Small Re flows, $\varepsilon = Re \ll 1$

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version 14 novembre 2019

Résumé

Creeping flows correspond to small Reynolds flows. Although every body knows the Stokes solution around a sphere of radius L moving at velocity U_0 : the force is $6\pi\mu L U_0$, few know that the next orders need Matched Asymptotic Expansion (MAE) to be computed. This is even more frustrating for the 2D flow around a circle, in this case there is **no** solution of the 2D Navier Stokes flow : this is the "Stokes Paradox". Fortunately, the MAE allows to compute the flow, which was done in 1957, and allows to estimate the drag on a cylinder.

This chapter is the "missing chapter" in all the courses of "microhydrodynamics" (because they are few references on this paradox on the web).

We present then the Hele-Shaw cuve with more terms than usually (and discuss the non linear convective terms).

1 Introduction

Small Reynolds flows receive a new impulse nowadays. There is a huge interest in "microhydrodynamics", which means that those flows are slow, and at a small scale, so the Reynolds

$$Re = U_0 L/\nu \ll 1$$

is small. This new interest comes from the fact that lot of applications in biological field have been studied. For example flow around blood cells, around spermatozoids, swimming of microorganisms. Or for example flows in small devices MEMS, lab on chip.... But Stokes flow can be at large scale with slow velocity and high viscosity (in geophysics, flow in porous media, flow of lava or ice (as a very first approximation)), flow of glass (window process) etc.

Those flows are called either "Stokes flow" either "low Reynolds flows", or also named "creeping flows" or "creeping motion", in french "écoulement rampant".

FIGURE 1 – A typical problem a body of length L in a uniform velocity U_0 ; the Reynolds number is small $Re = U_0 L/\nu << 1$.

Historically, it is one of the first solutions obtained by George Gabriel Stokes 1819-1903 : the flow around a sphere when advective inertial forces are small compared to viscous forces.

We will see that this solution exists by "chance" around a sphere, and that the viscous flow around a cylinder can not be computed leading to the Stokes paradox.

It puzzled Stokes himself in 1851 and later Oseen, 1910 ; Lamb, 1911... It can be understood through the use of matched asymptotic expansions (MAE) is one of the triumphs of perturbation theory. See Van Dyke (1964) who presents the work of Kaplun (1957) and Proudman and Pearson (1957) who found the solution.It seems that the ideas cames from systematic applications of MAE by Lagerstorm and its students. Proudman and Pearson were inspired by a paper of Lagerstorm & Cole (1955), and Kaplun (1957) found more terms than they did. Proudman and Pearson (1957) acknowledge the independent work of Kaplun in their paper (Saul Kaplun was a kind of genius of MAE, but he died at the age of 40 in 1964).

2 Small Reynolds flows

2.1 Incompressible Navier Stokes equations

The problem that we have to solve is the problem of the solution of Navier Stokes equations around a given body at small Reynolds number. Reynolds number Re is constructed with a velocity (U_0) and a typical length (L). We suppose that the flow is laminar,which is always the case in practice at small Renolds, or small enough. We will describe 2D or axi flows. The flow is supposed steady and incompressible.

So, we first non-dimensionalise the equations with L (the typical length of the body) and U_0 (the typical velocity) in all directions of space and velocity (with "bars" over the variables *i.e.* $\bar{x} = x/L$, $\bar{y} = y/L$, $\bar{u} = u/U_0$, $\bar{v} = v/U_0$ $p = p_0 + \bar{p}P_0$, the reference pressure is here taken to be p_0, this must be changed in compressible flows). For large Reynolds flows, we will take $P_0 = (\rho U_0^2)$, for small Reynolds flows, pressure scales with viscosity rather than with inertia :

$$P_0 = (\rho U_0^2)/Re = \mu U_0/L.$$

The drag is then obviously scaled by $P_0 L^2$ which is $\mu L U_0$ (in this chapter we will indeed find that the prefactor is 6π for a sphere, and a more complicated value for a cylinder).

Boundary conditions are no slip at the wall :
if $F_w(\bar{x}, \bar{y})$ is the implicit equation of the the wall : $\bar{u} = 0$ and $\bar{v} = 0$ on the body, and $u = 1$ and $\bar{v} = 0$ far away from the body. Incompressible steady adimensionalised Navier Stokes equations are :

$$\overrightarrow{\overline{\nabla}} \cdot \overrightarrow{\bar{u}} = 0$$

$$Re(\overrightarrow{\bar{u}} \cdot \overrightarrow{\overline{\nabla}}\, \overrightarrow{\bar{u}}) = -\overrightarrow{\overline{\nabla}} p + \overrightarrow{\overline{\nabla}}^2 \overrightarrow{\bar{u}}$$

These are the relevant non dimensional NS. The small parameter $\varepsilon = Re \ll 1$, at first glance, the ε does not remove second order derivatives. So we think that the problem will be regular. We will just put $Re = 0$ and try to solve the problem.

2.2 Stokes solution for a sphere

Stokes solution corresponds then to the solution of the purely viscous problem with $\varepsilon = 0$, so

$$\overrightarrow{\overline{\nabla}} \cdot \overrightarrow{\bar{u}} = 0 \quad \text{and } 0 = -\overrightarrow{\overline{\nabla}} p + \overrightarrow{\overline{\nabla}}^2 \overrightarrow{\bar{u}}$$

with no slip velocity on the body.

As says Milton Van Dyke [18] page 149 "Every high-school student learns that Millikan calculated the drag of an oil drop using the approximation developed by Stokes in 1851" (that gave to Robert Millikan Nobel prize in 1923). This exact solution is very classical (one of the first among the few). It leads to the famous force on the sphere $6\pi\mu R U_0$. As said by Milton, the solution is in every text book (Kundu [8], Guyon Hulin & Petit [7]... see Landau [12] for an elegant alternative), it is presented usually with the stream function for a flow in spherical coordinates :

$$\bar{u}_r = \frac{1}{\bar{r}^2 \sin\theta}\frac{\partial \bar{\psi}}{\partial \theta}, \quad \bar{u}_\theta = -\frac{1}{\bar{r}\sin\theta}\frac{\partial \bar{\psi}}{\partial \bar{r}}$$

as, to remove gradient of pressure, the rotational of equation ($0 = -\overrightarrow{\overline{\nabla}} p + \overrightarrow{\overline{\nabla}}^2 \overrightarrow{\bar{u}}$) gives the "Laplacian" of the "curl," the final equation is a "bilaplacian" :

$$\overrightarrow{\overline{\nabla}}^2 \overrightarrow{\overline{\nabla}}^2 \bar{\psi} = 0$$

which reads

$$[\frac{\partial^2}{\partial \bar{r}^2} + \frac{\sin\theta}{\bar{r}^2}\frac{\partial}{\partial\theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta})]^2\bar{\psi} = 0, \tag{1}$$

with on the circle, $\bar{r} = 1$ the no slip condition

$$\bar{\psi}(1,\theta) = \frac{\partial \bar{\psi}}{\partial \bar{r}}(1,\theta) = 0,$$

and far away the free stream of unit velocity

$$\bar{\psi}(\infty,\theta) = \frac{1}{2}\bar{r}^2 \sin^2\theta.$$

Solving a Laplacian is often done with separation of variables, $F(\bar{r})G(\theta)$. We guess that $G(\theta)$ will involve sine and cosine. One of the trick is to look at $\bar{\psi} = f(\bar{r})\sin^2\theta$, because the far field solution has this structure, and after substitution and integration, and use of B.C. and fact that $\bar{r}^4$ solutions are not possible, so the final solution of eq. 1 is :

$$\bar{\psi} = \bar{r}^2 \sin^2\theta(\frac{1}{2} - \frac{3}{4\bar{r}} + \frac{1}{4\bar{r}^3}).$$

The velocity derives from it :

$$u = \frac{1}{\bar{r}^2\sin\theta}\frac{\partial}{\partial\theta}\bar{\psi}, \text{ and } v = -\frac{1}{\bar{r}\sin\theta}\frac{\partial}{\partial\bar{r}}\bar{\psi},$$

which is :

$$\bar{u} = \cos\theta(1 - \frac{3}{2\bar{r}} + \frac{1}{2\bar{r}^3}) \text{ and } \bar{v} = \sin\theta(1 - \frac{3}{4\bar{r}} - \frac{1}{4\bar{r}^3})$$

then one computes the pressure and the viscous stress components on the sphere,

$$[-\bar{p}\cos\theta + \bar{\sigma}_{rr}\cos\theta - \bar{\sigma}_{r\theta}\sin\theta]_{\bar{r}=1},$$

with

$$\bar{p}(\bar{r},\theta) = -\frac{3\cos\theta}{2\bar{r}^2} \quad \text{on the sphere } \bar{p}(1,\theta) = -\frac{3\cos\theta}{2}.$$

So that the final total drag is 6π. With dimensions (multiplied by the scale of pressure which is $\mu U_0/L = \rho U_0^2/Re$ and by the scale of surface L^2), this gives the famous law for the drag :

$$D = 6\pi\mu L U_0$$

Notice that if the drag is scaled by the Bernoulli pressure $\rho U_0^2/2$ which is usual for high Reynolds flows, and the radius L of the sphere (projected area πL^2), at small Reynolds constructed on the diameter : $Re_D = 2UL/\nu$:

$$C_D = \frac{D}{\frac{\pi L^2}{2}\rho U_0^2} = \frac{24}{Re_D}.$$

See figure 8 which represents the C_D as function of Re.

Alternative way to solve the Stokes problem may be found, see Landau [11] §20 for the elegance of the use of invariances.

