KdV

waves, jumps, solitons & mascarets

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http://www.lmm.jussieu.fr/~lagree/COURS/M2MHP/kdv.pdf

Résumé

Free surface flows of water are clearly ubiquitous on Earth. As viscosity is small, the inviscid equations of water flow are presented. First, the case of small amplitude perturbations in small depth is presented, so linearized, this leads to the "d'Alembert equation" or "wave equation". Second, the case of small amplitude perturbations in any depth is presented, this allows to explain the dispersive behavior of water waves. This is called "Airy wave theory". Third the "shallow water theory" is presented, this corresponds to significant perturbations of the height of water, which is small compared to the length of the waves ; the equations are non linear ("Saint-Venant" equations). Then, if one considers in this latter framework small amplitude waves, and the first correction due to depth, one may have a balance between "non linearities" and "dispersion", or a balance between "steepening" and "spreading". This leads to the solitary wave solution of "KdV Equation" : the "soliton". The KdV (Korteweg–de Vries) equation is presented as an application of multiple scale analysis.

1 Introduction

1.1 Observations of the soliton

First observed by John Scott Russell in 1834, the "soliton" is a wave which has always the same shape even if it is not in the small perturbation regime. Russell was an engineer and scientist, he experimented Doppler effect with trains. He engineered and designed the "Great Eastern" the largest boat at the time in 1860. But, before he did many experiments on models. During one of those experiments, in 1834, on the Glasgow-Edinburgh channel, a wave was generated during the abrupt stop of the boat (drawn along a narrow channel and powered by horses). The wave moved with constant shape. He took his horse to follow it on several miles (see the text from Remoissenet [16], on page 319 of the `Report on Waves`, section I "The wave of translation"). He did after that a lot of experiments to reproduce it in a 30' wave tank in his back garden.

"This is a most beautiful and extraordinary phenomenon : the first day I saw it was the happiest day of my life. Nobody had ever had the good fortune to see it before or, at all events, to know what it meant. It is now known as the solitary wave of translation. No one before had fancied a solitary wave as a possible thing."

> ***I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped-not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, ...***
>
> **John Scott Russell, Report on Waves (1844)**

It was followed by other experiments by Henry Bazin and Henry Darcy and then by theoretical investigations by Joseph Boussinesq (≪ Théorie de l'intumescence liquide, appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire ≫, dans Comptes Rendus de l'Académie des Sciences, vol. 72, 1871, p. 755–759) and (≪ Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond ≫, dans Journal de Mathématique Pures et Appliquées, Deuxième Série, vol. 17, 1872, p. 55–108) and Rayleigh (1876) and, finally, Korteweg and De Vries in 1895. The equation that we will establish with asymptotic methods is :

$$\frac{\partial}{\partial \tau}\bar{\eta}_0 + \frac{3}{2}\bar{\eta}_0\frac{\partial \bar{\eta}_0}{\partial \xi} + \frac{1}{6}\frac{\partial^3 \bar{\eta}_0}{\partial \xi^3} = 0.$$

De Vries was the student of Korteweg, the title of the thesis *Bijdrage tot de kennis der lange golven*, in dutch "Contributions to the knowledge of long waves". Another famous student of Korteweg is A. Moens. They proposed the velocity in arterial flow "Moens-Korteveg" equation (arterial flows and water flows are very similar ; the elasticity of arteries is the gravitation in water flows). The KdV equation was not studied much after this until Fermi- Pasta- Ulam and Zabusky & Kruskal (1965). They wanted to study the heat transfer in a solid consisting in a crystal modeled by masses en springs in a periodic domain (one dimensional lattice). They discovered traveling waves which were not damped.

They rediscoverd the soliton.

FIGURE 1 – The Soliton reproduced in 1995 on the very same place than Scott Russell 'first' observed a solitary wave on the Union Canal near Edinburgh in 1834. (Photo from Nature v. 376, 3 Aug 1995, pg 373) `http://www.ma.hw.ac.uk/solitons/press.html`

See Dauxois, Newell [13], Remoissenet [16] and Maugin [15] for other historical details and a view of the fields of application. The fields are very large, from hydrodynamics, lattice waves, waves in electric lines, light waves in optic cables *etc.* but as Feynman says : *"Now, the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light ; they have all the complications that waves can have."* Lectures on Physics, chapitre 51-4 "Ondes".

1.2 Scope of the lecture : heuristical point of view, small dispersion on Shallow-Water

1.2.1 Considerations on equations

Waves in fluid are classical in mechanics of fluid courses (see M1 lecture by the same author [18] and [19]). What we want to do here is to restart from scratch this

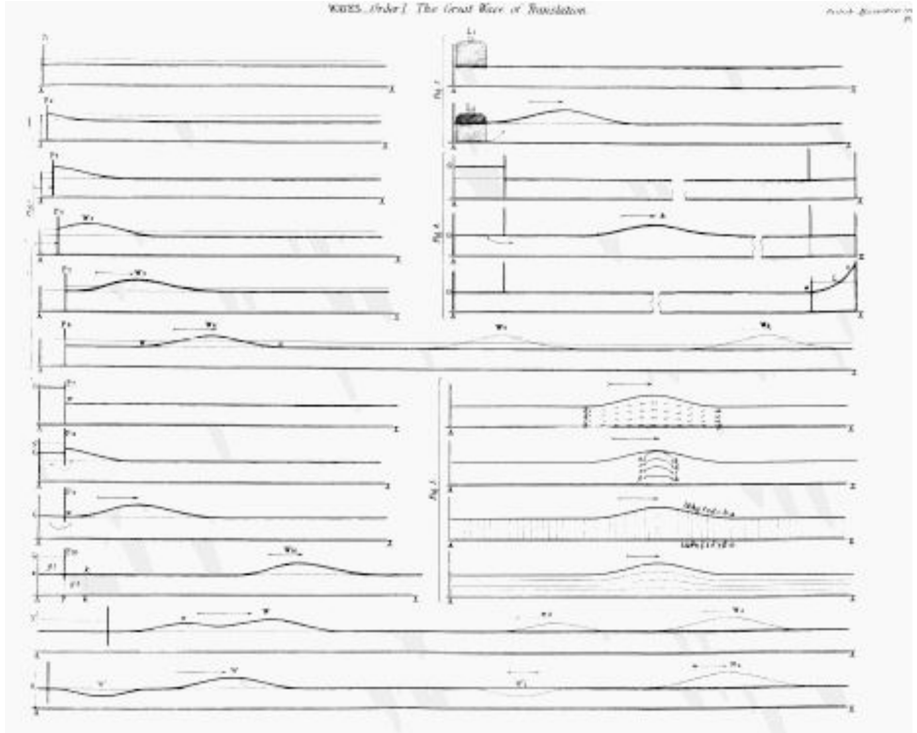

FIGURE 2 – The original sketches of Scott Russel. "The great wave of translation" `http://www.ma.hw.ac.uk/~chris/Scott-Russell/SR44.pdf`. The wave may be generated by a moving wall, top left or a falling weight top right, or opening a gate. The final result is a unique wave which translates without change in shape on very long distance.

study with a unified point of view in order to recover the ∂'Alembert Equation, the Saint -Venant and the Airy Wave theory all in the same theoretical framework. For instance, the ∂'Alembert Equation, is heuristically proved as follows, we suppose a plug flow $u(x,t)$, its acceleration is $\rho\partial u/\partial t$, the forces exerted are the pressure ones. Due to the elevation of the wave η (over the free level $y=0$), the pressure is simply $\rho g \eta$ for hydrostatic reason (we will see that in far more details later or see [18]). Then

$$\rho\frac{\partial u}{\partial t} = -\rho g\frac{\partial \eta}{\partial x}.$$

The thickness of the fluid layer is h_0, and η is smaller than h_0, the other needed equation is the conservation of mass, as there is no v in the equations, changes of the flux of u : $\frac{\partial h_0 u}{\partial x}$ are compensated by the displacement of the interface (see why in [18]) :

$$\frac{\partial \eta}{\partial t} + \frac{\partial h_0 u}{\partial x} = 0.$$

At this point, to eliminate u, one needs the above mentioned hypothesis : the velocity is almost constant across the layer (it looks like a plug flow, or in other words, the flow is irrotational). This gives the wave equation (∂'Alembert Equation) :

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0 \frac{\partial^2 \eta}{\partial x^2} = 0$$

with a celerity $c_0 = \sqrt{gh_0}$ and solution in $f(x - c_0 t) + g(x + c_0 t)$. To go further in complexity, we can put some non linearities in these equations but still suppose a plug flow ($\partial_y u = 0$). The acceleration is then $\rho \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}$, and flux will be $(h_0 + \eta)u$, this will give the Saint-Venant (Shallow Water) equations :

$$\rho \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\rho g \frac{\partial \eta}{\partial x}, \text{ and } \frac{\partial \eta}{\partial t} + \frac{\partial h u}{\partial x} = 0.$$

Instead of this description, we can imagine a linear small perturbation, with a flow which is no more a plug flow, but depends on x and y : $u(x, y, t)$, this will give the famous dispersion relation (again we will settle this again, see §3.3, or see as well [19]) :

$$\omega^2 = gk \tanh(kh_0).$$

For long waves, this equation gives again the ∂'Alembert celerity $c_0 = \omega/k = \sqrt{gh_0}$.

1.2.2 Adding independantly non linearities and dispersion

Having in mind those equations describing the waves, then the solitary wave will be presented as a mix of all this dispersion, kh_0 small and non linearity, η/h_0 small that we will look at now. There are simple argument to settle at this point KdV such as :

• from the wave equation the solution is $u = g\eta/c_0$, with $c_0^2 = gh_0$
from the wave equation the wave which travels from left to right, this solution satisfies

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} = 0$$

• from the linear Wave Theory we have the dispersion relation $\omega = \sqrt{gk \tanh(kh_0)}$, this is expanded in $\omega = \sqrt{gk^2 h_0 - \frac{1}{3} g k^4 h_0^3 + ...}$ for long waves, which is at order two : $\omega = \sqrt{gh_0} k(1 - \frac{1}{6} k^2 h_0^2 + ...)$, then for a wave $\eta = \eta_0 e^{i\omega t - ikx}$, as $-i\frac{\partial}{\partial t}\eta = \omega\eta$, we identify ω with $-i\frac{\partial}{\partial t}$, and k with $i\frac{\partial}{\partial x}$ therefore

$$-i\frac{\partial}{\partial t} = ic_0\left(\frac{\partial}{\partial x} - i^2 \frac{1}{6} h_0^2 \frac{\partial^3}{\partial x^3}\right) + ...$$

so this following equation has dispersion relation $\omega = \sqrt{gh_0} k(1 - \frac{1}{6} k^2 h_0^2)$:

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + c_0 \frac{1}{6} h_0^2 \frac{\partial^3 \eta}{\partial x^3} = 0.$$

• Shallow water equations may be re written in a different way. We define $c^2 = gh$, so $2cdc = gdh$, momentum is written with this new variable c :

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x} u + 2c \frac{\partial c}{\partial x} = 0$$

and mass conservation multiplied by g :

$$g\frac{\partial h}{\partial t} + ug\frac{\partial h}{\partial x} + gh\frac{\partial u}{\partial x} = 0 \text{ which is } 2c\frac{\partial c}{\partial t} + 2uc\frac{\partial c}{\partial x} + c^2\frac{\partial u}{\partial x} = 0 \text{ or : } \frac{\partial(2c)}{\partial t} + u\frac{\partial(2c)}{\partial x} + c\frac{\partial u}{\partial x} = 0$$

if we add and substract these equations with u et c, we obtain (more details in [18]) :

$$\left[\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right](u + 2c) = 0 \text{ and } \left[\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right](u - 2c) = 0.$$

This shows that along the lignes in the plane x, t defined by $dx/dt = u \pm c$ we have integrals of $u \pm 2c$ constants. Those lines are called "characteristics", and the integrals $u \pm 2c$ are the "Riemann invariants".

For a wave going to the right $(u - 2\sqrt{gh})$ is constant. If the surface is unperturbed far away ($u = 0$, $h = h_0$), then u is obtained thanks to conservation of the Riemann invariant :

$$u = 2\sqrt{gh} - 2\sqrt{gh_0}$$

the mass conservation :

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0$$

with $\eta + h_0 = h$ (η perturbation of free surface) and the previous u

$$\frac{\partial \eta}{\partial t} + (2\sqrt{g(h_0 + \eta)} - 2\sqrt{gh_0})\frac{\partial \eta}{\partial x} + (h_0 + \eta)\sqrt{(g)/((h_0 + \eta))}\frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial \eta}{\partial t} + (3\sqrt{g(h_0 + \eta)} - 2\sqrt{gh_0})\frac{\partial \eta}{\partial x} = 0$$

Linearisation around h_0 at small η :

$$(3\sqrt{g(h_0 + \eta)} - 2\sqrt{gh_0}) = \sqrt{gh_0}(3(1 + \frac{\eta}{2h_0} + ... - 2) = \sqrt{gh_0}(1 + \frac{3\eta}{2h_0} + ...)$$

The final (inviscid Bürgers) equation :

$$\frac{\partial \eta}{\partial t} + \sqrt{gh_0}(1 + \frac{3\eta}{2h_0})\frac{\partial \eta}{\partial x} = 0.$$

This equation leads to shocks as the higher the wave the faster it is.
• Then the final equation of perturbation of the right moving wave $\frac{\partial\eta}{\partial t}+c_0\frac{\partial\eta}{\partial x}=0$ is estimated as the sum of the two effects, the nonlinear steepening $c_0\frac{3\eta}{2h_0}\frac{\partial\eta}{\partial x}$ and the dispersive spreading $c_0\frac{1}{6}h_0^2\frac{\partial^3\eta}{\partial x^3}$, this is the KdV equation :

$$\frac{\partial\eta}{\partial t}+c_0\frac{\partial\eta}{\partial x}+c_0\frac{3\eta}{2h_0}\frac{\partial\eta}{\partial x}+c_0\frac{1}{6}h_0^2\frac{\partial^3\eta}{\partial x^3}=0.$$

We will use the longitudinal scale, say λ such that the dispersive term is small, it is $(h_0/\lambda)^2 \ll 1$ and the non linear term is small as well $\eta/h_0 \ll 1$, but :

$$\eta/h_0=(h_0/\lambda)^2 \ll 1$$

The Ursell number is the ratio $\frac{\eta/h_0}{(h_0/\lambda)^2}=\frac{\eta\lambda^2}{h_0^3}$. We will present the complete theory with small parameters $(h_0/\lambda)^2 \ll 1$ and $\eta/h_0 \ll 1$, (we will define $\delta = h_0/\lambda$ and $\varepsilon = \eta_0/h_0$ the small parameters) and using dominant balances and multiple scale. This, unfortunately, makes the ∂'Alembert and Shallow Water description less clear (the following pages are then obscure... maybe the previous ones as well...). That is the price to catch the Soliton.

So, we write the full system without dimension, and we show it contains Wave Equation, Shallow Water and Linear Wave theory of arbitrary depth. Those three "simple" solutions of the full problem will guide us to find the proper scales in terms on $\delta = h_0/\lambda$ and $\varepsilon = \eta_0/h_0$ only. So we will fight to find the final system (10) with δ and ε only. The developments to find the final system (10) are maybe obscure, but once we found the proper dominant balances, (10) is enlightening and obscurity goes away. Starting from this system, we show again that it contains Wave Equation (∂'Alembert), Shallow Water (Saint-Venant) and Linear Wave Theory of arbitrary depth (Airy).

The purpose of this lecture is to join three names ∂'Alembert, Saint-Venant, Airy and three models when $\varepsilon = \delta^2$ (non linearity balnces dispersion) in a new model : the Korteweg De Vries equation.

Let us do the reset/RAZ/ Crl Alt Del

2 Equations

2.1 Equations with dimensions

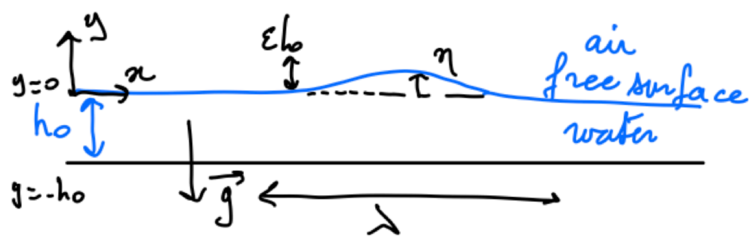

FIGURE 3 – Notations, $y = 0$ is the unperturbed level of water, h_0 depth of unperturbed water, λ characteristic length of the wave, δ ratio of these quantities; ε relative amplitude of the wave.

Let us do the reset. We start from scratch : Navier Stokes and try to identify all the small parameters to obtain KdV. Write Navier Stokes, without dimension, put $Re = \infty$ come back with dimensions, here is Euler incompressible and irrotational (remember conservation of vorticity in ideal fluids) :

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \text{ and } (\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x})=0$$

$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})=-\frac{\partial p}{\partial x},\quad \rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y})=-\frac{\partial p}{\partial y}-\rho g,$$

notice the irrotational hypothesis which is very strong, and which is not so much discussed in the literature. These equation are for $-h_0 < y < \eta$. Boundary conditions are the pressure $p(x, y = \eta) = p_0$ at the surface (we neglect here surface tension) and the relation linking the perturbation of the moving interface and the velocity of the water just at the surface :

$$v(x,\eta,t)=\frac{\partial\eta(x,t)}{\partial t}+u(x,\eta,t)\frac{\partial\eta(x,t)}{\partial x}$$

and slip conditions at $y = -h_0$.

