Equations de Navier Stokes Réduites

Systèmes d'équations simplifiées issues
de Navier Stokes:

Lagrée Pierre-Yves

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CNRS UMR 7607 -- Université Paris VI Jussieu

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Lagrée Pierre-Yves

Annemie van Hirtum, Sylvie Lorthois, Xavier Pelorson, Claire Ségoufin
Emanuel Berger, Bram de Bruin, Franz Chouly, Koen Gorman, Coriandre Vilain

But

- simplification des équations de Navier Stokes
- grâce aux théories asymptotiques de:

  “Couche Limite”

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- on simplifie NS en un sous sytème

  - qui contient les échelles

  - et les phénomènes principaux

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  Partant de Navier Stokes

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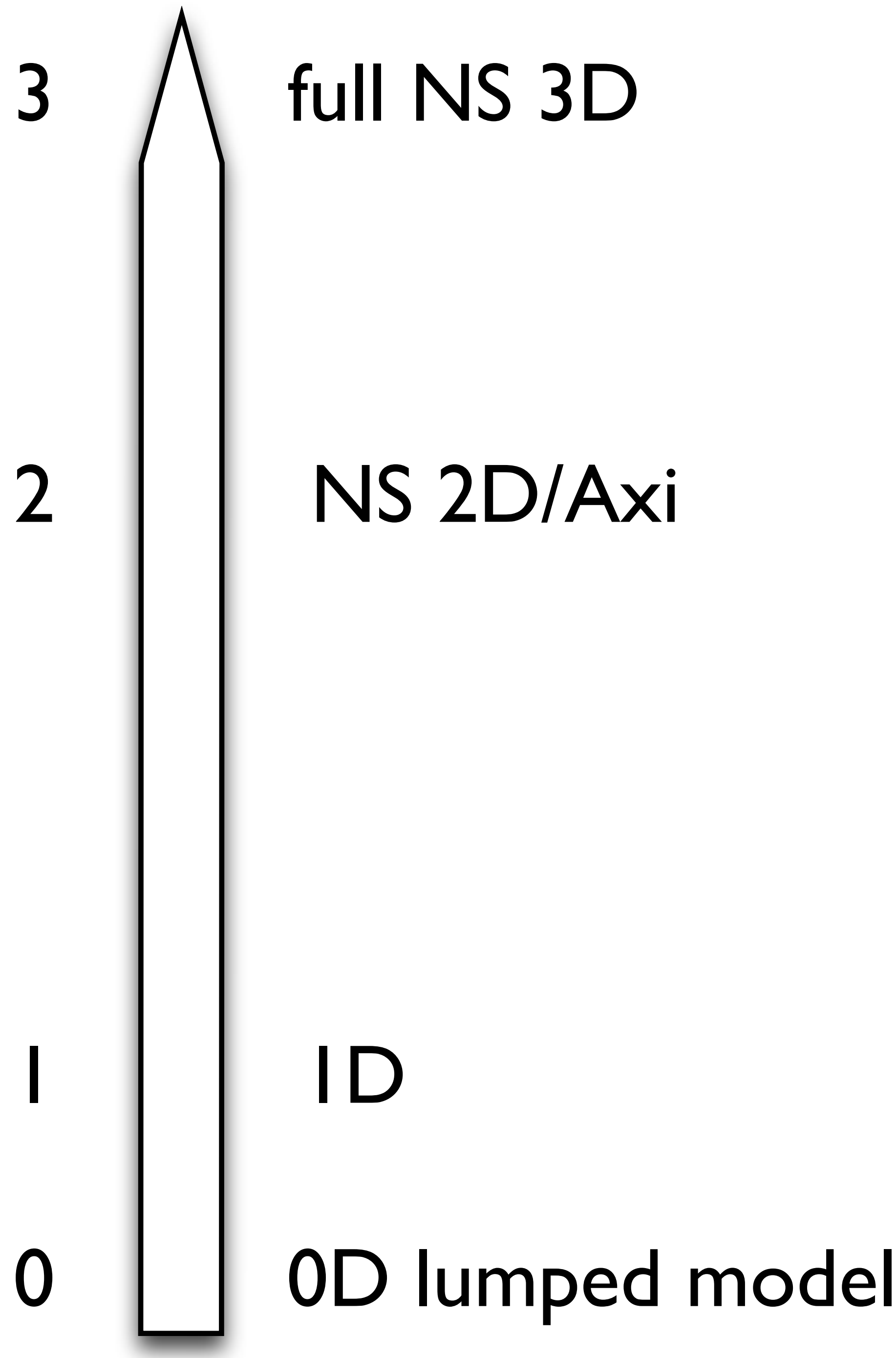
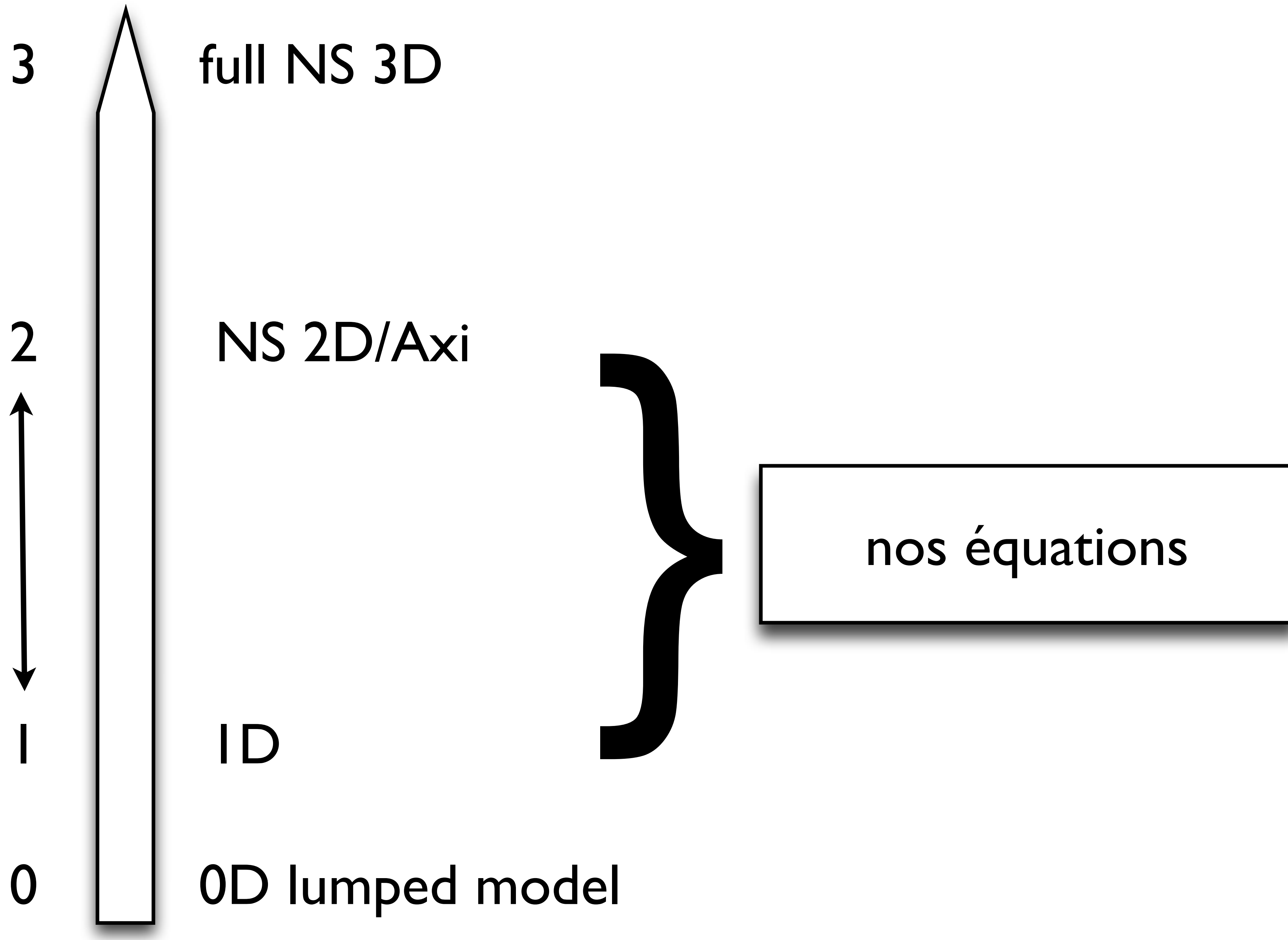
En simplifiant encore plus: système intégral

