SCIENCES SORBONNE UNIVERSITÉ

Multiscale Hydrodynamic Phenomena

M2, Fluid mechanics, MU5MEF15 2020/2021
Friday December 4th, 2020, 8 :30am - 12 :30pm, Room 24-34 201 Part I. : 80 minutes, NO documents

1. Quick Questions In few words and few formula :
1.1 What is "dominant balance" ?
1.2 Order of magnitude of drag on a sphere at small Re
1.3 Order of magnitude of drag on a cylinder at small Re
1.4 Write 2D Boundary layer equations (in x, y, u, v) in the case of Blasius problem, what is the scale of y compared to the scale of x ?
1.5 In which one of the 3 decks of Triple Deck is flow separation ?
1.6 What is the KDV equation ?
1.7 What is the natural selfsimilar variable for heat equation ?
1.8 ∂'Alembert equation : write the equation and the generic solution of it

2. Exercice
2.1 What is the name of the following equation (of course ε is a given small parameter)

$$(E_\varepsilon) \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}. \text{ with } u(-\infty) = 1, \quad u(\infty) = -1.$$

2.2 Say in few sentences what it represents
2.3 Let us define $(E_{S\varepsilon})$ the steady solution of (E_ε). We want to solve this steady problem with the Matched Asymptotic Expansion method.
2.4 Why is $(E_{S\varepsilon})$ problem singular ?
2.5 What is the outer problem and what is the possible general form of the outer solution ?
2.6 What is the inner problem of $(E_{S\varepsilon})$ and what is the inner solution ?
2.7 Solve the problem at first order (up to power ε^0).
2.8 Suggest the plot of the inner and outer solution.
2.9 What is the exact solution of $(E_{S\varepsilon})$ for any ε.
Hint : $\tanh'(x) = 1 - \tanh^2(x)$

3. Exercice
Let us look at the following ordinary differential equation : $(E_\varepsilon) \quad \dfrac{d^2y}{dt^2} + y = -\varepsilon^2 \dfrac{dy}{dt}, \quad$ valid for any $t > 0$ with boundary conditions $y(0) = 1$ and $y'(0) = 0$. Of course ε is a given small parameter.
We want to solve this problem with Multiple Scales Analysis.
3.1 Expand up to order ε^2 : $y = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t)$, show that there is a problem for long times.
3.2 Introduce two time scales, $t_0 = t$ and t_1, what is the relation between t, t_1 and ε ?
3.3 Compute $\partial/\partial t$ and $\partial^2/\partial t^2$
3.4 Solve the problem.
3.5 Suggest the plot of the solution.
3.6 What is the exact solution for any ε, compare.

4. Exercice
Solve with WKB approximation the problem

$$\varepsilon y'(x) + y(x) = 0 \text{ with } y(0) = 1$$

Compare with exact solution.

M2, **Fluid mechanics** 2020/2021
Friday, December 4th, 2020
Part II. : 1h 15 min all documents.

Multiscale Hydrodynamic Phenomena

Film Flow down an inclined plane

This is a part of "Modeling film flows down inclined planes" by C. Ruyer-Quil and P. Manneville. Eur. Phys. J. B 6, 277- 292 (1998). We consider the thin film 2D flow on an infinite inclined plate, see figure 1, and we aim to establish Shkadov's equation.
As all the results are more or less in the paper, be careful and rigorous to prove the results.

1.0 Write incompressible Navier Stokes equations, (1) and (2).
1.1 Write the kinematic condition at the interface, and no slip boundary condition. Which equations are they in the paper ?
1.2 Write $\underline{\underline{\sigma}}$, the stress tensor, what is its definition with derivatives of u, v and pressure.
1.3 How is the flow for $y > h$ (in air) in terms of pressure ? and in terms of viscosity ? (it is not clearly written in the paper, you must do some extra classical hypothesis).
1.4 Compute the normal $\underline{n}$ and tangent $\underline{t}$ to the surface and compute $\underline{\underline{\sigma}} \cdot \underline{n}$ (stress vector).
1.5 Compute $(\underline{\nabla} \cdot \underline{n})$.
1.6 Continuity of stress at the surface involves surface tension. Write the change of normal stress, $\underline{n}$ from media 1 to 2, at the interface :