2.2 Simple linearisation : ∂'Alembert equation

By simple linearisation around a basic state...

the scaling for x is say λ, the scaling for y is h_0, time is τ, U_0, V_0...
let us define ε the small parameters related to the variation of the water level εh_0.
let us define δ the small parameters ration of h_0 by λ

pressure variation is $\varepsilon\rho g h_0$

velocity at the surface

$$V_0 = \frac{\varepsilon h_0}{\tau}$$

incompressibility

$$\frac{U_0}{\lambda} = \frac{V_0}{h_0}$$

this gives

$$U_0 = \lambda\frac{V_0}{h_0} = \frac{\varepsilon\lambda}{\tau}$$

momentum, with $\bar{P}$ deviation from hydrostatic pressure at the surface $\bar{P} = \eta$

$$\rho\frac{U_0}{\tau} = \frac{\varepsilon\rho g h_0}{\lambda} \text{ then } \varepsilon\frac{\lambda^2}{\tau^2} = \varepsilon g h_0 \text{ then } \frac{\lambda}{\tau} = \sqrt{g h_0}$$

non linear terms are $O(\varepsilon)$

The scaling for u is $\varepsilon\sqrt{gh_0}$ and for v is $\varepsilon(h_0/\lambda)\sqrt{gh_0}$. Potential $\varepsilon\lambda\sqrt{gh_0}$ ...
momentum, with $\bar{P}$ deviation from hydrostatic pressure at the surface $\bar{P} = \eta$

$$\frac{\partial\bar{u}}{\partial\bar{t}} = -\frac{\partial\bar{P}}{\partial\bar{x}}$$

or $\frac{\partial\bar{u}}{\partial\bar{t}} = -\frac{\partial\bar{\eta}}{\partial\bar{x}}$, and

$$(\frac{h_0}{\lambda})^2\frac{\partial\bar{v}}{\partial\bar{t}} = \frac{\partial\bar{P}}{\partial\bar{y}}$$

as $\eta(x,t)$ $\bar{u}$ is not a function of y irrotationality is a result of $\delta = (h_0/\lambda)$ small. Hence $\bar{v} = -(\bar{y}+1)\frac{\partial\bar{u}}{\partial\bar{x}}$ and at surface $\bar{v} = -1\frac{\partial\bar{u}}{\partial\bar{x}} = \frac{\partial\bar{\eta}}{\partial\bar{t}}$ ...

as $\frac{\partial\bar{u}}{\partial\bar{t}} = -\frac{\partial\bar{\eta}}{\partial\bar{x}}$, and $\frac{\partial\bar{u}}{\partial\bar{x}} = -\frac{\partial\bar{\eta}}{\partial\bar{t}}$ we obtain ∂'Alembert equation...

$$\frac{\partial^2\bar{\eta}}{\partial\bar{x}^2} - \frac{\partial^2\bar{\eta}}{\partial\bar{t}^2} = 0$$

This is a fast way to find the scaling, we will do the same next but using the equations with potential.
We will play with the two parameters ε and δ.

2.3 Equations with the potential

2.3.1 Finding the Equations

From irrotational flow, one usually define a potential :

$$u = \frac{\partial\phi}{\partial x};\ v = \frac{\partial\phi}{\partial y}$$

and from incompressibility :

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0,$$

this Laplacian must be solved with boundary conditions at the free surface and at the bottom. At the free surface $v(x, y = \eta, t) = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x}$ which is

$$\frac{\partial\phi}{\partial y}|_{y=\eta} = \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}$$

and writing the momentum with irrotationality

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = \rho(\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} + \frac{1}{2}\frac{\partial v^2}{\partial x} + v(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}))$$

gives a kind of Bernoulli equation :

$$\rho(\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2)) = -\frac{\partial p}{\partial x},$$

with the pressure at the surface $p(x, y = \eta) = p_0$. Note here that the real jump relation in inviscid flow between two media 1 and 2 may be written with the surface tension :

$$p_1(x, \eta(x,t), t) - p_2(x, \eta(x,t), t) = \sigma\overrightarrow{\nabla}\cdot\overrightarrow{n}_{12}$$

the normal to the surface

$$\overrightarrow{n}_{12} = (-\partial\eta/\partial x, 1)/\sqrt{(1 + (\partial\eta/\partial x)^2}$$

Here we neglect $\sigma/(\rho U_0^2 h_0)$ (the inverse of the Weber number $We = (\rho U_0^2 h_0)/\sigma$).
At equilibrium, when there is no flow

$$p = p_0 + \rho g(-y)$$

Then $p_0 + \rho g(-y)$ is the "hydrostatic" pressure, we look at P the departure from it. Pressure is then

$$p = p_0 + \rho g(-y) + P.$$

At the interface

$$p(x, y = +\eta) = p_0, \text{ so } p(x, y = \eta) = p_0 = p_0 + \rho g(-\eta) + P(x, y = \eta)$$

We hence deduce

$$P(x, y = \eta) = \rho g \eta.$$

The active part of the pressure is related to variations of interface η :

$$\frac{\partial p}{\partial x} = \frac{\partial P}{\partial x} = \rho g \frac{\partial \eta}{\partial x}$$

At the free surface, the previous momentum with irrotationality and the pressure :

$$\rho \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + g\eta\right) = 0.$$

2.3.2 Equations

The final system that we have to solve is the Laplace equation for the potential ϕ in the domain of water :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

with an "unsteady Bernoulli" with potential ϕ at the free surface $y = \eta$:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}^2 + \frac{\partial \phi}{\partial y}^2\right) + g\eta = 0,$$

at the free surface we have as well $v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}$ which is the relation between potential and surface elevation η :

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}.$$

On bottom $y = -h_0$ $v = 0$ which is the slip condition

$$\frac{\partial \phi}{\partial y} = 0.$$

Far usptream, and maybe down stream every thing is 0. This is the full system we have to solve. note that the surface is an unknown of the problem, and that it is defined with a non linear equation.

First we write it without dimension.

2.4 Equations without dimension

It is here a good idea to distinguish between scale in x and scale in y, so we write $x = \lambda \bar{x}$ and $y = h_0 \bar{y}$, the surface elevation is $\eta = \eta_0 \bar{\eta}$. The potential is not known, as the time : $\phi = \varphi \bar{\phi}$, $t = \tau \bar{t}$. We define the ratio of scales $\delta = h_0/\lambda$ and $\varepsilon = \eta_0/h_0$. Then incompressibility :

$$\delta^2 \frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\phi}}{\partial \bar{y}^2} = 0 \quad (1)$$

momentum at the surface

$$\varepsilon \bar{\eta} + \frac{\varphi}{g\tau h_0}\frac{\partial \bar{\phi}}{\partial \bar{t}} + \frac{\varphi^2}{2gh_0^3}\left(\delta^2 \frac{\partial \bar{\phi}}{\partial \bar{x}}^2 + \frac{\partial \bar{\phi}}{\partial \bar{y}}^2\right) = 0 \quad (2)$$

velocity at the surface

$$\frac{\varphi \tau}{\varepsilon h_0^2}\frac{\partial \bar{\phi}}{\partial \bar{y}} = \frac{\partial \bar{\eta}}{\partial \bar{t}} + \frac{\varphi \tau \delta^2}{h_0^2}\frac{\partial \bar{\phi}}{\partial \bar{x}}\frac{\partial \bar{\eta}}{\partial \bar{x}} \quad (3)$$

at the bottom $\bar{y} = -1$

$$\frac{\partial \bar{\phi}}{\partial \bar{y}} = 0$$

we have to find the relations between :

$$\varphi, \tau, \lambda, \varepsilon \text{ and } \delta.$$

We will first explore three simplifications of this system (∂'Alembert, Airy swell, Saint-Venant). Those three simplifications will help us to understand two different regimes and find the pertinent system (the one with a box around 10).

As we do the same thing several times, this complicates the lecture, so **the reader may skip to read §4** and then come back here to be sure that the scales are OK.

3 Solutions of the Equations $\varepsilon \to 0$ and/or $\delta \to 0$

3.1 Fully linearised waves $\varepsilon \to 0$ and $\delta \to 0$

The first simple case is the case of fully linear waves in a shallow water, this leads to the ∂'Alembert equation as we just will see. First we start from the Laplacian Eq. 1

$$\frac{\partial^2 \bar{\phi}}{\partial \bar{y}^2} = -\delta^2 \frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2},$$

so $\frac{\partial^2\bar\phi}{\partial\bar y^2}=0$ at leading order, with a Poincaré expansion $\bar\phi=\bar\phi_0+\delta^2\bar\phi_1+\delta^4\bar\phi_2+...$ at first oder

$$\frac{\partial^2\bar\phi_0}{\partial\bar y^2}=0,\ \text{with in } \bar y=-1,\ \ \frac{\partial\bar\phi}{\partial\bar y}=0$$

so that $\phi_0(\bar x,\bar t)=f(\bar x,\bar t)$, we define $f'=\partial_{\bar x}\phi_0$, we note that $\phi(\bar x,\bar t)$ is only a function of $(\bar x,\bar t)$ at dominant order. If we now put the $O(\delta^2)$ term,

$$\frac{\partial^2\bar\phi_1}{\partial\bar y^2}=-\frac{\partial^2\bar\phi_0}{\partial\bar x^2}.$$

The solution at order 1, with $\frac{\partial\bar\phi_1}{\partial\bar y}|_{-1}=0$:

$$\frac{\partial^2\bar\phi_1}{\partial\bar y^2}=-f''\ \text{gives}\ \frac{\partial\bar\phi_1}{\partial\bar y}=-(\bar y+1)f''$$

from the value at the bottom -1 we have the expression of the transverse velocity, we note that this velocity is very small $O(\delta^2)$:

$$\frac{\partial\bar\phi}{\partial\bar y}=\frac{\partial\bar\phi_0}{\partial\bar y}+\delta^2\frac{\partial\bar\phi_1}{\partial\bar y}...=0-\delta^2\frac{\partial^2\bar\phi_0}{\partial\bar x^2}(\bar y+1)...$$

At this point, we have obtained the expression of the transverse velocity, we note that this velocity is very small $O(\delta^2)$:

$$\frac{\partial\bar\phi}{\partial\bar y}=\delta^2\frac{\partial\bar\phi_1}{\partial\bar y}=-\delta^2\frac{\partial^2\bar\phi_0}{\partial\bar x^2}(\bar y+1).$$

We have to look at the surface, it is in $\bar y=\varepsilon\eta$, as ε is small, this is $\bar y=0$. This is "flattening" of boundary conditions.

$$\frac{\partial\bar\phi}{\partial\bar y}=-\delta^2\frac{\partial^2\bar\phi_0}{\partial\bar x^2}(0+1).$$

The domain is $-1\le\bar y\le 0$. At the surface $\bar y=0$, the gradient is the variation of interface, hence equation 3 gives

$$\frac{\partial\bar\phi}{\partial\bar y}=\frac{\varepsilon h_0^2}{\varphi\tau}\frac{\partial\bar\eta}{\partial\bar t}+\delta^2\varepsilon\frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x}$$

Looking at those two expressions of $\frac{\partial\bar\phi}{\partial\bar y}$ by dominant balance $\delta^2=\frac{\varepsilon h_0^2}{\varphi\tau}$, and the last one is

$$\frac{\partial\bar\phi}{\partial\bar y}=\frac{\varepsilon h_0^2}{\varphi\tau}(\frac{\partial\bar\eta}{\partial\bar t}+\varepsilon\frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x})$$

we can neglect the non linear term

$$\frac{\partial\bar\phi}{\partial\bar y}=-\delta^2\frac{\partial^2\bar\phi_0}{\partial\bar x^2}(0+1)=\delta^2\frac{\partial\bar\eta}{\partial\bar t}+O(\delta^2\varepsilon)$$

so that and as by convention $\bar\phi_0=f$

$$\frac{\partial^2\bar f}{\partial\bar x^2}=\frac{\partial\bar\eta}{\partial\bar t}.$$

The momentum at the surface (Eq. 2) is (as we note that in $\delta^2\frac{\partial\bar\phi}{\partial\bar x}^2+\frac{\partial\bar\phi}{\partial\bar y}^2$, the first is $O(\delta^2)$ while the second is smaller $O(\delta^4)$) :

$$\varepsilon\bar\eta+\frac{\varphi}{g\tau h_0}\frac{\partial\bar\phi}{\partial\bar t}+O(\delta^2\frac{\varphi^2}{2gh_0^3})=0,$$

by dominant balance $\varphi=\varepsilon g\tau h_0$, gives (with remember $\bar\phi=\bar\phi_0+\delta^2\bar\phi_1+...$) :

$$\bar\eta+\frac{\partial\bar\phi}{\partial\bar t}+\ \text{smaller terms}=0,\ \text{or at dominant order}\ \bar\eta+\frac{\partial f}{\partial\bar t}=0.$$

So eliminating φ between this $\varphi=\varepsilon g\tau h_0$ and the previous $\delta^2=(\varepsilon h_0^2)/(\varphi\tau)$ and using the relevant scales $\lambda=h_0/\delta\gg h_0$, gives

$$\lambda/\tau=\sqrt{gh_0}.$$

Then, the non linear or smaller terms of (Eq. 2) are $O(\delta^2\frac{\varphi^2}{2gh_0^3})$ which is $O(\frac{\varepsilon^2gh_0\tau^2}{2\lambda^2})=O(\varepsilon^2)$, which is indeed small as claimed.

The expression $\sqrt{gh_0}$ is a velocity (say $c_0=\sqrt{gh_0}$). Hence $\varphi=\varepsilon\lambda\sqrt{gh_0}$. Then, eliminating f gives :

$$\frac{\partial^2 f}{\partial\bar x^2}=\frac{\partial^2 f}{\partial\bar t^2}\ \ \text{and}\ \ \frac{\partial^2\bar\eta}{\partial\bar x^2}=\frac{\partial^2\bar\eta}{\partial\bar t^2}$$

the ∂'Alembert wave equation of unit velocity (which is $c_0=\sqrt{gh_0}$ with dimensions). This is valid for waves of small amplitude $\varepsilon\ll 1$ in shallow water $\delta\ll 1$. In the next paragraph, we will study waves of small amplitude $\varepsilon\ll 1$ in deep water $\delta=1$. In the paragraph after, we will study waves in not small amplitude $\varepsilon=1$ in shallow water $\delta\ll 1$. Finally we will study waves of small amplitude $\varepsilon\ll 1$ in shallow water $\delta\ll 1$ but no so shallow.

For those various waves, we will follow the same way : do some dominant balance to take some terms, integrate the Laplacian from bottom (slip) to top (perturbation of free surface), which gives in fact the small transverse velocity and use of the momentum/ Bernoulli equation.

3.2 Scaling with ε,δ

This ∂'Alembert equation helps us to find the scalings. Having defined $\delta=h_0/\lambda$ and $\varepsilon=\eta_0/h_0$, we write $x=\lambda\bar x$ and $y=\delta\lambda\bar y$, the surface elevation is $\eta=\varepsilon\delta\lambda\bar\eta$. The incompressibility :

$$\delta^2\frac{\partial^2\bar\phi}{\partial\bar x^2}+\frac{\partial^2\bar\phi}{\partial\bar y^2}=0 \tag{4}$$

tells us that

$$\frac{\partial^2\bar\phi}{\partial\bar y^2} = O(\delta^2) \text{ or } \frac{1}{\delta^2}\frac{\partial\bar\phi}{\partial\bar y} = O(1). \tag{5}$$

This scaling was obtained in the linearised case (∂'Alembert). This scaling is a bit surprising, but comes from asymptotics. This is substituted in the velocity at the surface, as

$$\frac{\varphi\tau}{\varepsilon h_0^2}\frac{\partial\bar\phi}{\partial\bar y} = \delta^2\frac{\varphi\tau}{\varepsilon h_0^2}[\frac{1}{\delta^2}\frac{\partial\bar\phi}{\partial\bar y}] = \frac{\partial\bar\eta}{\partial\bar t} + \frac{\varphi\tau\delta^2}{h_0^2}\frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x} \tag{6}$$

the solution of the Laplace equation ($[\frac{1}{\delta^2}\frac{\partial\bar\phi}{\partial\bar y}] = O(1)$) suggested a dominant balance $\delta^2\frac{\varphi\tau}{\varepsilon h_0^2} = 1$. Then, the non linear term follows it is $\frac{\varphi\tau\delta^2}{h_0^2} = \varepsilon$. The equation is finally with δ and ε only :

$$\frac{1}{\delta^2}\frac{\partial\bar\phi}{\partial\bar y} = \frac{\partial\bar\eta}{\partial\bar t} + \varepsilon\frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x}. \tag{7}$$

Momentum at the surface,

$$\varepsilon\bar\eta + \frac{\varphi}{g\tau h_0}\frac{\partial\bar\phi}{\partial\bar t} + \frac{\varphi^2}{2gh_0^3}(\delta^2\frac{\partial\bar\phi}{\partial\bar x}^2 + \frac{\partial\bar\phi}{\partial\bar y}^2) = 0. \tag{8}$$

The scaling s obtained in the linearised case (∂'Alembert) is $\varepsilon = \frac{\varphi}{g\tau h_0}$ (pertubation of η are of same magnitude than $\frac{\partial\bar\phi}{\partial\bar t}$) : As seen just before $\frac{\varphi\tau}{\varepsilon h_0^2}\delta^2 = 1$, so that the non linear term :

$$\frac{\varphi^2}{gh_0^3} = \frac{\varphi}{g\tau h_0}(\frac{\varphi\tau}{h_0^2}) = \varepsilon(\frac{\varphi\tau}{h_0^2}) = \varepsilon(\varepsilon\frac{1}{\delta^2}).$$

Then, momentum at the surface reads :

$$\varepsilon\bar\eta + \varepsilon\frac{\partial\bar\phi}{\partial\bar t} + \frac{\varepsilon^2}{2\delta^2}(\delta^2\frac{\partial\bar\phi}{\partial\bar x}^2 + \frac{\partial\bar\phi}{\partial\bar y}^2) = 0 \tag{9}$$

As $\frac{\varphi\tau}{\varepsilon h_0^2}\delta^2 = 1$ and $\varepsilon = \frac{\varphi}{g\tau h_0}$, we have :

$$\lambda/\tau = \sqrt{gh_0}, \text{ and } \varphi = \varepsilon\lambda^2/\tau = \varepsilon\lambda\sqrt{gh_0}$$

this is as well

$$\tau = \sqrt{\frac{h_0}{\delta^2 g}}, \text{ and } \varphi = \varepsilon\sqrt{\frac{gh_0^3}{\delta^2}},$$