Comparaisons NS/RNSP/Intégral


3 full NS 3D
2 NS 2D/Axi
1 1D
0 0D lumped model


3
full NS 3D
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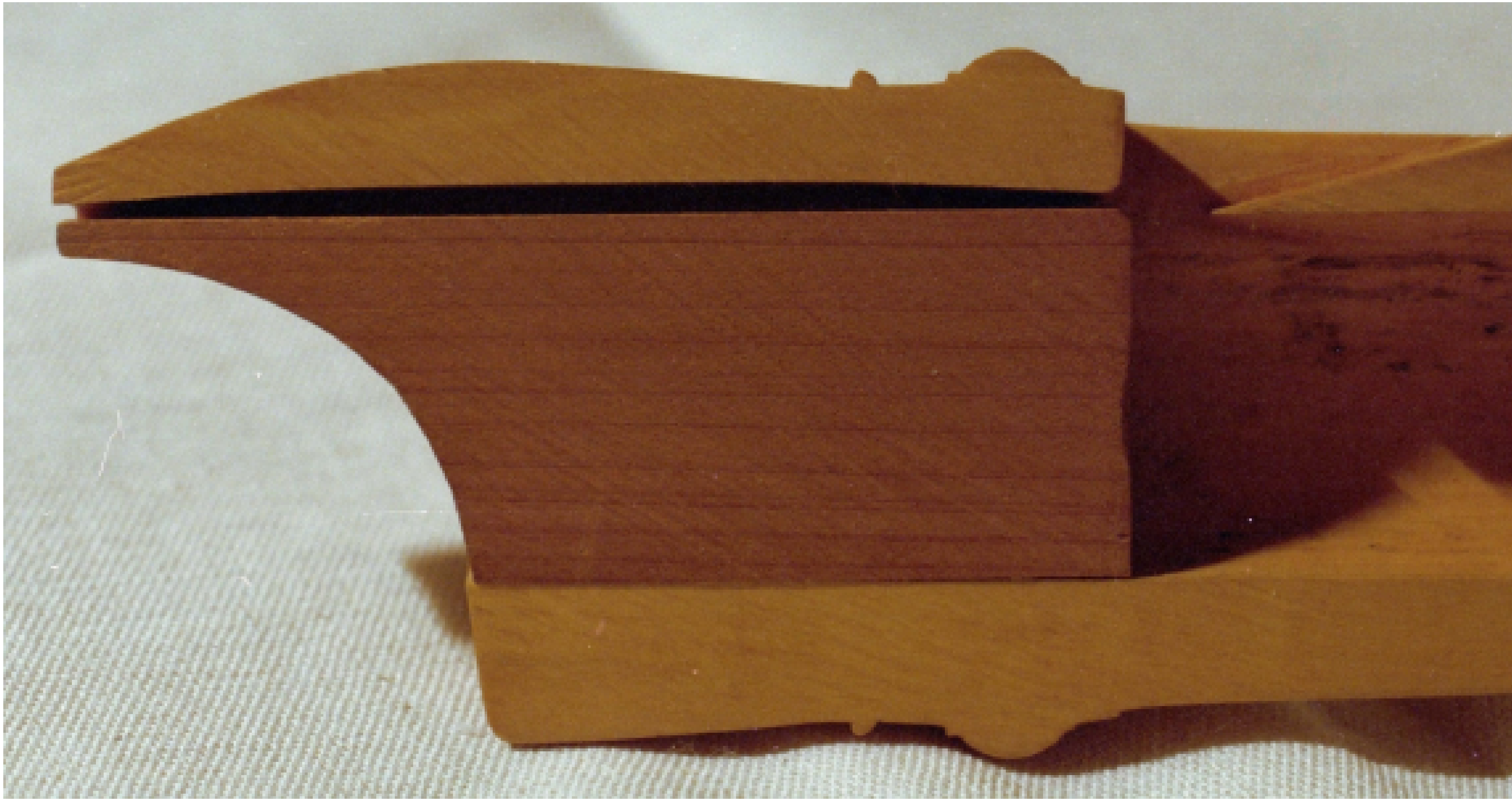
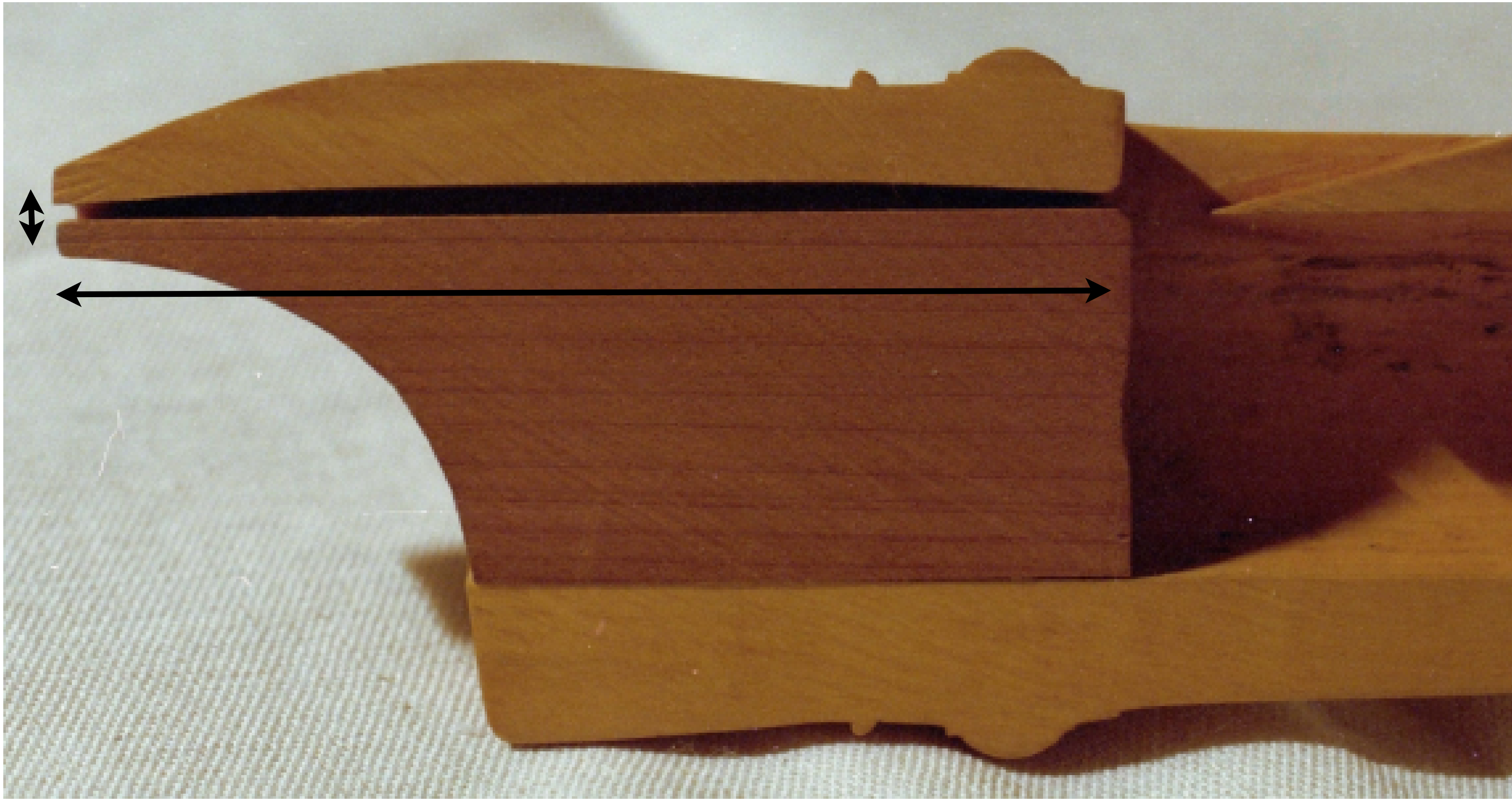
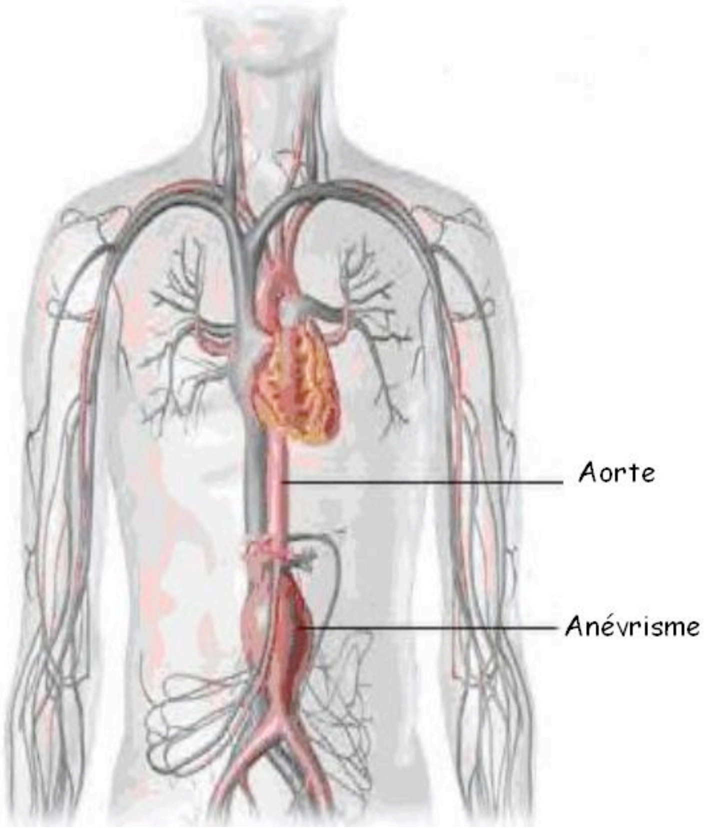
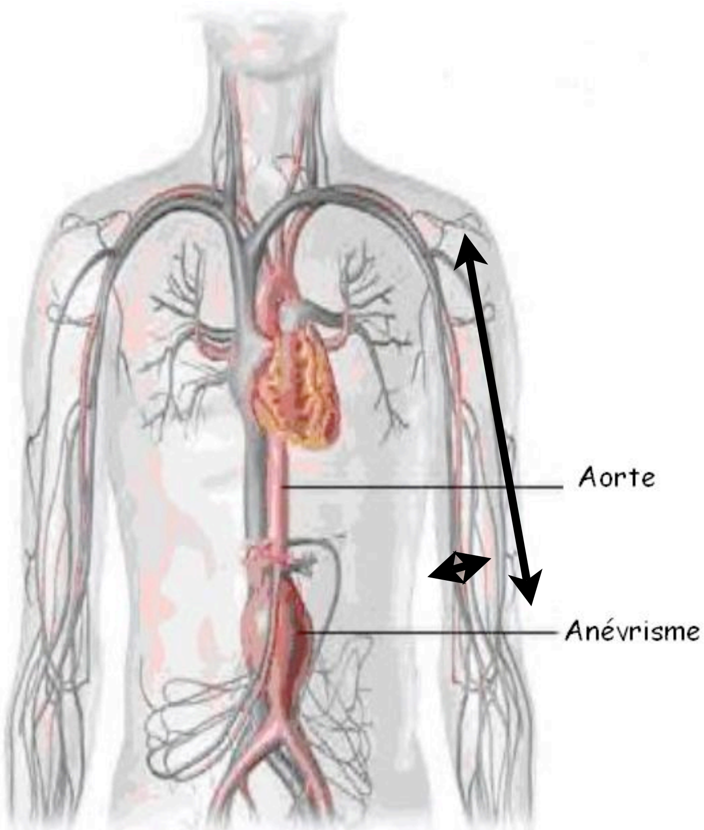
0D lumped model
nos équations

