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Multiscale Hydrodynamic Phenomena

M2, Fluid mechanics, MU5MEF15 2022/2023
Friday December 1nd 2023, 8 :30am - 12 :30pm, Salle : 23.24.201 Part I. : 75 minutes, NO documents

1. Quick Questions In few words and few formula :
1.1 Scale of pressure for Stokes flow.
1.2 Order of magnitude of drag on a cylinder at small Re.
1.3 Self similar solution of heat equation.
1.4 ∂'Alembert equation : write the equation and the generic solution of it in 1D and 3D.
1.5 Solution of $\nabla^2 p = 0$ in upper half domain ($\forall x$ and $y \geq 0$) with $-\partial p/\partial y|_0 = f'(x)$ and $p(\infty) \to 0$?
1.6 In which one of the 3 decks of Triple Deck is flow separation ?
1.7 What are Shallow Water Equations ? Main hypothesis ?
1.8 What is the KdV equation ? What balance is it ? Link with Saint-Venant ?
1.9 What is the Bürgers equation ? Link with Saint-Venant ?
1.10 RATP references to Asymptotics and/or Mechanics ?

2. Exercice
Consider the following equation (of course ε is a given small parameter)

$$(E_\varepsilon) \quad \varepsilon^2 \frac{d^2u}{dx^2} + \frac{du}{dx} = e^x \text{ with } u(0) = 0 \quad u(1) = e.$$

We want to solve this problem with the Matched Asymptotic Expansion method.
2.1 Why is this problem singular ?
2.2 What is the outer problem and what is the possible general form of the outer solution ?
2.3 What is the inner problem of (E_ε) and what is the inner solution ?
2.4 Suggest the plot of the inner and outer solution.
2.5 Composite expansion.
2.6 What is the exact solution for any ε.
2.7 Compare composite expansion and exact solution.

3. Exercice
Let us look at the following ordinary differential equation : $(E_\varepsilon) \quad \dfrac{d^2y}{dt^2} + \varepsilon\dfrac{dy}{dt} + \pi^2 y = 0,$ valid for any $t > 0$ with boundary conditions $y(0) = 1$ and $y'(0) = 0$. Of course ε is a given small parameter.
We want to solve this problem.
3.1 Solve with Feynman averaging method.
3.2 We want to solve this problem with Multiple Scales Analysis. Introduce two time scales, $t_0 = t$ and t_1, what is the relation between t, t_1 and ε ?
3.3 Compute $\partial/\partial t$ and $\partial^2/\partial t^2$
3.4 Solve the problem.
3.5 Suggest the plot of the solution.
3.6 What is the exact solution for any ε, compare.

4. Exercice
Solve with WKB approximation the Airy problem

$$\varepsilon y'' = xy,$$

Hint : show that $S_0 = \pm \int \sqrt{|x|} dx$ and $S_1 \propto ln(|x|)$

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M2, Fluid mechanics 2022/2023
Friday, December 1th, 2023
Part II. : 1h 15 min all documents.

Multiscale Hydrodynamic Phenomena

Flow of a viscous current down a slope

This is a part of "low and instability of a viscous current down a slope " by Herbert E. Huppert, Nature Vol. 300 2 December 1982.

We consider a broad band of viscous fluid, initialy uniform in depth across a slope (so 2D plane), released so as to flow down a constant slope of value α. This is a model for avalanche flow of more complex rheology, here the flow is viscous and newtonian (viscosity μ and $\nu = \mu/\rho$). The flowing layer is thin and long (ε is the ratio of depth by length), the flow is slow. Figure 1 shows the experimental flow down an inclined plate (flow from rear to front), we see a "front" of fluid flowing toward us. A sketch is on figure 2. The height of the flow is $h(x,t)$, see the paper for notations (note that direction normal to the plane is z).

As all the results are more or less in the paper, be careful and rigorous to prove the results. Question 1. 2. 3... are a bit independent. Numbers refer to equations in the papers (Eq. X) or questions (Q. 1.X).