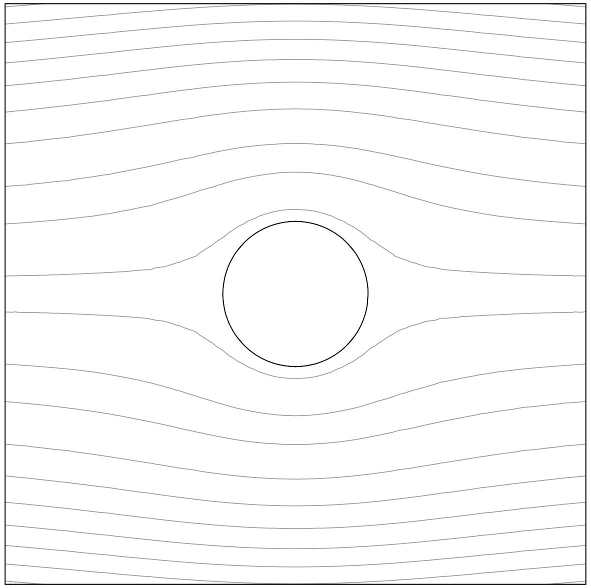

FIGURE 2 – flow around a sphere at $Re = 0$. iso stream function $\bar{\psi} = \bar{r}^2 \sin^2\theta(\frac{1}{2} - \frac{3}{4\bar{r}} + \frac{1}{4\bar{r}^3})$

2.3 Terminal velocity

This famous drag relation $D = 6\pi\mu L U_0$ gives us the terminal velocity in a gravity field g which corresponds to the balance of weight and viscous drag : $\rho_s(4/3)\pi R^3 = 6\pi\mu R U_0$ so that

$$U_0 = \rho_s \frac{g(2R)^2}{18\mu}.$$

In the buoyant case, one has to replace $\rho_s(4/3)\pi R^3$ by $(\rho_s - \rho)(4/3)\pi R^3$ so that

$$U_0 = \Delta\rho \frac{g(2R)^2}{18\mu}.$$

The Reynolds number

$$Re = \frac{\rho U_0 L}{\mu} \propto \rho \frac{g\Delta\rho(R)^3}{\mu^2}$$

with $A = \rho\frac{g\Delta\rho(R)^3}{\mu^2}$ the Archimedes number.

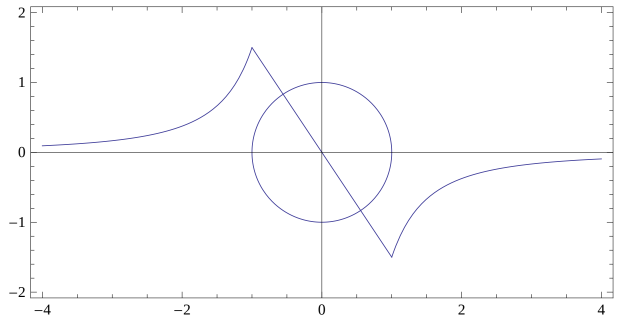

FIGURE 3 – Pressure along x for $Re = U_0 L/\nu = 0$ for flow around a sphere at $Re = 0$.

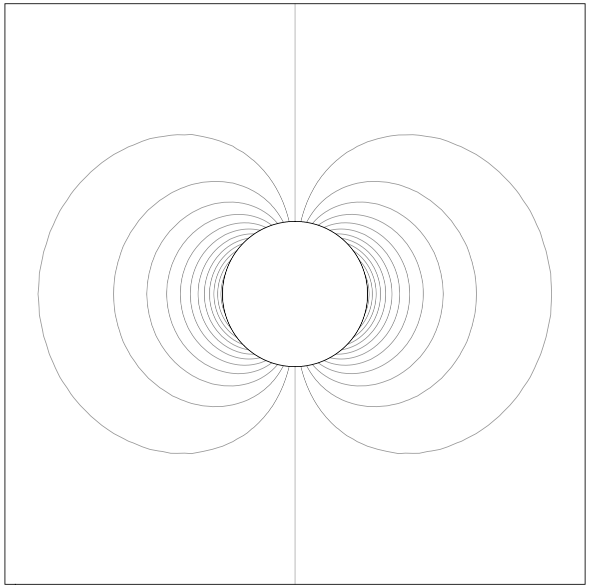

FIGURE 4 – iso Pressure for flow around a sphere at $Re = U_0 L/\nu = 0$.

2.4 Hadamard-Rybczynski Solution for rising bubbles

The general solution for a Stokes flow in a uniform stream is :

$$\bar{\psi} = \bar{r}^2 \sin^2\theta(\frac{1}{2} + \frac{A}{\bar{r}} + \frac{B}{\bar{r}^3}).$$

The Stokes flow corresponds to the flow around a sphere, were velocity is 0. The solution with no slip at the wall is $A = -3/4$, $B = 1/4$. If we compute the force it

FIGURE 5 – rising bubbles, those are too large to apply Hadamard-Rybczynski theory. Those are Taylor bubbles which are inertial, so that drag is scaled by $\rho U_0^2 L^2$.

is

$$D = \frac{4}{3}\pi\mu U_0 L(4 + 2A + 8B),$$

again, this gives for $A = -3/4$ and $B = 1/4$, the Stokes solution :

$$D = 6\pi\mu L U_0$$

There is another solution from Hadamard and Rybczynski (both in 1911) who modelised a bubble as a sphere with zero shear stress. Then $A = -1/2$ and $B = 0$ and the drag is

$$D = 4\pi\mu U_0 L.$$

In practice, a real bubble is in between (as the air is moving in it), and surfactants at the surface change the properties and the boundary condition.

One has to note that the potential flow solution is $A = 0$ and $B = -1/2$ and that the drag is

$$D = 0.$$

2.5 Oseen criticism (sphere at next order)

Carl Oseen (Lund 1879 Uppsala 1944) has remarked that far away from the sphere, at location $\bar{x} = O(1/Re)$, the neglected term of convection is no more negligible. Indeed, $Re(\overrightarrow{u} \cdot \overrightarrow{\nabla} \overrightarrow{u})$ which is almost $(Re\partial_{\bar{x}} \overrightarrow{u})$ is no more small compared to the viscous term $(\partial^2_{\bar{x}} + \partial^2_{\bar{y}} + \partial^2_{\bar{z}})\overrightarrow{u}$.

This is clear is we do the change of scale $\bar{x} = (1/Re)\tilde{x}$, so

$$Re\partial_{\bar{x}} = Re^2\partial_{\tilde{x}}$$

and

$$(\partial^2_{\bar{x}} + \partial^2_{\bar{y}} + \partial^2_{\bar{z}})\overrightarrow{u} = Re^2(\partial^2_{\tilde{x}} + \partial^2_{\tilde{y}} + \partial^2_{\tilde{z}})\overrightarrow{u}.$$

This is just as in the Graetz problem, so the inertia is as large as $(-\overrightarrow{\nabla} p + \overrightarrow{\nabla}^2 \overrightarrow{u})$. Hence, the inertial term is no more negligible...

He proposed to look at perturbation (with a prime) of a uniform flow along $\overrightarrow{e}_x$ which is the basic solution, so that

$$\overrightarrow{u} = \overrightarrow{e}_x + \overrightarrow{u}'.$$

The Oseen correction is linear :

$$\overrightarrow{\nabla} \cdot \overrightarrow{u}' = 0$$

$$Re(\frac{\partial}{\partial \bar{x}} \overrightarrow{u}') = -\overrightarrow{\nabla} p' + \overrightarrow{\nabla}^2 \overrightarrow{u}'$$

This can be solved (see Kundu [8], his Re is $2Re$)

$$\bar{\psi} = \bar{r}^2 \sin^2\theta(\frac{1}{2} + \frac{1}{4\bar{r}^3}) - \frac{3}{2Re}(1+\cos\theta)(1 - e^{-\frac{2Re\bar{r}}{4}(1-\cos\theta)})$$

If Re is small, the expansion of the exponential gives :

$$\frac{3}{2Re}(1+\cos\theta)(1 - e^{-\frac{2Re\bar{r}}{4}(1-\cos\theta)}) = -\frac{3}{Re}(1\cos^2\theta)\frac{Re\bar{r}}{4} = -\frac{3}{4\bar{r}}$$

so we have the Stokes solution from the Oseen solution when $Re \to 0$. See figure 7 for a comparison at $Re = 0.05$. One should nevertheless notice that $\bar{\psi}$ is not zero on the sphere with the Oseen solution (but is $O(Re)$).

After some algebra (Kundu [8], Lamb [10]), the correction to the drag is obtained from Oseen approximation :

$$D = 6\pi\mu L U_0(1 + \frac{3}{8}U_0 L/\nu)$$

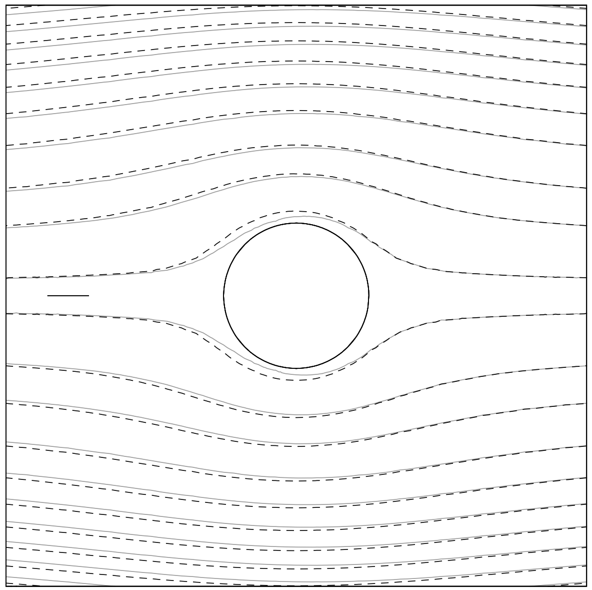

FIGURE 6 – flow around a sphere at $Re = 0$ (Stokes solution, dash), and at $Re = 0.05$ Oseen solution, plain. Note the small ($O(Re)$) transport of the stream lines.