The final equations without dimension with $\delta = h_0/\lambda$ and $\varepsilon = \eta_0/h_0$, and with no approximations

$$\boxed{\begin{array}{rcl} \delta^2\frac{\partial^2\bar\phi}{\partial\bar x^2} + \frac{\partial^2\bar\phi}{\partial\bar y^2} &=& 0,\\ \frac{1}{\delta^2}\frac{\partial\bar\phi}{\partial\bar y} &=& \frac{\partial\bar\eta}{\partial\bar t} + \varepsilon\frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x},\\ \bar\eta + \frac{\partial\bar\phi}{\partial\bar t} + \frac{\varepsilon}{2}(\frac{\partial\bar\phi}{\partial\bar x}^2 + \frac{1}{\delta^2}\frac{\partial\bar\phi}{\partial\bar y}^2) &=& 0,\\ \frac{\partial\bar\phi}{\partial\bar y}|_{\bar y=-1} &=& 0. \end{array}} \tag{10}$$

3.3 Linear dispersive solution, Airy 1845, $\varepsilon \to 0, \delta = O(1)$

We look here at the swell (*houle* in French) in open sea. This is a very classical solution (Lamb Chap IX, Landau & Lifshitz §12, etc) introduced by George Biddell Airy of the Linear Wave theory with arbitrary depth (between 1841 and 1845). The length and the depth are of same order. So we consider $\delta = 1$, we take the same scales in x and y, it is more simple to use h_0 here, so that the bottom will be in $\bar y = -1$. The equation (1) is then a full Laplacian. Taking both scales equals is of course the first simple possibility that we have to explore before looking at different scales. We consider small amplitude waves so that $\varepsilon \ll 1$ then from the same balance of the velocity at the surface (Eq (3)) and from the momentum (2) we have $\tau = \sqrt{h_0/g}$ then $\varphi = \varepsilon\sqrt{gh_0^3} = \eta_0\sqrt{gh_0}$

$$\frac{\partial^2\bar\phi}{\partial\bar x^2} + \frac{\partial^2\bar\phi}{\partial\bar y^2} = 0$$

$$\bar\eta + \frac{\partial\bar\phi}{\partial\bar t} + \varepsilon^2(\frac{\partial\bar\phi}{\partial\bar x}^2 + \frac{\partial\bar\phi}{\partial\bar y}^2) = 0 \text{ and as well at surface } \frac{\partial\bar\phi}{\partial\bar y} = \frac{\partial\bar\eta}{\partial\bar t} + \varepsilon\frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x}$$

at the bottom

$$\bar y = -1, \quad \frac{\partial\bar\phi}{\partial\bar y} = 0.$$

If course this is system (10) for $\delta = 1$.

We look at the solution of the linearised problem

$$\left\{\begin{array}{ll} \frac{\partial^2\bar\phi}{\partial\bar x^2} + \frac{\partial^2\bar\phi}{\partial\bar y^2} & = 0\\ \frac{\partial^2\bar\phi}{\partial\bar t^2}|_0 + \frac{\partial\bar\phi}{\partial\bar y}|_0 & = 0\\ \frac{\partial\bar\phi}{\partial\bar y}|_{-1} & = 0 \end{array}\right.$$

This problem as solution in $e^{i(\bar{k}\bar{x}-\bar{\omega}\bar{t})}$, with this ansatz, the solution of the Laplacian,

$$-\bar{k}^2\bar{\phi} + \frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} = 0 \text{ with B.C. } \frac{\partial\bar{\phi}}{\partial\bar{y}}|_{-1} = 0, \quad \frac{\partial\bar{\phi}}{\partial\bar{y}}|_0 = \bar{\omega}^2\bar{\phi}(0).$$

This gives a solution in $\bar{\phi}(0)\cosh(\bar{k}\bar{y}+\bar{k})/\cosh(\bar{k})$ which preserves the $\frac{\partial\bar{\phi}}{\partial\bar{y}}|_{-1} = 0$ boundary condition. So $\frac{\partial\bar{\phi}}{\partial\bar{y}}|_0 = \bar{\phi}(0)\bar{k}\tanh\bar{k}$ then, it gives the famous relation of dispersion :

$$\bar{\omega}^2 = \bar{k}\tanh(\bar{k})$$

with dimensions

$$\omega^2 = gk\tanh(kh_0),$$

the phase velocity $c(k) = \omega/k$ is function of k, this means that a signal will be changed as every space frequency has a different velocity :

$$c(k) = \sqrt{g\tanh(kh_0)/k}.$$

Looking a small depth

$$\omega^2 = gk(kh_0 - \frac{(kh_0)^3}{3} + ...)$$

This $\frac{(kh_0)^3}{3}$ will be important for solitons, we will discuss it after. Then, the phase velocity $c = \omega/k$:

$$c = \sqrt{gh_0}(1 - \frac{(kh_0)^2}{6} + ...)$$

gives at smalll kh_0, *ie* very very small depth,

$$c = \sqrt{gh_0}.$$

This is again the shallow water velocity we have already seen in the previous paragraph.

If depth is too small one has to take into account the surface tension : $\sigma/(\rho U_0^2 h_0)$ is here $\sigma/(\rho g h_0^2)$. So if the depth is of same order of magnitude than the capillary length $\lambda_c = \sqrt{\sigma/(\rho g)}$ then one has to use the surface tension jump. This gives the correct dispersion relation

$$\omega^2 = g(1 + k^2\lambda_c^2)k\tanh(kh_0).$$

Notice here that for surface tension wave, small wave length travel faster. But, the effect is reversed for Airy waves in shallow water.

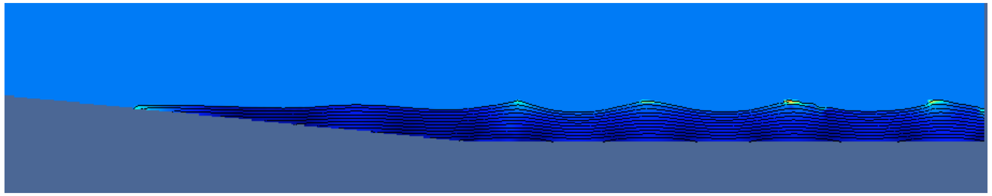

Figure 4 – Waves on a sloping beach with *Gerris*. Code it at the end.

3.4 Non linear waves : Shallow water, $\varepsilon = O(1), \delta \to 0$

There is another relevant simplification of the full problem which corresponds to flows in a small depth of water, so it is called "Shallow water". Or, flow with changes at a scale much larger than the depth. We call this also "Saint-Venant" system. So, if $\delta = h_0/\lambda \ll 1$, the length wave is long compared to the depth, and now we can allow $\varepsilon = 1$. The latter means that the wave may be large in height, so non linear phenomena appear. Let us look at this : a balance of terms with non linearities. At first order

$$\frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} = -\delta^2\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2}$$

but non linearities in Eq. 3 give the balance $\tau\varphi\delta^2/h_0^2 = 1$ which is $\tau\varphi/\lambda^2 = 1$

$$\frac{\varphi\tau}{h_0^2}\frac{\partial\bar{\phi}}{\partial\bar{y}} = \frac{\partial\bar{\eta}}{\partial\bar{t}} + \frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{\partial\bar{\eta}}{\partial\bar{x}}$$

In the momentum Eq.2, we have $\varepsilon\bar{\eta}$ which is $\bar{\eta}$, then $\frac{\varphi}{g\tau h_0}\frac{\partial\bar{\phi}}{\partial\bar{t}}$ that we compare to the next two which are, with $\tau\varphi\delta^2/h_0^2 = 1$, the non linear term $\frac{\varphi}{2g\tau h_0}(\frac{\partial\bar{\phi}}{\partial\bar{x}}^2 + \delta^2\frac{\partial\bar{\phi}}{\partial\bar{y}}^2)$. Hence, as δ is small :

$$\bar{\eta} + \frac{\varphi}{g\tau h_0}(\frac{\partial\bar{\phi}}{\partial\bar{t}} + \frac{1}{2}\frac{\partial\bar{\phi}}{\partial\bar{x}}^2) = 0$$

As we have $\tau\varphi/\lambda^2 = 1$ from the non linearities of 3 and $\varphi = g\tau h_0$ from the non linear terms of 2, this gives $\lambda/\tau = \sqrt{gh_0}$ and $\varphi = \lambda^2/\tau$.

With these scales, the scale $\frac{\varphi\tau}{h_0^2}$ in front of $\frac{\partial\bar{\phi}}{\partial\bar{y}}$ the velocity (of Eq. 3) is λ^2/h_0^2, which is large : $\delta^{-2} \gg 1$. This is the subtle part, as $\bar{\phi}$ does not change so much trough the layer (from the Laplacian, the variation in $\bar{y}$ are of order δ^2), the small variation is magnified by δ^{-2}, so that the result is of order one.

We have shown that $\lambda/\tau = \sqrt{gh_0}$ and $\varphi = \lambda^2/\tau$, so that $\varphi = \lambda\sqrt{gh_0}$ and $\tau = \lambda/\sqrt{gh_0}$, so with $\varepsilon = 1$ and $\delta \ll 1$, the final system reads :

$$\frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} = -\delta^2\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2}$$

$$\bar\eta + (\frac{\partial\bar\phi}{\partial\bar t} + \frac{1}{2}\frac{\partial\bar\phi}{\partial\bar x}^2) = 0$$

$$\frac{1}{\delta^2}\frac{\partial\bar\phi}{\partial\bar y} = \frac{\partial\bar\eta}{\partial\bar t} + \frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x}.$$

Coming back with $\bar u = \bar\phi_{\bar x}$ we have for momentum at surface (after derivation by $\bar x$), a balance between the total derivative of the velocity (which is a "plug" flow, $\bar u(\bar x, \bar t)$) influenced by the variations of pressure due to the changes of surfaces $\bar\eta$:

$$\frac{\partial\bar u}{\partial\bar t} + \bar u\frac{\partial\bar u}{\partial\bar x} = -\frac{\partial\bar\eta}{\partial\bar x}.$$

Then working on the continuity equation, we first we integrate the Laplacian

$$\frac{\partial\bar\phi}{\partial\bar y} = -\delta^2\frac{\partial^2\bar\phi}{\partial\bar x^2}(\bar y + 1)$$

and at $\bar y = \eta$, this gives $\frac{\partial\bar\phi}{\partial\bar y}$ at the interface which is $\delta^2\frac{\partial^2\bar\phi}{\partial\bar x^2}(\bar\eta + 1)$. Using the definition of $\bar u$ (note that $\bar u = \bar\phi_{\bar x}$ is function of x, t at dominant order), we inject this in relation of the surface velocity, δ disappears :

$$\frac{\partial\bar u}{\partial\bar x}(\bar\eta + 1) = \frac{\partial\bar\eta}{\partial\bar t} + \bar u\frac{\partial\bar\eta}{\partial\bar x}.$$

We can write this system in a more readable way, as we obtain the famous Shallow Water system (Saint-Venant, in French, `GfsRiver` in *Gerris* and `http://basilisk.fr/src/saint-venant.h` in *Basilisk*) :

$$\left\{\begin{array}{ll} \frac{\partial\bar u}{\partial\bar t} + \bar u\frac{\partial\bar u}{\partial\bar x} & = -\frac{\partial\bar\eta}{\partial\bar x}, \\ \frac{\partial\bar\eta}{\partial\bar t} + \frac{\partial}{\partial\bar x}((1+\eta)\bar u) & = 0. \end{array}\right.$$

This system gives advection, shocks, one example is on figure 5 were we see a moving hydraulic jump.

An alternate formulation suitable for Shallow Water only (Saint-Venant) is in `http://www.lmm.jussieu.fr/~lagree/COURS/MFEnv/MFEnv.pdf`.

4 Equations with ε and δ

4.1 Sum up of the scales

The three previous subsections we looked at the d'Alembert $\delta \ll 1$, $\varepsilon \ll 1$, then the case of dispersive linear waves, $\delta = 1$, $\varepsilon \ll 1$ and Saint-Venant $\delta \ll 1$, $\varepsilon = 1$. These are three fundamental points of view. We turn now to a case in between

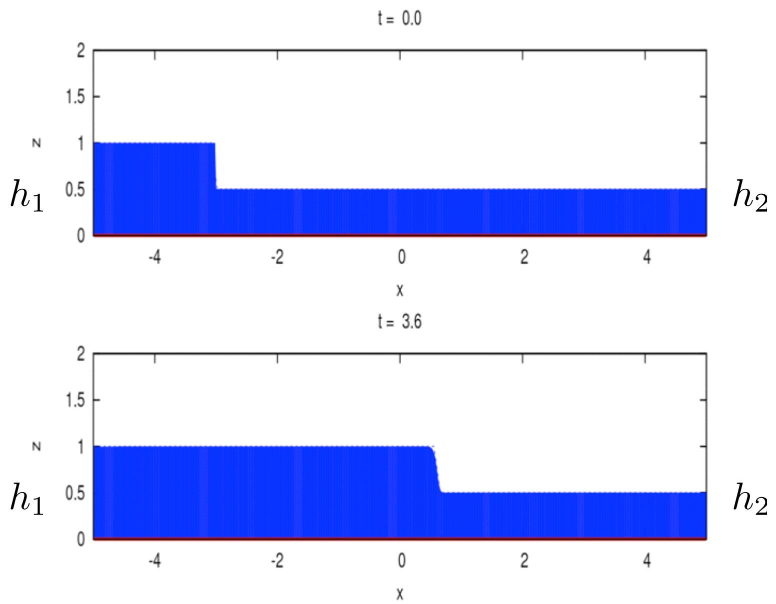

FIGURE 5 – here example of a moving hydraulic jump computed by *Gerris* solution of `GfsRiver` . Code it at the end. see `http://basilisk.fr/sandbox/M1EMN/Exemples/belanger.c` with *Basilisk*

with some dispersion and some nonlinearities in the set of equations (1, 2) first line and 3) second line.

Using the previous dominant balances necessary to obtain a displacement of the flow, each time we obtained the same scalings ($\delta^{-2}\frac{\partial\bar\phi}{\partial\bar y} \sim \frac{\partial\bar\eta}{\partial t}$ of the surface velocity Eq. 3 and Eq. 1 and $\bar\eta \sim \frac{\partial\bar\phi}{\partial t}$ of the momentum Eq. 2), we remind the scaling we have obtained

$$\varphi = \varepsilon\lambda^2/\tau = \varepsilon\lambda(\lambda/\tau), \quad \tau = \lambda/\sqrt{gh_0}$$

which is as well

$$\varphi = \varepsilon\lambda\sqrt{gh_0} \text{ and } \tau = \frac{\lambda}{\sqrt{gh_0}}, \quad \text{velocity is } \frac{\varphi}{\lambda} = \varepsilon\sqrt{gh_0}.$$

Remember that this comes from the scales we obtained in the fully linearized and Shallow Water cases. With those scales, we have already written the equations with ε and δ (10, the final system with ε and δ reads for Eq. (1), momentum at the surface Eq. (2), velocity at the surface Eq. (3), and at the bottom $\bar y=-1$:

$$\begin{array}{rcl} \delta^2\dfrac{\partial^2\bar\phi}{\partial\bar x^2}+\dfrac{\partial^2\bar\phi}{\partial\bar y^2} &=& 0\\ \bar\eta+\dfrac{\partial\bar\phi}{\partial\bar t}+\dfrac{\varepsilon}{2}(\dfrac{\partial\bar\phi}{\partial\bar x}^2+\dfrac{1}{\delta^2}\dfrac{\partial\bar\phi}{\partial\bar y}^2) &=& 0\\ \dfrac{1}{\delta^2}\dfrac{\partial\bar\phi}{\partial\bar y} &=& \dfrac{\partial\bar\eta}{\partial\bar t}+\varepsilon\dfrac{\partial\bar\phi}{\partial\bar x}\dfrac{\partial\bar\eta}{\partial\bar x}\\ \dfrac{\partial\bar\phi}{\partial\bar y}|_{\bar y=-1} &=& 0 \end{array}$$

If, in this set of equations we put $\varepsilon\ll 1$ and $\delta\ll 1$ we have the wave equation, and if, in this set of equations we put $\varepsilon=1$ and $\delta\ll 1$ we have the shallow water again, and if, in this set of equations we put $\varepsilon\ll 1$ and $\delta=1$ we have Airy linear dispersive solution again, we just look at what happens with a small wave in a not so shallow river.