1.1 Write incompressible full NS equations for oil (pay attention to projection of gravity).
1.2 Over the free surface, the influence of air is supposed to be negligible, pressure is supposed constant p_{atm}, what are the small parameters which allow those aproximations ?
1.3 Write boundary conditions at the surface (using hypotheses of Q 1.2), note that surface tension maybe present, comment the Bond number.
1.4 In a "boundary layer spirit" use h_0 the scale of the depth of the oil (z is scaled by h_0), use $L = h_0/\varepsilon$ ($L \gg h_0$), write NS without dimension (say that Π the scale of pressure, define a scale U_0 for velocity, time etc.)
1.5 From transverse momentum (along z) show that as ε is small the pressure is hydrostatic (value of Π ?).
1.6 From the slow flow thin layer analysis we are doing, show that the final equation is (Eq. 1) indeed (but of course without dimension). Comment the dominant balance done here.
1.7 Show that if α small then $\dfrac{\partial \bar{h}}{\partial \bar{x}}$ comes back in the game.
1.8 Comment the scale of the total derivative of velocity (longitudinal acceleration), introduce a number without dimension with g, ν and h_0. If we want to keep this term, what relation should we write between this number without dimension (with g, ν and h_0) and ε ?
1.9 As a summary, write all the hypotheses used to establish (Eq. 1).
1.10 As a summary write all the scales used to settle (Eq. 1).

2.1 Boundary conditions for $\bar{u}$?
2.2 Solution of (Eq. 1) is half a Poiseuille flow (or Nüβelt film flow), show it
2.3 From the half-Poiseuille solution compute $\bar{Q} = \int_0^{\bar{h}} \bar{u} d\bar{z}$, compute $\dfrac{\partial u}{\partial \bar{z}}|_0$ as function of $\bar{Q}$ and $\bar{h}$. Write them with dimensions.
2.4 What is Lubrication theory ?

3.1 From incompressibility find a relation between $\dfrac{\partial \bar{Q}}{\partial \bar{x}}$ and $\dfrac{\partial \bar{h}}{\partial \bar{t}}$. Establish (Eq. 2).
3.2 Link with Shallow Water (Saint-Venant) equations ?

4.1 Starting from equation (Eq. 2), comment (Eq 3).
4.2 Explain with your words (Eq. 4), (Eq. 5), (Eq. 6) and (Eq. 7).
4.3 We propose here an alternate method to solve (Eq. 2) : self similarity. Using streching invariances of (Eq. 2), show that the selfsimilar solution is such that $\bar{h} = \bar{t}^{-1/3}\mathcal{H}(\bar{x}/\bar{t}^{1/3})$. Write the equation for $\mathcal{H}$ issued

from (Eq. 2), solve it and show that $\mathcal{H}(\eta) = \sqrt{(\eta)}$.
4.4 Check that this self similar solution is the same than (Eq. 7).

5.1 The solution of (Eq. 2) is (Eq.7), plot the depth h as function of x at different times.
5.3 This solution has a jump (a discontinuity) at the front. This shows that a new scale must be found to describe this. Explain then equation (Eq. 8).
5.4 Comment the proposed choice of separated variables and check (Eq. 9)-(Eq. 11).

6.1 As said, solution (Eq. 7) has a jump at the front. Another way to look at this singularity is to reintroduce the pressure gradient in (Eq. 2) (we looked at this in question Q 1.6 and Q 1.7). This is the right regularisation process for avalanches as the length of Q 5.3 is small.
Do the analysis and show that the equation (counterpart of Eq. 2 and Eq. 8) with dimensions is now

$$\frac{\partial h}{\partial t} + \frac{g}{3\nu}\frac{\partial}{\partial x}[(\sin\alpha - \cos\alpha\frac{\partial h}{\partial x})h^3] = 0$$

6.2 We have a new problem : (Eq. 2) with an extra $\frac{\partial \bar{h}}{\partial \bar{x}}$ with a small parameter in front (if $\alpha = O(1)$). Explain that it is a singular problem. Show that the outer problem is the one we have solved with Huppert (Eq. 7). Introduce the inner problem with a change of scale in x. What is the length of the transition region ? Write the inner problem. Which matching shall we do ?
6.3 With more algebra the problem can be solved exactly. To your feeling : plot the shape of the front (as Q.5.1).