2.6 Criticism of Oseen criticism

In his book Lamb [10] did a paragraph called "Oseen criticism" explaining the above theory. In fact, this linearisation is not true, as the velocity changes all along distance. But by chance, the result, $(\frac{3}{8}U_0L/\nu)$, is good ! Whitehead wanted to solve the next order, he did not succeed to solve the problem by iteration (finding the $(U_0L/\nu)^2$ term), this is called the "Whitehead paradox" 1889. The problem has been solved by Proudman and Pearson [14] and simultaneously by Lagerstorm and Kaplun (1957) ; with the matched asymptotic expansion they obtained after some (long long) algebra :

$$D = 6\pi\mu L U_0(1 + \frac{3}{8}Re + \frac{9}{40}Re^2 Log(Re) + ...) \quad Re = U_0 L/\nu$$

notice that for $Re = 0.15$, we compute $3Re/8 = 0.055$ and $\frac{9}{40}Re^2 Log(Re) = -0.01$, which is small enough...

The expected development was Re^0, Re, Re^2, Re^3.... The fact that a new unexpected term arises $Re^2 Log(Re)$ and interplays $Re \gg Re^2 Log(Re) \gg Re^2$... was called "Switchback" by S. Kaplun [16] "in trying to find terms of a certain order one is forced to reconsider lower order terms." (in "fluid mechanics and singular perturbations collection of paper by Kaplun editor Lagerstrom)

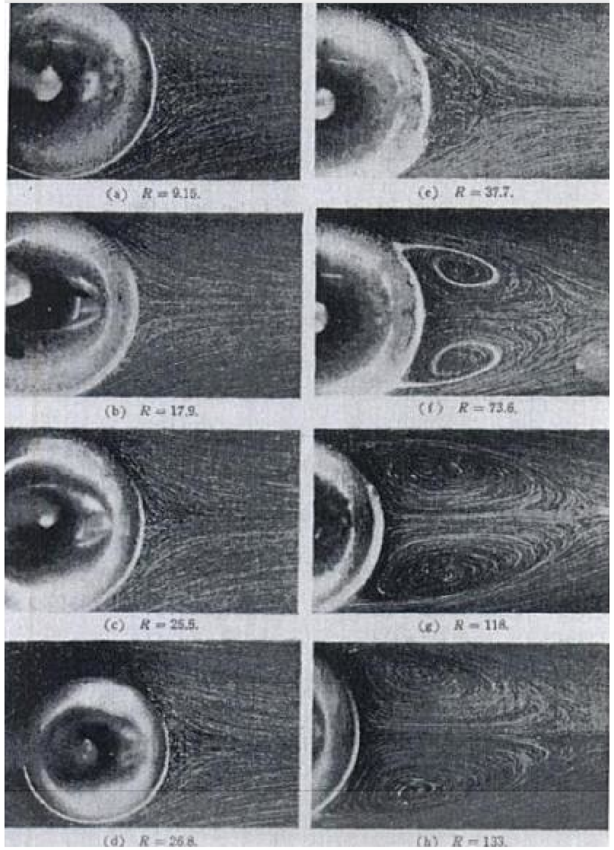

FIGURE 7 – Flow at various Re around a sphere from Taneda 1956

We will not present this matched asymptotic expansion in axi symmetrical flow around a sphere but we will present it in the next section all the details in 2D around a cylinder. Before, we plot the fields with a Navier Stokes solver.

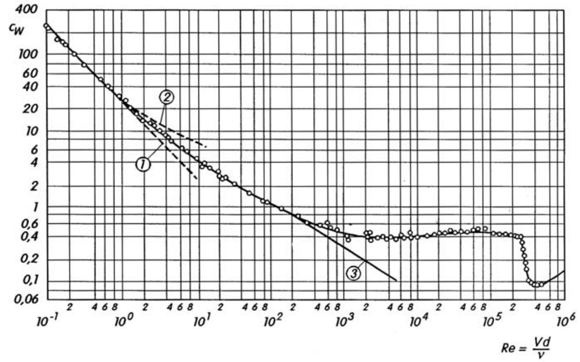

Bild 1.19. Widerstandsbeiwert von Kugeln in Abhängigkeit von der Reynolds-Zahl
Kurve 1: Theorie nach G.G. Stokes (1856), $c_W = 24/Re$
Kurve 2: Theorie nach C.W. Oseen (1911), $c_W = (24/Re)[1 + 3 Re/16]$
Zur Erweiterung dieser Gleichung für höhere Reynolds-Zahlen vgl. M. Van Dyke (1964b)
Kurve 3: Numerische Ergebnisse nach B. Fornberg (1988)

Einsetzen instationärer Strömung bei Re = 200, vgl. U. Dallmann et al. (1993)

FIGURE 8 – From Schlichting Gersten (Grenzsischt Theorie, Springer) c_D on a sphere as function of Reynolds based on diameter. Oseen and corrections are good for $Re < 5$, at $Re = 10^3$ the flow is turbulent and C_D remains constant, for $Re \sim 3\ 10^5$, C_D changes rapidly from about 0.4 to 0.1 : the flow pattern changes, the turbulent wake is narrower (this is strongly dependent of the defects on the sphere surface). This sudden drop is called "drag crisis".

2.7 With *Gerris*

```
 # PYL oct 2012
#gerris2D -DLEVEL=13 -DRe=0.1 sphre.gfs | gfsview2D v.gfv

1 0 GfsAxi GfsBox GfsGEdge {} {
    Time { end = 10 dtmax = 100.e-3}
    PhysicalParams { L = 200 }
    Refine 8
    Refine (LEVEL + 1./50.*(x*x + y*y)*(4. - LEVEL))
    Solid (ellipse (0., 0., 1, 1))
#   SourceViscosity 1. { beta = 1 }
#    SourceViscosity 1./Re { beta = 1 tolerance = 1e-4}
    SourceViscosity 1./Re { beta = 1 }
#   PhysicalParams { alpha = 1./Re }

    Init {} { U = 1.0 }
    AdaptGradient { istep = 1 } { cmax = 5e-2 minlevel = 5  maxlevel = LEVEL } U
    AdaptGradient { istep = 1 } { cmax = 5e-2 minlevel = 5  maxlevel = LEVEL } V
    AdaptFunction { istep = 1 } { cmax = 1e-2 minlevel = 5  maxlevel = LEVEL } {
return (fabs(dx("U"))+fabs(dy("U")))/fabs(U)*ftt_cell_size (cell);
    }

    EventStop { istep = 50 } U 1e-4 DU
    OutputTime { istep = 20 } stderr
    OutputSimulation { istep = 20 } stdout
    OutputSimulation { start = end } end.gfs

}
GfsBox {
    left = Boundary { BcDirichlet U 1.0 }
    right =  Boundary {
        BcNeumann U 0.
        BcNeumann V 0
#        BcDirichlet P 0
    }
    top =  Boundary {
        BcDirichlet U 1.0
        BcNeumann V 0
        BcDirichlet P 0
    }

    bottom = Boundary
}
```

put it in `sphre.gfs` and then put in `run.sh` the following script

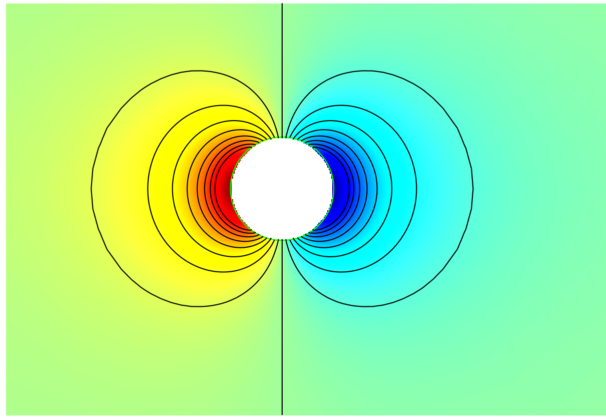

FIGURE 9 – iso Pressure $Re = U_0L/\nu = 0.05$ with *Gerris*, colors and exact solution $-\bar{3}x/(\bar{x}^2+\bar{y}^2)^{3/2}/Re$

```
#!/bin/sh
export LANG=C
for Re in 1.0 .10 0.01
do
 echo $Re
 gerris2D -DLEVEL=13 -DRe=$Re  sphre.gfs | gfsview2D v.gfv
 cp end.gfs end$Re.gfs

 gfs2oogl2D -c P -o -i < end.gfs | \
    awk '{print 3.14159265359 - atan2($2,$1),$4}' | \
    sort -k 1,2 > cp$Re.txt
done

cp end.gfs test.gfs

#######coupes en y=0
file="cuty0.dat"
    awk 'BEGIN{
      for (a = -100.; a <= 100; a += 0.01)
       { b=0;
        print a " " b " 0.0 "; }
    }' > $file
gerris2D -e "OutputLocation {  }  dat0.dat cuty0.dat" test.gfs > /dev/null
```

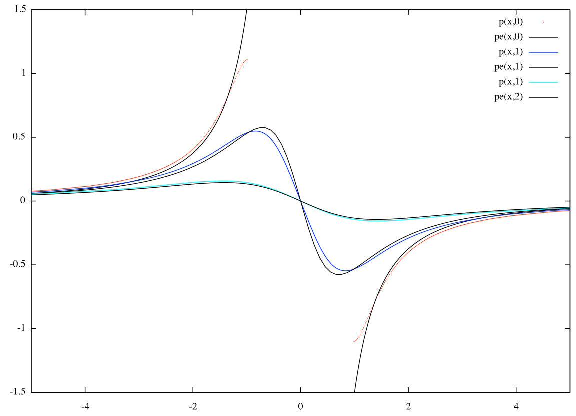

FIGURE 10 – pressures for $Re = U_0L/\nu = 0.005$ with *Gerris* and Stokes solution along the axis $y_1 = 0$, at a distance $y_1 = 1$ and $y_1 = 20$ from the axis : $\bar{p}_e = -3\bar{x}/(\bar{x}^2 + \bar{y}_1^2)^{3/2}$.

