

Homogenisation

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R sum 

This chapter is devoted to "homogenization" : an asymptotic method with two scales, the fast scale being averaged.

We consider as a practical example here a body which consists in a local structure which repeats periodically itself. On the global scale of the body, the number of small structures is huge. Homogenization is the study of the relationship between the local structure of a non-homogeneous medium and its macroscopic behavior. It allows a tremendous simplification : we do not need to compute all the fields at the small scale of the structure, but only once on the small scale which will give the behavior at the large scale of the body. We present "homogenization" theory as an application of the "multiple scales" technique. As usually, we propose first a simple example to understand the problem, next we present the theory in 1-D. Next we present briefly the more complex example of Darcy law linking the mean velocity and the mean pressure gradient in a porous media. The technique has been developed since 1978/80 by A. Bensoussan, J.-L. Lyons, G. Papanicolaou and Evariste Sanchez-Palencia. It is more used in solid mechanics, it is a routine tool for composites.

1 Introduction

Up to now, scales were obtained often by the geometry (aspect ratio...) or then by some dominant balance. Here, there is a fine structure due to a fine structure in the media. For example, the body is constituted of fibers, or lamellas, or cells, or grains, of small size compared to the body it self. This is the case for composite materia, for sand, for agglomerate wood,

grids, and so one. We define of course ε the small relative scale of the fine structure. Of course, if we solve at the level of these fine details, the problem is difficult (too many details to consider, think in term of discretization : it will be a too large problem with too many variables). We then want to obtain the mean influence of this small structure at the scale of the body, it means that we want to obtain the macroscopic, or homogenized or effective behavior. The homogenized system will be as simple as a homogenous system, which is more simple to solve.

This averaging process is called "from micro to macro" or "upscaling" or "multiple scale", depending on the people and on their sensibility.

2 A simple example in thermics, "Homogenisation for children"

2.1 Chain of composite walls

At first, we will look at a simple problem in thermics, to show the limit of simple approaches but to understand the concept. For example, if the media is constituted of parallel lamellas of low conductivity k_α and high conductivity $k_\beta \gg k_\alpha$, it is clear that the low conductivity (k_α) will rule the solution. The global conductivity will not be the simple mean conductivity $(k_\alpha + k_\beta)/2$. Let us show it.

The heat equation in 1D, in a fixed material, with no variations of volume

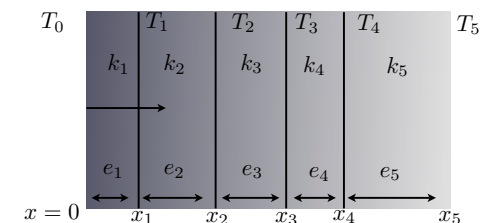


FIGURE 1 – A great number n of infinite walls (parallel lamellas); T_0 in $x = 0$ and T_n in $x = x_n$: there are n homogenous slides. We want to construct the total averaged solution : the "homogenised solution".

is

$$\rho \frac{\partial e}{\partial t} = -\frac{\partial q}{\partial x},$$

remember that the variation of energy $de = cdT$ in time t , is the divergence of the energy flux q , the constitutive law gives Fourier law $q = -k \frac{\partial T}{\partial x}$, where k is the function of the position, so that the heat equation in 1D is not $\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$, but indeed :

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial}{\partial x} \left(-k \frac{\partial T}{\partial x} \right).$$

Let us call e_i the distance between wall i and $i-1$, this slice has thermal conductivity k_i with $1 \leq i \leq n \dots$. Let define T_0, T_1 to T_n temperatures on the faces. Then, the steady heat equation is

$$-\frac{dq}{dx} = 0,$$

q is then constant in every slice. So, we have the linear solution in each slice, and the constant flux is the gradient :

$$\begin{aligned} q &= -k_1 \frac{T_1 - T_0}{e_1} \\ q &= -k_2 \frac{T_2 - T_1}{e_2} \\ &\dots \\ q &= -k_n \frac{T_n - T_{n-1}}{e} \end{aligned}$$

so, we have the following relations for all the temperatures :

$$0 < x < x_1, T = T_0 - \frac{x}{k_1} q \quad \text{and} \quad T_1 = T_0 - \frac{e_1}{k_1} q,$$

$$x_1 < x < x_2, T = T_1 - \frac{x - x_1}{k_2} q \quad \text{and} \quad T_2 = T_1 - \frac{e_2}{k_2} q, \text{ what}$$

$$\dots \\ x_{n-1} < x < x_n, T = T_{n-1} - \frac{x - x_{n-1}}{k_n} q \quad \text{and} \quad T_n = T_{n-1} - \frac{e_n}{k_n} q,$$

we see that in each cell

$$T_i = T_{i-1} - \frac{e_i}{k_i} q,$$

the increment of temperature is $-\frac{e_i}{k_i} q$, if we do the sum, we obtain

$$T_n = T_0 - q \left(\sum_1^n e_i / k_i \right), \quad \text{with} \quad e = \sum_1^n e_i.$$