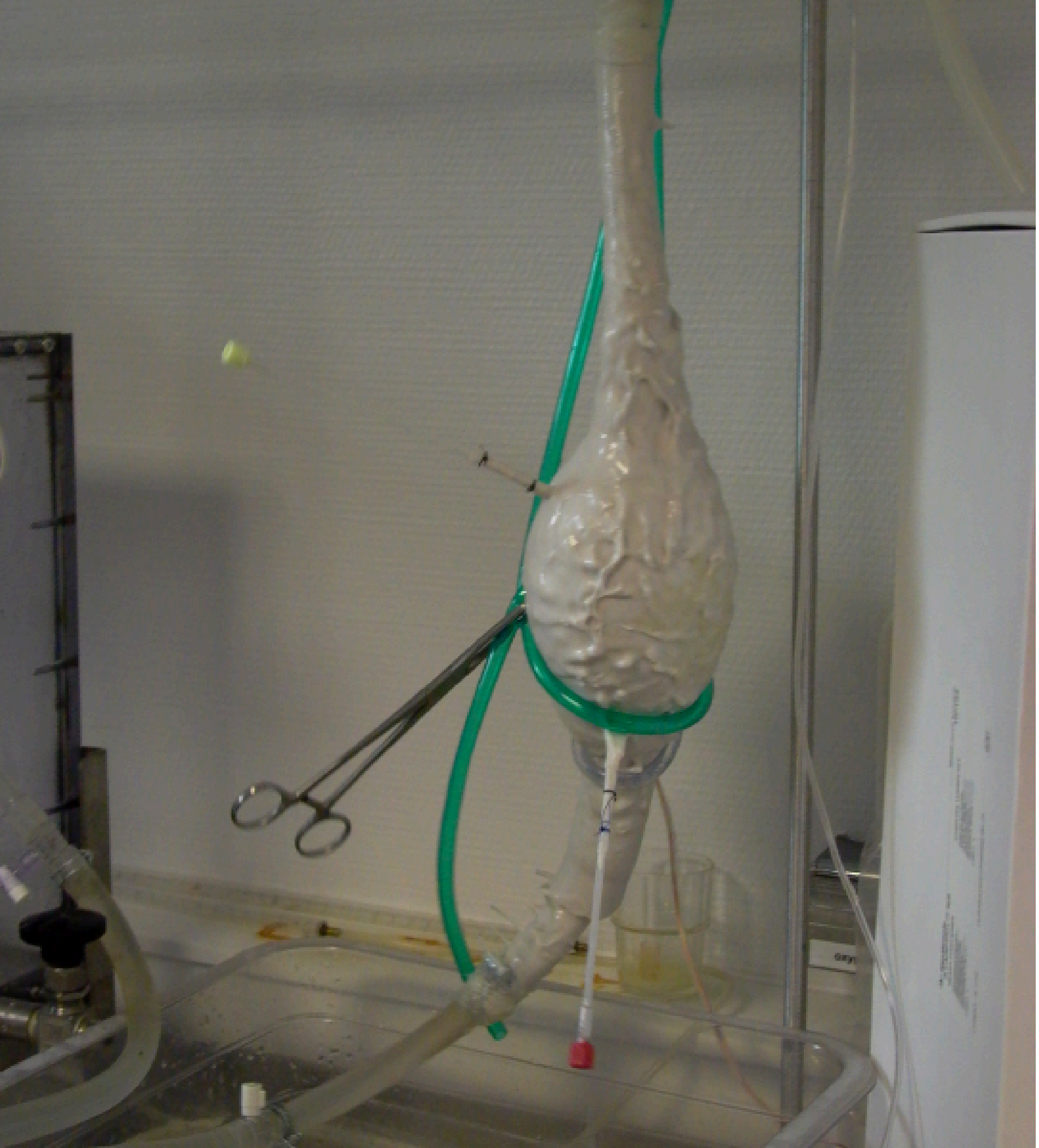


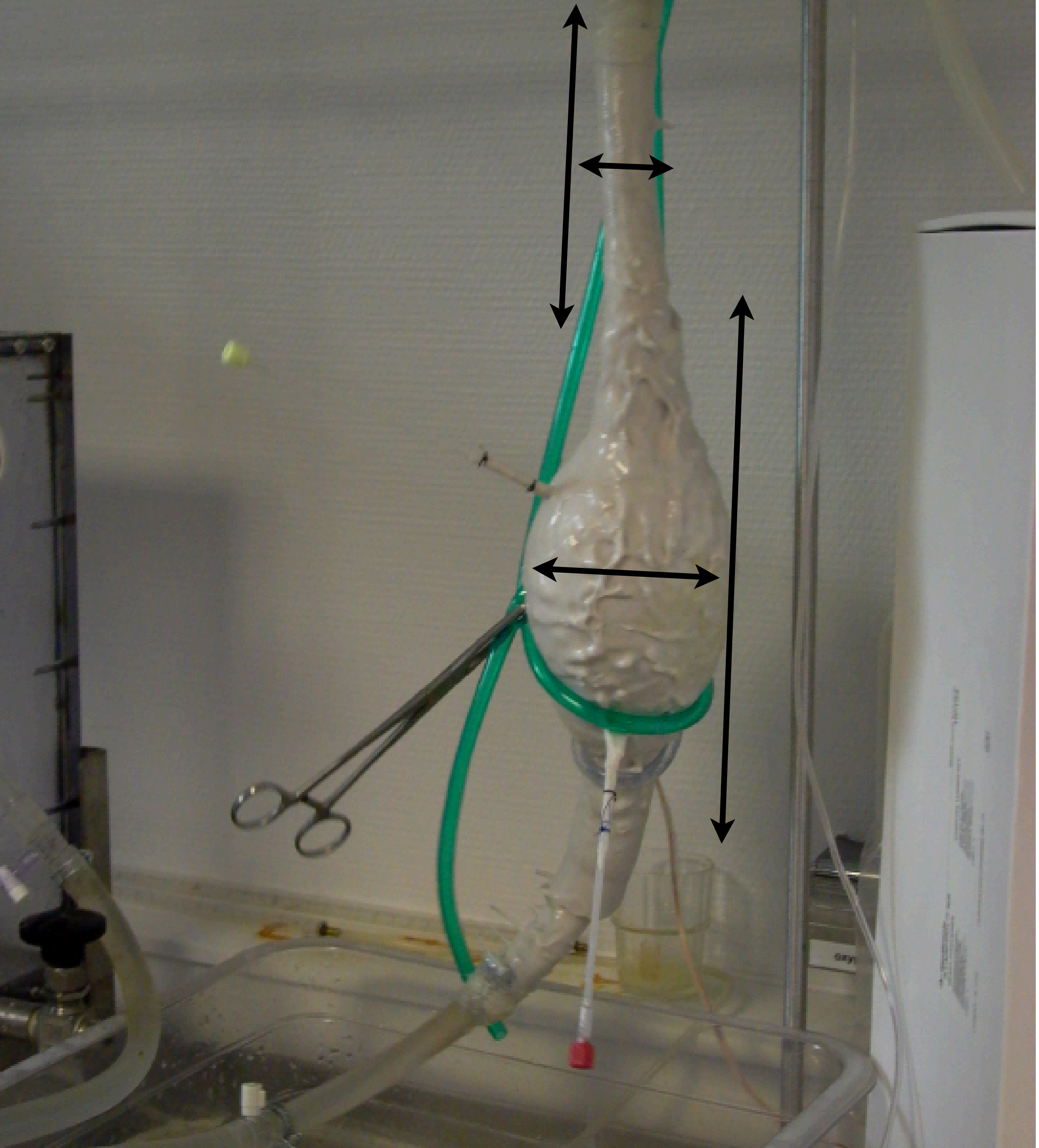


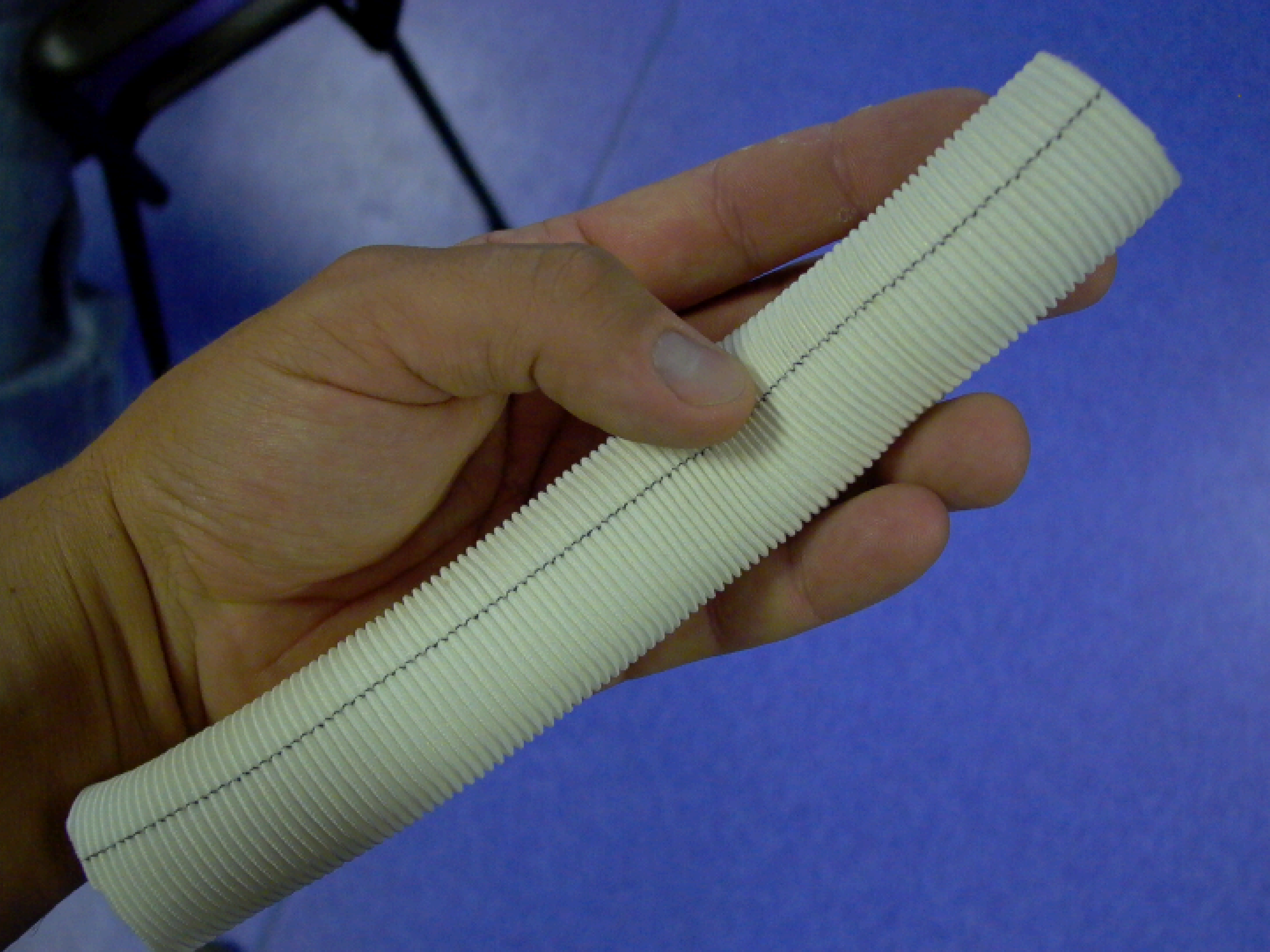