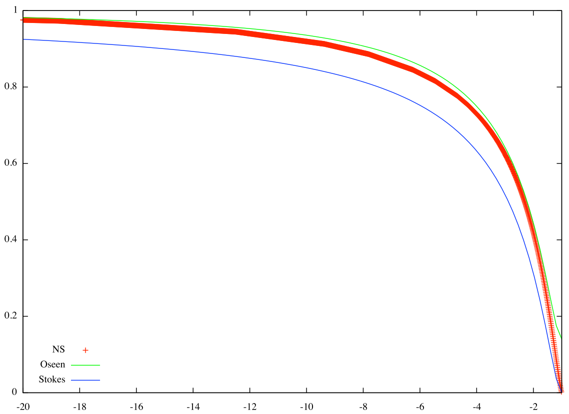

FIGURE 11 – Velocity $\bar{u}(\bar{r}, \theta = \pi)$ for $Re = U_0L/\nu = 0.1$ with *Gerris* and Oseen $-(-1 + 1/(2. * (x^3)) + 3/((x^2) * 2 * Re) - 3./(exp((-x * 2 * Re)/2.) * (x^2) * 2 * Re))$ and Stokes $(1 - 3/2./abs(x) + 1./2/abs(x)/x/x)$ solutions along the axis $y_1 = 0$, The Stokes solution is different, the Oseen and numerics are very close, the difference may be explained du to the fact that Oseen is an approximation. The decrease is very slow, the box is of size 200 compared to the unit radius.

3 Stokes paradox

3.1 The 2D Stokes problem, flow around a circle

3.1.1 Equations

In this subsubsection we will present the Stokes problem, how it leads to a paradox (sub subsection 3.1.2). Then we will show that there are two layers, and two problems in subsection 3.2 and the definite solution with MAE in subsection 3.3.

In 2D, let us start again just as in the case of sphere, but now we try to find the solution around a circle. We are indebted to François [3] who presents a clear synthetic solution that we will follow. Navier Stokes equations, scaled with L and U_0 are in 2D $(\bar{x}, \bar{y})$:

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial \bar{x}} + \dfrac{\partial \bar{v}}{\partial \bar{y}} = 0, \\ Re(\bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\dfrac{\partial \bar{u}}{\partial \bar{y}}) = -\dfrac{\partial \bar{p}}{\partial \bar{x}} + (\dfrac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{u}}{\partial \bar{y}^2}), \\ Re(\bar{u}\dfrac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\dfrac{\partial \bar{v}}{\partial \bar{y}}) = -\dfrac{\partial \bar{p}}{\partial \bar{y}} + (\dfrac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{v}}{\partial \bar{y}^2}). \end{cases} \quad (2)$$

let us introduce the stream function with straightforward scale

$$\psi = LU_0\bar{\psi}$$

and so that :

$$\bar{u} = \frac{\partial \bar{\psi}}{\partial \bar{y}}, \quad \bar{v} = -\frac{\partial \bar{\psi}}{\partial \bar{x}}.$$

Navier Stokes reads

$$Re((\frac{\partial \bar{\psi}}{\partial \bar{y}}\frac{\partial}{\partial \bar{x}} - \frac{\partial \bar{\psi}}{\partial \bar{x}}\frac{\partial}{\partial \bar{y}})\overline{\overrightarrow{\nabla}}^2\bar{\psi}) = \overline{\overrightarrow{\nabla}}^2\overline{\overrightarrow{\nabla}}^2\bar{\psi}.$$

It seems simple to write the expansion

$$\psi = \psi_0 + Re\psi_1 + ...$$

the problem at order 0 is the Stokes flow around a circle

$$\overline{\overrightarrow{\nabla}}^2\overline{\overrightarrow{\nabla}}^2\bar{\psi}_0 = 0,$$

and the order one

$$((\frac{\partial \bar{\psi}_0}{\partial \bar{y}}\frac{\partial}{\partial \bar{x}} - \frac{\partial \bar{\psi}_0}{\partial \bar{x}}\frac{\partial}{\partial \bar{y}})\overline{\overrightarrow{\nabla}}^2\bar{\psi}_0) = \overline{\overrightarrow{\nabla}}^2\overline{\overrightarrow{\nabla}}^2\bar{\psi}_1,$$

may be obtained from the order 0. This is exactly as in 3D.

3.1.2 The paradox

But unfortunately, there is no solution in 2D of the Stokes problem :

$\overline{\overrightarrow{\nabla}}^2\overline{\overrightarrow{\nabla}}^2\bar{\psi}_0 = 0$ with, on the circle $\bar{\psi}_0 = \partial_{\bar{n}}\bar{\psi}_0 = 0$ and far from the circle $\bar{\psi}_0 = y$.

This is the **Stokes Paradox.**

More precisely the bi-Laplacian : $\overline{\overrightarrow{\nabla}}^2\overline{\overrightarrow{\nabla}}^2\bar{\psi}_0 = 0$ with on the circle $\bar{\psi}_0 = \partial_{\bar{n}}\psi_0 = 0$, in polar coordinates, and with a good guess (as in 3D) $\bar{\psi}_0 = f(\bar{r})\sin\theta$ has the general solution :

$$f(\bar{r}) = A\bar{r}^3 + B\bar{r}Log(\bar{r}) + C\bar{r} + \frac{D}{\bar{r}}.$$

After removing spurious solutions and using the no slip condition it is :

$$\bar{\psi}_0 = D\sin\theta(2\bar{r}Log(\bar{r}) - \bar{r} + \frac{1}{\bar{r}}),$$

but we can not fit the boundary condition far away, due to the logarithmic term. If now, we try to fit the condition at infinity and on the circle $f(1) = 0$, we have :

$$f(\bar{r}) = C\bar{r} - \frac{C}{\bar{r}}$$

but now we do not have $f'(1) = 0$.

This the Stokes paradox, there is no possible set of values of constants to obtain the good boundary conditions.

This paradox has puzzled people during 100 years, for some it was a proof that 2D flow can not exist, or that steady flow can not exit... But we will see that the paradox may be solved in the sequel of this course.

3.2 The two problems

3.2.1 Observation of the scaling of ψ and of space

All the problem arises because the sequence is not in powers of the small parameter Re. Here we just look at what happens if we do not take LU_0 to scale ψ so say $\psi = \Psi_0\bar{\psi}$ and as well let us use $x = \ell\bar{x}$, $y = \ell\bar{y}$ then we have :

$$\frac{\Psi_0}{\nu}((\frac{\partial \bar{\psi}}{\partial \bar{y}}\frac{\partial}{\partial \bar{x}} - \frac{\partial \bar{\psi}}{\partial \bar{x}}\frac{\partial}{\partial \bar{y}})\overline{\overrightarrow{\nabla}}^2\bar{\psi}) = \overline{\overrightarrow{\nabla}}^2\overline{\overrightarrow{\nabla}}^2\bar{\psi}$$

with Ψ_0/ν is without dimension as we have defined $Re = U_0L/\nu$, so $\Psi_0/\nu = \Psi_0 Re/(U_0L)$. Note that the scale ℓ is not in the Stokes equation at all, any scale is possible. Different cases may arise :

• either $\Psi_0/\nu = \Psi_0 Re/(U_0 L) \ll 1$ or $\Psi_0 \ll (U_0 L)/Re$, so that

$$\overline{\overrightarrow{\nabla}}^2\overline{\overrightarrow{\nabla}}^2\bar{\psi} = 0$$

we will see that we will take $\ell = L$ there.

• either $\Psi_0/\nu = \Psi_0 Re/(U_0 L) = 1$ or $\Psi_0 = (U_0 L)/Re$, so that we have the full Navier Stokes, we will see that we will have to take $\ell >> L$ there.

• either $\Psi_0 Re/(U_0 L) \gg 1$ or $\Psi_0 \gg (U_0 L)/Re$, so that we have a Euler equation

$$\frac{\Psi_0}{\nu}((\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial}{\partial\bar{x}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial}{\partial\bar{y}})\overline{\overrightarrow{\nabla}}^2\bar{\psi}) = 0,$$

we will see that we will have to take $\ell >> L$ there.

It is interesting to notice that Ψ_0 is the relevant scale for the problem, the longitudinal scale ℓ disappears in the equations. The boundary condition far away is $\psi = U_0 y$ so that $\bar{\psi} = (U_0\ell/\Psi_0)\bar{y}$ so that we guess that it is impossible to have only one scale for the length, there is a scale of the size of the radius near the body $\ell = L$ and another one far from it $\ell \gg L$. Furthermore ψ has not the same scale in those layers (see final figure 13 as summary).
Far away is a Euler region (negligible obstacle), and a Navier Stokes around the cylinder. This layer will include a viscous layer (Stokes problem) around the cylinder, the outer Navier Stokes and the Euler regions are the "Oseen" layer.
So we construct two problems, one far, the other near the circle.