2.2 Final effective conductivity

Then, the flux, which is constant, is linked with the total variation of temperature :

$$q = -\frac{T_n - T_0}{e_1/k + \dots + e_n/k_n},$$

it can be identified as $q = -k_H \frac{T_n - T_0}{e}$, were

$$k_H = \frac{\sum_1^n e_i}{\sum_1^n e_i / k_i}$$

the effective coefficient of thermal diffusion is a kind of geometrical average, it is not the averaged value $\frac{\sum_1^n k_i e_i}{\sum_1^n e_i}$.

Indeed if $k_{2i} = k_\alpha$ and $k_{2i+1} = k_\beta$; $e_i = e/n$ with $k_\beta \gg k_\alpha$, $k_H = \frac{2k_\beta k_\alpha}{k_\beta + k_\alpha}$, not $(k_\beta + k_\alpha)/2$. For $k_\beta \gg k_\alpha$, we have $k_H \sim 2k_\alpha$.

See section 3.7 for a numerical example.

2.3 Heuristical view

After this simple way to present homogenisation, let us introduce a second simple approach (quoted by Caillerie [5], said from Sanchez and said from Suquet 1982 "thèse d'Etat", but I did not found it in it). Let us take a media with periodicity e in which k changes at this period. In the slice the temperature is $T^{slice}(x)$, the global mean temperature is $T(x)$, so from one slice to the other, if we forget the small details in the slice, the change of temperature is more or less the final global gradient :

$$T^{slice}(x+e) - T^{slice} \simeq e \frac{dT}{dx}.$$

But as well, locally in the slice

$$\frac{dT^{slice}}{dx} = -\frac{q}{k(x)}$$

so by integration over the length of the slice :

$$T^{slice}(x+e) - T^{slice}(x) = \int_x^{x+e} -\frac{q}{k(x)} dx$$

compared with the previous expression of $T^{slice}(x+e) - T^{slice}(x)$:

$$\int_x^{x+e} -\frac{q}{k(x)} dx \simeq e \frac{dT}{dx}.$$

For the overall media, let us define k_H as :

$$q = -k^H \frac{dT}{dx}$$

where k^H is the effective heat transfer coefficient we are looking for. Then as $dT/dx = -q/k^H$,

$$\int_x^{x+e} -\frac{q}{k(x)} dx \simeq e \frac{q}{k^H}.$$

so we find again (if we simplify by q) that the harmonic averaged value of k gives k^H

$$\frac{1}{k^H} = \frac{1}{e} \int_x^{x+e} \frac{1}{k(x)} dx.$$

We will now look at the full problem with the "homogenization theory" and see whether we can find again this results as a particular case of a more complicated theory. Then, section 3.7 we will compare with a numerical example.

3 The theory of Homogenisation

3.1 Example in heat conduction

Again, we study the simple model of steady heat conduction in a one D media.

$$-\frac{\partial q}{\partial x} = 0$$

with the constitutive relation :

$$q = -k \frac{\partial T}{\partial x}$$

where k is a function which varies quickly at the small scale of the fine structure. It changes at a short scale variable x/ε , in fact, at each beginning x_i of a cell i , the variable is $(x - x_i)/\varepsilon$, or with other words the periodicity is ε . We will follow the Sanchez-Palencia [12, 13] and [11] developments.

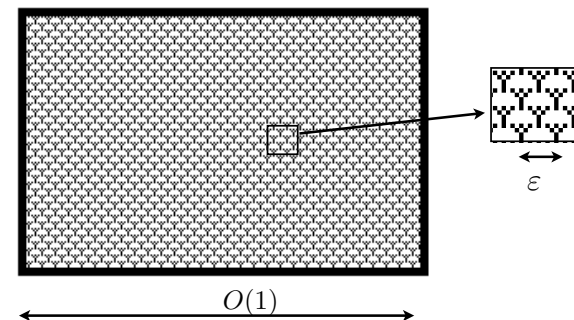


FIGURE 2 – a media Ω of size $O(1)$ constituted of small structure ("elementary cell") Υ of relative size ε . We look at a mean behavior of the media filtering the rapid small scale. Note that the small cell definition is not unique, and that it can be several times the one presented here....