$$\left(\underline{\underline{\sigma}} \cdot \underline{n}\right)_2 - \left(\underline{\underline{\sigma}} \cdot \underline{n}\right)_1 = \gamma(\underline{\nabla} \cdot \underline{n})\underline{n}$$

due to surface tension, then obtain (5) and (6).
1.7 It is said that system admits a trivial solution : the Nusselt solution. Check that this is the case (remind all the hypothesis to obtain this simple solution). What link with Poiseuille ?
1.8 Compute wall shear stress $\tau_w = \mu \partial u / \partial y$. Write it as function of q_N.
1.9 Compute $\int_0^h u^2 dy$. Write it as function of q_N^2.
1.10 Show that (3), (4) give (9), note that there is a derivation of an integral (Leibnitz integral rule).
1.11 Do a full dominant balance analysis of Navier Stokes equations using a length L, a velocity U and a time T. Note that $B = O(1)$, why ?
1.12 Show that it leads indeed to (10) (11) and (12).
2. In this part we consider (38)-(40) and (41)-(43). Those equations are presented in this paper without expanding in ε. We want to put back some ε in those equations to be sure of approximations presented here. We will write (38)-(40) and (41)-(43) starting form (10)-(12) with tildes to show and emphasize the change of scales. We will do a small layer analysis : the longitudinal scale is $1/\varepsilon$ compared to the transversal one.
2.1 Show that if we define $\tilde{x}, \tilde{y}$ as : $x = \tilde{x}/\varepsilon$, $y = \tilde{y}$, it corresponds to a long wave analysis. From dominant balance of (9) show that $t = \tilde{t}/\varepsilon$.
2.2 As we keep $u = \tilde{u}$, what is the new scales for v in order that (3) is invariant (and gives (40) but with tildes over the variables) ?
2.3 Question 2.3 and 2.4 are strongly coupled. The pressure remains $p = \tilde{p}$, show that (38) corresponds to (10) with an error of $O(\varepsilon^2)$ (show that the convective term is $O(\varepsilon)$).
2.4 Starting from (11) we obtain (39) (and that indeed $p = \tilde{p}$), what is the error in term of order of magnitude in ε ?
2.5 Do the same analysis for (41)-(43) : check the linearisation in (41)-(43). Note that Γ is large, what is $O(\Gamma)$ in order of magnitude in ε so that it surface tension plays a role and appears in the pressure gradient ?
2.6 Write (44) and (45) with the ε and tilde variables.
3.1 Using the same (Leibnitz integral rule) than in 1.10, integrate the momentum (44) to obtain (46) and (47).
3.2 Identify r and τ_w from question (1.8) and (1.9).
3.3 Obtain the final Shkadov equation (49)-(50).

Eur. Phys. J. B **6**, 277–292 (1998)

Modeling film flows down inclined planes

The European Physical Journal B

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Abstract. A new model of film flow down an inclined plane is derived by a method combining results of the classical long wavelength expansion to a weighted-residuals technique. It can be expressed as a set of three coupled evolution equations for three slowly varying fields, the thickness h, the flow-rate q, and a new variable τ that measures the departure of the wall shear from the shear predicted by a parabolic velocity profile. Results of a preliminary study are in good agreement with theoretical asymptotic properties close to the instability threshold, laboratory experiments beyond threshold and numerical simulations of the full Navier–Stokes equations.

1 Introduction

In addition to being involved in a wide variety of technical applications (chemical reactors, evaporators, *etc.*), the dynamics of fluid films is an interesting topic in itself. As a matter of fact, thin films flowing down inclined surfaces exhibit a rich phenomenology [1] and offer a good testing ground for the study of the transition to turbulence. Instabilities take place at low flow rates, which gives a unique opportunity to analyze the development of waves at the surface of the fluid into large-amplitude strongly nonlinear localized structures such as solitary pulses and further to study their disorganization into developed spatio-temporal chaos *via* secondary instabilities.