3.2.2 First problem or "Oseen problem"

First, from this analysis, the so called Oseen problem (we will understand why latter), far from the body, in which inertia and viscosity are playing a role :

$$\psi = \frac{U_0 L}{Re}\tilde{\psi}, \quad (x, y) = \frac{L}{Re}(\tilde{x}, \tilde{y})$$

we are in $(\tilde{x}, \tilde{y}) = O(1)$, at this scale the cylinder is a point ($(\tilde{x}, \tilde{y})$ are the Oseen variables),

$$((\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial}{\partial\tilde{x}} - \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial}{\partial\tilde{y}})\tilde{\overrightarrow{\nabla}}^2\tilde{\psi}) = \tilde{\overrightarrow{\nabla}}^2\tilde{\overrightarrow{\nabla}}^2\tilde{\psi}$$

with boundary $\tilde{\psi} = \tilde{y}$ far away. Notice that viscosity becomes negligible far from the body, and we may re obtain Euler.
For small $(\tilde{x}, \tilde{y})$ we have to match with the inner problem which comes next.

3.2.3 Second problem or "Stokes Problem"

Second the Stokes problem, near the body (variables scaled by L), were inertia is small

$$\psi = \Psi_0\bar{\psi}, \ (x, y) = L(\bar{x}, \bar{y})$$

Ψ_0 is not known, and we define $(\bar{x}, \bar{y}) = \frac{1}{Re}(\tilde{x}, \tilde{y})$ now, we are in $(\bar{x}, \bar{y}) = O(1)$, the scale of the cylinder ($(\bar{x}, \bar{y})$ are the Stokes variables), $\Psi_0 Re/(U_0 L) \ll 1$ or $\Psi_0 \ll (U_0 L)/Re$,

$$\frac{\Psi_0 Re}{U_0 L}((\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial}{\partial\bar{x}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial}{\partial\bar{y}})\overline{\overrightarrow{\nabla}}^2\bar{\psi}) = \overline{\overrightarrow{\nabla}}^2\overline{\overrightarrow{\nabla}}^2\bar{\psi} \qquad (3)$$

with, as the convection is small compared to the diffusion :

$$\frac{\Psi_0 Re}{U_0 L} \ll 1$$

On the circle $\bar{\psi} = \partial_{\bar{n}}\psi = 0$ and far away we match to the first problem for large $(\tilde{x}, \tilde{y})$.

3.3 Solving the two problems

3.3.1 First problem : Oseen

It seems possible to expand

$$\tilde{\psi} = \tilde{y} + \delta\tilde{\psi}_1 + ...$$

where δ is not known up to now. The flow is nearly not perturbed by the point, the stream remains parallel to x axis.

3.3.2 Second problem : Stokes

Looking at an expansion

$$\bar{\psi} = \bar{\psi}_0 + ...$$

gives the bi-Laplacian :

$$\overline{\overrightarrow{\nabla}}^2\overline{\overrightarrow{\nabla}}^2\bar{\psi}_0 = 0$$

with on the circle $\bar{\psi}_0 = \partial_{\bar{n}}\psi_0 = 0$

The best way is to be in polar coordinates. and a good guess (as in 3D) is to try $\bar{\psi}_0 = f(\bar{r})\sin\theta$ so the general solution is :

$$f(\bar{r}) = A\bar{r}^3 + B\bar{r}Log(\bar{r}) + C\bar{r} + \frac{D}{\bar{r}}$$

we then obtain from the no slip condition on the circle (and as $\bar{r}^3$ is too large at infinity) :

$$\bar{\psi}_0 = D\sin\theta(2\bar{r}Log(\bar{r}) - \bar{r} + \frac{1}{\bar{r}})$$

so we have the Stokes paradox, there is no possible matching with a uniform flow, in this layer. But we have to match with the layer of the first problem.

3.3.3 Van Dyke Matching

Let us write the solution of the Stokes problem $\Psi_0\bar{\psi}_0$ in the Oseen $\tilde{r}$ variable which is $\bar{r} = \frac{\tilde{r}}{Re}$, with $\tilde{r} = O(1)$

$$\Psi_0\bar{\psi}_0 = D\Psi_0\sin\theta(2\frac{\tilde{r}}{Re}Log(\tilde{r}) - 2\frac{\tilde{r}}{Re}Log(Re) - \frac{\tilde{r}}{Re} + \frac{Re}{\tilde{r}})$$

the larger term when Re is small is the term with $Log(Re)$ as $|Log(Re)| \to \infty$ for $Re \to 0$. So that the behavior is :

$$\Psi_0\bar{\psi}_0 = -2D\Psi_0\sin\theta\frac{\tilde{r}}{Re}Log(Re).$$

The solution of the outer problem (first problem or Oseen problem $\tilde{\psi} = \tilde{y} + ...$) written with its scales

$$U_0L\tilde{\psi}_0/Re = U_0L\tilde{y}/Re + ...$$

or in axi coordinates :

$$\frac{U_0L}{Re}\tilde{\psi}_0 = U_0\frac{L}{Re}\tilde{r}\sin\theta + ...$$

hence the $pq - qp$ rule gives

$$-2D\Psi_0\frac{1}{Re}Log(Re) = \frac{U_0L}{Re}$$

so that $D = 1/2$ and

$$\Psi_0 = -\frac{LU_0}{LogRe}$$

we have exhibited the relevant scale for the stream function. Notice that the sign $-LogRe > 0$ for $Re < 1$ and we verify that as expected $\frac{\Psi_0 Re}{U_0L} = \frac{Re}{LogRe} \ll 1$ as $Re \to 0$.

3.3.4 Final first order solution

Near the cylinder, the scale of space is L (Stokes scale) and the stream function has scale $-\frac{LU_0}{LogRe}$ the solution is

$$\bar{\psi}_0 = \sin\theta(\bar{r}Log(\bar{r}) - \frac{\bar{r}}{2} + \frac{1}{2\bar{r}})$$

far from the cylinder the scale of space is L/Re (Oseen scale) and the stream function has scale $\frac{LU_0}{Re}$ the solution is

$$\tilde{\psi}_0 = \tilde{r}\sin\theta$$

The Stokes paradox is solved thanks asymptotics, it was due to this logarithmic term.

3.4 Next order

3.4.1 Next order Stokes variable

If we wish more precision, the work is not finished and is harder and harder.

Near the cylinder, one can compute the next order from the momentum equation with $\Psi_0 = -\frac{LU_0}{LogRe}$ so eq. 3 is with Ψ_0/ν in front of the viscous term :

$$\frac{Re}{-LogRe}((\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial}{\partial\bar{x}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial}{\partial\bar{y}})\vec{\bar{\nabla}}^2\bar{\psi}) = \vec{\bar{\nabla}}^2\vec{\bar{\nabla}}^2\bar{\psi}.$$

The solution is then with the expansion with $\frac{Re}{-LogRe}$ as small parameter :

$$\bar{\psi} = \bar{\psi}_0 + \frac{Re}{-LogRe}\bar{\psi}_1 + ...$$

Unfortunately, the solution of the problem of $\bar{\psi}_1$ is not possible (see François), then, it means that the expansion is not good, it should be with an intermediate term of order ν with $1 \gg \nu \gg \frac{Re}{-LogRe}$. This is the Kaplun's "Switchback" : "in trying to find terms of a certain order one is forced to reconsider lower order terms."

We write

$$\bar{\psi} = \bar{\psi}_0 + \nu\bar{\psi}_1 + O(\frac{Re}{-LogRe})...$$

We have again a bi Laplacian to solve for $\bar{\psi}_1$, the shape of the solution is the same function $\sin\theta(\bar{r}Log(\bar{r}) - \frac{\bar{r}}{2} + \frac{1}{2\bar{r}})$ at a multiplicative constant.

3.4.2 Next order Oseen variable

Far from the cylinder : $\tilde{y}$ is $\tilde{\psi}_0$:

$$\tilde{\psi} = \tilde{y} + \delta\tilde{\psi}_1 + ...$$

so NS are at order δ

$$\frac{\partial}{\partial\tilde{x}}\vec{\tilde{\nabla}}^2\tilde{\psi}_1 = \vec{\tilde{\nabla}}(\vec{\tilde{\nabla}}^2\tilde{\psi}_1)$$

This is the "Oseen approximation" that we described for the sphere! The solution is really complicated in involves modified Bessel function K_n. In fact it has been computed by Lamb in 1911 (for the cylinder ,not by Oseen, who did the sphere in 1910). After algebra, one of the first is a transform (Goldstein transform) of

$$\frac{\partial}{\partial\tilde{x}}\tilde{\phi} = \vec{\vec{\nabla}}\tilde{\phi} \text{ in to } \frac{1}{4}\tilde{\varphi} = \vec{\vec{\nabla}}\tilde{\varphi}$$

thanks to $\tilde{\varphi} = e^{\tilde{x}/2}\tilde{\phi}$ and then by Fourier transform, after some algebra with Bessel functions :

$$\tilde{\psi}_1 = B\tilde{r}(Log(\tilde{r}) - Log(4) + \gamma - 1)$$

we obtain as in the example in the first chapter the Euler constant γ.