3.2 Developements

There is a change of notation compared to the chapter on "Multiple scales", we wrote with a subscript; now we put a subscript :

$$T = T^0(x^0) + \varepsilon T^1(x^0, x^1) + O(\varepsilon^2),$$

so that we can use the tensorial notation thereafter. Most of all, there is a change of point of view between fast and slow. In the oscillators of the "Multiple scales" examples, there was the "normal" time, at human scale and then the slow time (secular time, long long time, the time of the planets). Here the human scale is the slow one (the size of the macroscopic body), and the local scale is the fast one ("man is the measure of all things") :

$$x^0 = x \text{ and } x^1 = \frac{x}{\varepsilon}.$$

The variables x^0 is the "slow" (macroscopic) scale and x^1 is the "fast" (microscopic) scale. When looking at the problem at the microscopic scale, the changes in x^1 are so fast that x^0 seems to be constant. So the first order T^0 is a function of x^0 , not of x^1 . Scale separation is exploited : x^0 and x^1 are treated as independent variables :

$$\frac{d}{dx} = \frac{\partial}{\partial x^0} + \frac{1}{\varepsilon} \frac{\partial}{\partial x^1}$$

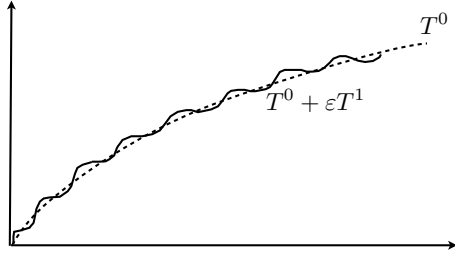


FIGURE 3 – The mean global solution, T^0 and the small scale solution T^1 (inspired from Sanchez-Palencia sketches), see indeed section 3.7 for a numerical example showing those small oscillations at the small structure size.

so that $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i^0} + \frac{1}{\varepsilon} \frac{\partial}{\partial x_i^1}$ for tensors. We start first, for simplicity, with scalar fields.

3.3 Multiple scale expansion

From the constitutive equation linking the flux and the derivative of temperature $q = -k \frac{d}{dx} T$ with the expansion $T = T^0(x^0) + \varepsilon T^1(x^0, x^1) + \varepsilon T^2(x^0, x^1) + O(\varepsilon^3)$ we have

$$\frac{d}{dx} T = \left(\frac{\partial}{\partial x^0} T^0(x^0) + \frac{\partial}{\partial x^1} T^1(x^0, x^1) \right) + \varepsilon \left(\frac{\partial}{\partial x^0} T^1(x^0, x^1) + \frac{\partial}{\partial x^1} T^2(x^0, x^1) \right) + \dots$$

The expression of the energy density flux q is an expansion

$$q = q^0(x^0, x^1) + \varepsilon q^1(x^0, x^1) + O(\varepsilon^2)$$

with, by identification, q^0 and q^1 :

$$q^0 = k(x^1) \left(\frac{\partial}{\partial x^0} T^0(x^0) + \frac{\partial}{\partial x^1} T^1(x^0, x^1) \right)$$

$$q^1 = k(x^1) \left(\frac{\partial}{\partial x^0} T^1(x^0, x^1) + \frac{\partial}{\partial x^1} T^2(x^0, x^1) \right),$$

where indeed k changes abruptly with space, so that, the steady 1D energy equation

$$-\frac{\partial}{\partial x} q = 0$$

gives by substitution :

$$\left(-\frac{\partial}{\partial x^0} - \frac{1}{\varepsilon} \frac{\partial}{\partial x^1} \right) \left(q^0(x^0, x^1) + \varepsilon q^1(x^0, x^1) + \dots \right) = 0,$$

at order $O(1/\varepsilon)$ we have

$$\left(-\frac{\partial}{\partial x^1} q^0 \right) = 0$$

this equation corresponds to the microscopic behavior of the media. The next order $O(1)$

$$\left(-\frac{\partial}{\partial x^0} q^0 - \frac{\partial}{\partial x^1} q^1 \right) = 0$$

this equation corresponds to the macroscopic behavior of the media. As we will see next.