After this, from this base flow, Huppert looks at the destabilisation of the front in several "fingers", that is another story...

biblio

http://www.itg.cam.ac.uk/people/heh/Paper49.pdf
https://web.mit.edu/1.63/www/Lec-notes/chap2_slow/2-4spread-mud.pdf

Nature Vol. 300 2 December 1982

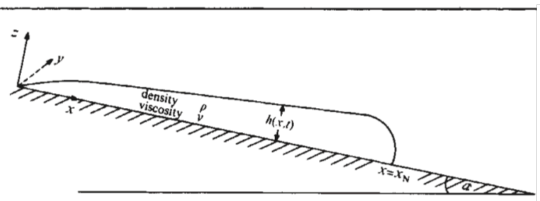

Fig. 2 A sketch of the flow and coordinate system.

Flow and instability of a viscous current down a slope

Herbert E. Huppert

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, UK

If viscous fluid is released on a horizontal surface it rapidly takes up a circular plan form as it spreads. This form is observed[1,2] to be stable to any small disturbances which are initiated on the front due, for example, to irregularities in the horizontal surface or to chance perturbations. Alternatively, if some fluid is released onto a sloping surface—for example, some liquid detergent on a slanted plate—a quite different plan form occurs. One, two or more extended regions of fluid develop downslope, as shown in Fig. 1*a*, *b*. A situation intermediate between these two is now discussed. Consider a broad band of viscous fluid, uniform in depth across a slope, released so as to flow down a constant slope. By following the motion, which is initially independent of the cross-slope coordinate, the speed of advance and the depth of the flow before it breaks up into a series of waves of ever increasing amplitude can be determined. I present an expression for the wavelength of the front, which is determined by surface tension and is independent of the coefficient of viscosity.

With the coordinate system depicted in Fig. 2 and use of the approximations of lubrication theory[3], the y-independent, down-slope momentum equation can be written as

$$0 = g \sin \alpha + \nu u_{zz} \quad (1)$$

In deriving equation (1) we have neglected both surface tension, the effects of which will be analysed below, and contact line effects[4], which are negligible[1] under the assumption that the Bond number $B = \rho g l^2/T \gg 1$, where ρ is the fluid density, l a representative length scale of the current and T the surface tension. Use of the equation of continuity then leads to

$$h_t + (g \sin \alpha/\nu)h^2 h_x = 0 \quad (2)$$

as the nonlinear partial differential equation for the unknown free surface $h(x, t)$. To equation (2) must be added the global continuity equation

$$\int_0^{x_N(t)} h(x, t)\mathrm{d}x = A \quad (3)$$

where $x_N(t)$ is the value of x at the front of the current and A is the initial cross-sectional area. The particular solution of equations (2) and (3) is sought in the range $0 \leq x \leq x_N(t)$.

Equation (2) shows that h is constant along characteristics given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (g \sin \alpha/\nu)h^2 \quad (4)$$

Thus if initially $h = f(x)$, say, the equation of the characteristics is given by

$$x = x_0 + (g \sin \alpha/\nu)f^2(x_0)t \quad (5)$$

where x_0 is the initial value of the characteristic. The solution of equation (2) is thus

$$h = [\nu(x - x_0)/g \sin \alpha]^{1/2} t^{-1/2} \quad (6a)$$

$$\rightarrow (\nu/g \sin \alpha)^{1/2} x^{1/2} t^{-1/2} \quad (x \gg x_0) \quad (6b)$$

independent of the initial conditions. When combined with equaton (3), in order to evaluate the length of the current, equation (6*b*) proves that some time after the initiation of the current, no matter what the initial shape, the solution takes the form

$$h = (\nu/g \sin \alpha)^{1/2} x^{1/2} t^{-1/2}$$
$$0 \leq x \leq x_N = (9A^2 g \sin \alpha/4\nu)^{1/3} t^{1/3} \quad (7)$$

Expressed alternatively, equation (7) represents the unique similarity solution of equations (2) and (3). The profile of the predicted current ends abruptly at $x = x_N$ with $h = h_N(t) = 1.5A/x_N$ there. The profile can be smoothed off at x_N by including the effects of surface tension. This will be done below after a discussion of an experimental investigation of the validity of the truncated profile expressed by equation (7).

