Stewartson (1974) for details) the governing equations of the triple deck may be reduced to (1.4) with $\rho=1$ together with

$$p=-A'(x) \text{ if } M_\infty>1, \tag{2.4a}$$

$$p=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A'(x_1)\,dx_1}{x-x_1} \quad \text{if } M_\infty<1. \tag{2.4b}$$

form of these equations (Schneider (1974)). Lastly, the generalization to include three-dimensional effects changes both the lower-deck and upper-deck equations. The scale of $z$, the variable in the third direction, along the wall but perpendicular to the direction of the main stream, is also $R^{-3/8}$, and if $w$ is the reduced velocity in that direction (1.4) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2},$$
$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{\partial^2 w}{\partial y^2}, \tag{3.5}$$

with boundary conditions

$$u-y\to A(x,z), \qquad zw\to D(x,z) \quad \text{as } y\to\infty, \tag{3.6a}$$

where $\partial D/\partial x=-\partial p/\partial z$ (Smith, Sykes and Brighton (1977)). In the upper deck the

The generalization of the two-dimensional triple-deck theory to include three-dimensional effects is obviously important but difficult in both the lower- and the upper-deck calculations. Following on the successful studies on two-dimensional humps in boundary layers (Smith (1973), Napolitano, Werle and Davis (1978)), some extensions to three dimensions have been made. The lower-deck equations are given in (3.5) and the Hilbert integral of the upper deck is replaced by

$$p=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int\frac{\partial^2 A/\partial\xi^2\,d\xi\,d\eta}{[(x-\xi)^2+(y-\eta)^2]^{1/2}}. \tag{4.6}$$

**correction Ex 2**

Exactly the curse with coefficient 2, so that the solution is $y = e^{-t_1} sin(t_0)$

**correction Ex 3**

If wi put $\varepsilon = 0$, we haev an order one problem with 2 BC, so singular.

We find $u_{out} = e^x$ (note $u_{out} = e^x - 1$ is possible, but is not a good idea, check it !). We note that $u_{out}(0) = 1$, so we have to introduce an inner layer to full fit the 0 BC.

Change of scale $x = \delta\tilde{x}$, by dominant balance $\varepsilon = \delta$, the problem is

$$\frac{d^2\bar{u}}{d\bar{x}^2} + \frac{d\bar{u}}{d\bar{x}} = \varepsilon e^{\varepsilon\bar{x}}$$

as $\varepsilon \to 0$ then $u''_{in} + u'_{in} = 0$ zolution is $u_{in} = A(1 - exp(-\tilde{x}))$. Matching gives $A = 1$. Hence composite expansion is

$$u_{composite} = exp(x) - exp(-x/\varepsilon).$$

**correction Ex 4**

Two cases :

- $(S_0')^2 = x$ if $x > 0$ then $S_0 = \pm \int \sqrt{x} dx = \pm \frac{2}{3} x^{3/2}$
- $S_0'^2 = x$ if $0 > x$ then $S_0 = \pm i \int \sqrt{|x|} dx = \pm i \frac{2}{3} (-x)^{3/2}$ solution will be with cosines and sines

and in both cases $S_1 = -\frac{1}{4} ln(|x|)$

see the curse for a plot of the solution.