4.2 Non linearity balances dispersion : small ε equals δ^2

We are now fully convinced that system (10) is the good system with the pertinent scales. As $\delta^2\frac{\partial^2\bar\phi}{\partial\bar x^2}+\frac{\partial^2\bar\phi}{\partial\bar y^2}=0$, by integration we have $\frac{\partial\bar\phi}{\partial\bar y}$. The velocity at the surface (Eq. 3) says

$$\frac{1}{\delta^2}\frac{\partial\bar\phi}{\partial\bar y}=\frac{\partial\bar\eta}{\partial\bar t}+\varepsilon\frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x},$$

at the bottom $\bar y=-1$

$$\frac{\partial\bar\phi}{\partial\bar y}=0.$$

Solving the Laplacian (1)

$$\delta^2\frac{\partial^2\bar\phi}{\partial\bar x^2}+\frac{\partial^2\bar\phi}{\partial\bar y^2}=0,$$

with a Poincaré expansion $\bar\phi=\bar\phi_0+\delta^2\bar\phi_1+\delta^4\bar\phi_2+...$ at first oder

$$\frac{\partial^2\bar\phi_0}{\partial\bar y^2}=0,\ \text{with in } \bar y=-1,\ \ \frac{\partial\bar\phi}{\partial\bar y}=0$$

so that $\phi_0(\bar x,\bar t)=f(\bar x,\bar t)$, we define $f'=\partial_{\bar x}\phi_0$. Hence the various orders solve the recurrence

$$\frac{\partial^2\bar\phi_1}{\partial\bar y^2}=-\frac{\partial^2\bar\phi_0}{\partial\bar x^2}\ \text{and}\ \frac{\partial^2\bar\phi_2}{\partial\bar y^2}=-\frac{\partial^2\bar\phi_1}{\partial\bar x^2},....$$

The solution at order 1, with $\frac{\partial\bar\phi_1}{\partial\bar y}|_{-1}=0$:

$$\frac{\partial^2\bar\phi_1}{\partial\bar y^2}=-f''\ \text{gives}\ \frac{\partial\bar\phi_1}{\partial\bar y}=-(\bar y+1)f''$$

and by integration

$$\bar\phi_1=-f''(\bar x,\bar t)(\bar y+\frac{1}{2}\bar y^2)+K(\bar x,\bar t)$$

the K is 0 as if we suppose that the condition for $(\bar x,\bar t)$ are in $f(\bar x,\bar t)$. At next order, with $\frac{\partial\bar\phi_2}{\partial\bar y}=0$:

$$\frac{\partial^2\bar\phi_2}{\partial\bar y^2}=f''''(x)(\bar y+\frac{1}{2}\bar y^2)\ \text{gives}\ \frac{\partial\bar\phi_2}{\partial\bar y}=(\frac{1}{2}\bar y^2+\frac{1}{6}\bar y^3-\frac{1}{3})f''$$

then, if we suppose again a 0 constant of integration :

$$\bar\phi_2=f''''(\bar x,\bar t)(\frac{1}{6}\bar y^3+\frac{1}{24}\bar y^4-\frac{1}{3}\bar y)$$

Then $\bar\phi$ is a polynom in $\bar y$ with coefficients functions of derivatives of f,

$$\bar\phi=f(\bar x,\bar t)-\delta^2f''(x,\bar t)(\bar y+\frac{1}{2}\bar y^2)+\delta^4f''''(\bar x,\bar t)(\frac{1}{6}\bar y^3+\frac{1}{24}\bar y^4-\frac{1}{3}\bar y)+...$$

Then $\frac{\partial\bar\phi}{\partial\bar y}$ is a polynom in $\bar y$ with coefficients functions of derivatives of f,

$$\frac{\partial\bar\phi}{\partial\bar y}=-\delta^2f''(x,\bar t)(1+\bar y)+\delta^4f''''(\bar x,\bar t)(\frac{1}{3}\bar y^2+\frac{1}{6}\bar y^3-\frac{1}{3})+...$$

so that if we substitute in expression of the velocity and perturbation of surface : $\frac{1}{\delta^2}\frac{\partial\bar\phi}{\partial\bar y}=\frac{\partial\bar\eta}{\partial\bar t}+\varepsilon\frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x}$ we have in $\bar y=\varepsilon\bar\eta$

$$-f''(\bar x,\bar t)(1+\varepsilon\bar\eta)+\delta^2f''''(\bar x,\bar t)\frac{-1}{3}=\frac{\partial\bar\eta}{\partial\bar t}+\varepsilon\frac{\partial\bar\phi}{\partial\bar x}\frac{\partial\bar\eta}{\partial\bar x}.$$

Let us define the longitudinal velocity by $\bar u=f'$. Other choices are possible, not only the value at surface $\bar y=0$ but the mean value :

$$\bar u_b=\int_{-1}^{0}\partial_{\bar x}\bar\phi d\bar y=f'+\frac{\delta^2}{3}f'''+...$$

depending of the choice, the various Boussinesq systems, see after, are possible). With $\bar u=f'$ this equation is

$$\frac{\partial\bar\eta}{\partial\bar t}+\frac{\partial\bar u}{\partial\bar x}=-\varepsilon\bar u\frac{\partial\bar\eta}{\partial\bar x}-\varepsilon\bar\eta\frac{\partial\bar u}{\partial\bar x}-\frac{1}{3}\delta^2\frac{\partial^3\bar u}{\partial\bar x^3}.$$

By dominant balance, we guess here that if we want non linear terms of order ε and variation across the layer of order δ^2, then

$$\varepsilon=\delta^2,$$

this is the fundamental balance for solitary wave.

The momentum follows from the scale of $\bar\phi$, we notice that

$$\frac{\partial\bar\phi}{\partial\bar y} = 0 + \delta^2\frac{\partial\bar\phi_1}{\partial\bar y} + ..$$

so we can again neglect it in momentum at the surface (2)

$$\bar\eta + \frac{\partial\bar\phi}{\partial\bar t} + (\varepsilon/2)(\frac{\partial\bar\phi}{\partial\bar x}^2 + \delta^{-2}(O(\delta^4))) = 0.$$

Which is at leading order

$$\bar\eta + \frac{\partial\bar\phi}{\partial\bar t} + (\varepsilon/2)\frac{\partial\bar\phi}{\partial\bar x}^2 = 0.$$

Remember that $\bar u = f'$ and at leading order, it is the usual inertial-pressure balance

$$\frac{\partial\bar u}{\partial\bar t} + \varepsilon\bar u\frac{\partial\bar u}{\partial\bar x} = -\frac{\partial\bar\eta}{\partial\bar x}.$$

At this point, as we want a dominant balance of the perturbative terms, wet take $\varepsilon = \delta^2$, this is $\eta_0/h_0 = (h_0/\lambda)^2$, the ratio :

$$\frac{\eta_0\lambda^2}{h_0^3}$$

is called the the fundamental parameter in the theory of water waves, this is the Ursell number $\frac{\eta/h_0}{(h_0/\lambda)^2} = \frac{\eta\lambda^2}{h_0^3}$.

So the final system of interest is :

$$\begin{cases} \frac{\partial\bar\eta}{\partial\bar t} + \frac{\partial\bar u}{\partial\bar x} & = -\varepsilon\bar u\frac{\partial\bar\eta}{\partial\bar x} - \varepsilon\bar\eta\frac{\partial\bar u}{\partial\bar x} - \frac{1}{3}\varepsilon\frac{\partial^3\bar u}{\partial\bar x^3}, \\ \frac{\partial\bar u}{\partial\bar t} + \varepsilon\bar u\frac{\partial\bar u}{\partial\bar x} & = -\frac{\partial\bar\eta}{\partial\bar x}. \end{cases}$$

We may write it :

$$\begin{cases} \frac{\partial\bar\eta}{\partial\bar t} + \frac{\partial}{\partial\bar x}((1+\varepsilon\bar\eta)\bar u) & = -\frac{1}{3}\varepsilon\frac{\partial^3\bar u}{\partial\bar x^3} \\ \frac{\partial\bar u}{\partial\bar t} + \varepsilon\bar u\frac{\partial\bar u}{\partial\bar x} & = -\frac{\partial\bar\eta}{\partial\bar x}. \end{cases}$$

Before looking at KdV, we see that this system has the non linear shallow water terms, plus an extra term which comes from the depth which is not so shallow. Considering linear waves, gives

$$\frac{\partial\bar\eta}{\partial\bar t} + \frac{\partial\bar u}{\partial\bar x} = -\frac{1}{3}\varepsilon\frac{\partial^3\bar u}{\partial\bar x^3} \text{ and } \frac{\partial\bar u}{\partial\bar t} = -\frac{\partial\bar\eta}{\partial\bar x}$$

the plane wave solution $e^{i(\bar k\bar x - \bar\omega\bar t)}$ gives

$$\bar\omega^2 = \bar k^2 - \frac{\varepsilon\bar k^4}{3}$$

remember that Airy wave gave the dispersion relation

$$\omega^2 = \bar k\tanh\bar k$$

which gives for long wave expansion $\omega^2 = \bar k(\bar k - \frac{\bar k^3}{3} + ...)$, so we find without surprise the same relation (after change of scale which implies the δ^2).

4.3 KdV ($\varepsilon = O(\delta^2) \ll 1$)

Let us no present the final canonical form of KdV with the final balance of unsteadiness, non linearity and dispersion, written in a moving frame. The Ursell number ($\frac{\eta/h_0}{(h_0/\lambda)^2} = \frac{\eta\lambda^2}{h_0^3}$) is then one. The obtained system

$$\begin{cases} \frac{\partial\bar\eta}{\partial\bar t} + \frac{\partial}{\partial\bar x}((1+\varepsilon\bar\eta)\bar u) & = -\frac{1}{3}\varepsilon\frac{\partial^3\bar u}{\partial\bar x^3} \\ \frac{\partial\bar u}{\partial\bar t} + \varepsilon\bar u\frac{\partial\bar u}{\partial\bar x} & = -\frac{\partial\bar\eta}{\partial\bar x} \end{cases}$$

has ε terms, so we look at an expansion :

$$\bar u = \bar u_0 + \varepsilon\bar u_1 + ...$$

$$\bar v = \bar v_0 + \varepsilon\bar v_1 + ...$$

The solution at order 0

$$\begin{cases} \frac{\partial\bar\eta_0}{\partial\bar t} + \frac{\partial}{\partial\bar x}(\bar u_0) & = 0 \\ \frac{\partial\bar u_0}{\partial\bar t} & = -\frac{\partial\bar\eta_0}{\partial\bar x} \end{cases}$$

will clearly imply ∂'Alembert equation...

$$\frac{\partial^2}{\partial\bar x^2}\bar u_0 - \frac{\partial^2}{\partial\bar t^2}\bar u_0 = 0, \quad \frac{\partial^2}{\partial\bar x^2}\bar\eta_0 - \frac{\partial^2}{\partial\bar t^2}\bar\eta_0 = 0,$$

say that $\xi = \bar x - \bar t$ and $\zeta = \bar x + \bar t$ to classically solve the wave equation so

$$\frac{\partial}{\partial\bar x} = \frac{\partial}{\partial\xi} + \frac{\partial}{\partial\zeta}, \quad \frac{\partial}{\partial\bar t} = -\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\zeta}$$

then at order 0

$$\frac{\partial}{\partial\xi}(-\bar\eta_0 + \bar u_0) + \frac{\partial}{\partial\zeta}(\bar u_0 + \bar\eta_0) = 0$$

$$\frac{\partial}{\partial \xi}(-\bar{u}_0+\bar{\eta}_0)+\frac{\partial}{\partial \zeta}(\bar{u}_0+\bar{\eta}_0)=0$$

or by sum and substraction

$$\frac{\partial}{\partial \xi}(-\bar{\eta}_0+\bar{u}_0)=0,\;\; \frac{\partial}{\partial \zeta}(\bar{u}_0+\bar{\eta}_0)=0$$

so that

$$-\bar{\eta}_0+\bar{u}_0=F(\zeta) \text{ and } \bar{u}_0+\bar{\eta}_0=G(\xi).$$

We will focus on a wave going to the right (no information in $\zeta=\bar{x}+\bar{t}$), and follow it with our horse. We deduce that $\bar{\eta}_0=\bar{u}_0$ and that this is a function of ξ, a moving wave to the right. We prefer now to be in the moving frame, so that $\xi=\bar{x}-\bar{t}$.

If we specify only the right moving wave and do not care about the other, then in this moving frame $\frac{\partial}{\partial \bar{x}}=\frac{\partial}{\partial \xi}$, and $\frac{\partial}{\partial \bar{t}}=-\frac{\partial}{\partial \xi}$. But, due to the ε terms, we guess that it will create cumulative terms. So we do a "multiple scale analysis" (`http://www.lmm.jussieu.fr/~lagree/COURS/M2MHP/MEM_GB.pdf`), with a slow time $\tau=\varepsilon\bar{t}$ we then have for the time an extra slow term,

$$\frac{\partial}{\partial \bar{x}}=\frac{\partial}{\partial \xi},\quad \frac{\partial}{\partial \bar{t}}=-\frac{\partial}{\partial \xi}+\varepsilon\frac{\partial}{\partial \tau}$$

so the momentum equation,

$$\frac{\partial \bar{u}}{\partial \bar{t}}+\varepsilon\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}=-\frac{\partial \bar{\eta}}{\partial \bar{x}} \text{ is } (-\frac{\partial \bar{u}}{\partial \xi}+\varepsilon\frac{\partial \bar{u}}{\partial \tau})+\varepsilon\bar{u}\frac{\partial \bar{u}}{\partial \xi}=-\frac{\partial \bar{\eta}}{\partial \xi}$$

after expansion $\bar{u}=\bar{u}_0+\varepsilon\bar{u}_1+...$ and $\bar{v}=\bar{v}_0+\varepsilon\bar{v}_1+...$ gives after substitution :

$$\frac{\partial}{\partial \xi}(-\bar{u}_0+\bar{\eta}_0)+\varepsilon(\frac{\partial}{\partial \tau}\bar{u}_0-\frac{\partial}{\partial \xi}\bar{u}_1+\frac{\partial}{\partial \xi}\bar{\eta}_1+\bar{u}_0\frac{\partial \bar{u}_0}{\partial \xi})=0,$$

whereas the mass conservation gives

$$\frac{\partial}{\partial \xi}(\bar{u}_0-\bar{\eta}_0)+\varepsilon(\frac{\partial}{\partial \tau}\bar{\eta}_0-\frac{\partial}{\partial \xi}\bar{\eta}_1+\frac{\partial}{\partial \xi}\bar{u}_1+\bar{u}_0\frac{\partial \bar{\eta}_0}{\partial \xi}+\bar{\eta}_0\frac{\partial \bar{u}_0}{\partial \xi}+\frac{1}{3}\frac{\partial^3 \bar{u}_0}{\partial \xi^3})=0.$$

From the first, we have

$$\frac{\partial}{\partial \xi}\bar{u}_1-\frac{\partial}{\partial \xi}\bar{\eta}_1=\frac{\partial}{\partial \tau}\bar{u}_0+\bar{u}_0\frac{\partial \bar{u}_0}{\partial \xi},$$

we substitute in the second

$$(\frac{\partial}{\partial \tau}\bar{\eta}_0+\frac{\partial}{\partial \tau}\bar{u}_0+\bar{u}_0\frac{\partial \bar{u}_0}{\partial \xi}+\bar{u}_0\frac{\partial \bar{\eta}_0}{\partial \xi}+\bar{\eta}_0\frac{\partial \bar{u}_0}{\partial \xi}+\frac{1}{3}\frac{\partial^3 \bar{u}_0}{\partial \xi^3})=0,$$

but the wave solution $\frac{\partial}{\partial \xi}(\bar{u}_0-\bar{\eta}_0)=0$ gives $\bar{u}_0=\bar{\eta}_0+F(\tau)$ so

$$(\frac{\partial}{\partial \tau}F+2\frac{\partial}{\partial \tau}\bar{\eta}_0+\bar{\eta}_0\frac{\partial \bar{\eta}_0}{\partial \xi}+2(\bar{\eta}_0+F(\tau))\frac{\partial \bar{\eta}_0}{\partial \xi}+\frac{1}{3}\frac{\partial^3 \bar{\eta}_0}{\partial \xi^3})=0.$$

This unknown function $F(\tau)$ is interpreted as a "secular term", it must be always 0, hence we finally obtain the KdV equation :

$$\boxed{\frac{\partial}{\partial \tau}\bar{\eta}_0+\frac{3}{2}\bar{\eta}_0\frac{\partial \bar{\eta}_0}{\partial \xi}+\frac{1}{6}\frac{\partial^3 \bar{\eta}_0}{\partial \xi^3}=0.}$$

"Le chemin qui y conduit semble long car nous avons détaillé chaque étape. On peut trouver des solutions élégantes pour établir rapidement l'équation KdV mais le prix à payer est que l'on ne contrôle pas les approximations" as says Dauxois [14], who takes another tortuous path.

4.4 Boussinesq Equation

From the point of view we developed :

$$\begin{cases} \frac{\partial \bar{\eta}}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}}((1+\varepsilon\bar{\eta})\bar{u}) &= -\frac{1}{3}\varepsilon\frac{\partial^3 \bar{u}}{\partial \bar{x}^3}+O(\varepsilon^2) \\ \frac{\partial \bar{u}}{\partial \bar{t}}+\varepsilon\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} &= -\frac{\partial \bar{\eta}}{\partial \bar{x}}+O(\varepsilon^2) \end{cases}$$

We write it with dimensions (and forget the $O(\varepsilon^2)$) :

$$\begin{cases} \frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x} &= -\frac{h^3}{3}\frac{\partial^3 u}{\partial x^3} \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} &= -g\frac{\partial h}{\partial x} \end{cases}$$

Depending on the choice of the horizontal fluid velocity given at some definite height in the fluid column, we can change the system. For example, if we define a new velocity $\bar{u}_b=\bar{u}+\frac{1}{3}\varepsilon\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}$ (the mean value of velocity), the system is now at the same order ε, because of course ε^2 terms are different.

$$\begin{cases} \frac{\partial \bar{\eta}}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}}((1+\varepsilon\bar{\eta})\bar{u}_b) &= 0+O(\varepsilon^2) \\ \frac{\partial \bar{u}_b}{\partial \bar{t}}-\frac{1}{3}\varepsilon\frac{\partial^3 \bar{u}_c}{\partial \bar{t}\partial \bar{x}^2}+\varepsilon\bar{u}_c\frac{\partial \bar{u}_b}{\partial \bar{x}} &= -\frac{\partial \bar{\eta}}{\partial \bar{x}}+O(\varepsilon^2) \end{cases}$$

We write it with dimensions, using this new velocity :

$$\begin{cases} \frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x} &= 0 \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} &= -g\frac{\partial h}{\partial x}+\frac{h^2}{3}\frac{\partial^3 u}{\partial x^2 \partial t} \end{cases}$$

this system is better as it is "more" conservative : mass conservation is full fitted.