Some 30 experiments were conducted using a pentagonal sheet of Perspex. The longest side was 101.7 cm and from it two sides of length 82 cm emanated at right angles. The remaining two sides were 60 cm. The sheet was surrounded by a Perspex wall 5 cm high to make a tray which was firmly attached to a rigid, flat board. Before each experiment the board was tilted by raising the 101.7-cm long side by the required amount. A removable Perspex gate was fitted 5.0 cm from the back edge of the tray and fluid poured into the space behind the gate. Three different fluids, whose physical properties are listed in Table 1, were used. After the fluid behind the gate had been left a sufficiently long time that it had become quite stationary, the gate was raised and the fluid proceeded to pour down the slope.

At first the motion was virtually independent of the cross-slope coordinate except near the walls where viscous drag retarded the flow. This form of motion is clearly evident in Fig. 1*c*. Observations of the length of the current as a function of time were taken and some typical results plotted in Fig. 3. It can be seen that the agreement between the experimental observations and the theoretical prediction (7) is good. The agreement justifies the neglect of both surface tension and contact line effects in predicting the temporal development of the two-dimensional current.

After the two-dimensional motion had continued for some time, the flow front seemed spontaneously to develop a series of small amplitude waves of fairly constant wavelength across the slope, as seen in Fig. 1*d*. The amplitude of the waves increased in time as the maxima (points furthest down the slope) travelled faster than the minima, as seen in Fig. 1*e*. The wavelength remained unaltered. The long-time shape of the flow front was different for the two fluids used. Both silicone oils developed the form shown in Fig. 1*f*. This consisted of a periodic, triangular front with tightly rounded maxima. The maxima were connected by very straight portions at an angle to the slope to extremely pointed minima. The glycerine developed the form shown in Fig. 1*g*. This was also periodic, though with much less tightly rounded maxima. These were again connected by extremely straight portions, but in this case they were almost directly down slope and connected to very broad minima.

Table 1 The fluids used, their viscosity and surface tension

Fluid	Viscosity at 17 °C ($cm^2 s^{-1}$)	Surface tension at 17 °C ($dyn\ cm^{-1}$)
Silicone oil MS200/100	1.15	20.6
Silicone oil MS200/1000	12.8	22.0
Glycerine	9.8	60.2

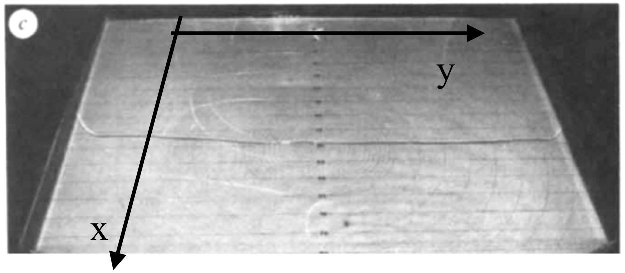

Fig .1 flow of MS200/1,000 oil down an inclined plane12° 80s after release showing the broadly two-dimensional front

(...)

Experiments with silicone oils whose coefficients of viscosity differ by an order of magnitude, although their surface tensions are almost equal, indicated that the wavelength of the instability is independent of the coefficient of viscosity. Experiments with the two fluids of comparable viscosity yet different surface

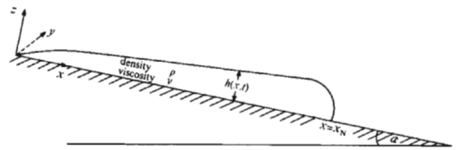

Fig. 2 A sketch of the flow and coordinate system.

tensions indicated that the wavelength is a function of the surface tension. Finally, observations taken during experiments using the same fluid but with amounts differing by up to a factor of 10 indicated that the wavelength is weakly dependent on the initial cross-sectional area A.

The form of the quasi-steady, two-dimensional tip, which is determined by including surface tension and by matching the tip onto the main flow given by equation (7), can be easily determined. The addition to equation (2) of the terms due to surface tension T leads to

$$h_t + (g \sin \alpha/\nu)h^2 h_x - 1/3(T/\rho\nu)h^3 h_{xxxx} = 0 \quad (8)$$

In the tip the dominant balance is between the second and third terms of equation (8), the solution to which can be written as

$$h = h_N(t)H(\xi) \qquad \xi = (\rho g \sin \alpha/T h_N)^{1/3}(x_N - x) \quad (9)$$

where $H(\xi)$ satisfies

$$H^3 H''' + H^3 = 1 \quad (10)$$

$$H \rightarrow \left(\frac{16}{15}\right)^{1/4} \xi^{3/4} \quad (\xi \rightarrow 0) \quad (11)$$

$$\rightarrow 1 \quad (\xi \rightarrow \infty) \quad (12)$$

The length scale of the tip is thus given by $(T h_N/\rho g \sin \alpha)^{1/3}$.