Matching both gives $\nu_1 = 1/LogRe$ and $\delta = \frac{1}{LogRe}$ as well, and we have

$$\bar{\psi} = \sin\theta(\bar{r}Log(\bar{r}) - \frac{\bar{r}}{2} + \frac{1}{2\bar{r}})(1 + \frac{Log(4) - \gamma + 1/2}{Log(Re)} + ...),$$

and

$$\tilde{\psi} = \frac{1}{Re}(\tilde{r}sin\theta + \frac{1}{LogRe}\tilde{r}(Log(\tilde{r}) - Log(4) + \gamma - 1).$$

3.5 Drag on a Cylinder at small Re

From the scale of $\Psi_0 = -U_0L/LogRe$ we deduce that the total stress will be $\mu\Psi_0/L^2$ so that the force over the circle will be $\mu\Psi_0/L$ which is :

$$D \sim \frac{\mu U_0}{-Log(\frac{U_0L}{\nu})}$$

In fact, after lot of algebra, the final result (after computing the pressure and the shear at the wall from the function $\bar{\psi}_1$ at the wall) for the drag on a cylinder at small Re is then

$$D = \frac{4\pi\mu U_0}{\frac{1}{2} - \gamma - Log(\frac{U_0L}{4\nu})}$$

Again, this formula as been obtained by Lamb, but in the wrong framework. It has been re formulated by Proudman & Pearson and Kaplun & Lagerstorm who fixed the right framework : Matched Asymptotic Expansions. Most of the ideas were from Kaplun, he disseminate them in conferences, so that Proudman & Pearson found in parallel the final result. Saul Kaplun died at the age of 40 in 1964.

We did not give all of the complicated details, they can be found in those papers. The complete analysis has nearly never been explained in books of fluid mechanics.

We notice :

• that for $Re = 0$ there is no movement at all, but remember that we look at

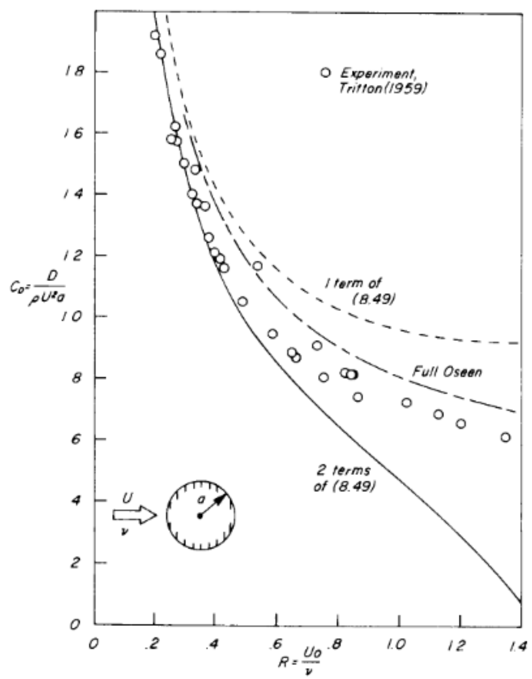

FIGURE 12 – From Van Dyke [18] page 164, drag function of Reynolds for a Cylinder, formula (8.49) in Van Dyke [18] : $C_D = \frac{4\pi}{Re}[\Delta 1 - 0.87\Delta_1^3 + O(\Delta_1^4)]$ with $\Delta_1 = 1/(Log(4/Re) - \gamma - 1/2)$. "Full Oseen" refers to the solution of the Oseen problem $(Re\frac{\partial}{\partial x} - \nabla^2)\nabla^2\psi = 0$ by Tomotika and Aori 1950.

$Re \to 0$, so that $1/LogRe$ is larger than Re, so an even small effect in $1/LogRe$ is not so small compared to Re

• that the sequence is in $1/LogRe$, so the convergence is slow.

• in fact all the Stokes terms are the same in the development so no asymmetry is introduced by the various orders near the cylinder.

• the development fails for $Re = 1$ which is bad!

• the formula is written using $\rho U_0^2/2$ as scale in Van Dyke [18]. The experimental datas and the drag is on figure 12, it reads :

$$C_D = \frac{4\pi}{Re}[\Delta_1 - 0.87\Delta_1^3 + O(\Delta_1^4)] \text{ with } \Delta_1 = \frac{1}{Log(4/Re) - \gamma - 1/2}$$

As says Moffat in the "cours des Houches" 1973 "The complexity of the formula is indicative of the complexity of the underlying analysis".

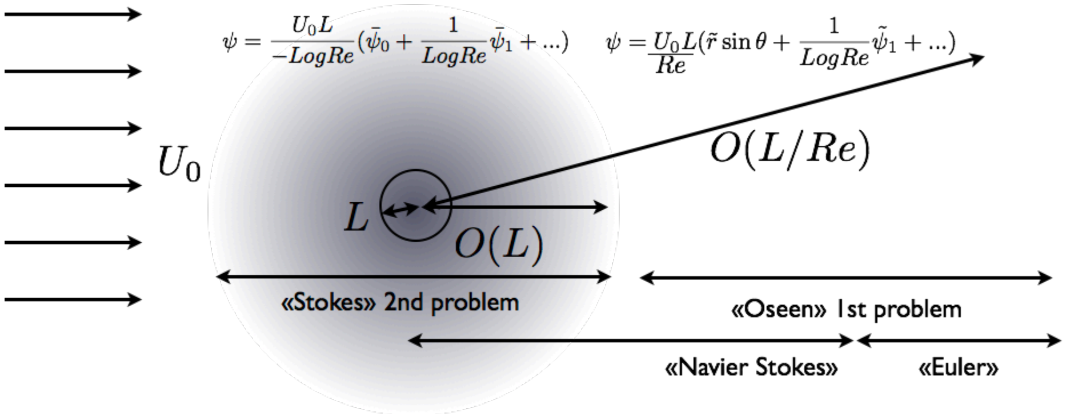

FIGURE 13 – The various regions around the cylinder of radius L in a stream U_0. Far away is a Euler region (negligible obstacle), and a Navier Stokes around the cylinder. This layer includes a "Stokes" layer around the cylinder, the outer Navier Stokes and the Euler regions are the "Oseen" layer. Near the cylinder, we are in the "Stokes" region were $\psi/(U_0L) = \frac{1}{-LogRe}(\bar\psi_0 + \frac{1}{LogRe}\bar\psi_1 + ...)$ and $r = L\bar r$. Far from the cylinder, we are in the "Oseen " region were $\psi/(U_0L) = \frac{1}{Re}(\tilde\psi_0 + \frac{1}{LogRe}\tilde\psi_1 + ...)$ and $r = \frac{L}{Re}\tilde r$

4 Unsteady Stokes

The unsteady equations, $\bar{x} = x/L$, $\bar{y} = y/L$, $\bar{u} = u/U_0$, $\bar{t} = ReU_0 t/L$:

$$\overrightarrow{\bar{\nabla}} \cdot \overrightarrow{\bar{u}} = 0,$$

$$\frac{\partial}{\partial \bar{t}} \overrightarrow{\bar{u}} = -\overrightarrow{\bar{\nabla}} \bar{p} + \overrightarrow{\bar{\nabla}}^2 \overrightarrow{\bar{u}},$$

see Landau §24... ex 5... In Fourier space the solution gives (Graebel [6]/ Landau) : we recognise a fractional derivative when we come back in real time domain, the term is "Basset " and we have added mass.

Added mass or virtual mass is the inertia added to a system because an accelerating or decelerating body must move some volume of surrounding fluid as it moves through it.

5 The Hele Shaw cuve

We will present here the Hele Shaw cuve (Henry Selby Hele-Shaw (1854-1941)). This cuve is a channel which is thin. It is a kind of pipe, as there is no free surface. This pipe has a rectangular section of size $L \times h$, the vertical section h is far smaller than L. A pressure drop produces the flow, as it is a flow in a kind of pipe, we guess that this is a kind of Poiseuille flow. In this pipe, an obstacle will be introduced. As a final result the mean flow around the obstacle behaves as an ideal fluid (even if the flow is very viscous).

See figure 15 for an original picture from Hele-Shaw and 14 for a sketch.

The coordinate x is along the pipe and y and z are in the cross section. The longitudinal and transverse scales (for x and y) are supposed of the same L. We then write $x = L\bar{x}$, and $y = L\bar{y}$ and of course $z = h\bar{z}$. The ratio of scales $\varepsilon = h/L$ is small. The Reynolds number is defined as $Re = U_0 h/\nu$. The pressure drop is ΔP_0, it is imposed to promote the flow. we have $w = \varepsilon U_0 \bar{w}$ by dominant balance of incompressibility.

Equations without dimension :

$$\left\{\begin{array}{l} \dfrac{\partial \bar{u}}{\partial \bar{x}} + \dfrac{\partial \bar{v}}{\partial \bar{y}} + \dfrac{\partial \bar{w}}{\partial \bar{z}} = 0, \\ \varepsilon Re(\bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\dfrac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\dfrac{\partial \bar{u}}{\partial \bar{z}}) = -\varepsilon^2 Re\dfrac{\Delta P_0}{\rho U_0^2}\dfrac{\partial \bar{p}}{\partial \bar{x}} + \varepsilon^2(\dfrac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{u}}{\partial \bar{y}^2}) + \dfrac{\partial^2 \bar{u}}{\partial \bar{z}^2}, \\ \varepsilon Re(\bar{u}\dfrac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\dfrac{\partial \bar{v}}{\partial \bar{y}} + \bar{w}\dfrac{\partial \bar{v}}{\partial \bar{z}}) = -\varepsilon^2 Re\dfrac{\Delta P_0}{\rho U_0^2}\dfrac{\partial \bar{p}}{\partial \bar{y}} + \varepsilon^2(\dfrac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{v}}{\partial \bar{y}^2}) + \dfrac{\partial^2 \bar{v}}{\partial \bar{z}^2}, \\ \varepsilon Re(\bar{u}\dfrac{\partial \bar{w}}{\partial \bar{x}} + \bar{v}\dfrac{\partial \bar{w}}{\partial \bar{y}} + \bar{w}\dfrac{\partial \bar{w}}{\partial \bar{z}}) = -\varepsilon Re\dfrac{P_0}{\rho U_0^2}\dfrac{\partial \bar{p}}{\partial \bar{z}} + \varepsilon^2(\dfrac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{v}}{\partial \bar{y}^2}) + \dfrac{\partial^2 \bar{w}}{\partial \bar{z}^2}. \end{array}\right. \quad (4)$$