A trivial solution to the flow equations is easily found in the form of a steady uniform parallel flow with parabolic velocity profile, often called Nusselt's solution, where the work done by gravity is exactly consumed by viscous dissipation. Thin films at low flow rate over sufficiently steep surfaces turn out to be unstable against long wavelength infinitesimal perturbations, *i.e.* wavelength large when compared to the thickness of the flow. This is confirmed by a general study of the relevant Orr–Sommerfeld equation which shows that short-wavelength shear instabilities of the Tollmien–Schlichting type are only relevant for flows over planes at vanishingly small inclination angles and very high flow rates [2].

In the following we will thus be concerned with long wavelength interfacial instability modes, the dynamics of which is essentially controlled by viscosity and surface tension effects. **(···)**

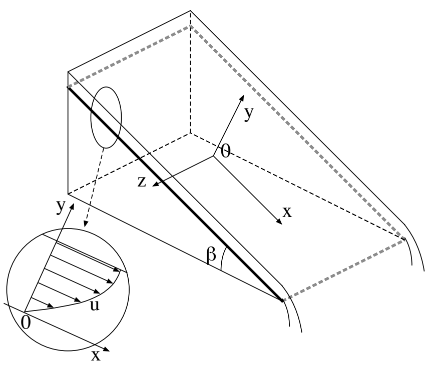

Fig. 1. Fluid film flowing down an inclined plane: definition of the geometry.

(···)

2 Governing equations

The geometry is defined in Figure 1: the inclined plane makes an angle β with the horizontal. As usual, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors in the stream-wise, cross-stream, and span-wise directions respectively. Here we only consider the two-dimensional case where the solution is independent of coordinate z, the extension to the full three-dimensional case does not present conceptual difficulties.

The basic (2D) equations read

$$\rho\left[\partial_t u + u\,\partial_x u + v\,\partial_y u\right] = -\partial_x p + \rho g \sin\beta + \mu\left(\partial_{xx} + \partial_{yy}\right) u, \tag{1}$$

$$\rho\left[\partial_t v + u\,\partial_x v + v\,\partial_y v\right] = -\partial_y p - \rho g \cos\beta + \mu\left(\partial_{xx} + \partial_{yy}\right) v, \tag{2}$$

$$\partial_x u + \partial_y v = 0, \tag{3}$$

where u and v denote x and y velocity components, and p the pressure. ρ is the density, μ the viscosity, and g the intensity of the gravitational acceleration.

These equations must be completed with boundary conditions at $y = 0$ or $y = h$. They will be denoted as $w\big|_0$ or $w\big|_h$ where $w(x, y, t)$ is a generic name for the pressure field, the velocity components and their derivatives. The first such condition:

$$\partial_t h + u\big|_h\,\partial_x h = v\big|_h, \tag{4}$$

simply expresses the fact that the interface $h(x, t)$ is a material line. The continuity of the stress at $y = h$ adds two more equations. The normal component reads

$$\frac{\gamma\,\partial_{xx} h}{\left[1 + (\partial_x h)^2\right]^{3/2}} + \frac{2\mu}{1 + (\partial_x h)^2}\left[\partial_x h\left(\partial_y u\big|_h + \partial_x v\big|_h\right) - (\partial_x h)^2\,\partial_x u\big|_h - \partial_y v\big|_h\right] + p\big|_h - p_a = 0 \tag{5}$$

where coefficient γ is the surface tension and the term in $\partial_{xx} h$ describes the curvature of the interface (p_a is the atmospheric pressure). For the tangential component one gets

$$0 = 2\partial_x h\left(\partial_y v\big|_h - \partial_x u\big|_h\right) + \left[1 - (\partial_x h)^2\right]\left(\partial_y u\big|_h + \partial_x v\big|_h\right). \tag{6}$$