(...)

Laplace, Maison des Examens (et aussi rue Cauchy à Arcueil)
Madame du Châtelet (and Schrödinger Equation by Pierre-Yves Trémois)
Gare du Nord (Equations de Manabe Prix Nobel 2021 avec Hasselmann et Parisi)
Pierre et Marie Curie ligne 7 du métro de Paris,
Place Monge
Réaumur

- **correction Ex 2**

here there is a trap, $x = \delta\bar{x}$ gives by dominant balance $\delta = \varepsilon^2$

```
se = DSolve[{e u''[y] + u'[y] == E^y, u[1] == E, u[0] == 0},    u[y], {y, 0, 1}];
s = DSolve[{u'[y] == E^y, u[1] == E}, u[y], {y, 0, 1}];
Plot[{0, u[y] /. se /. e -> .25, u[y] /. se /. e -> .125,
                                         u[y] /. se /. e -> .05, u[y] /. s}, {y, 0, 1}, Frame
 FrameLabel -> {"x", "u(x)"}]
```

$$\frac{ee^{\frac{e-x+1}{e}}\left(e^{\frac{x}{e}}-1\right)-e^{\frac{1-x}{e}}+e^{\frac{1}{e}+x}-e^{x}+1}{(e+1)\left(e^{\frac{1}{e}}-1\right)}$$

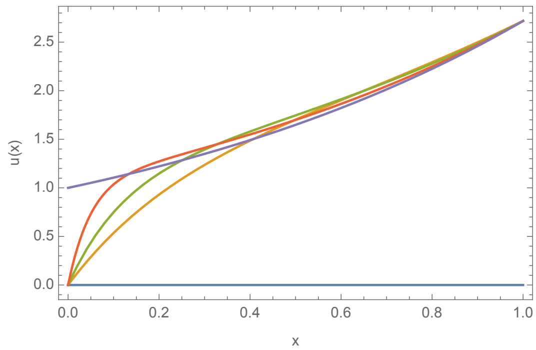

FIGURE 1 –

correction Ex 3

Solution with $\varepsilon = 0$ is $y = A\cos(\pi t)$, and $y' = -\pi A\sin(\pi t)$,.
Mean value

$$< \sin^2 t >= \frac{1}{2\pi}\int_0^{2\pi}\sin^2 t dt = \frac{1}{2} \qquad < \cos^2 t >= \frac{1}{2\pi}\int_0^{2\pi}\cos^2 t dt = \frac{1}{2}$$

hence as the period is now 2, $y = A\cos(\pi t)$, $< y^2 >= A^2/2$ and $y' = -\pi A\sin(\pi t)$, $< y'^2 >= \pi^2 A^2/2$

Energy

$$\frac{d}{dt} < y^2/2 + \pi^2 y^2/2 >= -\varepsilon < y'^2 >$$

$y = A\cos(\pi t)$, $< y^2 >= A^2/2$
$y' = -\pi A\sin(\pi t)$, $< y'^2 >= \pi^2 A^2/2$

$$\frac{d}{dt}(\pi^2 A^2/2 + \pi^2 A^2/2) = -\varepsilon\pi^2 A^2$$

the π disappears

$$\frac{d}{dt}A^2 = -\varepsilon A^2$$

...