Aorte
Anévrisme

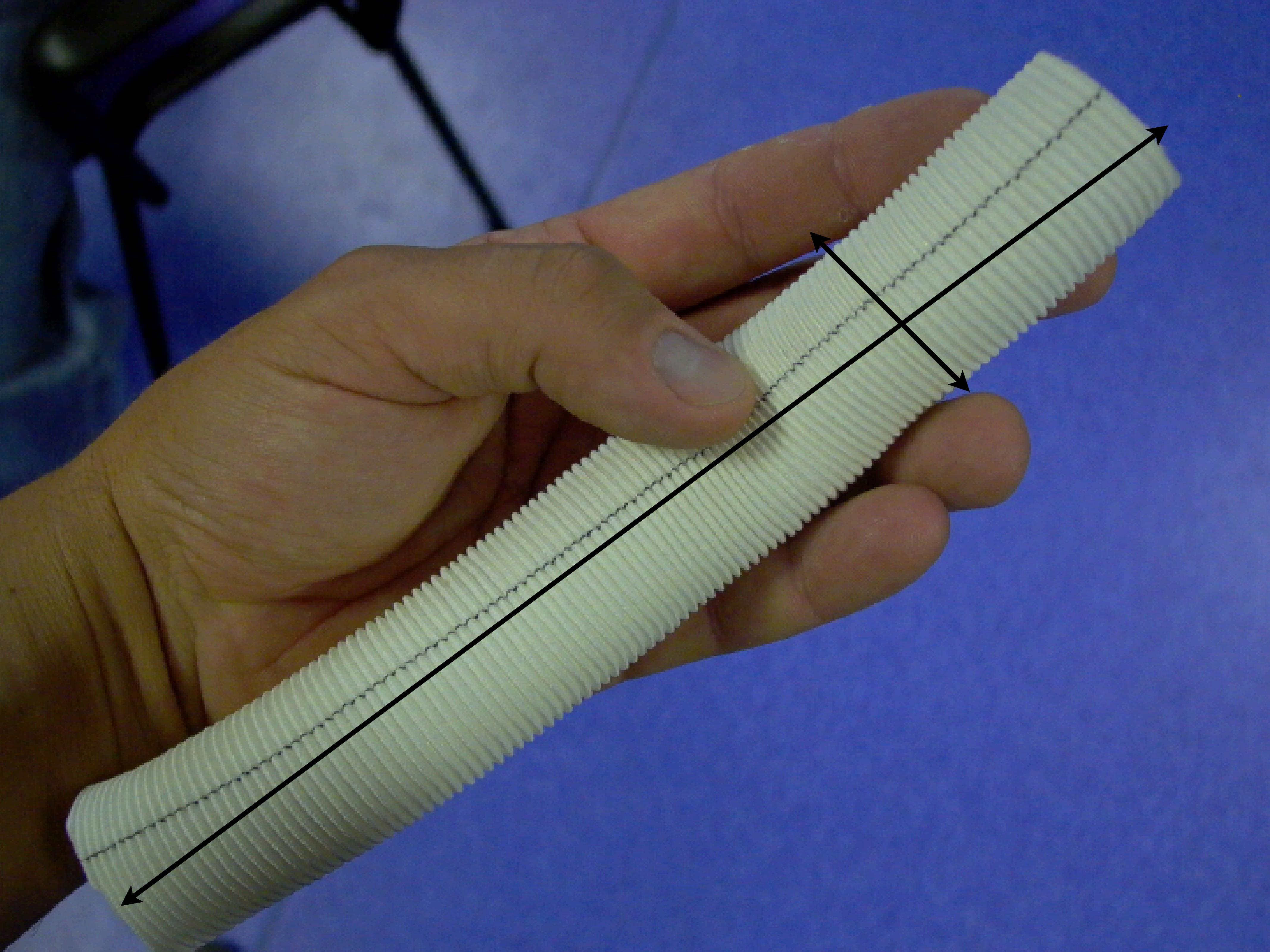


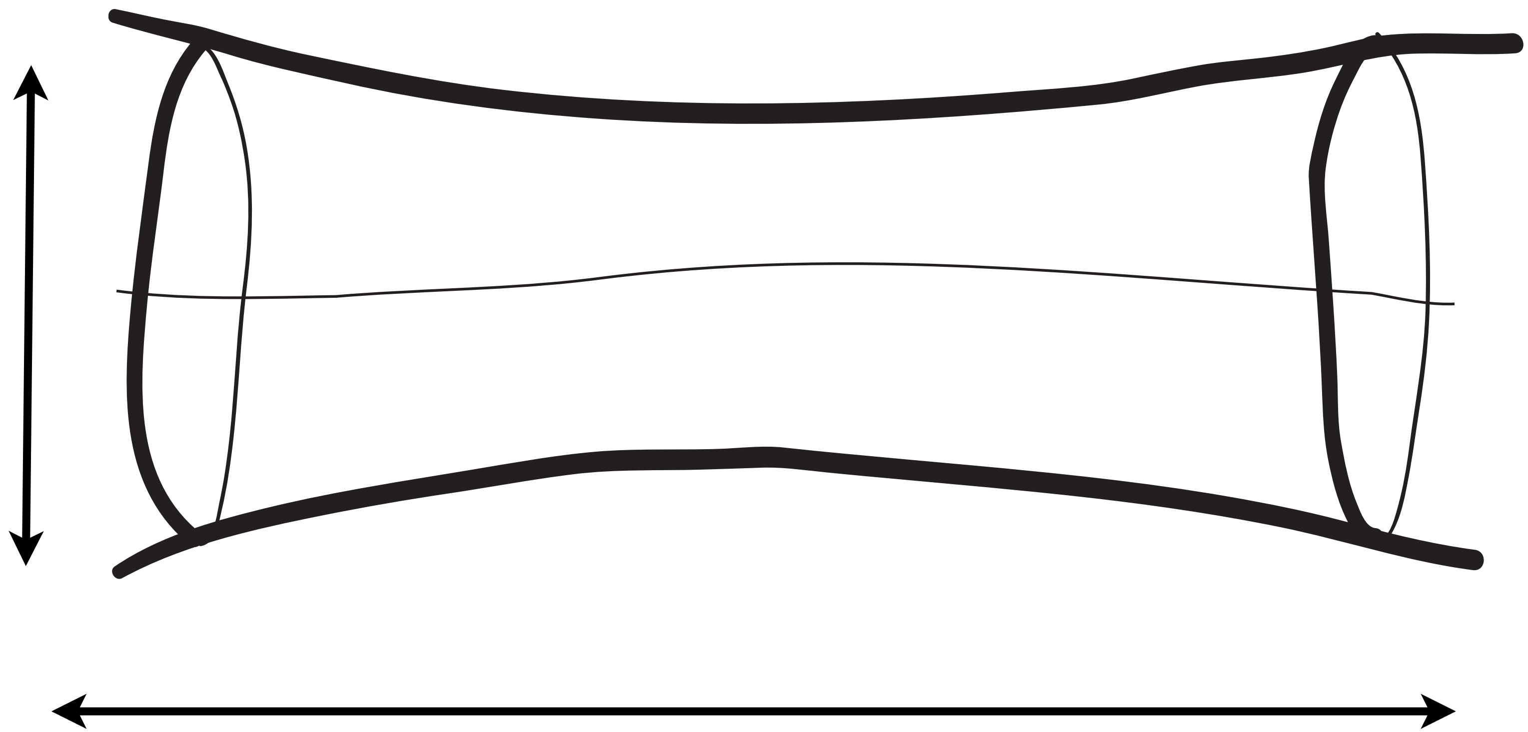
Aorte
Anévrisme