Finally, the no-slip condition at the rigid bottom, $y = 0$, reads:

$$u\big|_0 = v\big|_0 = 0. \tag{7}$$

It will turn interesting to replace the kinematic condition (4) at $y = h$ by an equivalent equation derived from the continuity condition. Integrating (3) over the interval $[0,h]$ we obtain:

$$0=\int_0^h(\partial_x u+\partial_y v)\,dy=\int_0^h\partial_x u\,dy+v\big|_h-v\big|_0$$
$$=\partial_t h+\left[u\big|_h\partial_x h+\int_0^h\partial_x u\,dy\right]=\partial_t h+\partial_x\int_0^h u\,dy$$

using $v\big|_h$ given by (4) and $v\big|_0=0$ from (7). Defining the local instantaneous flow rate as

$$q(x,t)=\int_0^{h(x,t)}u(x,y,t)dy, \tag{8}$$

we arrive at the integral condition

$$\partial_t h+\partial_x q=0. \tag{9}$$

System (1-7) admits a trivial solution corresponding to a steady constant-thickness film, often called the Nusselt solution (hence the subscript "N" in the following). Assuming $\partial_t\equiv 0$ and $\partial_x\equiv 0$, one simply gets $v\equiv 0$, $p\big|_h=p_a$ and

$$\mu\partial_{yy}u+\rho g\sin\beta=0,\;\;\partial_y p=-\rho g\cos\beta,\;\;u\big|_0=0,\;\;\partial_y u\big|_h=0,$$

which, for a film of thickness h_N yields:

$$u(y)=\frac{\rho g\sin\beta}{2\mu}y(2h_N-y),\quad p(y)=p_a+\rho g\cos\beta(h_N-y),$$

where the atmospheric pressure p_a is set to zero in the following. The corresponding flow rate is given by

$$q_N=\int_0^h u(y)dy=\frac{\rho g\sin\beta h_N^3}{3\mu},$$

from which an average velocity u_N can be defined by $q_N=h_N u_N$, *i.e.* $u_N=\rho g\sin\beta h_N^2/3\mu$.

At this stage, it is usual to turn to dimensionless equations. Different scalings can be used. The first and most obvious one takes h_N and h_N/u_N as length and time units, see note [26]. Here, we will take another scaling defined without reference to the flow by constructing the length and time units from g (LT^{-2}) and the kinematic viscosity $\nu=\mu/\rho$ (L^2T^{-1}). Taking for convenience $g\sin\beta$ instead of g, this yields $L=\nu^{2/3}(g\sin\beta)^{-1/3}$ and $T=\nu^{1/3}(g\sin\beta)^{-2/3}$. The velocity unit is then $U=LT^{-1}=(\nu g\sin\beta)^{1/3}$. For the pressure, we get $\rho(\nu g\sin\beta)^{2/3}$. The surface tension is then measured by the Kapitza number $\mathit{\Gamma}=\gamma\big/\left[\rho\nu^{4/3}(g\sin\beta)^{1/3}\right]$. In fact, Kapitza was concerned with vertical planes for which $\beta=\pi/2$ so that the factor $\sin\beta$ did not appear in his definition. It is a matter of convenience to include it or not. The two numbers, with and without, are of the same order of magnitude as long as one does not consider nearly horizontal planes.

Inserting the corresponding variable changes we obtain

$$\partial_t u+u\,\partial_x u+v\,\partial_y u=-\partial_x p+1+(\partial_{xx}+\partial_{yy})\,u, \tag{10}$$
$$\partial_t v+u\,\partial_x v+v\,\partial_y v=-\partial_y p-B+(\partial_{xx}+\partial_{yy})\,v, \tag{11}$$

where $B=\cot\beta$ and, for the normal-stress boundary condition at $y=h$

$$\frac{\mathit{\Gamma}\,\partial_{xx}h}{\left[1+(\partial_x h)^2\right]^{3/2}}+\frac{2}{1+(\partial_x h)^2}\Big[\partial_x h\Big(\partial_y u\big|_h+\partial_x v\big|_h\Big)$$
$$-(\partial_x h)^2\partial_x u\big|_h-\partial_y v\big|_h\Big]+p\big|_h\;=\;0, \tag{12}$$

while the continuity condition (3), the kinematic condition (4) at $y=h$ and the remaining boundary conditions (6,7) are left unchanged. In this unit system where $g\sin\beta=\nu=\rho=1$, the Nusselt flow rate given by $q_N=u_N h_N=\frac{1}{3}h_N^3$ is numerically equal to the Reynolds number R as defined in note [26].