correction Ex 4

Airy

$$\varepsilon y''(x) = xy(x),$$

WKB expansion

$$(\varepsilon/\delta^2)S_0'^2 + (2\varepsilon/\delta)S_0'S_1' + (\varepsilon/\delta)S_0'' = x,$$

so $\delta = \sqrt{\varepsilon}$ et $S_0'^2 = x$ this is the "eikonale" (phase)

$$S_0 = \pm \int^x \sqrt{|x|}dx$$

next order

$$S_0'' + 2S_0'S_1' = 0$$

writes $S_1' = -(1/2)S_0''/S_0'$, c'est aussi $S_1 = (-1/2)LogS_0'$ or $S_0' = \sqrt{|x|}$ we have $S_1 = -(1/4)Log|x|$ solution is superposition

$$y(x) = |x|^{-1/4}(C_1 e^{\frac{1}{\sqrt{\varepsilon}}\int^x \sqrt{|x|}dx} + C_2 e^{\frac{-1}{\sqrt{\varepsilon}}\int^x \sqrt{|x|}dx})$$

C_1 et C_2 from BC (not given)

For $x \to \infty$

$$Ai(x) = Cx^{-1/4}e^{-\frac{2}{3}x^{3/2}}$$

(on peut montrer par ailleurs $C = 1/(2\sqrt{\pi})$.

for $x \to -\infty$

$$Ai(x) = C_1 x^{-1/4}\cos\frac{2}{3}(-x)^{3/2} + C_2 x^{-1/4}\sin\frac{2}{3}(-x)^{3/2}$$

1.1
1.2 air pressure small, density and viscosity of air small, small surface tension... $p_{air}/(\rho g h_0) \ll 1$, $\rho_{air}/\rho_{water} \ll 1$, $\mu_{air}/\mu_{water} \ll 1$, $B \ll 1$
1.5 Transverse pressure il large due to a $1/h_0$, the transverse velocity terms are small (like in BL Theory), we do a balance with the gravity, so with $\Pi = \rho g h_0$ dominant part of transverse momentum

$$0 = -(\frac{\partial \tilde{p}}{\partial \tilde{z}}) - \cos\alpha$$

this gives hydrostatic pressure after integration

$$\tilde{p} = \cos\alpha(\tilde{h} - \tilde{z})$$

1.6 dominant part of longitudinal momentum

$$0 = -(\rho g h_0/L)(\frac{\partial \tilde{p}}{\partial \tilde{x}}) + (\rho \nu U_0/h_0^2)(\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}) + \rho g \alpha$$

viscous/ gravity balance (brake/motor) in a small slope $\sin\alpha \sim \alpha$, but not so small

$$\nu U_0/h_0^2 = g\alpha$$

so that

$$0 = -(\varepsilon/\alpha)(\frac{\partial \tilde{p}}{\partial \tilde{x}}) + (\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}) + 1$$

if $\alpha \gg \varepsilon$ this is Eq. 1 of paper :

$$0 = (\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}) + 1$$

1.7 as $\cos\alpha \sim 1$ and $U_0 = g h_0^2 \alpha/\nu$, the longitudinal momentum with pressure is

$$0 = -(\varepsilon/\alpha)(\frac{\partial \tilde{h}}{\partial \tilde{x}}) + (\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}) + 1$$

if slope is large enough, the pressure term is not present in the equation, as proposed by Huppert and just seen before. If $\alpha \sim \varepsilon$, we have the three terms : pressure comes back in the game...

1.8 compare inertia U_0^2/L and viscous $\nu U_0/h_0^2$, full longitudinal equation is :

$$(gL^3/\nu)(h_0/L)^4 \frac{d\tilde{u}}{d\tilde{t}} = -(\varepsilon/\alpha)(\frac{\partial \tilde{h}}{\partial \tilde{x}}) + \frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2} + 1 + \varepsilon^2 \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}$$

jjjjjj

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial \tilde{x}} + \dfrac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \\ Fr^2\varepsilon^5 \left(\dfrac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\dfrac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\dfrac{\partial \tilde{u}}{\partial \tilde{y}} \right) = -\dfrac{\partial \tilde{p}}{\partial \tilde{x}} + (\varepsilon^2 \dfrac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \dfrac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}) \\ Fr^2\varepsilon^6 \left(\dfrac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u}\dfrac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\dfrac{\partial \tilde{v}}{\partial \tilde{y}} \right) = -\dfrac{\partial \tilde{p}}{\partial \tilde{y}} + (\varepsilon^3 \dfrac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \varepsilon^2 \dfrac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}) - 1. \end{cases} \tag{1}$$

Remarquons que le terme $Ga = Fr^2\varepsilon^5 = \frac{gL^3}{\nu^2}\varepsilon^5$