3.4 Averaging

Let us introduce the definition of an averaged value over a representative volume Y of length/surface/volume $mes Y$, depending of dimension, here it is a length :

$$\langle g \rangle = \frac{1}{mes Y} \int_Y g(x^1) dx^1$$

so, $\left(-\frac{\partial}{\partial x^0} q^0 - \frac{\partial}{\partial x^1} q^1 \right) = 0$ is averaged over the small cell in

$$\langle -\frac{\partial}{\partial x^0} q^0 \rangle - \langle \frac{\partial}{\partial x^1} q^1 \rangle = 0$$

but the second $\int \frac{\partial}{\partial x^1} q^1 dx^1$ is zero by periodicity. So that the first is

$$-\frac{\partial}{\partial x^0} \langle q^0 \rangle = 0$$

which is a macroscopic relation (as we claimed before) as it depends only on x^0 as $\langle q^0 \rangle$ is a function of x^0 only.

This can be interpreted as well as the secularity condition : no forcing influence.

So, the mean temperature $\langle T^0(x^0) \rangle$ is such that $T^0(x^0)$ is constant at leading order over the local microstructure. We claimed that $\left(-\frac{\partial}{\partial x^1} q^0 \right) = 0$

corresponds to the microscopic behavior of the media. Indeed, from the expression of q^0 we have :

$$\left(-\frac{\partial}{\partial x^1} q^0\right) = -\frac{\partial}{\partial x^1} \left(k(x^1) \left(\frac{\partial}{\partial x^0} T^0(x^0) + \frac{\partial}{\partial x^1} T^1(x^0, x^1)\right)\right) = 0$$

but as $\frac{\partial}{\partial x^0} T^0(x^0)$ and $T^0(x^0)$ do not depend on x^1 , we write it :

$$-\frac{\partial}{\partial x^1} \left(k(x^1) \frac{\partial}{\partial x^1} T^1(x^0, x^1)\right) = \frac{\partial k(x^1)}{\partial x^1} \frac{\partial}{\partial x^0} T^0(x^0).$$

this is an equation for the unknown T^1 which is periodic in x^1 . We want to solve for T^1 and we note that the equation is linear in $\frac{\partial}{\partial x^0} T^0(x^0)$ so we guess that we can write :

$$T^1(x^0, x^1) = w(x^1) \frac{\partial}{\partial x^0} T^0(x^0),$$

were w is a fast function which is solution of the problem (the "cell-problem")

$$-\frac{\partial}{\partial x^1} \left(k(x^1) \frac{\partial}{\partial x^1} w(x^1)\right) = \frac{\partial k(x^1)}{\partial x^1}.$$

This problem is solved numerically in practice. Often with finite elements. The function $w(x^1)$ depends only in the function $k(x^1)$ which depends on the characteristics of the media, at the very small scale x^1 of the cell. This dependence is non-local, the homogenized heat coefficient depends on the value of the gradients of w . Note that the w can be computed once, as they do not change with the Boundary Conditions, or they do no change if we change the total media.

The w is defined at an additive constant, we can impose a mean zero value. If the zero of x^1 is at the center of the cell, w may be interpreted as the distance (far away from the center of the cell, the temperature is the distance by the flux :

$$T^1(x^0, x^1 \rightarrow \infty) \simeq x^1 \frac{\partial}{\partial x^0} T^0(x^0).$$

3.5 The homogenised media

Now, suppose we found w , let us put all the results together, the macroscopic equation is

$$-\frac{\partial}{\partial x^0} \langle q^0 \rangle = 0,$$

with

$$q^0 = -k(x^1) \left(\frac{\partial}{\partial x^0} T^0(x^0) + \frac{\partial}{\partial x^1} T^1(x^0, x^1)\right),$$

which, thanks to the analysis is

$$q^0 = -k(x^1) \left(1 + \frac{\partial}{\partial x^1} w(x^1)\right) \frac{\partial}{\partial x^0} T^0(x^0)$$

and the average $\langle q^0 \rangle$ is

$$\langle q^0 \rangle = -k_H \frac{\partial}{\partial x^0} T^0(x^0) \text{ with } k_H = \frac{1}{mes\Upsilon} \int_{\Upsilon} \left(k(x^1) + k(x^1) \frac{\partial}{\partial x^1} w(x^1)\right) dx^1,$$

the expression of k_H is the "homogenised constitutive relation", or effective constitutive relation :

$$k_H = \langle \left(k(x^1) + k(x^1) \frac{\partial}{\partial x^1} w(x^1)\right) \rangle .$$

This is the result. It is more complicated than the simple averaged value of $k(x^1)$, there is and extra term which takes into account the details of the microstructure, those details are in the function w . Again, the dependance is non-local, the homogenized coefficient depends on the value of the gradients of w .