Depending on the choice of the horizontal fluid velocity given at some definite height in the fluid column, see [1], in fact different types of Boussinesq equations have been introduced. Note that Boussinesq himself did not present exactly those equations in 1871, CR Acad. Sci. Paris, "Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire". Nevertheless, one writes :

$$\left\{\begin{array}{ll} \frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x} & =\frac{h^3}{2}(\theta^2-\frac{1}{3})\frac{\partial^3 h}{\partial x^2\partial t} \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} & =-g\frac{\partial h}{\partial x}+(1-\theta^2)\frac{h^2}{2}\frac{\partial^3 u}{\partial x^2\partial t}\end{array}\right.$$

depending on θ and notice that $\frac{\partial h}{\partial t}=\frac{\partial \eta}{\partial t}=-c_0\frac{\partial u}{\partial x}$. These are other common Shallow Water equation with a dispersive term (see literature)

$$\left\{\begin{array}{ll} \frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x} & =\frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} & =-g\frac{\partial h}{\partial x}+\frac{h^2}{2}\frac{\partial^3 u}{\partial x^2\partial t}\end{array}\right.$$

They give all KdV so that they are a bit more universal.

For $\theta^2=1$ we have our previous expression, for $\theta^2=1/3$, the conservation of mass is the standard one, which is a good thing

$$\left\{\begin{array}{ll} \frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x} & =0 \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} & =-g\frac{\partial h}{\partial x}+\frac{h^2}{3}\frac{\partial^3 u}{\partial x^2\partial t}\end{array}\right.$$

Linearisation of this last set of equations gives with $h=h_0+\varepsilon h_1+...$ and $u=0+\varepsilon u_1+...$:

$$i\omega h_1=ih_0ku_1,\;\; i\omega u_1=+igkh_1-i\omega k^2\frac{h_0^2}{3}u_1$$

so that $\omega^2(1+k^2\frac{h_0^2}{3})=gh_0k^2$, the dispersion relation is :

$$\omega=\frac{\sqrt{gh_0}k}{\sqrt{1+k^2\frac{h_0^2}{3}}},$$

close to 0 we expect dispersion (i.e. ω/k function of k) :

$$\omega=\sqrt{gh_0}(k-k^3\frac{h_0^2}{6}+...)$$

it allows a closer behavior of the exact dispersive wave solution (Airy Swell)

$$\omega=\sqrt{gk\tanh(kh_0)}\simeq\sqrt{gk(kh_0-k^3\frac{h_0^3}{3}+...)}\simeq k\sqrt{gh_0}(1-k^2\frac{h_0^2}{6}+...)$$

so that up to order 3 we have the right dispersion relation.

4.5 Serre-Green-Naghdi equations, $(\varepsilon=O(1),(\delta^2)\ll 1)$

At this point, we are close to a more improved model of Boussineq system : the Serre-Green-Naghdi equations. They were derived by Serre (1953, independently rediscovered by Su and Gardner (1969) and again by Green, Laws and Naghdi (1974) (see D. Duthyk HDR 2010 for derivation with a Lagrangian and many other expansions). Lannes and Bonneton Phys Fluids 21 2009 give a sophisticated derivation. We prefer to give here a more simple description which follows closely Bonneton's lecture in Cargese (06/2017) [2].

The important point in the Saint-Venant description is that the pressure is purely hydrostatic,

$$\frac{\partial p}{\partial y}=-\rho g,\;\text{ pressure is }\; p(x,y,t)=\rho g(h(x,t)-y),$$

thus the pressure gradient is $\rho g\partial h/\partial x$. The previous Boussinesq equation show the influence of the gradient of the non hydrostatic part, we found : $-\frac{h^2}{3}\frac{\partial^3 u}{\partial x^2\partial t}$.
This part of the pressure may be reobtained, even more precisely, starting from Euler description. Let us define the transverse acceleration γ, so that the transverse equation is

$$\frac{\partial\bar v}{\partial\bar t}+\bar u\frac{\partial\bar v}{\partial\bar x}+\bar v\frac{\partial\bar v}{\partial\bar y}=-\frac{\partial\bar p}{\partial\bar y}-1,\;\text{ is, say }\gamma=-\frac{\partial\bar p}{\partial\bar y}-1.$$

We integrate the pressure gradient, which is no more hydrostatic due to "transverse acceleration" γ :

$$\frac{\partial\bar p}{\partial\bar y}=-1-\gamma.$$

The pressure is indeed

$$\bar p(\bar x,\bar y,\bar t)-\bar p(\bar x,\bar y=\varepsilon\bar\eta,\bar t)=[-y]_{\varepsilon\bar\eta}^{\bar y}-\int_{\varepsilon\bar\eta}^{\bar y}\gamma d\zeta$$

at the surface $\bar p(\bar x,\bar y=\varepsilon\bar\eta,\bar t)=0$, so the pressure is the hydrostatic one, plus the acceleration (correction)

$$\bar p(\bar y)=-y+\varepsilon\bar\eta-\int_{\varepsilon\bar\eta}^{\bar y}\gamma d\zeta$$

we integrate a second time across the whole layer, as

$$\int_{-1}^{\varepsilon\bar\eta}(-y+\varepsilon\bar\eta)d\bar y = \frac{-(\varepsilon\bar\eta)^2+1}{2}+(\varepsilon\bar\eta)(1+\varepsilon\bar\eta)=\frac{(1+\varepsilon\bar\eta)^2}{2}$$

then

$$\int_{-1}^{\varepsilon\bar\eta}\bar p(\bar y)d\bar y=\frac{(1+\varepsilon\bar\eta)^2}{2}-\int_{-1}^{\varepsilon\bar\eta}(\int_{\varepsilon\bar\eta}^{\bar y}\gamma d\zeta)d\bar y$$

the tricky part is the double integral corresponding to the non hydrostatic part of pressure

$$I_2=\int_{-1}^{\varepsilon\bar\eta}(\int_{\bar y}^{\varepsilon\bar\eta}\gamma(\zeta)d\zeta)d\bar y$$

let us define $\Gamma(\bar y)=\int_{\bar y}^{\varepsilon\bar\eta}\gamma(\zeta)d\zeta$, so that $\frac{d\Gamma}{d\bar y}=-\gamma(\bar y)$

$$I_2=\int_{-1}^{\varepsilon\bar\eta}\Gamma(\bar y)d\bar y,\ \text{is by parts}\ \ I_2=\int_{-1}^{\varepsilon\bar\eta}(\frac{d}{d\bar y}(\bar y\Gamma))d\bar y-\int_{-1}^{\varepsilon\bar\eta}\bar y\frac{d\Gamma}{d\bar y}d\bar y,$$

integrating the first, and using the definition of Γ

$$I_2=\Gamma(-1)+\int_{-1}^{\varepsilon\bar\eta}\bar y\gamma(\bar y)d\bar y=\int_{-1}^{\varepsilon\bar\eta}\gamma(\zeta)d\zeta+\int_{-1}^{\varepsilon\bar\eta}\bar y\gamma(\bar y)d\bar y$$

finally the non hydrostatic correction is exactly

$$I_2=\int_{-1}^{\varepsilon\bar\eta}(1+\bar y)\gamma(\bar y)d\bar y.$$

Remember $\frac{\partial\bar\phi}{\partial\bar y}$ is a polynom in $\bar y$ with coefficients functions of derivatives of f,

$$\frac{\partial\bar\phi}{\partial\bar y}=-\delta^2 f''(x,\bar t)(1+\bar y)+\delta^4 f''''(\bar x,\bar t)(\frac{1}{3}\bar y^2+\frac{1}{6}\bar y^3-\frac{1}{3})+...$$

so that as $\bar v=\delta\frac{\partial\bar\phi}{\partial\bar y}$ and $\bar u=f'$ then

$$\bar v=-\delta\frac{\partial\bar u}{\partial\bar x}(x,\bar t)(1+\bar y)+O(\delta^2)$$

as $\gamma=\frac{\partial\bar v}{\partial\bar t}+\bar u\frac{\partial\bar v}{\partial\bar x}+\bar v\frac{\partial\bar v}{\partial\bar y}$, we take the time derivative, and the two space derivatives of $\bar v$, then by substitution of this derivatives of the transverse velocity and at leading order

$$\gamma=-(\delta(1+\bar y))[\frac{\partial^2\bar u}{\partial\bar t\partial\bar x}+\bar u\frac{\partial^2\bar u}{\partial\bar x^2}-(\frac{\partial\bar u}{\partial\bar x})^2+O(\delta)].$$

This is substituted in I_2, then as $\int_{-1}^{0}(1+\bar y^2)d\bar y=1/3$, the contribution of the non hydrostatic pressure is

$$I_2=-\frac{\delta^2\bar h^2}{3}[\frac{\partial^2\bar u}{\partial\bar t\partial\bar x}+\bar u\frac{\partial^2\bar u}{\partial\bar x^2}-(\frac{\partial\bar u}{\partial\bar x})^2+O(\delta)],$$

finally the Serre-Green-Naghdi non linear weakly dispersive equations are :

$$\left\{\begin{array}{ll}\frac{\partial\bar h}{\partial\bar t}+\frac{\partial}{\partial\bar x}(\bar h\bar u) & =0\\ \frac{\partial\bar u}{\partial\bar t}+\bar u\frac{\partial\bar u}{\partial\bar x}, & =-\frac{\partial\bar\eta}{\partial\bar x}+\frac{\delta^2}{3\bar h}\frac{\partial}{\partial\bar x}\left(\bar h^2(\frac{\partial^2\bar u}{\partial\bar t\partial\bar x}+\bar u\frac{\partial^2\bar u}{\partial\bar x^2}-(\frac{\partial\bar u}{\partial\bar x})^2)\right)+O(\delta)^4.\end{array}\right.$$

The non linear term of $O(\delta)^2$ is more precise than in Boussinesq description. In the case of weak non linearity, this is the same : $\frac{\delta^2}{3}\frac{\partial}{\partial\bar x}\frac{\partial^2\bar u}{\partial\bar t\partial\bar x}$

5 Some Solutions of KdV equation

5.1 KdV equation :

Coming back with variables, assuming that the fundamental parameter in the theory of water waves is of order one (Ursell number) :

$$\frac{\eta_0\lambda^2}{h_0^3} = O(1),$$

the KdV equation reads with scales :

$$\frac{\partial}{\partial t}\eta + c_0\frac{\partial}{\partial x}\eta + \frac{3c_0}{2h_0}\eta\frac{\partial\eta}{\partial x} + \frac{c_0h_0^2}{6}\frac{\partial^3\eta}{\partial x^3} = 0$$

with $c_0 = \sqrt{gh_0}$. this equation represent the dominant balance between non linearities that will create a hydraulic jump and dispersion that destroys the wave in several waves. When there is balance, a special wave, the "soliton", exists. It does not change in shape.

5.2 Solution of linearized KdV equation :

The linearized dispersion equation is that corresponding to the following problem (it is called linearized KdV equation) :

$$\frac{\partial\eta}{\partial t} + c_0\frac{\partial\eta}{\partial x} + \frac{c_0h_0^2}{6}\frac{\partial^3\eta}{\partial x^3} = 0.$$

This equation has as dispersion relation : $i\omega = c_0(ik)(1 + \frac{(h_0)^2}{6}(ik)^2)$.

Note $kc_0(1 - \frac{(h_0)^2}{6}(k)^2)$ is exactly the first two terms of the expansion on $\omega = \sqrt{gkth(kh_0)}$.

We will solve this equation assuming that the displaced surface $\int_{-\infty}^{\infty}\eta dx$ is given. The first idea is to move with speed c_0 and put $\xi = x - c_0t$ and $\alpha = c_0h_0^2$,so that the equation becomes :

$$\frac{\partial\eta}{\partial t} = -\frac{\alpha}{6}\frac{\partial^3\eta}{\partial\xi^3}.$$

This equation is solved using the similar solutions technique...

$$\text{change of scales}\begin{cases} t = T\hat{t} \\ \xi = X\hat{\xi} \\ \eta = H\hat{\eta} \end{cases} \qquad (11)$$

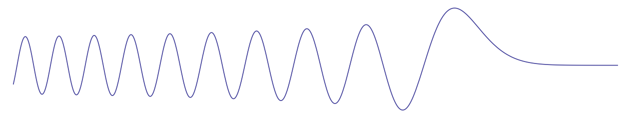

FIGURE 6 – Dispersive wave at t fixed, function of x (fonction de Airy)

Conservation of total mass $\int_{-\infty}^{\infty}\eta d\xi$ becomes $HX\int_{-\infty}^{\infty}\hat{\eta}d\hat{\xi}$ but as we want invariance $\int_{-\infty}^{\infty}\eta d\xi = \int_{-\infty}^{\infty}\hat{\eta}d\hat{\xi}$, hence $HX = 1$ preserves the conservation of the displaced surface $\int_{-\infty}^{\infty}\eta dx = 1$.

Likewise for the equation itself , $T = X^3$ preserves the invariance of the equation which is written identically

$$\frac{\partial\hat{\eta}}{\partial\hat{t}} = -\frac{\alpha}{6}\frac{\partial^3\hat{\eta}}{\partial\hat{\xi}^3}.$$

using the classical trick of invariance of the implicit solution, it is straightforward to obtain the similarity variable ζ and the surface η of the form :

$$\zeta = \frac{\xi}{t^{1/3}}, \text{ and } \eta = t^{-1/3}f(\frac{\xi}{t^{1/3}}).$$

By substitution and derivation the function $f(\zeta)$ checks

$$-\alpha/6f''' = -\zeta f'/3 - f/3.$$

By integrating, and since f is zero at infinity, we have :

$$\alpha f'' = 2\zeta f.$$

The solution of $y''(x) = xy(x)$ with $y(\infty) = 0$ is $y = Ai(x)$ the Airy function, moreover $\int_{-\infty}^{\infty}Ai(x)dx = 1$ (and notice $\int_0^{\infty}Ai(x)dx = 1/3$). We therefore have the solution for $f(\zeta) = (2/\alpha)^{1/3}Ai((2/\alpha)^{1/3}\zeta)$, since $\xi = x - c_0t$ The solution is ultimately this f divided by two :

$$\eta(x,t) = \frac{1}{2}(\frac{2}{c_0h_0^2t})^{1/3}Ai[(\frac{2}{c_0h_0^2})^{1/3}\frac{(x-c_0t)}{t^{1/3}}].$$

Warning!!! "The factor 1/2 appears because these represent only the waves moving to the right; those moving to the left complete the full initial condition" Whitham [17] page 443.

See figure 14 for a plot of Airy's function

5.3 Solution of KdV equation with no dispersion : Bürgers

If we neglect the dispersion (third order derivative) the equation at long times and in the frame of the moving wave becomes the so-called Bürgers equation (of inviscid Bürgers) :

$$\frac{\partial \bar\eta}{\partial \tau} + \frac{3}{2}\bar\eta\frac{\partial \bar\eta}{\partial \xi} = 0.$$

• From characteristic theory is it evident that this equation has ξ/τ solutions.... It is also trivial that this nonlinear equation tends to create a shock because the waves of greater height catch up with the lower ones.

• Indeed it is also written $\frac{\partial \bar\eta}{\partial \tau} + \frac{3}{4}\frac{\partial \bar\eta^2}{\partial \xi} = 0$. Its associated shock velocity is therefore $\frac{[|\frac{3}{4}\bar\eta^2|]}{\bar\eta} = \frac{3}{4}[|\bar\eta|]$. The total velocity of the discontinuity, into the laboratory frame and putting the dimensions back, is

$$W = c_0(1 + \frac{3[|\ \eta|].}{4h_0}). \qquad (12)$$

This is the velocity of a discontinuity.

• We found the asymptotic selfsimilar solution for a given wave in terms of Airy's function. Note that we we can construct an analogue to this Airy solution for this kdV with only non linearities (assuming that the displaced surface $\int_{-\infty}^{\infty}\eta dx$ is given. $\frac{\partial \bar\eta}{\partial \tau} + \frac{3}{2}\bar\eta\frac{\partial \bar\eta}{\partial \xi} = 0$. This equation is solved using the similar solutions technique...

$$\text{change of scales} \begin{cases} t = T\hat t \\ \xi = X\hat\xi \\ \eta = H\hat\eta \end{cases} \qquad (13)$$

Conservation of total mass is preserved by $HX = 1$ Likewise for the equation itself, $T = X^2$ preserves the invariance of the equation which is written identically

$$\frac{\partial \hat\eta}{\partial \hat t} = -\frac{3\hat\eta}{2}\frac{\partial \hat\eta}{\partial \hat\xi}.$$

using the classical trick of invariance of the implicit solution, it is straightforward to obtain the similarity variable ζ and the surface η of the form :

$$\zeta = \frac{\xi}{t^{1/2}},\ \text{and}\ \eta = t^{-1/2}f(\frac{\xi}{t^{1/2}}).$$

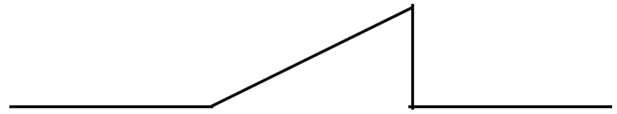

FIGURE 7 – A wave is transformed in a triangle due to breaking

By substitution and derivation the function $f(\zeta)$ checks

$$(-\zeta f' - f)/2 = 3ff'$$

As $(-\zeta f' - f) = -(\zeta f)'$ by integration and since f is zero at infinity, we have : $\zeta f = 3f^2$. This gives $f = \zeta/3$. This gives

$$\eta = \frac{\xi}{3\tau},\ \text{with}\ \xi_{max} = \sqrt{6}\tau$$

Any wave is transformed in a triangle whose shape decreases with time.

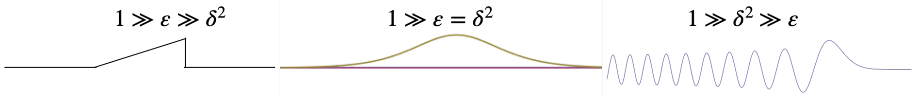

FIGURE 8 – Center the solitary wave, it can be transformed in a triangle due to breaking, or in many waves by dispersion

5.4 One famous solution of KdV equation : the Solitary Wave

Now we mix both nonlinear terms and dispersive terms into one the KdV, we will have the competition between the steepening of the triangle of figure 8 and the dispersion of figure 14.