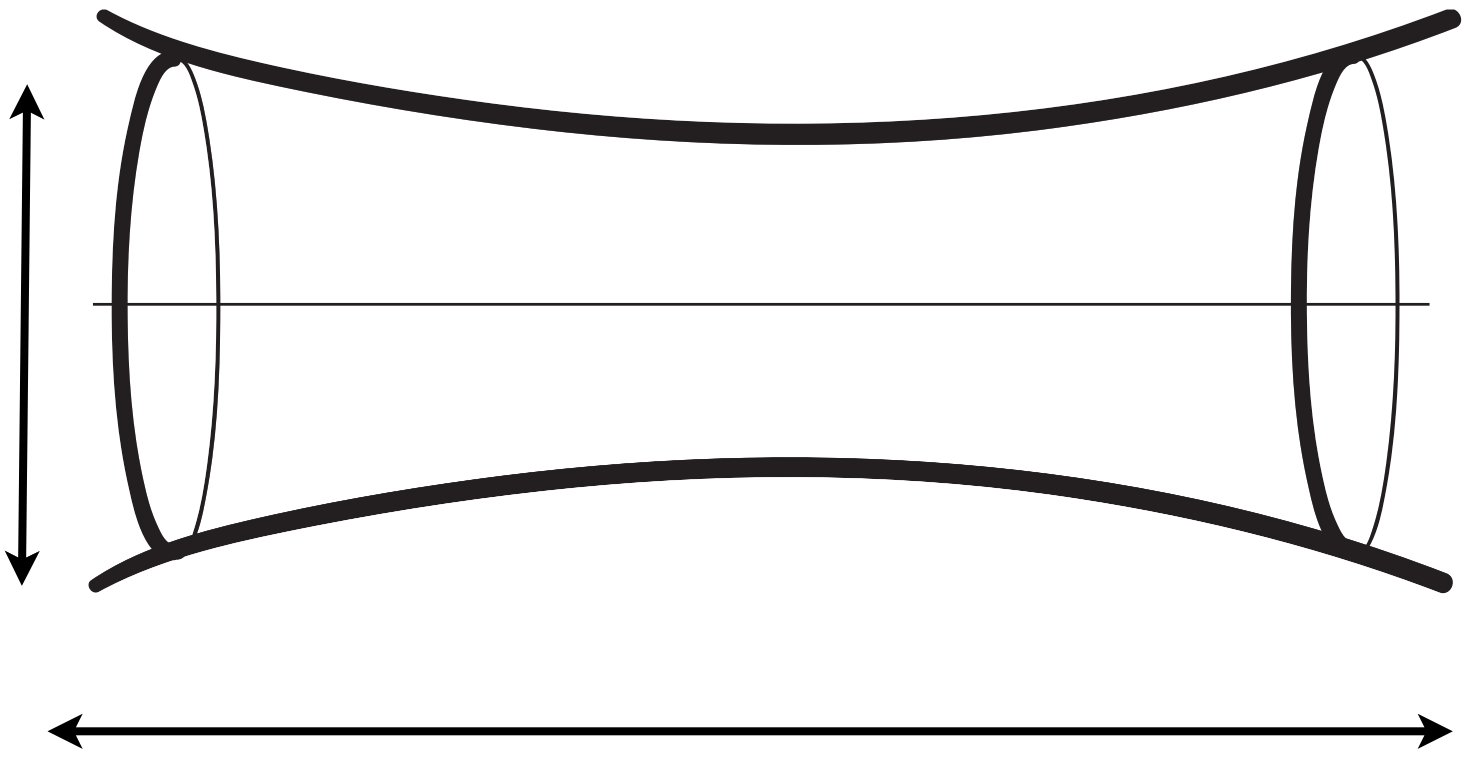




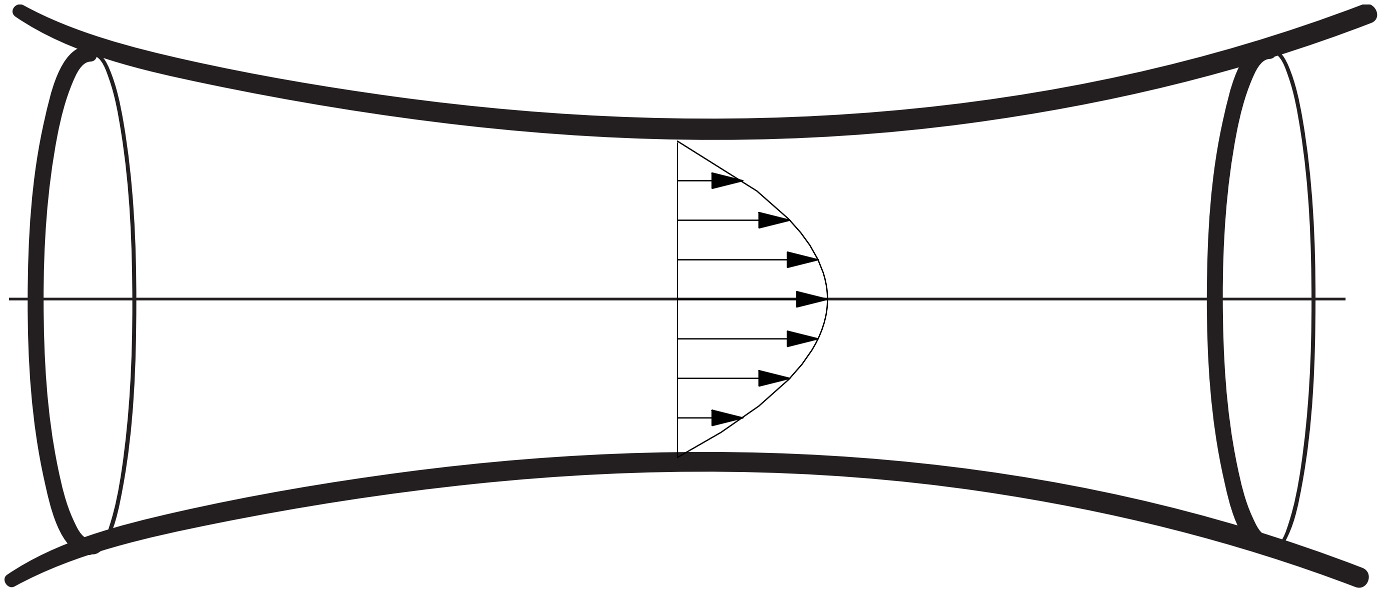




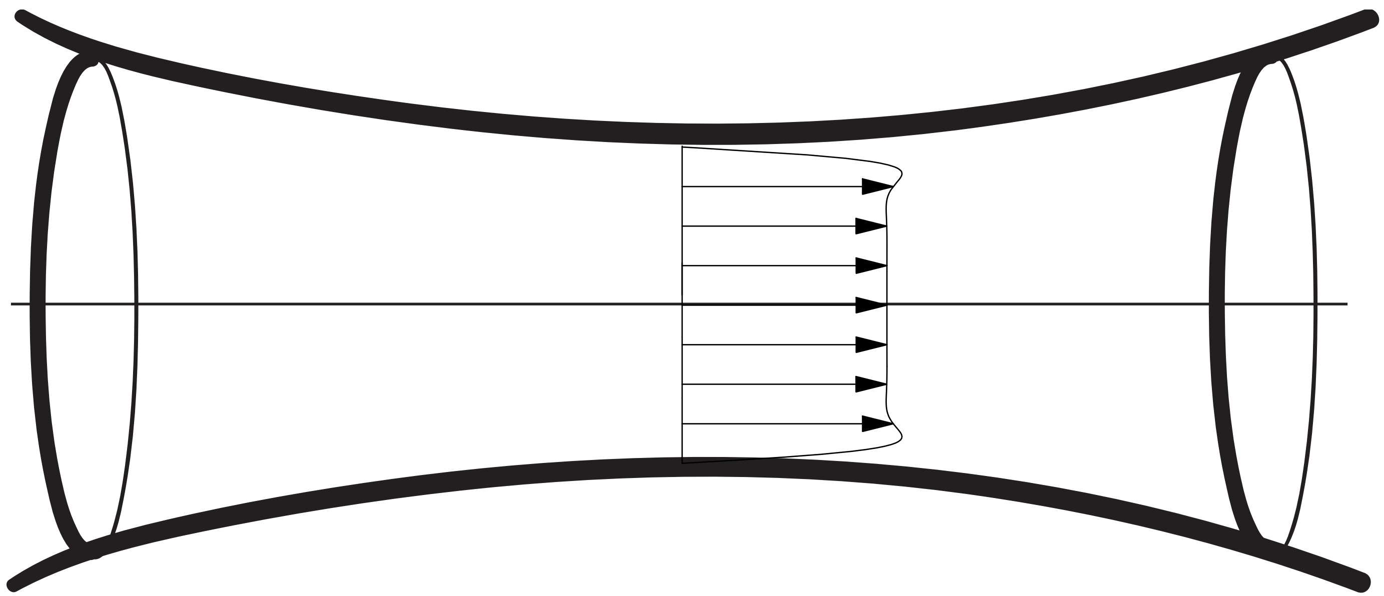
réalité?