Once k_H is obtained, the computation is done at the slow scale x^0 as usually (there is no more fast variation in x^1).

3.6 Example of application : feedback with "example for children"

Look at the "cell-problem", in one dimension, we can go further in this case and obtain a simple expression for w :

$$-\frac{\partial}{\partial x^1} \left(k(x^1) \frac{\partial}{\partial x^1} w(x^1)\right) = \frac{\partial k(x^1)}{\partial x^1}.$$

By integration

$$k(x^1) \frac{\partial}{\partial x^1} w(x^1) = -k(x^1) + K,$$

where K is a constant of integration, so

$$\frac{\partial}{\partial x^1} w(x^1) = -1 + \frac{K}{k(x^1)}$$

then, by average, as $\langle \frac{\partial}{\partial x^1} w(x^1) \rangle = 0$

$$0 = -1 + K \langle \frac{1}{k(x^1)} \rangle,$$

so K is the inverse of the mean value of the inverse of k :

$$K = \frac{1}{\langle \frac{1}{k(x^1)} \rangle},$$

which is what we found from the simple analysis (Kevorkian & Cole [10] or E [6]). So, the effective coefficient :

$$k_H = \langle (k(x^1) + k(x^1)) \frac{\partial}{\partial x^1} w(x^1) \rangle$$

is then

$$k_H = \langle (k(x^1) + k(x^1))(-1 + \frac{K}{k(x^1)}) \rangle = K,$$

the homogenized coefficient is the inverse of the harmonic average,

$$k_H = \frac{1}{\langle \frac{1}{k(x^1)} \rangle},$$

as expected from the simple analysis presented at first. Of course, in 2D or 3D, one has to solve a more complex system !

3.7 Numerical example

3.7.1 Full solution

In order to check on an example the previous concepts, let us consider the 1D heat equation with a given heat conductivity which changes on a small scale ε . Let us give :

$$k(x^1) = .04 + 50 \sin(\pi x^1)^4,$$

a priori given function of the microstructure variable x^1 (plot on figure 4 left), so it is $k(x/\varepsilon)$ with $\varepsilon = 0.05$. We first numerically solve the full problem

$$\frac{\partial}{\partial x} \left(k\left(\frac{x}{\varepsilon}\right) \frac{\partial}{\partial x} T(x) \right) = 0$$

with

$$k(0) \frac{\partial}{\partial x} T(0) = 1 \text{ and } T(1) = 0$$

and say $\varepsilon = 0.05$ this gives the following figure (4 right). The zig zags seen on figure (4 right, red curve) correspond to the changes in $k(x_1)$.

The averaged homogenized result from the previous theory is

$$k_{AH} = \frac{1}{\int_{x^1=0}^{x^1=1} 1/k(x^1) dx^1}$$

the numerical value is $k_{AH} = 0.3915$ so that the homogenized solution for the temperature is a decrease with a slope $1/k_H$:

$$T = (1 - x)/k_{AH},$$

it is on figure (4 right, blue line).

3.7.2 homogenization

Let us now follow the procedure of homogenization. The homogenized problem is (see subsection Averaging) :

$$\frac{\partial}{\partial x^1} \left(k(x^1) \frac{\partial}{\partial x^1} w(x^1) \right) = -\frac{\partial}{\partial x^1} k(x^1)$$

with $w(x^1 = 0) = w(x^1 = 1) = 0$

Then we solve this local problem numerically we obtain $w(x^1)$ plotted in blue on figure 5 left. We obtain as well $w'(x^1)$ plotted in red on figure 5 left. From these, we compute the effective coefficient :

$$k_H = \int_{x^1=0}^{x^1=1} (k(x^1) + k(x^1)) \frac{\partial}{\partial x^1} w(x^1) dx^1$$

whose numerical value is $k_H = 0.3915$ (which is exactly k_{AH} of course).

For sake of final comparison, on figure 5 right, we plot the numerical solution and the homogenized solution at the first cell $T(x) + x/k_H - T[0], -1.\varepsilon/k_H w(x^1)$, they are indeed superposed.