Coming back with variables, assuming that the fundamental parameter in the theory of water waves is of order one (Ursell number) :

$$\frac{\eta_0\lambda^2}{h_0^3} = O(1),$$

the KdV equation reads with scales :

$$\frac{\partial}{\partial t}\eta + c_0\frac{\partial}{\partial x}\eta + \frac{3c_0}{2h_0}\eta\frac{\partial \eta}{\partial x} + \frac{c_0h_0^2}{6}\frac{\partial^3\eta}{\partial x^3} = 0$$

with $c_0 = \sqrt{gh_0}$. this equation represent the dominant balance between non linearities that will create a hydraulic jump and dispersion that destroys the wave in several waves. When there is balance, a special wave, the "soliton", exists. It does not change in shape.

The solution with no elevation of the surface up and down stream needs some algebra. We look at traveling waves $f(x - Ct) = f(s)$, so that

$$\frac{\partial}{\partial t}f(x,t) = -C\frac{\partial}{\partial s}f(s) = -Cf'(s) \text{ and } \frac{\partial}{\partial x}f(x,t) = \frac{\partial}{\partial s}f(s) = f'(s)$$

then :

$$\frac{\partial}{\partial t}f(x,t) + \frac{3}{2}f(x,t)\frac{\partial f(x,t)}{\partial x} + \frac{1}{6}\frac{\partial^3 f(x,t)}{\partial x^3} = 0, \text{ is } -Cf' + \frac{3}{4}(f^2)' + \frac{f'''}{6} = 0,$$

hence, it can be integrated once, the integration constant is such for $s \to \pm\infty$ perturbations of f are zero, so there is no constant. Then multiply by f' and integrate again :

$$\frac{(f')^2}{3} - 2Cf^2 + f^3 = 0,$$

by separation of variables

$$\int \frac{df}{f\sqrt{6C - 3f}} = ds$$

by change of variable, and few extra manipulations, we can find the solution. Indeed, if (yes ! if...) we notice that

$$(\text{Arctanh}(z)' = 1/(1-z^2), \quad \text{we have } (\text{Arctanh}((1-z)^n)' = -n(1-z^{-1+n})/(1-(1-z)^{2n}).$$

This gives us ($n = 1/2$) the solution of

$$\int \frac{dz}{z\sqrt{1-z}} = -2(\text{Arctanh}((1-z)^{1/2}).$$

The inverse function of

$$x = -2(\text{Arctanh}((1-z)^{1/2}) \text{ is } z = 1 - \tanh^2(x/2) = 1/\cosh(x/2)^2$$

note that $\text{sech}(x) = 1/\cosh(x)$

$$f = \frac{1}{\cosh^2(\sqrt{3}s/2)}$$

so that the final perturbation of the free surface is exactly :

$$\eta = \frac{\eta_0}{\cosh^2\left(\frac{1}{2h_0}\sqrt{\frac{3\eta_0}{h_0}}(x - c_0(1 + \frac{\eta_0}{2h_0})t)\right)}.$$

This is the "Soliton" or Solitary Wave solution. It has a lot of properties... Other solutions exist such as cnoidal waves (see literature, Whitham Lighthill, Debnath...) John Scott Russell found the $1/\cosh^2$ form experimental fit only. He obtained experimentally for velocity : $c_{JSR} = \sqrt{g(h_0 + \eta_0)}$ this is consistent with velocity of characteristics, so that it is a clever guess. The final exact velocity is $\sqrt{gh_0}(1+\eta_0/(2h_0)$, the two velocity are close : $\sqrt{g(h_0 + \eta_0)} = \sqrt{gh_0}(1+\eta_0/(2h_0)+...)$... so that John Scott Russell experimental result is not so wrong.

5.5 Other solutions of KdV equation : the Cnoidal Waves

Solutions of $\frac{(f')^2}{3} - 2Cf^2 + f^3 = 0$, can be examined in a phase plane, with $g = df/ds$ and

$$\frac{dg}{df} = \frac{6Cf + 9f^2/2}{g} = 0.$$

Critical points are $f = g = 0$ and $f = 4C/3$ and $g = 0$, the drawing in the phase plane shows closed curves. For a given C it involves cnoidal functions $cn(s)$ ...

$$\eta = \eta_0 \,\text{cn}^2((2K(m)/\lambda)(x - Ut)$$

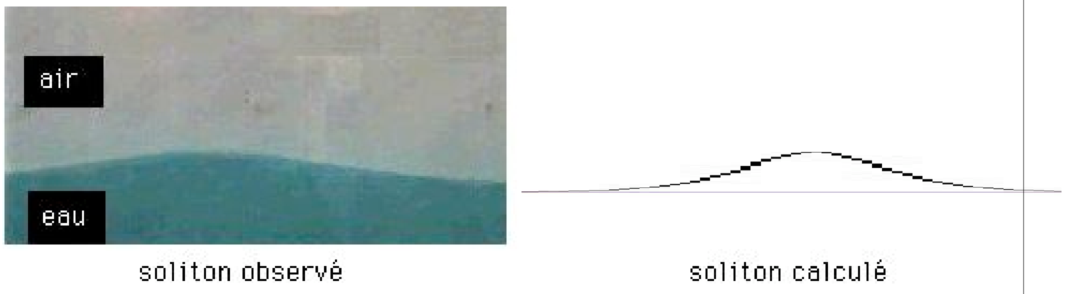

FIGURE 9 – Soliton in a flume at Palais de la Découverte(†), photo PYL `http://www.lmm.jussieu.fr/ lagree/SIEF/SIEF97/solitongd.mov` , right the $1/cosh^2$ solution.

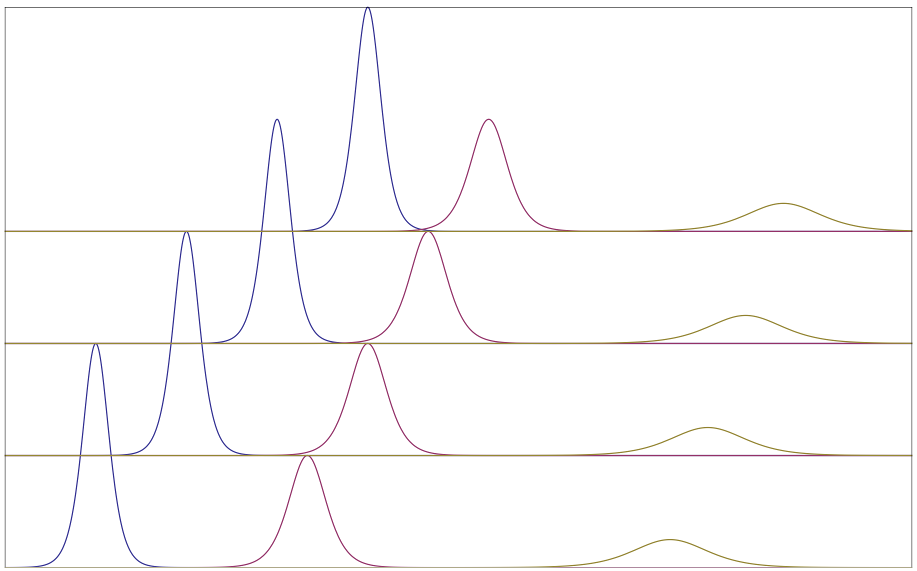

FIGURE 10 – 3 solitons of various selfsimilar shape $\epsilon_0/\cosh^2\left(\frac{1}{2}\sqrt{3\epsilon_0}x\right)$, they are in the moving frame. The larger the height, the thinner the width, the faster the wave. Time increases from bottom to top.

6 What about viscosity ?

6.1 *adhoc* viscosity

Of course the viscosity plays a role and destroys the waves. Let us look at the influence of the up to now neglected viscous term, first in a crude and false way.

According to Whitham [17] the ondular bore equation is the kdV with an *ad hoc* viscous coefficient necessary for the model but without real physical significance.

$$\frac{\partial}{\partial t}\eta + c_0\frac{\partial}{\partial x}\eta + \frac{3c_0}{2h_0}\eta\frac{\partial \eta}{\partial x} + \frac{c_0 h_0^2}{6}\frac{\partial^3\eta}{\partial x^3} - \nu\frac{\partial^2\eta}{\partial x^2} = 0$$

with $c_0 = \sqrt{gh_0}$ and ν this fake viscosity.

6.2 Bürgers equation

In the Navier Stokes equations, the second derivative term (∂_x^2) has always been neglected, claiming that it was negligible. However, this term is sometimes added in an *ad hoc* manner. It allows precisely to smooth the shocks, over a distance that is not very physical, but useful in practice the invicid burgers :

$$\frac{\partial \eta}{\partial \tau} + \frac{3}{2}\eta\frac{\partial \eta}{\partial \xi} = \nu\frac{\partial^2\eta}{\partial \xi^2}$$

Removing the 3/2 and changing η, ν, τ, ξ we have the generic Bürgers model equation :

$$\frac{\partial h}{\partial t} + h\frac{\partial h}{\partial x} = \nu\frac{\partial^2 h}{\partial x^2}.$$

To solve this equation (Bürgers equation) we do the so-called Hopf transformation by setting $\eta = -2\nu\frac{\partial log(F)}{\partial \xi}$ from which

$$\frac{\partial F}{\partial \tau} - \nu\frac{\partial^2 F}{\partial \xi^2} = g(t)F, \quad \text{We find} \quad F(x,t) = \frac{1}{4\pi\nu t}\int_{-\infty}^{\infty} exp(-\frac{(x-\zeta)^2}{4\nu t})F(\zeta,0)d\zeta.$$

Another solution of this equation, by setting $\zeta = (\xi - c\tau)/\nu$ to make the artificial viscosity disappear and by looking for a solution $F(\zeta)$: $-cF' + FF' + F'' = 0$, soit $-cF + F^2/2 + F' = cst$ solution is then

$$c = \frac{h_2+h_1}{2}, \quad \eta = \frac{h_2+h_1}{2} + \frac{h_2-h_1}{2}\tanh((h_1-h_2)\frac{(x-ct)}{4\nu})$$

the shape moves at celerity $c = (h_1+h_2)/2$, goes from h_1 to h_2 on a thickness $\nu/(h_1-h_2)$.

But again this viscosity has no asymptotic settlement.

6.3 Kakutani & Matsuuchi equation with fractional derivative

The viscosity plays a role and destroys the solitary wave. Let us look asymptotically at the influence of the up to now neglected viscous term. This is more complicated than adding a false longitudinal term.

At dominant order, we have computed the ideal fluid solution, for the boundary layer, we must add the dominant viscous term $\frac{1}{Re}\frac{\partial^2 \bar{u}_0}{\partial \bar{y}^2}$ in the momentum equation

$$\frac{\partial \bar{u}_0}{\partial \bar{t}} = -\frac{\partial \bar{\eta}_0}{\partial \bar{x}} + \frac{1}{Re}\frac{\partial^2 \bar{u}_0}{\partial \bar{y}^2}$$

with $\bar{y} = -1 + \frac{1}{\Re^{1/2}}\tilde{y}$, and $\tilde{u}_0 = \bar{u}_0$ we change the scales to be in the boundary layer as usual. We refer to the moving frame $\frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial \xi}$, and $\frac{\partial}{\partial \bar{t}} = -\frac{\partial}{\partial \xi}$. we obtain

$$\frac{\partial}{\partial \xi}\tilde{u}_0 + \frac{\partial^2 \tilde{u}_0}{\partial \tilde{y}^2} = \frac{\partial}{\partial \xi}\eta_0$$

with boundary conditions, first the no slip $\tilde{u}_0 = 0$ in $\tilde{y} = 0$, second the matching

$$\tilde{u}_0(\tilde{y} \to \infty) = \bar{u}_0(\bar{y} \to -1) = \bar{\eta}_0(-1).$$

the resolution has been proposed by Kakutani & Matsuuchi in 1971 ([9]. The problem is to solve for $f = \tilde{u}_0 - \bar{\eta}_0(\xi, -1)$

$$\frac{\partial}{\partial \xi}f + \frac{\partial^2}{\partial \tilde{y}^2}f = 0$$

with boundary conditions, first $f = -\bar{\eta}_0$ in $\tilde{y} = 0$, second $f(\tilde{y} \to \infty) = 0$ in fourier space $ik\hat{f} + \hat{f}'' = 0$, so the solution is in $e^{-\sigma\tilde{y}}$ with $\sigma = \frac{(1-i)}{\sqrt{2}}(k\ sgn(k))^{1/2}$, then

$$\tilde{u}_0 = \bar{\eta}_0 - \int \hat{\eta}_0 e^{-\sigma} e^{ik\xi} dk$$

from this expression, we compute the transverse velocity using the trick of the velocity in the boundary layer (second order effect, the blowing of the displacement thickness in the ideal fluid) see `http://www.lmm.jussieu.fr/~lagree/COURS/CISM/blasius_CISM.pdf`

$$\bar{v}_1 = -\frac{\partial \bar{u}_0}{\partial \xi}\bar{y} + \frac{1}{Re^{1/2}}\int_{-1}^{\infty}(\frac{\partial}{\partial \xi}(\bar{u}_0 - \tilde{u}_0))d\tilde{y}$$

the corrective term, due to the blowing is rewritten after integration

$$\bar{v}_{BL} = \frac{1}{(2Re)^{1/2}}\int_{-\infty}^{\infty}(-1 + isgn(k)|k|^{1/2}\hat{\eta}_0 e^{ik\xi}dk$$

by convolution, K& M wrote

$$\bar{v}_{BL} = \frac{1}{(\pi Re)^{1/2}} \int_{\xi}^{\infty} \frac{\partial \eta_0}{\partial \xi'} \frac{d\xi'}{(\xi' - \xi)^{1/2}}$$

this velocity is inserted in the $\frac{1}{\delta^2}\frac{\partial \bar{\phi}}{\partial \bar{y}} = \frac{\partial \bar{\eta}}{\partial \bar{t}} + \varepsilon \frac{\partial \bar{\phi}}{\partial \bar{x}}\frac{\partial \bar{\eta}}{\partial \bar{x}}$

$$-f''(\bar{x},\bar{t})(1+\varepsilon\bar{\eta}) + \delta^2 f''''(\bar{x},\bar{t})\frac{-1}{3} - \bar{v}_{BL} = \frac{\partial \bar{\eta}}{\partial \bar{t}} + \varepsilon \frac{\partial \bar{\phi}}{\partial \bar{x}}\frac{\partial \bar{\eta}}{\partial \bar{x}}.$$

The final Kakutani & Matsuuchi [9] is

$$\frac{\partial}{\partial \tau}\bar{\eta}_0 + \frac{3}{2}\bar{\eta}_0\frac{\partial \bar{\eta}_0}{\partial \xi} + \frac{1}{6}\frac{\partial^3 \bar{\eta}_0}{\partial \xi^3} = \frac{1}{(\pi Re)^{1/2}} \int_{\xi}^{\infty} \frac{\partial \eta_0}{\partial \xi'} \frac{d\xi'}{(\xi' - \xi)^{1/2}}$$

In the integral one recognises a "fractional" derivative. As the Fourier transform of f is $\hat{f}$, the the Fourier transform of $\frac{d^n f}{dx^n}$ is $(-ik)^n\hat{f}$. Here, in this problem we have $(-ik)^{1/2}$, so a 1/2 derivative! by inverse transform and convolution this $(-ik)^{1/2}$ gives the part $\int_{\xi}^{\infty}\frac{d\xi'}{(\xi'-\xi)^{1/2}}$

The final

$$\frac{\partial}{\partial \tau}\bar{\eta}_0 + \frac{3}{2}\bar{\eta}_0\frac{\partial \bar{\eta}_0}{\partial \xi} + \frac{1}{6}\frac{\partial^3 \bar{\eta}_0}{\partial \xi^3} + \frac{1}{(\pi Re)^{1/2}}\frac{\partial^{1/2}\bar{\eta}_0}{\partial \xi^{1/2}} = 0$$

note that the coefficients are maybe wrong (check, it depends on the definition of the 1/2 derivative). See le Meur `https://hal.archives-ouvertes.fr/hal-00826564/document` for discussion and bibliography, and controversy of the use of Fourier transform, Laplace transform must be better to take into account the history of the development of the boundary layer.

We note that the KM equation is not only local : with ∂_ξ and ∂_τ derivatives. This equation is as well non-local : $\int_d \xi'$. The mix of properties makes it difficult to solve and interesting to study.