tuyau droit, murs lisses, symétrie

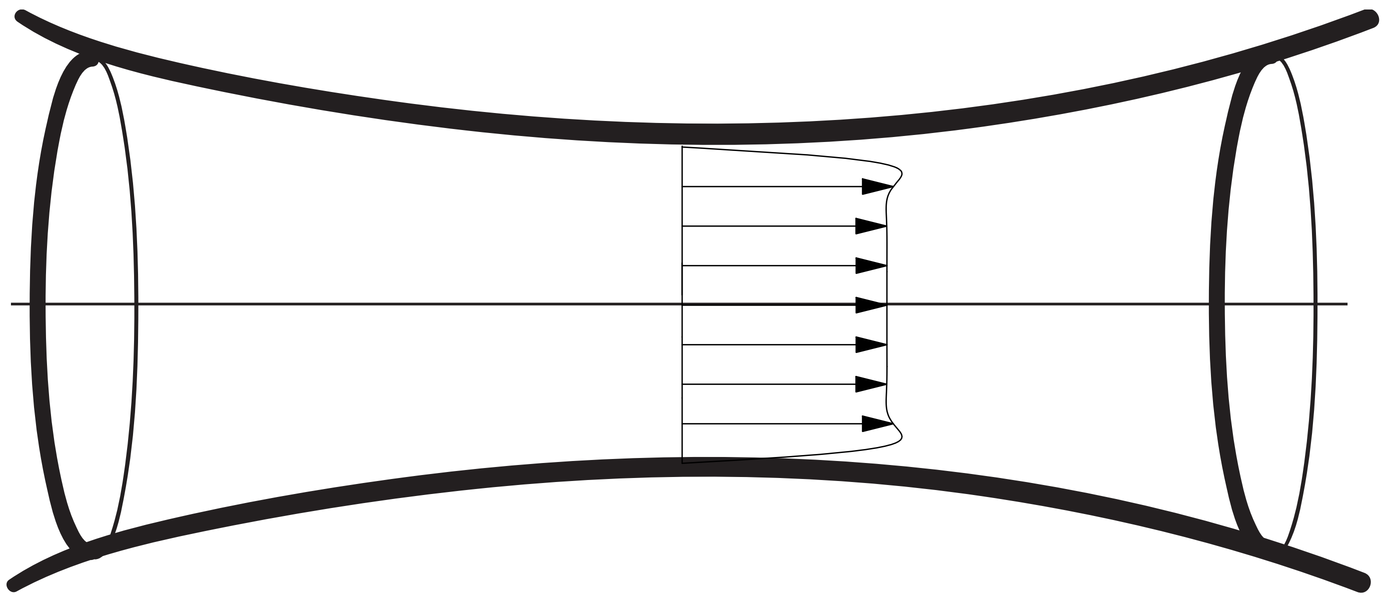


profil de vitesse

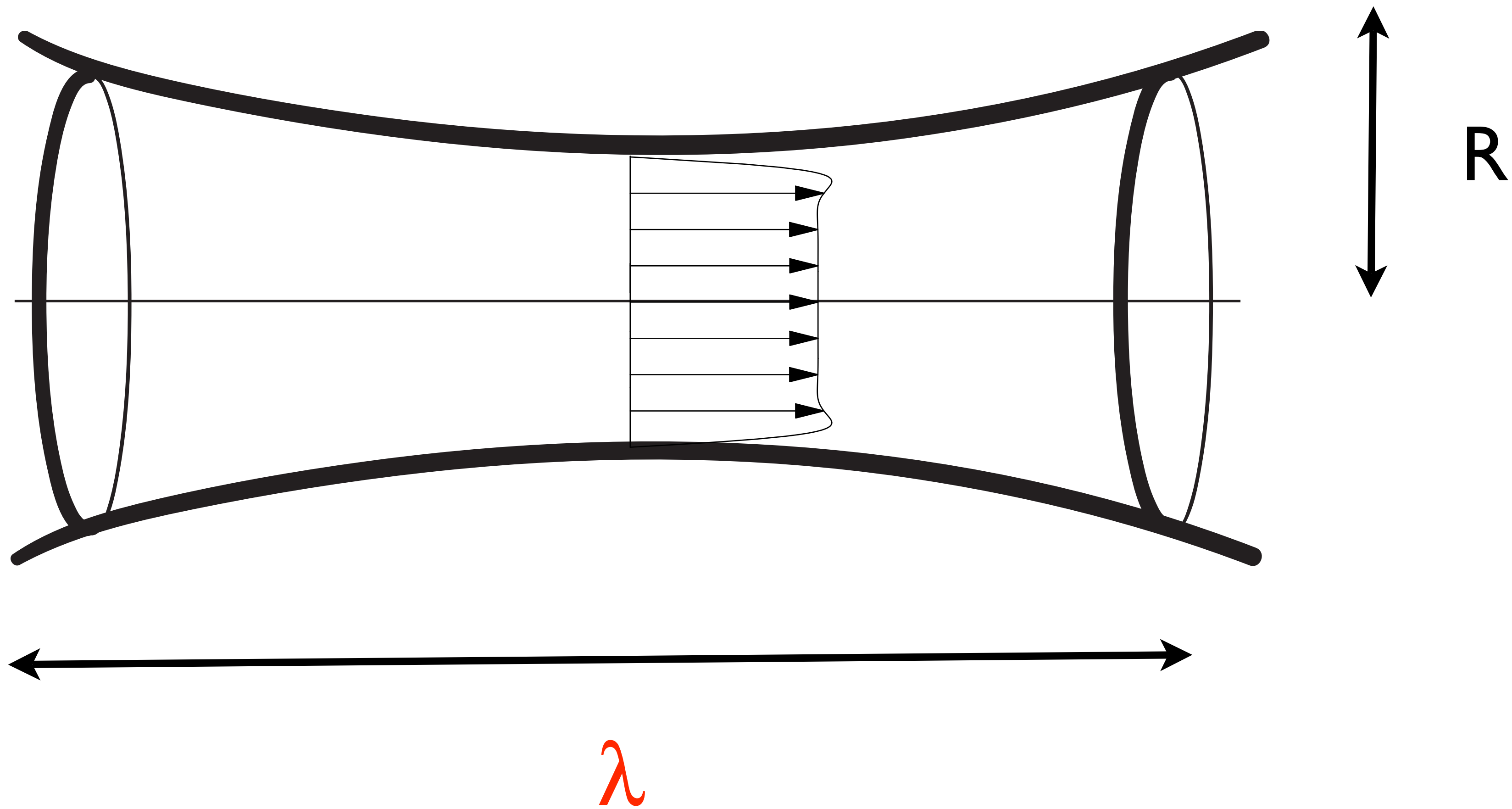


profil de vitesse

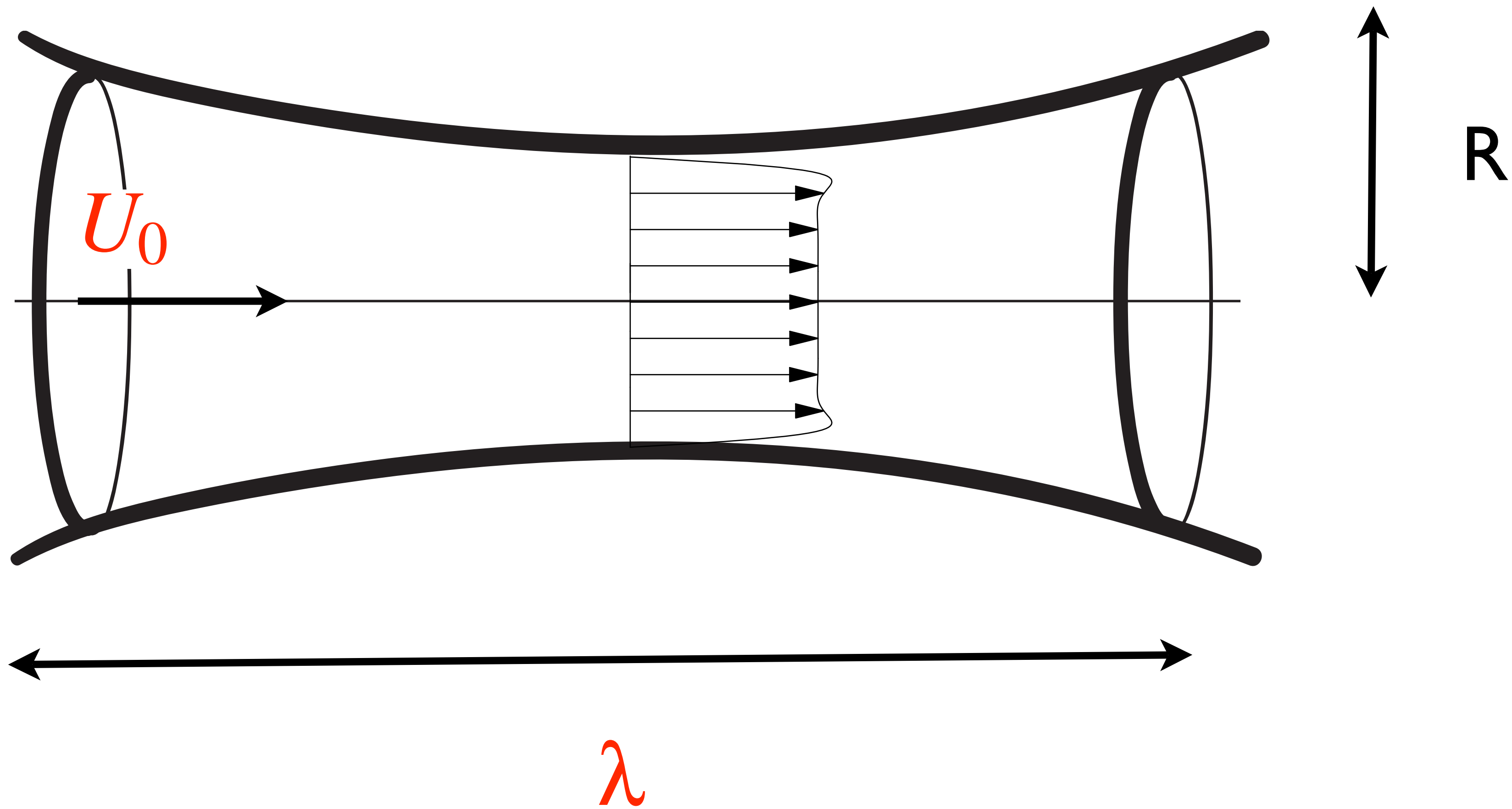
Equations



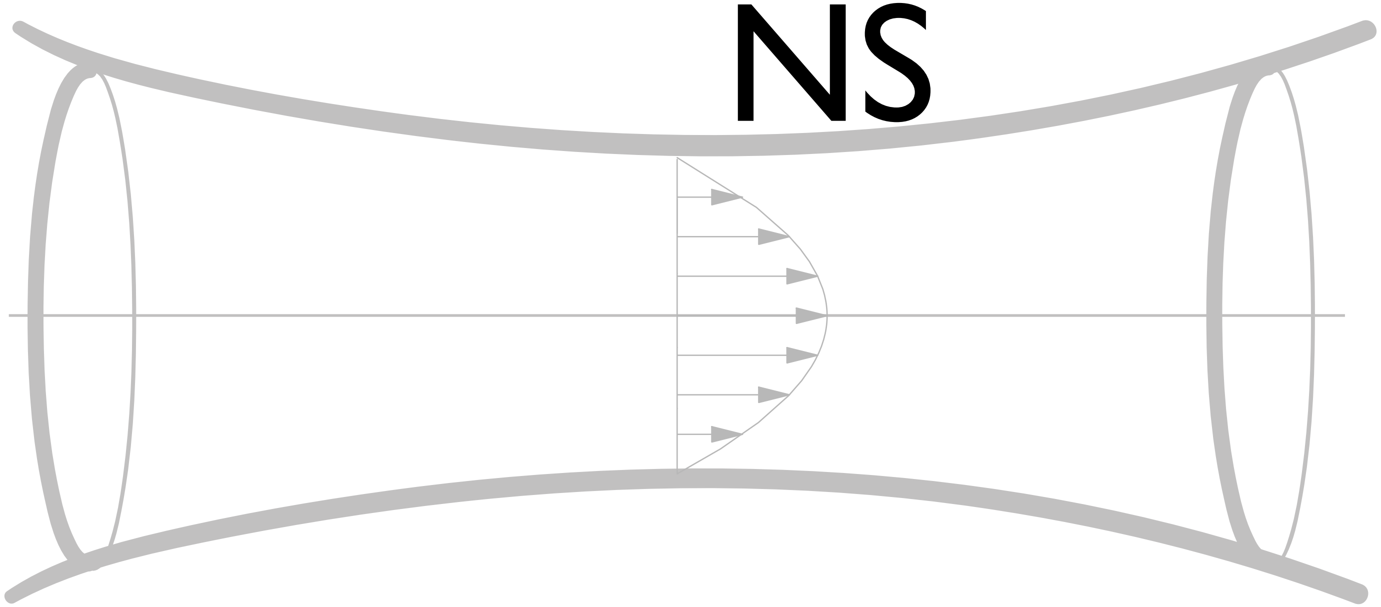
- simplifiées
- déduites d’ordres de grandeur