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4 First-order model

In the gradient expansion, the flow variables are supposed to be strictly enslaved to the local thickness h which plays the role of an effective degree of freedom governed by a Benney-like evolution equation. Another approach is then needed to deal with the dynamics of the film in a context where this enslaving is partly relaxed and other effective degrees of freedom are introduced, under the constraint that these new variables should remain slowly variable in x and t and that exact results of the gradient expansion should be recovered in the appropriate limit. Up to now, the hydrodynamic fields (u, v, p) could be expanded on a special set of polynomials in y with slowly varying coefficients functions of $h(x, t)$ and its derivatives. If the flow modulations are sufficiently slow, these fields should not be far from their estimates obtained by the gradient expansion. In other terms, the residue of a Galerkin expansion —or of an approximation derived from a more general weighted residual method— based on these polynomials should be intrinsically small. The coefficients of the expansion would then be considered as the sought-after effective degrees of freedom, and they would be governed by equations generalizing the expressions asymptotically valid when modulations are infinitely slow. The required extension would give some latitude of evolution to these coefficients around the asymptotic value obtained from the gradient expansion. The model developed below is an attempt to implement this general idea in the most "economical" way.

Let us begin with the set of equations consistent at first order except for surface tension effects that, though formally of higher order, are included here owing to their gradient-limiting role, as discussed above. The problem to be solved reads:

$$\partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_{yy} u - 1 = 0, \qquad (38)$$

$$\partial_y p + B - \partial_{yy} v = 0, \qquad (39)$$

$$\partial_x u + \partial_y v = 0, \qquad (40)$$

with boundary conditions

$$p\big|_h + \Gamma\partial_{xx} h - 2\partial_y v\big|_h = 0, \qquad (41)$$

$$\partial_y u\big|_h = 0, \qquad (42)$$

$$u\big|_0 = 0, \qquad v\big|_0 = 0, \qquad (43)$$

and of course the kinematic condition at the interface which, in integral form (9), accounts for mass conservation on average over the thickness.

Integrating (39) with the help of boundary conditions (41–42) we get $p = B(h - y) + \partial_y v + \partial_y v\big|_h - \Gamma\partial_{xx} h$ and further eliminate $\partial_x p$ from (38). Because $\partial_y v = -\partial_x u$ is a first order term, its derivative is of second order and can be dropped of. Therefore, our set of equations read

$$\partial_t u + u\partial_x u + v\partial_y u - \partial_{yy} u = 1 - B\partial_x h + \Gamma\partial_{x^3} h, \qquad (44)$$

$$\partial_x u + \partial_y v = 0, \qquad (45)$$

with boundary conditions (42–43). (44–45) is sometimes called boundary-layer equations (BL).

Let us now consider the averaging of equation (44) that gives the balance of x-momentum (von Kármán's equation in the context of boundary layers). We obtain:

$$\int_0^h [\partial_t u + u\partial_x u + v\partial_y u - \partial_{yy} u]\, dy = h + \Gamma h\partial_{x^3} h - Bh\partial_x h, \qquad (46)$$

which can be transformed into

$$\partial_t \int_0^h u\, dy + \partial_x \int_0^h u^2\, dy = h - \partial_y u\big|_0 + \Gamma h\partial_{x^3} h - Bh\partial_x h. \qquad (47)$$

Transformation of the l.h.s. is similar to that leading to (9). The term $\partial_y u\big|_0$, representing the shear at the wall, will be denoted τ_w in the following. On the l.h.s. we recognize $q = \int_0^h u(y)\, dy$ and we can define a new averaged field $r = \int_0^h u^2(y)\, dy$. With these notations (47) reads:

$$\partial_t q + \partial_x r = h\,(1 + \Gamma\partial_{x^3} h - B\partial_x h) - \tau_w. \qquad (48)$$