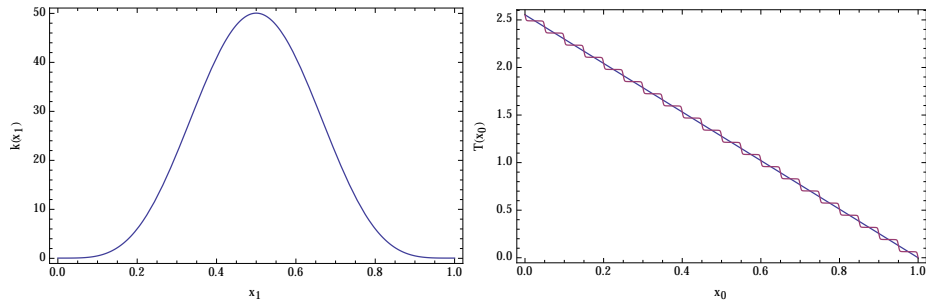


FIGURE 4 – Left the function k . Right Zig-zag : the full solution of the non homogenized problem, here $k(x^1) = .04 + 50 \sin(\pi x^1)^4$ and $\varepsilon = 0.05$, line the homogenized one.

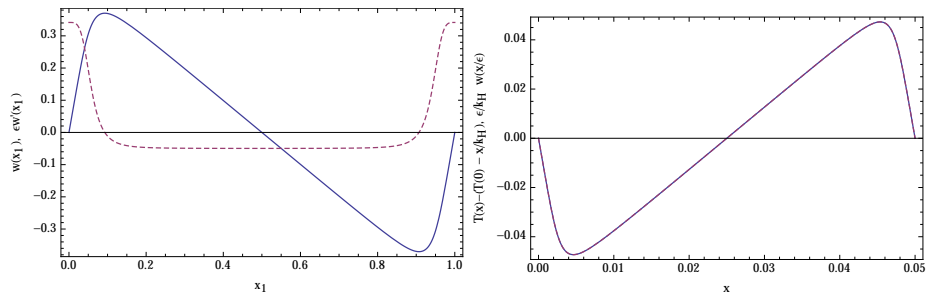


FIGURE 5 – Left : blue line the homogenized solution of the homogenized problem w in x^1 variables (betwee 0 and 1), red dashing is the solution $\varepsilon w'(x_1)$. Right the full solution for $0 < x < \varepsilon$ in which we subtract the homogenized solution $(1 - x)/k_H$, compared to the local solution (curves superposed at the small structure size : 0, 0.05).

4 Other examples

The method is general for elliptic PDEs, we have seen it in heat transfer, but it can be done in porous media or in electrostatics (see Sanchez-Palencia papers for various applications).

4.1 Darcy law

This theory may be applied to create an equivalent media (coordinates x_1^0, x_2^0, x_3^0) for a porous media (scale L). A porous media is a media constituted of a small structure repeated a lot of time. For exemple, the sand is a porous media, it is constituted of small grains of size of around $400\mu\text{m}$. If we are interested in rain infiltration, or mechanical properties for construction of a building on sand, we consider the sand at scale 1 m. at this scale, we do not see the grains, and consider a mean media. We call "pores" the small structure. At this small scale, we guess that viscosity is dominant for the flow. Looking at the slow flow in the small pores (x_1^1, x_2^1, x_3^1) of scale d , the Stokes equations are relevant, with no slip conditions :

$$\vec{\nabla} \cdot \vec{u} = 0 \quad 0 = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u} + \vec{f}$$

without dimension, if L is the macroscopic size, $\bar{x}_i = x/L$, $u_i = U_0 \bar{u}_i$ with $p = \mu U_0 / L \bar{p}$ and $f_i = \mu L^2 U_0 \bar{f}_i$. Note that in practice Π_x the pressure gradient is given. It is the good quantity so that the characteristic velocity follows from the pressure gradient : $U_0 = L^2 \Pi_x / \mu$. The Stokes equations :

$$\vec{\nabla} \cdot \vec{u} = 0 \quad 0 = -\vec{\nabla} \bar{p} + \vec{\nabla}^2 \vec{u} + \vec{f}$$

notice that d the small scale is such that $d = \varepsilon L$, we use this ε for the development

$$\bar{p} = \bar{p}^0 + \varepsilon \bar{p}^1 + \varepsilon^2 \bar{p}^2 + \dots$$

$$\bar{u} = \bar{u}^0 + \varepsilon \bar{u}^1 + \varepsilon^2 \bar{u}^2 + \dots$$