$R << \lambda$


R
U_0
λ

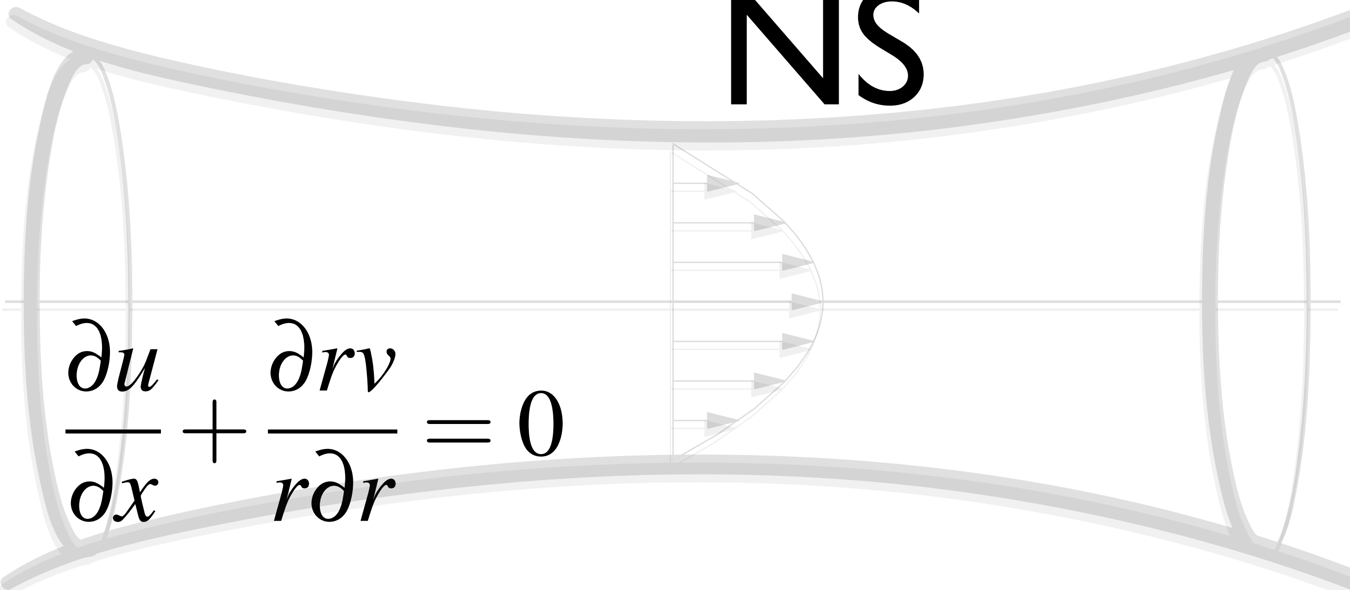
NS



$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu\frac{\partial^2}{\partial x^2}u + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial r}v = -\frac{\partial p}{\rho \partial r} + \nu\frac{\partial^2}{\partial x^2}v + \nu\frac{\partial}{r\partial r}r\frac{\partial v}{\partial r}$$

NS

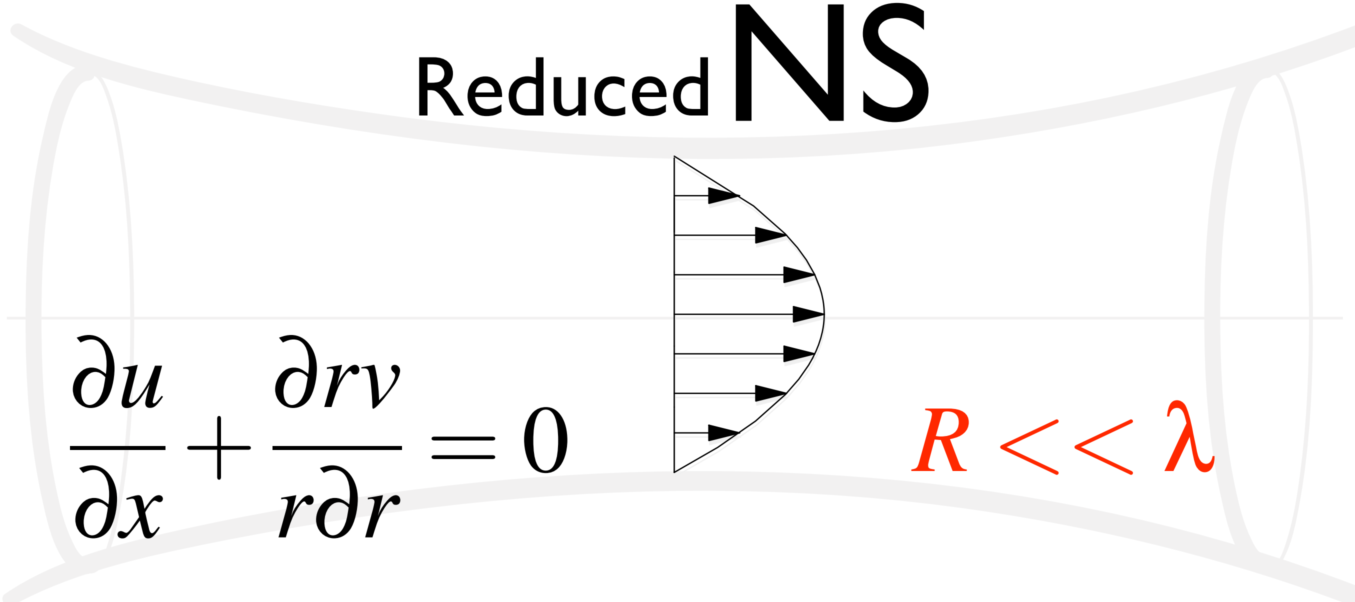


$$\frac{\partial u}{\partial x}+\frac{\partial rv}{r\partial r}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial r}u=-\frac{\partial p}{\rho\partial x}+\nu\frac{\partial^2}{\partial x^2}u+\nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial}{\partial x}v+v\frac{\partial}{\partial r}v=-\frac{\partial p}{\rho\partial r}+\nu\frac{\partial^2}{\partial x^2}v+\nu\frac{\partial}{r\partial r}r\frac{\partial v}{\partial r}$$

Reduced NS



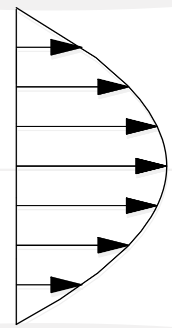
$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$R << \lambda$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \cancel{\frac{\partial^2}{\partial x^2}} u + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial r}v = -\frac{\partial p}{\rho \partial r} + \nu \cancel{\frac{\partial^2}{\partial x^2}} v + \nu \frac{\partial}{r\partial r} r \frac{\partial v}{\partial r}$$

RNS/P



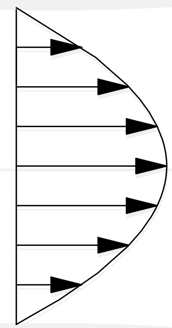
$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$R << \lambda \qquad V \sim U_0 \frac{R}{\lambda}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u = -\frac{\partial p}{\rho \partial x} + \nu \xcancel{\frac{\partial^2}{\partial x^2}} u + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial r} v = -\frac{\partial p}{\rho \partial r} + \nu \xcancel{\frac{\partial^2}{\partial x^2}} v + \nu \frac{\partial}{r \partial r} r \frac{\partial v}{\partial r}$$

RNS/P



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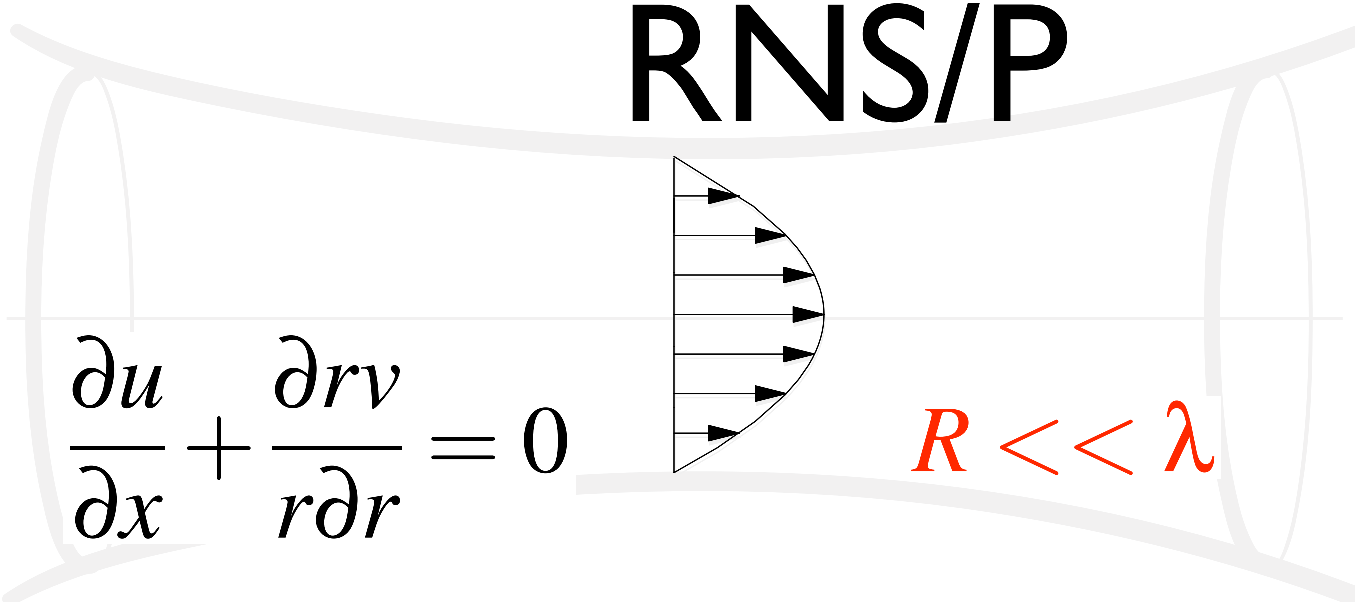
RNS/P