Assuming a given velocity profile, one arrives at a set of two equations (9) and (48) for two unknowns h and q, since r can then be computed from q. Simply taking Kapitza's parabolic profile $u(y) \propto \frac{1}{2}\zeta(2-\zeta)$ where $\zeta = y/h$, we have $r = \frac{6}{5}(q^2/h)$, and $\tau_w = 3q/h^2$. Inserting these estimates in (48) we obtain Shkadov's model [22]:

$$\partial_t h = -\partial_x q, \qquad (49)$$

$$\partial_t q = h - 3\,\frac{q}{h^2} - \frac{12}{5}\,\frac{q}{h}\,\partial_x q + \left(\frac{6}{5}\frac{q^2}{h^2} - Bh\right)\partial_x h + \Gamma h\partial_{x^3} h. \qquad (50)$$

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Part I

• **correction Ex 2**
Steady problem :

$$(E_{S\varepsilon}) \quad u\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}. \text{ with } u(-\infty) = 1, \quad u(\infty) = -1.$$

The external problem is

$$(E_{S0}) \quad u\frac{\partial u}{\partial x} = 0$$

so the solution is $u(x < 0) = 1$ and $u(x > 0) = -1$, solution is discontinuous in $x = 0$, and this gives the limits for the matching $u(0^-) = 1$ and $u(0^+) = -1$

By dominant balance we find the scale of the solution is with $x = \varepsilon\tilde{x}$ so that

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}. \text{ with } \tilde{u}(-\infty) = u(0^-), \quad \tilde{u}(\infty) = u(0^+)$$

first integral

$$\frac{\partial}{\partial \tilde{x}}(\frac{\tilde{u}^2}{2} - \frac{1}{2}) = \frac{\partial \tilde{u}}{\partial \tilde{x}} + 0$$

$$\frac{d\tilde{u}}{1-\tilde{u}^2} = d\tilde{x}/2 \text{ so that } \frac{d\tilde{u}}{1-\tilde{u}} + \frac{d\tilde{u}}{1+\tilde{u}} = d\tilde{x}/2 \text{ hence } -Log(1-\tilde{u}) + Log(1+\tilde{u}) = \tilde{x}/2$$

so $\tilde{u} = \tanh(-\tilde{x}/2)$

• **correction Ex 3**
trap : ε^2 is the small parameter : $t_0 = t$ and $t_1 = \varepsilon^2 t$...

• **correction Ex 4**

• with $\delta = \varepsilon$, the eikonal $S_0' = -1$ hence the solution is $y(x) = e^{-x/\varepsilon}$. c'est exactement la solution exacte !

Part II normal tangent vectors :

$$\underline{n} = \frac{1}{\sqrt{1+(\partial_x h)^2}} \begin{pmatrix} -\partial_x h \\ 1 \end{pmatrix}, \quad \underline{t} = \frac{1}{\sqrt{1+(\partial_x h)^2}} \begin{pmatrix} 1 \\ \partial_x h \end{pmatrix}$$

without dimension teh stresstensor $\underline{\underline{\sigma}}$:

$$\begin{pmatrix} 2\partial_{\bar{x}}\bar{u} - \bar{p} & (\partial_{\bar{x}}\bar{v} + \partial_{\bar{y}}\bar{u}) \\ (\partial_{\bar{y}}\bar{u} + \partial_{\bar{x}}\bar{v}) & 2\partial_{\bar{y}}\bar{v} - \bar{p} \end{pmatrix}$$

compute $\underline{\underline{\sigma}} \cdot \underline{n}$ and next $\underline{n} \cdot (\underline{\underline{\sigma}} \cdot \underline{n})$ and $\underline{t} \cdot (\underline{\underline{\sigma}} \cdot \underline{n})$

$$\underline{n} \cdot (\underline{\underline{\sigma}} \cdot \underline{n}) = \frac{1}{1+(\partial_x \eta)^2} (-2\mu \left((\partial_x v + \partial_y u) - (\partial_x \eta)^2 \partial_x u - \partial_y v \right))$$

$$\underline{t} \cdot (\underline{\underline{\sigma}} \cdot \underline{n}) = \frac{1}{1+(\partial_x \eta)^2} (\mu \left(2(\partial_x \eta)(\partial_y v - \partial_x u) + (1-(\partial_x \eta)^2)(\partial_x v + \partial_y u) \right))$$