each function \bar{p}^k and \bar{u}^k is function of both the slow variables $\bar{x}_1^0, \bar{x}_2^0, \bar{x}_3^0$ and the fast $\bar{x}_1^1, \bar{x}_2^1, \bar{x}_3^1$. The macroscopic media is the "slow" $\bar{x}^0 = \bar{x}$ and the small structure is represented by $\bar{x}^1 = \bar{x}/\varepsilon$ the "fast" one... By chain rule derivative

$$\frac{\partial}{\partial \bar{x}_i} = \frac{\partial \bar{x}_i^0}{\partial \bar{x}_i} \frac{\partial}{\partial \bar{x}_i^0} + \frac{\partial \bar{x}_i^1}{\partial \bar{x}_i} \frac{\partial}{\partial \bar{x}_i^1}$$

the gradient will be

$$\frac{\partial}{\partial \bar{x}_i} = \frac{\partial}{\partial \bar{x}_i^0} + \frac{1}{\varepsilon} \frac{\partial}{\partial \bar{x}_i^1}$$

and the Laplacian

$$\frac{\partial^2}{\partial \bar{x}_i^2} = \frac{\partial^2}{\partial \bar{x}_i^{02}} + \frac{2}{\varepsilon} \frac{\partial}{\partial \bar{x}_i^1} \frac{\partial}{\partial \bar{x}_i^0} + \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial \bar{x}_i^{12}}$$

- at order ε^{-2} , Stokes equation gives $\frac{1}{\varepsilon^2} \frac{\partial}{\partial \bar{x}_i^1 \partial \bar{x}_i^1} u^0 = 0$, by no slip boundary condition on the solid and periodicity, it gives $\bar{u}^0 = 0$.

- at order ε^{-1} , Stokes equation gives

$$0 = -\frac{\partial}{\partial \bar{x}_i^1} \bar{p}^0 + \frac{\partial^2}{\partial \bar{x}_i^{12}} \bar{u}_i^1, \quad \text{and} \quad \frac{\partial}{\partial \bar{x}_i^1} \bar{u}_i^1 = 0,$$

so that again by BC $\bar{u}^1 = 0$. We deduce that $0 = -\frac{\partial}{\partial \bar{x}_i^1} \bar{p}^0 = 0$ so that pressure is independent of the fast variable. Pressure is only function of slow variable : $\bar{p}^0(\bar{x}_i^0)$.

- at order ε^0 momentum and incompressibility :

$$0 = -\frac{\partial}{\partial \bar{x}_i^1} \bar{p}^1 + \frac{\partial^2}{\partial \bar{x}_i^{12}} \bar{u}_i^2 + (\bar{f}_i - \frac{\partial}{\partial \bar{x}_i^0} \bar{p}^0), \quad \text{and} \quad \frac{\partial}{\partial \bar{x}_i^1} \bar{u}_i^2 = 0,$$

We note that $(\bar{f}_i - \frac{\partial}{\partial \bar{x}_i^0} \bar{p}^0)$ is a given source term. Given the periodic test function \bar{v} ,

$$0 = -\int \frac{\partial}{\partial \bar{x}_i^1} \bar{p}^1 \bar{v}_i d\bar{\tau}^1 + \int \frac{\partial^2}{\partial \bar{x}_i^{12}} \bar{u}_i^2 \bar{v}_i d\bar{\tau}^1 + \int (\bar{f}_i - \frac{\partial}{\partial \bar{x}_i^0} \bar{p}^0) \bar{v}_i d\bar{\tau}^1$$

The integrals correspond to the sum in the fluid part (where the fluid is let us call it Υ_f). Integrating by parts and by periodicity and no slip condition the pressure disappears

$$0 = -\int \frac{\partial}{\partial \bar{x}_i^1} \bar{u}_i^2 \frac{\partial}{\partial \bar{x}_i^1} \bar{v}_i d\bar{\tau}^1 + \int (\bar{f}_i - \frac{\partial}{\partial \bar{x}_i^0} \bar{p}^0) \bar{v}_i d\bar{\tau}^1$$

this allows us to exhibit the \bar{w}_{ij} tensor :

$$\bar{u}_i^2 = (\bar{f}_k - \frac{\partial}{\partial \bar{x}_k^0} \bar{p}^0) \bar{w}_{ik}$$

so that \bar{w}_{ij} is solution of :

$$\int \frac{\partial \bar{w}_{ki}}{\partial \bar{x}_i^1} \frac{\partial \bar{v}_k}{\partial \bar{x}_i^1} d\bar{\tau}^1 = \int \bar{v}_i d\bar{\tau}^1$$

the mean value $\langle \cdot \rangle$ is taken on the fluide volume of the cell $\bar{\Upsilon}_f$:

$$\langle \bar{u}_i^2 \rangle = \left(\frac{1}{mes\bar{\Upsilon}_f} \int \bar{w}_{ik} d\bar{\tau}^1 \right) \left(\bar{f}_k - \frac{\partial}{\partial \bar{x}_k^0} p^0 \right)$$