the driving pressure ΔP_0 is counter balanced by the viscous gradient across the depth $(\rho U_0^2)/(\varepsilon^2 Re)$, so that $\varepsilon^2 Re\frac{\Delta P_0}{\rho U_0^2} = 1$. It gives the value of the flow velocity in the channel $U_0 = \frac{\Delta P}{\mu L}h^2$. this is exactly the Poiseuille scaling. Hence NS without dimension is :

$$\left\{\begin{array}{l} \dfrac{\partial \bar{u}}{\partial \bar{x}} + \dfrac{\partial \bar{v}}{\partial \bar{y}} + \dfrac{\partial \bar{w}}{\partial \bar{z}} = 0, \\ \varepsilon Re(\bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\dfrac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\dfrac{\partial \bar{u}}{\partial \bar{z}}) = -\dfrac{\partial \bar{p}}{\partial \bar{x}} + \varepsilon^2(\dfrac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{u}}{\partial \bar{y}^2}) + \dfrac{\partial^2 \bar{u}}{\partial \bar{z}^2}, \\ \varepsilon Re(\bar{u}\dfrac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\dfrac{\partial \bar{v}}{\partial \bar{y}} + \bar{w}\dfrac{\partial \bar{v}}{\partial \bar{z}}) = -\dfrac{\partial \bar{p}}{\partial \bar{y}} + \varepsilon^2(\dfrac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{v}}{\partial \bar{y}^2}) + \dfrac{\partial^2 \bar{v}}{\partial \bar{z}^2}, \\ \varepsilon^3 Re(\bar{u}\dfrac{\partial \bar{w}}{\partial \bar{x}} + \bar{v}\dfrac{\partial \bar{w}}{\partial \bar{y}} + \bar{w}\dfrac{\partial \bar{w}}{\partial \bar{z}}) = -\dfrac{\partial \bar{p}}{\partial \bar{z}} + \varepsilon^3(\dfrac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{v}}{\partial \bar{y}^2}) + \varepsilon\dfrac{\partial^2 \bar{w}}{\partial \bar{z}^2}. \end{array}\right. \quad (5)$$

for small enough ε and εRe, then $\frac{\partial \bar{p}}{\partial \bar{z}} = 0$, so $\bar{p} = \bar{P}(\bar{x}, \bar{y})$

$$0 = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}, \quad \text{and } 0 = -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2},$$

give

$$\bar{u} = -\frac{\partial \bar{P}}{\partial \bar{x}}\bar{z}(1-\bar{z}) \quad \text{and } \bar{v} = -\frac{\partial \bar{P}}{\partial \bar{y}}\bar{z}(1-\bar{z})$$

incompressibility gives by integration from the lower boundary were $\bar{w}(\bar{x}, \bar{y}, 0) = 0$:

$$\bar{w} = (\frac{\partial^2 \bar{P}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{P}}{\partial \bar{y}^2})(\bar{z}^2/2 - \bar{z}^3/3)$$

The upper boundary condition ($\bar{w}(\bar{x}, \bar{y}, 1) = 0$) is not reached. This is impossible, so it means that $\bar{w}$ is always 0 :

$$(\frac{\partial^2 \bar{P}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{P}}{\partial \bar{y}^2}) = 0$$

The theory of Hele-Shaw cuve is as follow, the mean velocity $\bar{U} = \int_0^1 \bar{u}d\bar{z}$ is such that

$$\bar{U} = -\frac{1}{6}\frac{\partial \bar{P}}{\partial \bar{x}} \quad \text{and } \bar{V} = -\frac{1}{6}\frac{\partial \bar{P}}{\partial \bar{y}}$$

so that

$$(\frac{\partial^2 \bar{P}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{P}}{\partial \bar{y}^2}) = 0$$

this is a pre `freefem++` tool to solve Laplacians.... and that is indeed the figure 15 from Hele-Shaw were we see the flow around an airfoil.

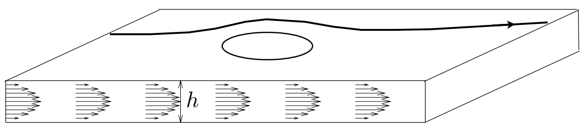

FIGURE 14 – A cuve, the flow across the small dimension is a Poiseuille profile.

5.1 Next orders, averaged method

The influence of the next term may be interesting. As it is complicated, a good idea is to say that the velocity has always the Poiseuille shape, so that

$$\bar u = 6\bar z(1-\bar z)\bar U(\bar x,\bar y) \text{ and } \bar v = 6\bar z(1-\bar z)\bar V(\bar x,\bar y) \text{ and } \bar w = 0$$

were $\bar U(\bar x,\bar y)$ and $\bar V(\bar x,\bar y)$ are the deflected velocity. As $\int_0^1 \bar z(1-\bar z)d\bar z = 1/6$ and $\int_0^1 \bar z^2(1-\bar z)^2 d\bar z = 1/30$, we compute and $\partial_{\bar z}\bar u = \bar U 6(1-2\bar z)$ and we have $\partial_{\bar z}\bar u(\bar{,}\bar y,1)-\partial_{\bar z}\bar u(\bar{,}\bar y,0) = -\bar U$

The final approximative system that we have to consider is :

$$\left\{\begin{array}{l} \dfrac{\partial \bar U}{\partial \bar x} + \dfrac{\partial \bar V}{\partial \bar y} = 0, \\ \dfrac{6}{5}\varepsilon Re(\bar U\dfrac{\partial \bar U}{\partial \bar x} + \bar V\dfrac{\partial \bar U}{\partial \bar y} = -\dfrac{\partial \bar P}{\partial \bar x} + \varepsilon^2(\dfrac{\partial^2 \bar U}{\partial \bar x^2} + \dfrac{\partial^2 \bar U}{\partial \bar y^2}) - 6\bar U \\ \dfrac{6}{5}\varepsilon Re(\bar U\dfrac{\partial \bar V}{\partial \bar x} + \bar V\dfrac{\partial \bar V}{\partial \bar y}) = -\dfrac{\partial \bar P}{\partial \bar y} + \varepsilon^2(\dfrac{\partial^2 \bar V}{\partial \bar x^2} + \dfrac{\partial^2 \bar V}{\partial \bar y^2}) - 6\bar V, \end{array}\right. \tag{6}$$

See Gondret & Rabaud, Plouraboué & Hinch, Loiseleux & Doppler.

Note that the viscous term has an equivalent Reynolds number $1/\varepsilon^2$. And that the convective term is maybe not so small as it is a product of ε which is small, but we can increase Re if we increase the pressure drop. Hence εRe is maybe not small.

5.2 Euler Like

Of course, a good choice of parameters ε the small geometrical ratio and Re the viscosity & pressure drop is to adjust $\varepsilon Re = 1$ in order to have a kind of NS problem (with an equivalent Reynolds $1/\varepsilon^2$) with an extra friction term opposed to the velocity. Hece, at small ε we have kind of ideal fluid equations (no viscous terms with Laplacian) :

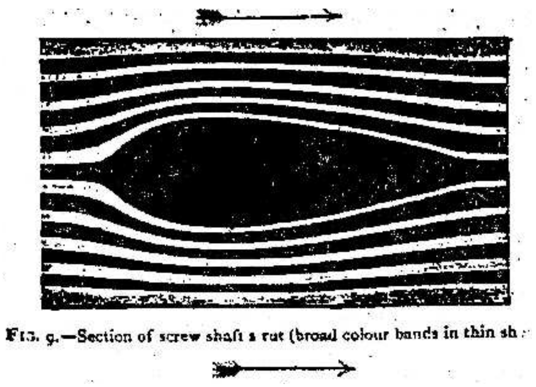

FIGURE 15 – An original photo of Hele-Shaw

$$\left\{\begin{array}{l} \dfrac{\partial \bar U}{\partial \bar x} + \dfrac{\partial \bar V}{\partial \bar y}) = 0, \\ \dfrac{6}{5}(\bar U\dfrac{\partial \bar U}{\partial \bar x} + \bar V\dfrac{\partial \bar U}{\partial \bar y}) = -\dfrac{\partial \bar P}{\partial \bar x} - 6\bar U \\ \dfrac{6}{5}(\bar U\dfrac{\partial \bar V}{\partial \bar x} + \bar V\dfrac{\partial \bar V}{\partial \bar y}) = -\dfrac{\partial \bar P}{\partial \bar y} - 6\bar V, \end{array}\right. \tag{7}$$