Condition at the interface (pressure is 0 in the air, where there is no vsicosity as well) :

$$-p\underline{n} + \mu(\underline{\nabla}\, \underline{u} + \underline{\nabla}\, \underline{u}^t) \cdot \underline{n} = -\gamma(\underline{\nabla} \cdot \underline{n})\underline{n} \text{ without dimension } -\bar{p}\underline{n} + \mu(\bar{\underline{\nabla}}\, \bar{\underline{u}} + \bar{\underline{\nabla}}\, \bar{\underline{u}}^t) \cdot \underline{n} = -\frac{\gamma}{\rho g L^2}(\bar{\underline{\nabla}} \cdot \underline{n})\underline{n}$$

$$\text{Kapitza number : } \Gamma = \frac{\gamma}{\rho g L^2} = \frac{\ell_c^2}{L^2}, \text{ avec la longueur capillaire } \ell_c = \sqrt{\gamma/(\rho g)}$$

$\underline{n} = (-h', 1)/(1+h'^2)^{1/2}$ donc $(\bar{\underline{\nabla}} \cdot \underline{n}) = -h''/(1+h'^2)^{3/2}$

Scaling

to take all the terms in NS : $L = \nu^{2/3}(g \sin\beta)^{1/3}$, $T = \nu^{1/3}(g\sin\beta)^{-2/3}$, $U = L/T$ adn $P = \rho(\nu g \sin\beta)^{2/3}$

Thin layer

if we define $\tilde{x}, \tilde{y}$ as : $x = \tilde{x}/\varepsilon$, $y = \tilde{y}$, it corresponds to a long wave analysis ; (3) or (40)is :

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \tag{1}$$

(10) is (38) which is

$$\varepsilon \left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} \right) = -\varepsilon \frac{\partial \tilde{p}}{\partial \tilde{x}} + 1 + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} + O(\varepsilon^2) \tag{2}$$

(11) is (39) which is

$$0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + -B + \varepsilon \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} + O(\varepsilon^2) \tag{3}$$

pressure is then

$$\tilde{p} = B(\tilde{h} - \tilde{y}) + (\varepsilon^2 \Gamma) \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} + O(\varepsilon)$$

we have to suppose that $(\varepsilon^2 \Gamma)$ is of order one to be larger that the neglected $O(\varepsilon)$ terms of the velocity. Hence

$$\varepsilon \left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} \right) = +1 - B\varepsilon \frac{\partial \tilde{h}}{\partial \tilde{x}} + \varepsilon(\varepsilon^2\Gamma)\frac{\partial^3 \tilde{h}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} + O(\varepsilon^2) \tag{4}$$

by integration

$$\varepsilon \left(\frac{\partial}{\partial \tilde{t}} \int_0^{\tilde{h}} \tilde{u} d\tilde{y} + \frac{\partial}{\partial \tilde{x}} \int_0^{\tilde{h}} \tilde{u}^2 d\tilde{y} \right) = \tilde{h} - B\varepsilon \tilde{h} \frac{\partial \tilde{h}}{\partial \tilde{x}} + \varepsilon \tilde{h}(\varepsilon^2\Gamma)\frac{\partial^3 \tilde{h}}{\partial \tilde{x}^2} - \frac{\partial \tilde{u}}{\partial \tilde{y}}|_0 + O(\varepsilon^2) \tag{5}$$

by hypothesis : we are close to a Nusselt film ; so Shkadov equations (50) are, as proposed

$$\frac{\partial}{\partial \tilde{t}}\tilde{h} + \frac{\partial}{\partial \tilde{x}}\tilde{q} = 0, \quad \varepsilon \left(\frac{\partial}{\partial \tilde{t}}\tilde{q} + \frac{\partial}{\partial \tilde{x}} \frac{6\tilde{q}^2}{5\tilde{h}} \right) = \tilde{h} - B\varepsilon \tilde{h} \frac{\partial \tilde{h}}{\partial \tilde{x}} + \varepsilon \tilde{h}(\varepsilon^2\Gamma)\frac{\partial^3 \tilde{h}}{\partial \tilde{x}^2} - 3\frac{\tilde{q}}{\tilde{h}^2} + O(\varepsilon^2) \tag{6}$$