We note that if we define $\varphi = \left(\frac{mes\bar{\Upsilon}_f}{\bar{V}} \right)$ where \bar{V} is the total volume of the cell, then φ is the porosity of the material. It is the fraction of volume accessible by the fluid, it is the "porosity", the "compacity" $\phi = 1 - \varphi$.

Coming back with dimensions, L^2 and μ appear back, and remember that the velocity is at order ε^2 , then

$$u_i = \frac{\varepsilon^2 L^2}{\mu \varphi} \left[\frac{1}{\bar{V}} \int \bar{w}_{ik} d\bar{\tau}^1 \right] \left(f_k - \frac{\partial}{\partial x_k^0} p^0 \right)$$

we note that indeed $u_i = u_i^{fluid}$ is the velocity of the fluid (the grains are fixed) :

$$\varphi u_i^{fluid} = \frac{1}{\mu} \left[\frac{d^2}{\bar{V}} \int \bar{w}_{ik} d\bar{\tau}^1 \right] \left(f_k - \frac{\partial}{\partial x_k^0} p^0 \right)$$

where $\left(\frac{d^2}{\bar{V}} \int \bar{w}_{ik} d\bar{\tau}^1 \right)$ is a 3x3 tensor (ie a table of real numbers) corresponding to the integration at size εL , the detail of the small structure. as $d = \varepsilon L$

$$\varphi u^{fluide} = -\frac{1}{\mu} k_{ij} \left(-f_k + \frac{\partial}{\partial x_k^0} p^0 \right)$$

where k_{ij} is the permeability tensor obtained from the local problem (d is in the bar variables) :

$$k_{ij} = \left(\frac{1}{\bar{V}} \int w_{ik} d\tau^1 \right)$$

The Darcy Law as defined in books is with a q_i vector : the flux. The flux or is defined $q_i = \varphi u_i$, Darcy law is then :

$$q_i = -\frac{1}{\mu} k_{ij} \left(\frac{\partial}{\partial x_j^0} p^0 - f_j \right)$$

as the body force is often gravity $g_i = -\delta_{3i}g$:

$$q_i = -K_{ij} \left(\frac{\partial}{\partial x_j^0} \left(\frac{p^0}{\rho g} \right) + \delta_{3j} \right) \text{ with } K_{ij} = \rho g \frac{k_{ij}}{\mu}$$

k in m^2 is permeability, unit D, the Darcy, corresponds 1 cm/s velocity for a pressure gradient of 0,1 atmosphere per cm. Viscosity of water $10^{-3} Pas$ so 1D = $10^{-12} m^2$

Sand k is almost 1D, granite μD , gravels 100D.

K is hydraulic conductivity. H hydraulic head ; h pressure of water in the soil, $H = h + z$ or $H = p/(\rho g) + z$ charge hydraulique

4.2 Mixture

The Einstein law may be reobtained.

4.3 Elasticity

The Homogenisation allows to have effective media... there is a huge number of applications in Mechanics of solids.

5 Conclusion

The method of homogenisation is a power full tool used mostly in elasticity (materials with a micro structure, concrete, composites...). But, it is used in fluid mechanics as well, mainly for porous flows. This method was developed by various researchers in the 70s with significant contributions from Sanchez-Palenzia, It is well suited for variational problems and resolution with finite elements that is why it is a routine tool in solids. There are other methods to do the averaging, like "mean field method" (which is not so far in fact), and many others.

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FIGURE 6 – Homogenization for children, "L'homogénéisation? Un jeu d'enfant!" Evariste Sanchez Palencia, ~ 2010 in Jussieu

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This course is a part of a larger set of files devoted on perturbations methods, asymptotic methods (Matched Asymptotic Expansions, Multiple Scales) and boundary layers (triple deck) by *P.-Y. Lagrée* .

The web page of these files is <http://www.lmm.jussieu.fr/~lagree/COURS/M2MHP>.

/Users/pyl/ ... /homo.pdf



Raymond Subes "Sans Titre" 1961 (entrée de Jussieu Quai Saint Bernard)