6.4 Effet faible de la profondeur : solution lointaine

6.4.1 A partir de l'équation de dispersion tronquée (KdV linéarisé)

On a trouvé des solutions en ondes telles que $\omega^2 = gk\tanh(kh_0)$ si on développe dans le cas peu dispersif de l'eau peu profonde, à grande longueur d'onde $kh_0 \to 0$, on fait un développement limité en kh_0 de la tangente puis de la racine, (et $c_0^2 = gh_0$) :

$$\omega^2 = gk(kh_0 - \frac{(kh_0)^3}{3} + ...) = (gh_0)k^2(1 - \frac{(kh_0)^2}{3} + ...), \quad \text{dont les racines sont } \omega = \pm c_0 k(1 - \frac{(kh)^2}{6} + ...).$$

On en déduit en prenant la valeur + des ondes qui se déplacent vers la droite. On la forme d'onde $\eta = \eta_0 exp(i(\omega t - kx))$, avec $i\omega = (ik)c_0(1 + \frac{(h_0)^2}{6}(ik)^2 + ...)$ ce qui veut dire que puisque $\partial_t \eta = i\omega\eta$ et que $\partial_x \eta = -ik\eta$ l'équation de dispersion linéarisée est celle correspondant au problème suivant (elle s'appelle équation de KdV linéarisée) :

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \frac{c_0 h_0^2}{6}\frac{\partial^3 \eta}{\partial x^3} = 0.$$

Cette équation s'obtient aussi à partir du système de Boussinesq obtenu en corrigeant Saint-Venant et en considérant une onde qui va vers la droite. Nous allons résoudre cette équation en supposant que la surface déplacée $\int_{-\infty}^{\infty} \eta dx$ est une donnée. La première idée est de se déplacer avec la vitesse c_0 et de poser $\xi = x - c_0 t$ et $\alpha = c_0 h_0^2$, l'équation devient :

$$\frac{\partial \eta}{\partial t} = -\frac{\alpha}{6}\frac{\partial^3 \eta}{\partial \xi^3}.$$

Cette équation, se résout par la technique des solutions semblables. Par invariances par dilatations on cherche des solutions semblables... Consulter

$$\text{le changement d'échelle} \begin{cases} t = T\hat{t} \\ \xi = X\hat{\xi} \\ \eta = H\hat{\eta} \end{cases} \quad (14)$$

La conservation de la masse totale $\int_{-\infty}^{\infty} \eta d\xi$ devient $HX\int_{-\infty}^{\infty} \hat{\eta} d\hat{\xi}$ mais comme on veut l'invariance $\int_{-\infty}^{\infty} \eta d\xi = \int_{-\infty}^{\infty} \hat{\eta} d\hat{\xi}$, donc $HX = 1$ préserve la conservation de la surface déplacée $\int_{-\infty}^{\infty} \eta dx = 1$. De même pour l'équation, si $T = X^3$ cela préserve l'invariance de l'équation qui s'écrit identiquement $\frac{\partial \hat{\eta}}{\partial \hat{t}} = -\frac{\alpha}{6}\frac{\partial^3 \hat{\eta}}{\partial \hat{\xi}^3}$. La variable de similitude est $\zeta = \frac{\xi}{t^{1/3}}$ et la surface est de la forme : $\eta = t^{-1/3} f(\frac{\xi}{t^{1/3}})$. Par substitution et dérivation la fonction $f(\zeta)$ vérifie $-\alpha/6 f''' = -\zeta f'/3 - f/3$. En intégrant, et comme f est nulle à l'infini, on a : $\alpha f'' = 2\zeta f$. Introduisons une

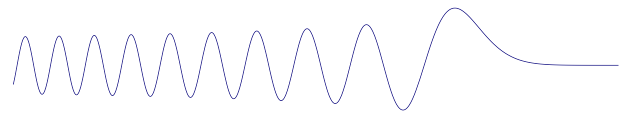

Figure 11 – Onde dispersive à t fixé, fonction de x (fonction de Airy).

nouvelle fonction qui est la fonction d'Airy (et qui n'a pas de rapport en soit avec la Houle de Airy)

La solution de $y''(x) = xy(x)$ avec $y(\infty) = 0$ et avec $\int_{-\infty}^{\infty} Ai(x)dx = 1$ est $y = Ai(x)$ la f

Nous allons montrer dans le paragraphe suivant par la méthode de la "phase stationnaire" ou de "plus grande descente" t Hinch page 34, Erdély page 41 (et voir plus loin) que

$$\text{pour } z < 0 \; Ai(z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4} \sin(\frac{2}{3}|z|^{3/2} + \pi/4), \text{ et pour } z > 0 \quad \text{on a } Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4}$$

Le développement de Taylor : $Ai(x) = \frac{1 + \frac{x^3}{2*3} + \frac{x^{(2*3)}}{2*3*5*6} + \frac{x^{3*3}}{2*3*5*6*8*9} + ... + \frac{a^+_{k-1}}{(3k-1)3k} x^{3k} + ...}{3^{2/3}\Gamma(\frac{2}{3})} - \frac{x + \frac{x^4}{3*4} + \frac{x^7}{3*4*6*7} + \frac{x^{10}}{3*4*6*7*9*10} + ... + \frac{a^-_{k-1}}{3k(3k+1)} x^{3k+1} + ...}{\sqrt[3]{3}\Gamma(\frac{1}{3})}$

On a donc la solution pour $f = (2/\alpha)^{1/3} Ai((2/\alpha)^{1/3}\zeta)$, puisque $\xi = x - c_0 t$ la solution est au final :

$$\eta(x,t) = \frac{1}{2}(\frac{2}{c_0 h_0^2 t})^{1/3} Ai[(\frac{2}{c_0 h_0^2})^{1/3}\frac{(x - c_0 t)}{t^{1/3}}]. \quad (15)$$

Attention il a fallu rajouter un facteur 1/2 car comme le rappelle Whitham [?] page 443, on ne considère que la moitié des vagues, celles qui vont vers la droite !

Nous allons retrouver ce résultat autrement dans la suite.

Pour mémoire quelques lignes `Mathematica` sur `raspberry`

```
DSolve[y''[x] - x y[x] == 0, y[x], x]
y[x_, t_] :=  F[(2^(1/3))  x/t^(1/3)] t^(-1/3)
FullSimplify[D[y[x, t], t] + 1/6  D[y[x, t], x, x, x]]
intF = Integrate[%, x];
intFe = Simplify[% /. x -> eta t^(1/3) (2^(-1/3))]
DSolve[{% == 0, F[Infinity] == 0}, F[eta], eta]
Integrate[(2^(1/3))  AiryAi[(2^(1/3)) x], {x, -Infinity, Infinity}]

Series[AiryAi[x], {x, 0, 4}]

Series[AiryAi[x], {x, -\[Infinity], 0}] // Normal
```

Cette manière est la plus simple de trouver la forme au loin des ondes dispersives. Nous allons voir deux autres méthodes classiques mais plus compliquées.

6.4.2 Plus compliqué : paquet au loin par la méthode de la phase stationnaire, méthode générale

On a vu le déplacement d'un paquet d'ondes sous la forme $\eta = \int_{-\infty}^{\infty} F(k)e^{i(kx-\omega t)}dk$ dans les cas simplifiés de deux ondes puis d'une porte (plus avant en §**??**). Cela nous a permis d'introduire la vitesse de groupe. Dans le cas où F est réduite à la fonction porte $F(k) = 1$ pour $k_0 - \Delta k_0/2 < k < k_0 + \Delta k_0/2$, et 0 sinon :

$$\int_{-\infty}^{\infty} F(\kappa)e^{-i\kappa(x-v_g t)}d\kappa = \Delta k_0 sinc((x - v_g t)\Delta k_0/2)$$

la fonction sinus cardinal $sinc(x) = sin(x)/x$ est une fonction "piquée" en 0. Ce calcul était un peu simpliste, reprenons le. On part de la perturbation de surface libre décomposée suivant tous les modes spatiaux de Fourier :

$$\eta = \int_{-\infty}^{\infty} F(k)e^{i(kx-\omega t)}dk$$

Cette intégrale est en fait difficile à calculer car on l'a vu il y a beaucoup d'oscillations qui se compensent mutuellement (d'où les simplifications : deux ondes puis une porte en §**??**). On va appliquer l'idée de la phase stationnaire ("méthode du col", ou "méthode de la phase stationnaire", ou encore *steepest descent* cf Hinch "Perturbation methods" page 30, cf Erdélyi 1956), pour cela on commence par privilégier un rayon $x = Vt$, ce qui permet d'éliminer la dépendance en x. L'onde η se développe alors en :

$$\eta = \int F(k)exp(i\varphi)dk \quad \text{où la phase } \varphi = Vk - \omega(k) = Vk - kc(k).$$

La contribution principale de l'intégrale est donc lorsque la phase φ varie peu avec k, c'est à dire lorsque $\partial\varphi/\partial k = 0$. La dérivée s'annule justement pour un certain k_0 tel que :

$$V = v_g = \partial\omega/\partial k.$$

On retrouve donc la vitesse de groupe. On développe en série au voisinage de ce k_0 la relation de dispersion $\omega(k)$, on note $\omega_0'' = \partial^2\omega/\partial k^2$:

$$\omega = \omega_0 + v_g(k - k_0) + \frac{1}{2}\omega_0''(k - k_0)^2 + ... :$$

et on injecte dans l'intégrant,

$$\eta = \int exp(i(kv_g - \omega_0)t)F(k)exp(-\omega_0''((k - k_0)/(\sqrt{(-2i)})^2 + ...))dk$$

(astuce $i = -1/i$ et $(\sqrt{2})^2 = 2$). Or, seule la fréquence k_0 est sélectionnée, en effet $F(k) = F(k_0) + (k - k_0)\partial F/\partial k + ...$, les termes autres que $F(k_0)$ ont une contribution négligeable, il vient :

$$\eta \sim exp(i(k_0 v_g - \omega_0)t)F(k_0) \int exp(-\omega_0''((k - k_0)/(\sqrt{(-2i)})^2))dk$$

L'intégrale de $exp(-\omega_0''((k - k_0)/(\sqrt{(-2i)})^2)))dk$ est réécrite comme l'intégrale de $(-2i/(\omega_0''t))^{1/2}exp(-s^2)ds$ par changement de variable en définitive puisque $\int_{-\infty}^{\infty} exp(-s^2)ds = \sqrt{\pi}$, on obtient l'estimation de la perturbation de surface libre

$$\eta \sim i\sqrt{\frac{2\pi}{\omega_0''t}}e^{i\pi/4}(F(k_0)exp(i(k_0x - \omega_0 t)), \quad \text{autour du rayon principal} \quad x/t = v_g.$$

Si $\omega_0'' < 0$ alors en le remplace par $-\omega_0''$ dans l'expression et on change le signe $e^{-i\pi/4}$. L'amplitude de la perturbation décroit au loin en $t^{-1/2}$, et ce le long du rayon v_g, en dehors de ce rayon, les ondes sont inexistantes. Le paquet d'onde se déplace bien à la vitesse v_g, l'amplitude décroît en $1/\sqrt{t}$.

6.4.3 Plus compliqué : paquet au loin par la méthode de la phase stationnaire, cas des vagues

Le calcul précédent est général, il est utilisé dans d'autres branches de la mécanique des fluides (en stabilité par exemple). Malheureusement, si on part directement de $\omega = k\sqrt{gh_0}(1 - \frac{1}{6}k^2h_0^2 + ...)$, on a alors $v_g = c_0 - \frac{h_0^2 c_0}{2}k^2$ donc $\omega_0'' = -h_0^2c_0k$. il n'y a pas de terme ω_0'' à première vue quand $k = 0$.

Pour rattraper le coup, en fait, il y en a un si on suppose k petit et fini, et $\omega_0'' = -h_0^2c_0k$ est donc petit mais pas nul. D'où $k = \pm\sqrt{(c_0t - x)/(\frac{h_0^2c_0}{2}t)}$, en mettant dans l'équation de la phase stationnaire vue à l'instant $\eta \sim i\sqrt{\frac{2\pi}{\omega_0'' t}} e^{i\pi/4}(F(k_0)exp(i(k_0x-\omega_0 t))$, ces deux racines on fait apparaître le sinus, le terme $(k_0x-\omega_0 t)$ fait apparaître $(x-c_0t)$ en puissance 3/2 dans le sinus et ω_0'' qui est en k est donc en $\sqrt{c_0t-x}$. On obtient après calcul la forme de l'onde

$$\eta \sim 2\sqrt{\pi}F(0)\left(\frac{2}{(h_0^2c_0t)(c_0t-x)}\right)^{1/4}\sin\left(\frac{2}{3}\frac{(c_0t-x)^{3/2}}{(h_0^2c_0t/2)^{1/2}}+\frac{\pi}{4}\right)$$

6.4.4 Toujours aussi compliqué : paquet au loin sans la méthode de la phase stationnaire

Sinon, si on ne veut pas faire ce calcul, on peut partir directement de la composition des ondes

$$\eta = \int_{-\infty}^{\infty} F(k)e^{i(kx-\omega t)}dk$$

et en effectuant le développement au voisinage de $k=0$ dans l'exponentielle

$$\eta = F(0)\int exp(ik(x-c_0t)+\frac{ic_0h_0^3k^3t}{6}+...)dk$$

Ensuite, on pose (cf Mei p 30) $z^3=\frac{2[x-c_0t]^3}{c_0h_0^2t}$ et $k[x-c_0t]=z\alpha$ la partie réelle

$$\eta = F(0)\int cos(ik(x-c_0t)+\frac{ic_0h_0^3k^3t}{6}+...)dk$$

devient avec ce changement de variables

$$\eta = F(0)\frac{2^{1/3}}{(c_0h_0^2t)^{1/3}}\int cos(z\alpha+\frac{\alpha^3}{3})d\alpha$$

or il est "bien connu" qu'une définition de la fonction de Airy est :

$$Ai(z)=\frac{1}{\pi}\int_0^{\infty} cos(sz+\frac{sz^3}{3})ds$$

en fait on retombe sur la solution d'Airy que l'on a déjà vue avec des solutions semblables (à partir de la relation de dispersion de KdV linéarisé Eq. 15), mais obtenue ici par le passage par la définition intégrale de l'équation d'Airy :

$$\eta \sim \pi F(0)\frac{2^{1/3}}{(h_0^2c_0t)^{1/3}}Ai\left(2^{1/3}\frac{(x-c_0t)}{(h_0^2c_0t)^{1/3}}\right).$$

On se demande quelle est la relation entre cette description avec la fonction de Airy et la méthode de la phase stationnaire , c'est ce que l'on va voir maintenant.

6.4.5 Lien final entre ces différentes approches

La fonction d'Airy a un comportement asymptotique [que l'on obtient justement par la méthode de la phase stationnaire dans Hinch page 34, Erdély page 41]

$$Ai(z)\sim\frac{1}{\sqrt{\pi}}z^{-1/4}\sin(\frac{2}{3}|z|^{3/2}+\pi/4) \text{ pour } z<0 \text{ ou pour } z>0 \quad \text{on a } Ai(z)\sim\frac{1}{2\sqrt{\pi}}z^{-1/4}e$$

et on fait la substitution dans l'expression précédente du paquet au loin sans la méthode de la phase stationnaire

$$\eta \sim 2\sqrt{\pi}F(0)\ ((h_0^2c_0t/2)(c_0t-x))^{-1/4}\sin\left(\frac{2}{3}\frac{(c_0t-x)^{3/2}}{(h_0^2c_0t/2)^{1/2}}+\frac{\pi}{4}\right)$$

on retrouve bien la même forme obtenue à partir de la phase stationnaire à k petit.

Au final, on voit quelle est la forme de la perturbation de surface libre : elle devient exponentiellement petite pour $x>c_0t$, elle est maximale en $x=c_0t$ (l'approximation avec $(c_0t-x)^{-1/4}$ y diverge, il faut garder Airy), ensuite pour $x<c_0t$ elle a un caractère ondulatoire, l'amplitude décroit lentement en $(c_0t-x)^{-1/4}$ au fur et à mesure que l'on s'éloigne du front.

Il faut alors retourner aux comparaisons issues de Noda [?] "Water waves generated by a local surface disturbance". Il compare, des expériences, aux deux approximations proposées, Airy complet, la solution en phase stationnaire asymptotique donc avec la puissance -1/4 qui fait que l'onde diverge en $x=c_0t$.

Des exemples de calcul avec Basilisk :
`http://basilisk.fr/sandbox/M1EMN/Exemples/boussinesqc.c` résolution en C des équations de Boussinesq,
`http://basilisk.fr/sandbox/M1EMN/Exemples/airy_watertrainfront.c` résolution Multilcouche Euler Lagrange ou Green Naghdi avec Basilisk
`http://basilisk.fr/sandbox/M1EMN/Exemples/ressaut_mascaret.c`*Poor's man dispersive model*

Water Waves 7395

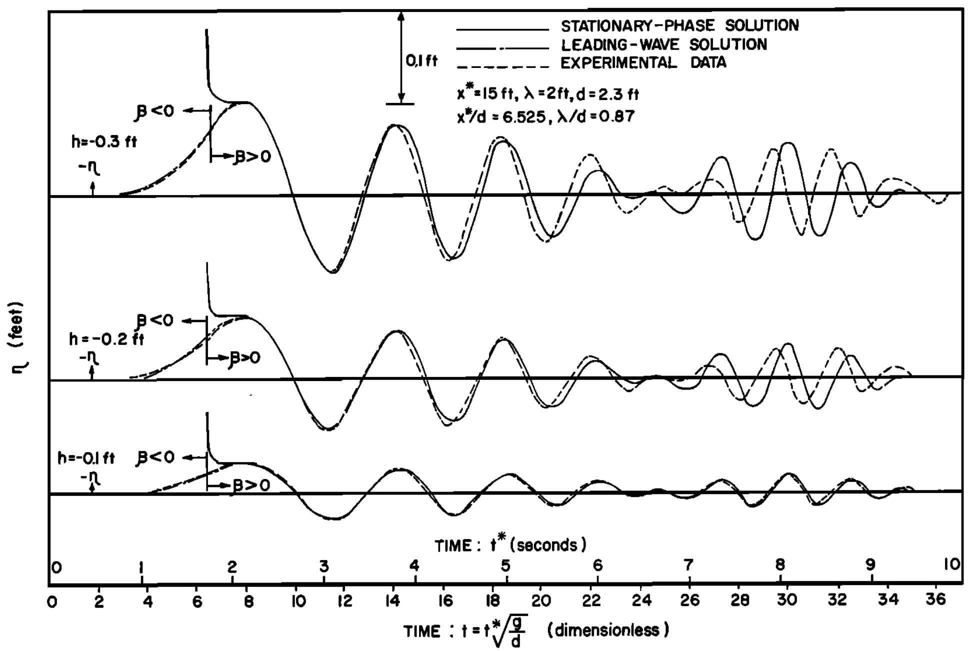

Fig. 6. Experimental and analytical amplitude–time relationships.

FIGURE 12 – Comparaisons issues de Noda [?]. Comparaison des expériences, aux deux approximations proposées, celle de la phase stationnaire $\eta \sim i\sqrt{2\pi}e^{i\pi/4}(\omega_0'' t)^{-1/2}F(k_0)exp(i(k_0x-\omega_0 t))$ que l'on va voir plus loin à celle de la solution $\eta = \frac{1}{h_0^2 c_0 t/2}^{1/3} Ai(2^{1/3}\frac{x-c_0}{(h_0^2 c_0 t)^{1/3}})$ que l'on vient de voir (en fait il y a un $t^{2/3}$ au lieu de $t^{1/3}$, cela est certainement une typo.