5.3 Next orders, residual method

This is the classical simple point of view, more recently, an alternative method has been proposed (popularised by C. Ruyer Quill and others). The change will be in the non linear term. We keep εRe in factor to insist on the perturbative point of view valid for any given Re. Neglecting ε^2 terms in equation (5), we have :

$$\left\{\begin{array}{l} \dfrac{\partial \bar u}{\partial \bar x} + \dfrac{\partial \bar v}{\partial \bar y} + \dfrac{\partial \bar w}{\partial \bar z} = 0, \\ \varepsilon Re(\bar u\dfrac{\partial \bar u}{\partial \bar x} + \bar v\dfrac{\partial \bar u}{\partial \bar y} + \bar w\dfrac{\partial \bar u}{\partial \bar z}) = -\dfrac{\partial \bar p}{\partial \bar x} + \dfrac{\partial^2 \bar u}{\partial \bar z^2}, \\ \varepsilon Re(\bar u\dfrac{\partial \bar v}{\partial \bar x} + \bar v\dfrac{\partial \bar v}{\partial \bar y} + \bar w\dfrac{\partial \bar v}{\partial \bar z}) = -\dfrac{\partial \bar p}{\partial \bar y} + \dfrac{\partial^2 \bar v}{\partial \bar z^2}, \\ \varepsilon^3 Re(\bar u\dfrac{\partial \bar w}{\partial \bar x} + \bar v\dfrac{\partial \bar w}{\partial \bar y} + \bar w\dfrac{\partial \bar w}{\partial \bar z}) = -\dfrac{\partial \bar p}{\partial \bar z} + \varepsilon^3(\dfrac{\partial^2 \bar v}{\partial \bar x^2} + \dfrac{\partial^2 \bar v}{\partial \bar y^2}) + \varepsilon\dfrac{\partial^2 \bar w}{\partial \bar z^2}. \end{array}\right. \tag{8}$$

we define the velocity as Poiseuille one plus a correction, which is small, at fixed $Re = O(1)$. This correction is of order ε and comes from the correction of inertia.

$$\bar u = \bar U 6\bar z(1-\bar z) + \bar u_c \text{ and } \bar v = \bar V 6\bar z(1-\bar z) + \bar v_c$$

this correction being small, let say $O(u_c) = O(v_c)$ which is of order ε, incompressibility gives

$$\bar w = (\frac{\partial \bar U}{\partial \bar x} + \frac{\partial \bar V}{\partial \bar y})(3\bar z^2 - 2\bar z^3) + O(u_c)$$

so as the correction is small a good idea is to keep a global incompressibility equations for $\bar U$ and $\bar V$:

$$(\frac{\partial \bar U}{\partial \bar x} + \frac{\partial \bar V}{\partial \bar y}) = 0$$

and the $\bar w$ is of order $O(u_c)$. Then, the trick is to look at the correction $\bar u_c$ from the momentum. so that the velocity, comes from :

$$\frac{\partial^2 \bar u}{\partial \bar z^2} = \varepsilon Re(\bar u\frac{\partial \bar u}{\partial \bar x} + \bar v\frac{\partial \bar u}{\partial \bar y} + \bar w\frac{\partial \bar u}{\partial \bar z}) + \frac{\partial \bar p}{\partial \bar x}$$

this is integrated twice. We integrate each of the terms. First we start by inertia

$$(\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}+\bar{w}\frac{\partial\bar{u}}{\partial\bar{z}})=36\bar{z}^2(1-\bar{z})^2(\bar{U}\frac{\partial\bar{U}}{\partial\bar{x}}+\bar{V}\frac{\partial\bar{V}}{\partial\bar{y}})+O(u_c)$$

integrated twice $\int_0\int_0(36\bar{z}^2(1-\bar{z})^2)d\bar{z}d\bar{z}=3\bar{z}^4-18\frac{\bar{z}^5}{5}+\frac{6}{5}\bar{z}^6$.

Now looking at the second term, the pressure, $\frac{\partial\bar{p}}{\partial\bar{x}}$, integrated twice $\frac{\partial\bar{p}}{\partial\bar{x}}\frac{\bar{z}^2}{2}$.

We have computed

$$\bar{u}=(\bar{z}^4-\frac{18}{5}\bar{z}^5+\frac{6}{5}\bar{z}^6)(\varepsilon Re)(\bar{U}\frac{\partial\bar{U}}{\partial\bar{x}}+\bar{V}\frac{\partial\bar{V}}{\partial\bar{y}})+\frac{\partial\bar{p}}{\partial\bar{x}}\frac{\bar{z}^2}{2}$$

we substract from this $\bar{U}6\bar{z}(1-\bar{z})$ to obtain the correction field

$$\bar{u}_c=-\bar{U}6\bar{z}(1-\bar{z})+\varepsilon Re(3\bar{z}^4-\frac{18}{5}\bar{z}^5+\frac{6}{5}\bar{z}^6)(\bar{U}\frac{\partial\bar{U}}{\partial\bar{x}}+\bar{V}\frac{\partial\bar{U}}{\partial\bar{y}})+\frac{\partial\bar{p}}{\partial\bar{x}}\frac{\bar{z}^2}{2}$$

we impose that the total flux of u_c is zero $\int_0^1 u_c d\bar{z}=0$ so

$$-\bar{U}+\varepsilon Re(\frac{6}{35})(\bar{U}\frac{\partial\bar{U}}{\partial\bar{x}}+\bar{V}\frac{\partial\bar{V}}{\partial\bar{y}})+(\frac{1}{6})\frac{\partial\bar{p}}{\partial\bar{x}}=0$$

and it gives the final new averaged system :

$$\left\{\begin{array}{l}\frac{\partial\bar{U}}{\partial\bar{x}}+\frac{\partial\bar{V}}{\partial\bar{y}}=0,\\ \frac{36}{35}\varepsilon Re(\bar{U}\frac{\partial\bar{U}}{\partial\bar{x}}+\bar{V}\frac{\partial\bar{U}}{\partial\bar{y}})=-\frac{\partial\bar{P}}{\partial\bar{x}}-6\bar{U}\\ \frac{36}{35}\varepsilon Re(\bar{U}\frac{\partial\bar{V}}{\partial\bar{x}}+\bar{V}\frac{\partial\bar{V}}{\partial\bar{y}})=-\frac{\partial\bar{P}}{\partial\bar{y}}-6\bar{V},\end{array}\right. \quad (9)$$

note that $\frac{36}{35}=1.02$ is not one, and it was $\frac{6}{5}=1.2$ in the simple previous method.

See Gondret & Rabaud, Ruyer Quill, Plouraboué & Hinch, Loiseleux & Doppler, Lagrée

6 Kelvin Helmholtz in Hele Shaw

Rabaud & Gondret constructed such Hele Shaw cuves, with two fluids (a liquid and a gas), see figure 16). So they did a Kelvin Helmholtz configuration, as the gas is injected. So, one can observe waves at the interface between the fluids. They performed the stability analysis of this system (Plouraboué & Hinch).

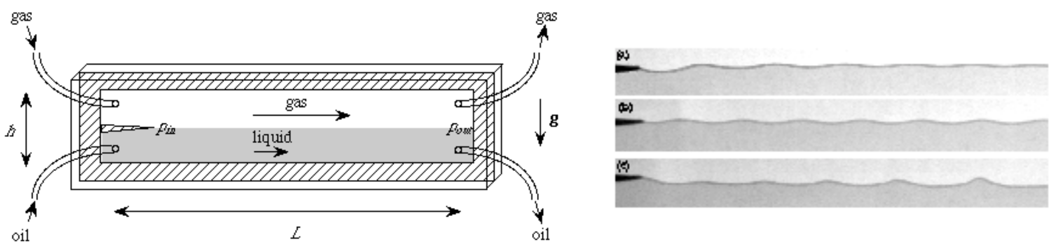

FIGURE 16 – Hele-Shaw set up(L = 1,2 m, H = 0,1 m et h = 0,35 mm).., right, depending of the gas velocity , the regime is stable, neutral, or instable (size 1 cm 7 cm), from Gondret Rabaud

7 Other Topics and conclusion

Small Reynolds flow is a new "à la mode" subject (even if it is old). Looking at courses in micro hydrodynamics, there are some constants
• unicity, linearity of solutions.
• reciprocal theorem....
• property of minimal dissipation of energy for $Re = 0$.
• friction depends weakly on the exact shape...
• viscosity of suspensions...
• Moffat vorticies (see chapter on similitude)
• painting brush (Taylor's scraper)....
• Lubrication equations between plates, between cylinders

Most of the courses on micro hydrodynamics do not present the complete solution of the flow around a sphere (except the Stokes drag in 6π) or a circle. So, this chapter is the missing one, even if all the details are not given.

The fields covered is the interaction of small structures with a flow, for example the study of swimming of micro-organisms. It is called Stokesian locomotion. Those small organisms change there shape and move cilia and flagella. The aim of this waving of organelles is usually to move the organisms. Indeed time-reversal symmetry plays an key role in the selection of swimming strategies. There is then the famous scallop theorem (Coquille Saint-Jacques, the best one are from Erquy in the Bay of Saint Brieuc) [2] (from Purcell 78) " Suppose that a small swimming body in an infinite expanse of fluid is observed to execute a periodic cycle of configurations, relative to a coordinate system moving with constant velocity U relative to the fluid at infinity. Suppose that the fluid dynamics is that of Stokes flow. If the sequence of configurations is indistinguishable from the time reversed sequence, then $U = 0$ and the body does not locomote."

The small Reynolds world is a new striking world...

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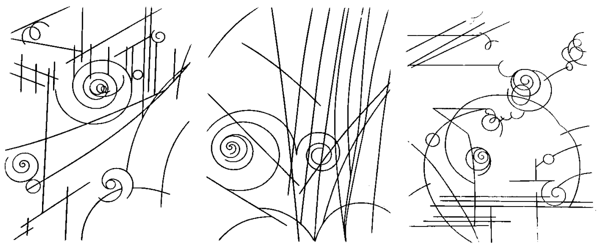
créé 09/12
up to date 14 novembre 2019

This course is a part of a larger set of files devoted on perturbations methods, asymptotic methods (Matched Asymptotic Expansions, Multiple Scales) and boundary layers (triple deck) by $\mathscr{P}$.-$\mathscr{Y}$. $\mathscr{L}$agrée .
The web page of these files is `http://www.lmm.jussieu.fr/∼lagree/COURS/M2MHP`.

`/Users/pyl/    ...    /petitRe.pdf`



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Raymond Subes "Sans Titre" 1961 (entrée de Jussieu Quai Saint Bernard)