7 The ondular bore or Mascaret

Hydraulic jumps arise some time in rivers due to the elevation of the sea level due to the tide. The simple bore (hydraulic jump) is solution of Shallow water equations, but if the river is with enough water depth, dispersion occurs, then the bore breaks and create an "ondular bore" (a "Mascaret", a "Poroqua", see Chanson [3] definite book : "Tidal Bores, Aegir, Eagre, Mascaret, Pororoca : Theory and Observations"). It is present in some rivers in the world and it is due to the high tide. The mascaret on the "Severn River" (Lighthill book [12]), is famous. But the mascaret on the Dordogne in Saint Pardon is spectacular (of course the ondular bores of China and Amazonia are the largest in the world), see figure 13. Far upstream, it breaks in solitary waves.

FIGURE 13 – Mascaret or Ondular Bore at Saint Pardon on the Dordogne, Photo PYL See other photos : `http://www.lmm.jussieu.fr/~lagree/SIEF/SIEF97/sieft97m.html`

To define a model, we notice that the boundary conditions are different from KdV as the levels are not the same downstream and upstream.

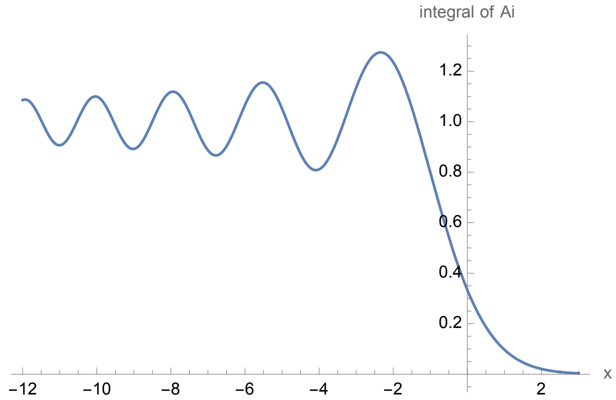

FIGURE 14 – Dispersive infinitesimal small jump : integral of Airy).

If we neglect non linear terms, the KdV equation is

$$\frac{\partial}{\partial t}\eta + c_0\frac{\partial}{\partial x}\eta + \frac{c_0 h_0^2}{6}\frac{\partial^3\eta}{\partial x^3} - \nu\frac{\partial^2\eta}{\partial x^2} = 0$$

with $c_0 = \sqrt{gh_0}$. It is a first good model for ondular bore, as the solution of this equation is with $\int Ai$ which is the solution that has two different level at $\pm\infty$, see figure 14

If we take the kdV with an initial jump as initial condition

$$\frac{\partial}{\partial t}\eta + c_0\frac{\partial}{\partial x}\eta + \frac{3c_0}{2h_0}\eta\frac{\partial\eta}{\partial x} + \frac{c_0 h_0^2}{6}\frac{\partial^3\eta}{\partial x^3} = 0$$

with $c_0 = \sqrt{gh_0}$, there is the formation of a train of solitary waves.

According to Whitham [17] the ondular bore equation is then the modified kdV with an extra *ad hoc* term

$$\frac{\partial}{\partial t}\eta + c_0\frac{\partial}{\partial x}\eta + \frac{3c_0}{2h_0}\eta\frac{\partial\eta}{\partial x} + \frac{c_0 h_0^2}{6}\frac{\partial^3\eta}{\partial x^3} - \nu\frac{\partial^2\eta}{\partial x^2} = 0$$

with $c_0 = \sqrt{gh_0}$ and ν an *ad hoc* viscous coefficient necessary for the model but without real physical significance. This dissipation avoids the escape of the train of solitary waves.
This is a dissipation which prevent the formation of a train of solitons.

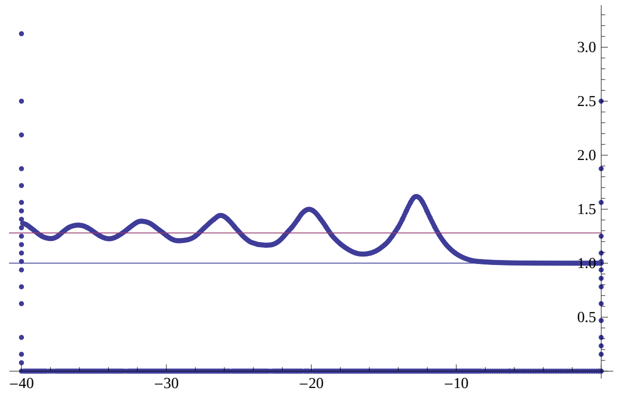

FIGURE 15 – A "Mascaret" with *Gerris*.

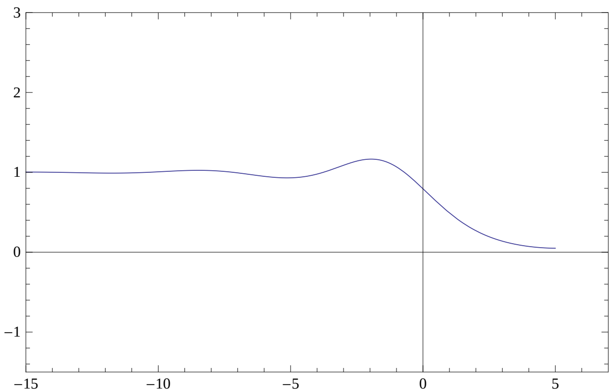

FIGURE 16 – A "Mascaret" in the moving frame, numerical solution of the reduce equation $y''(x) - my'(x) - y(x) + y(x)^2 = 0$, "Model bore structure" see figure 13.6 page 484 of Whitham [17]

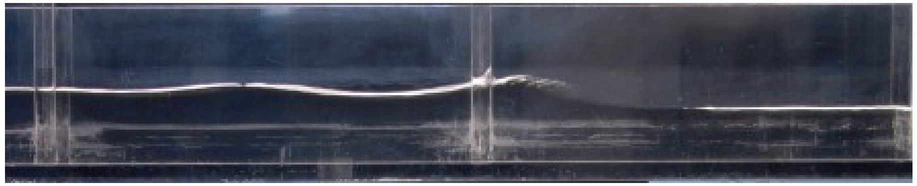

FIGURE 17 – ondular bore in a channel ENSTA experimental lab. Batterie de l'Yvette, photo PYL. See other films : `http://www.lmm.jussieu.fr/~lagree/SIEF/SIEF97/MAQUETTE/mascaret.html`

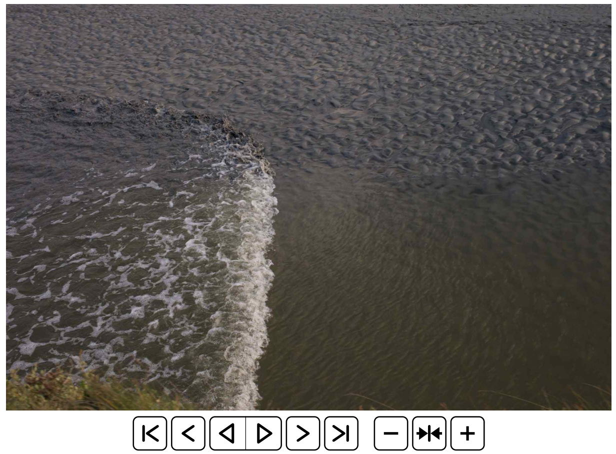

FIGURE 18 – A hydraulic jump is metamorphosed in a undular bore due to a small increase in depth. photo PYL, Baie de la Fresnaye (22) Port à la Duc 2010. [click to launch the movie, Adobe Reader required]

FIGURE 19 – Some meters down stream, the hydraulic jump changes ... into an undular bore photo PYL, Baie de la Fresnaye (22) Port à la Duc.

 

FIGURE 20 – left a very non linear wave Miami 2016, right a mascaret Saint Pardon 1997...

8 Conclusion

In this chapter, we observed waves in water. First, we study waves of small amplitude $\varepsilon \ll 1$ in shallow water $\delta \ll 1$. This gives the ∂'Alembert wave equation. Second, we study waves of small amplitude $\varepsilon \ll 1$ in deep water $\delta = 1$. This is Airy wave theory. Third, we study waves in not small amplitude $\varepsilon = 1$ in shallow water $\delta \ll 1$. This is shallow water. Finally we study waves of small amplitude $\varepsilon \ll 1$ in shallow water $\delta \ll 1$ but no so shallow, with $\varepsilon = \delta^2 \ll 1$. This is Boussinesq KdV theory.

The Soliton and the Ondular Bore are nice examples of waves.

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[18] http://www.lmm.jussieu.fr/~lagree/COURS/MFEnv/MFEnv.pdf M1 lecture on Shallow Water

[19] http://www.lmm.jussieu.fr/~lagree/COURS/MFEnv/MFEhoule.pdf M1 lecture on Airy swell

http://en.wikipedia.org/wiki/Korteweg-de_Vries_equation
http://en.wikipedia.org/wiki/Airy_wave_theory

up to date 12 février 2025

This course is a part of a larger set of files devoted on perturbations methods, asymptotic methods (Matched Asymptotic Expansions, Multiple Scales) and boundary layers (triple deck) by $\mathscr{P}$.-$\mathscr{Y}$. $\mathscr{L}$agrée .
The web page of these files is http://www.lmm.jussieu.fr/∼lagree/COURS/M2MHP.

/Users/pyl/ ... /kdv.pdf

54 *Sir James Lighthill and Modern Fluid Mechanics*

Fig. 3.2 At the University of Central Florida, October 1995. Left to right: Lokenath Debnath, Sir James Lighthill and Lady Nancy Lighthill.

FIGURE 21 – From the book "Sir James Lighthill and Modern Fluid Mechanics" by Lokenath Debnath

Codes *Gerris Basilisk*

Gerris

Some *Gerris* code for water waves

```
  mkdir SIM
rm SIM/sim*

gerris2D -m hydrolicjump3Bv.gfs | gfsview2D



# Title: Airy waves
#
# Description:
#
# Author: PYL
Define Uhoul  0.25*sin(omega*t + 2*pi*x/lambda)*cosh(2*pi*y/lambda)/cosh(2*pi/lambda)*(y<1.1)
Define Vhoul  -0.25*cos(omega*t + 2*pi*x/lambda)*sinh(2*pi*y/lambda)/cosh(2*pi/lambda)*(y<1.1)
Define LEVEL2 ((LEVEL-2) *(y<h0+.3)+(LEVEL-4)*(y>=h0+.3))
Define LEVEL1 (((LEVEL-2)*(y<=h0-.3))+(LEVEL*(y>(h0-.3)&&(y<h0+.4)))+(LEVEL-3)*(y>=h0+.4))
Define Nraf 9
# suffit 8 pour houle simple


3 2 GfsSimulation GfsBox GfsGEdge {
# shift origin of the domain
    x = 0.5 y = 0.5 } {
    Global {
      #define LEVEL Nraf
      #define h0 1
      #define RATIO (1.2/1000.)
      #define VAR(T,min,max) (min + CLAMP(T,0,1)*(max - min))
      #define pi 3.141516
      #define eps 1.e-6
      #define lambda 4.0
      #define omega sqrt(2*pi/lambda*tanh(2*pi/lambda))
    }
    PhysicalParams { L = 10 }


    Refine LEVEL2
    VariableTracerVOF T
    VariableFiltered T1 T 1
    Time {end = 100 }

     InitFraction T ((h0  - y))
     Init { } {U = Uhoul*0 V = Vhoul*0  }

# air/water density ratio si T1=0 RATIO  si T1=1 1
    PhysicalParams { alpha = 1./VAR(T1,RATIO,1.) }

    AdaptGradient { istep = 1 } { cmax = 0.0 maxlevel = LEVEL1 } U*T

    ProjectionParams { tolerance = 1.e-3 }
    ApproxProjectionParams { tolerance = 1.e-3 }


    RefineSolid Nraf
    Solid ( y + 0.1*(x-30./2))
    Source V -1.
    Source U 0.0
    RemoveDroplets { istep = 1 } T -2

    OutputTime { step = 2 } stderr
   OutputSimulation { istep = 25} stdout
# noter le format 000
    OutputSimulation { step = 0.25 } SIM/sim-%06.2f.gfs
}

GfsBox {
    left = Boundary {
     BcNeumann U 0
     BcNeumann T 0   }
    top = Boundary
```

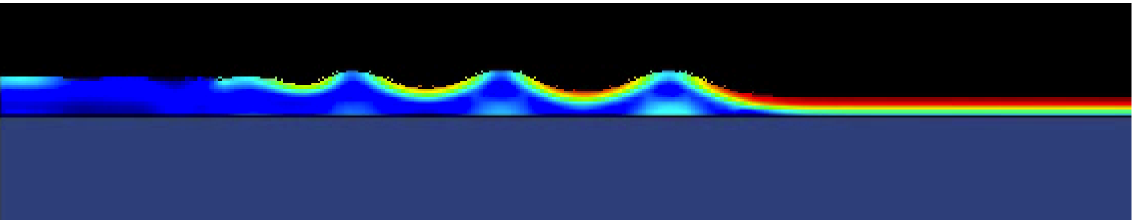
```
    bottom = Boundary  {
    BcDirichlet V 0
}}

GfsBox {
top = Boundary
    bottom = Boundary  {
     BcDirichlet V 0
    }}

GfsBox {
    top = Boundary
    bottom = Boundary  {
BcDirichlet V 0
    }
    right = Boundary {
         BcDirichlet U Uhoul
         BcDirichlet V Vhoul
      BcNeumann T 0
    }
}

1 2 right
2 3 right
```

Improve this code, verify that the dispersion relation works, try to do a solitary wave and a mascaret.



Here code for Saint-Venant hydraulic jump (the bore)

```
# Title: Steady Hydraulic Jump
#
# Description:
#
# Author: PYL
# Command: gerris2D dam.gfs
# Required files: dam.plot
# Generated files: jump.gif
##
#  F1^2  0.375
#  F2^2  0.
#  h1 1
#  h2 0.5
#  W 1.22474
#
#Define L0 10
#
# Use the GfsRiver Saint-Venant solver
1 0 GfsRiver GfsBox GfsGEdge {} {
    PhysicalParams { L = 10 }
    RefineSolid 9
    # Set a solid boundary close to the top boundary to limit the
    # domain width to one cell (i.e. a 1D domain)
    Solid (y/10. + 1./pow(2,9) - 1e-5 - 0.5)
    # Set the topography Zb and the initial water surface elevation P
    Init {} {
Zb = 0
U = 0.387632*(x<-3)+(-.22474*0.5)*(x>-3)
P = {
     double p = x < -3 ? 1  : 0.5;
//      p = 1+(1.30277563773199-1)*(1+tanh(x))/2;
    return MAX (0., p - Zb);
}
    }
    PhysicalParams { g = 1. }
    # Use a first-order scheme rather than the default second-order
    # minmod limiter. This is just to add some numerical damping.
    AdvectionParams {
       # gradient = gfs_center_minmod_gradient
gradient = none
    }
```

```
    Time { end = 7}
    OutputProgress { istep = 10 } stderr
    # Save a text-formatted simulation
    OutputSimulation { step = 0.1 } sim-%g.txt { format = text }
    # Use gnuplot to create gif images
    EventScript { step = 0.1 } {
time=`echo $GfsTime | awk '{printf("%4.1f\n", $1);}'`
cp sim-$GfsTime.txt sim.txt
cat <<EOF | gnuplot
load 'dam.plot'
set title "t = $time"
set term postscript eps color 14
set output "sim.eps"
h(x)= 1-(0.5)*(x>-3+1.*$time)
plot [-5.:5.][0:2]'sim-$GfsTime.txt' u 1:7:8 w filledcu lc 3, 'sim-0.txt'
EOF
time=`echo $GfsTime | awk '{printf("%04.1f\n", $1);}'`
convert -density 300 sim.eps -trim +repage -bordercolor white -border 10 -
rm -f sim.eps
    }
# 1:x 2:y 3:z 4:P 5:U 6:V 7:Zb 8:H 9:Px 10:Py 11:Ux 12:Uy 13:Vx 14:Vy 15:Z
# Combine all the gif images into a gif animation using gifsicle
    EventScript { start = end } {
gifsicle --colors 256 --optimize --delay 25 --loopcount=0 sim-*.gif > mjum
rm -f sim-*.gif sim-*.txt
    }
}
GfsBox {
    left =  Boundary { BcNeumann U 0 }
    right = Boundary { BcNeumann U 0 }
}
```

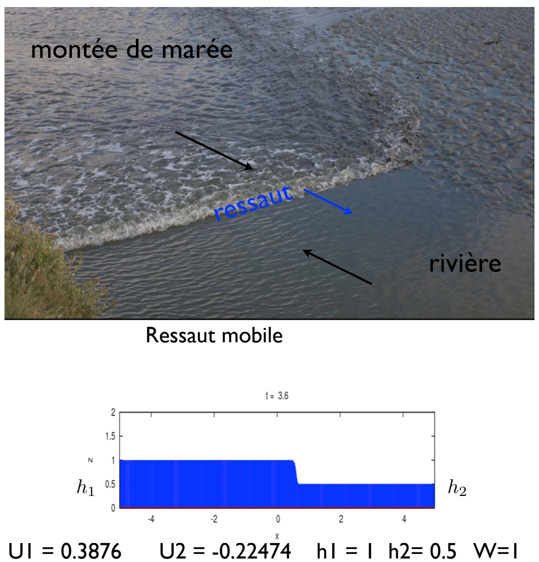


FIGURE 22 – bore at Port à la Duc, baie de la Fresnaye. Photo PYL and with *Gerris*

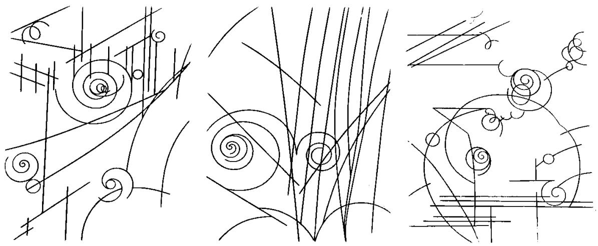
Basilisk

see web

Annex

Multilayer codes...

. .



Raymond Subes ”Sans Titre” 1961 (entrée de Jussieu Quai Saint Bernard)