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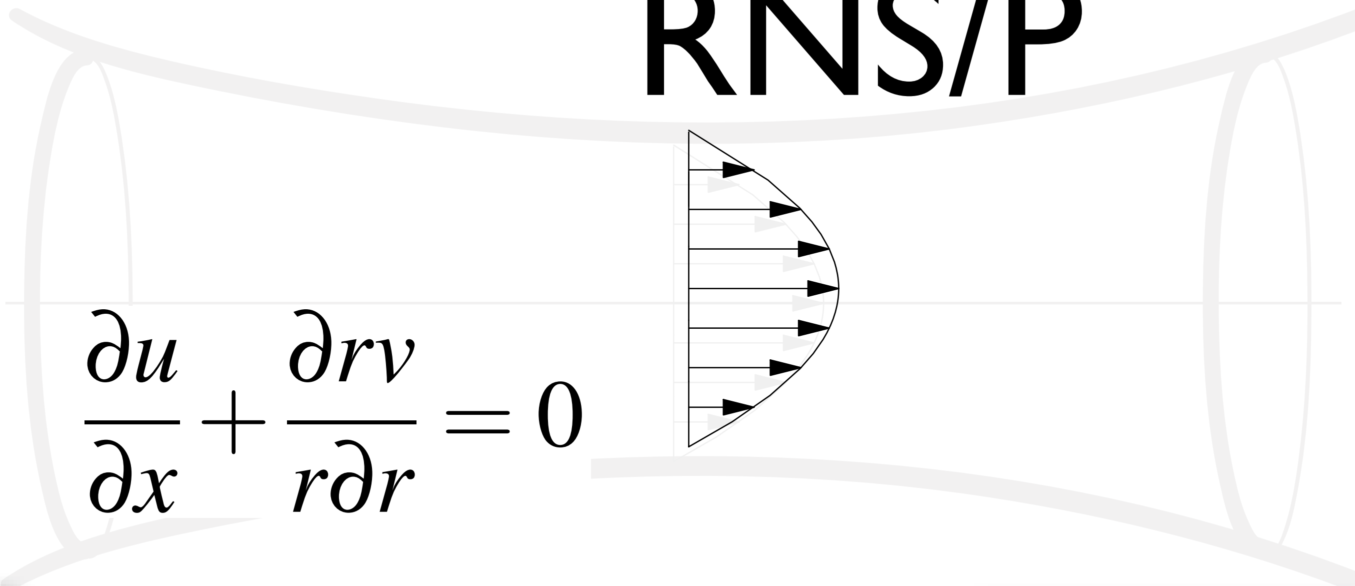
RNS/P

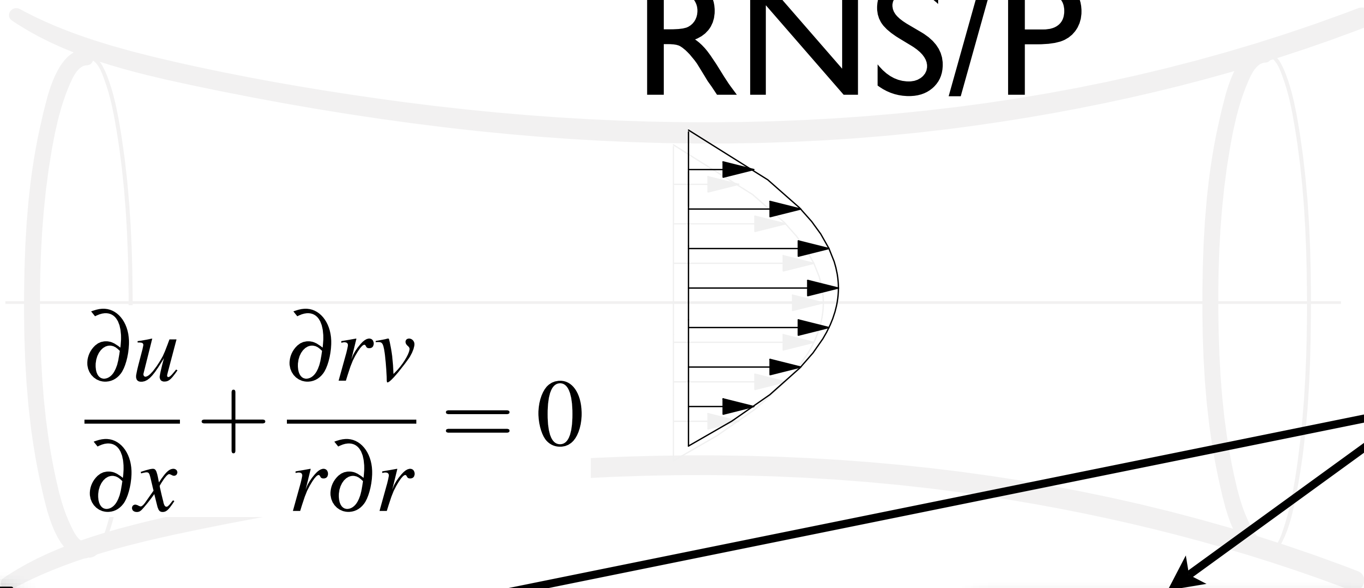


$$\frac{\partial u}{\partial x}+\frac{\partial rv}{r\partial r}=0$$

$$\boxed{\frac{\partial u}{\partial t}}+u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial r}u=-\frac{\partial p}{\rho\partial x}+\boxed{\nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}}$$

$$0=-\frac{\partial p}{\rho\partial r}$$

RNS/P



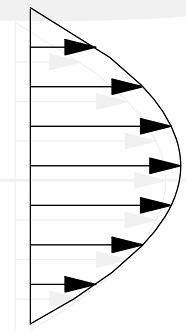
$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r\partial r} = 0$$

$$\nu \frac{1}{\omega R^2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho\partial x} + \nu\frac{\partial}{r\partial r} r\frac{\partial u}{\partial r}$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

RNS/P



$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\alpha = R\sqrt{\frac{\omega}{\nu}}$$

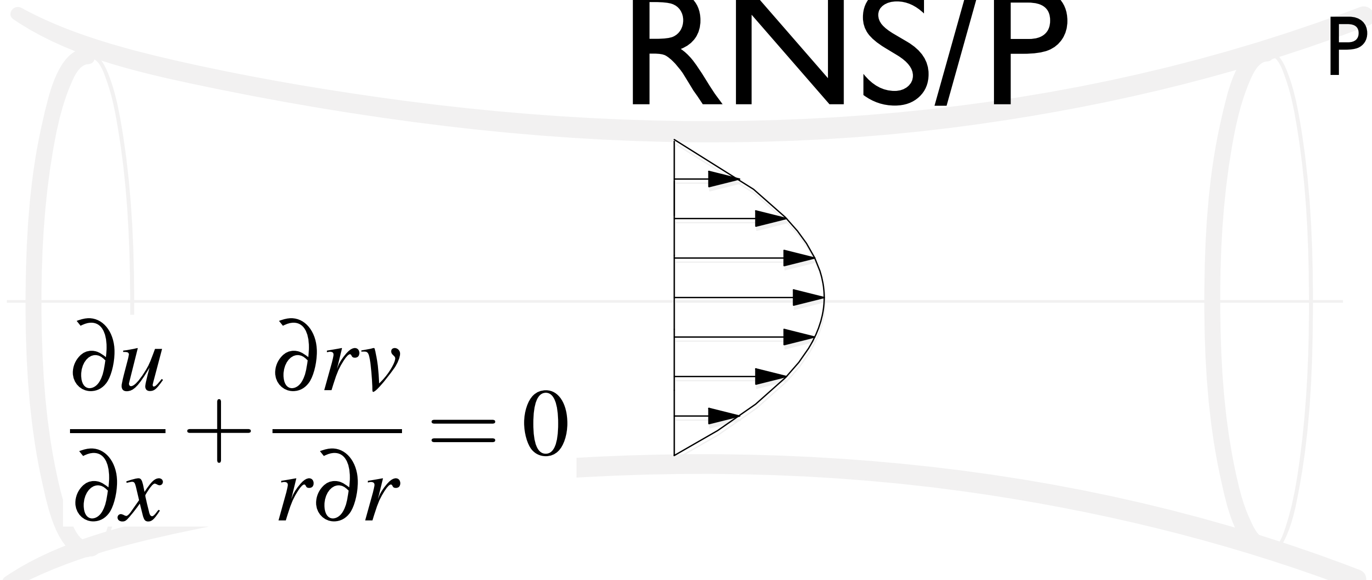
$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu\frac{\partial}{r \partial r}r\frac{\partial u}{\partial r}$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

$1/(\text{Womersley})^2$

RNS/P

Prandtl

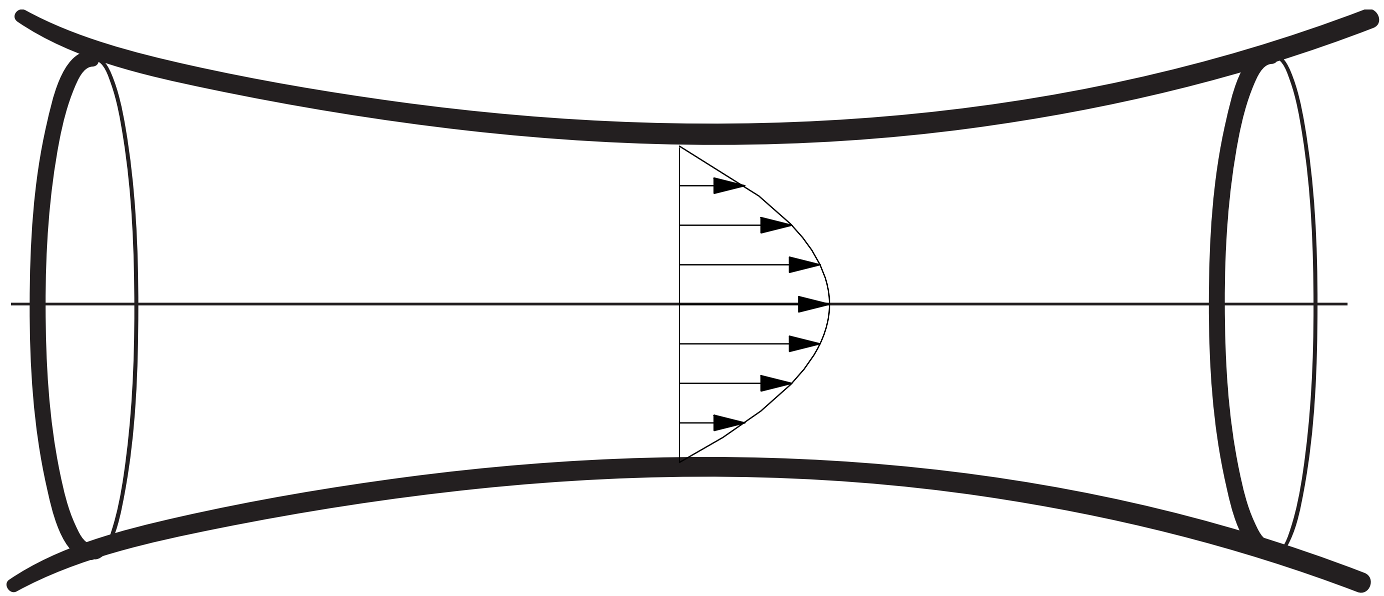


$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}$$

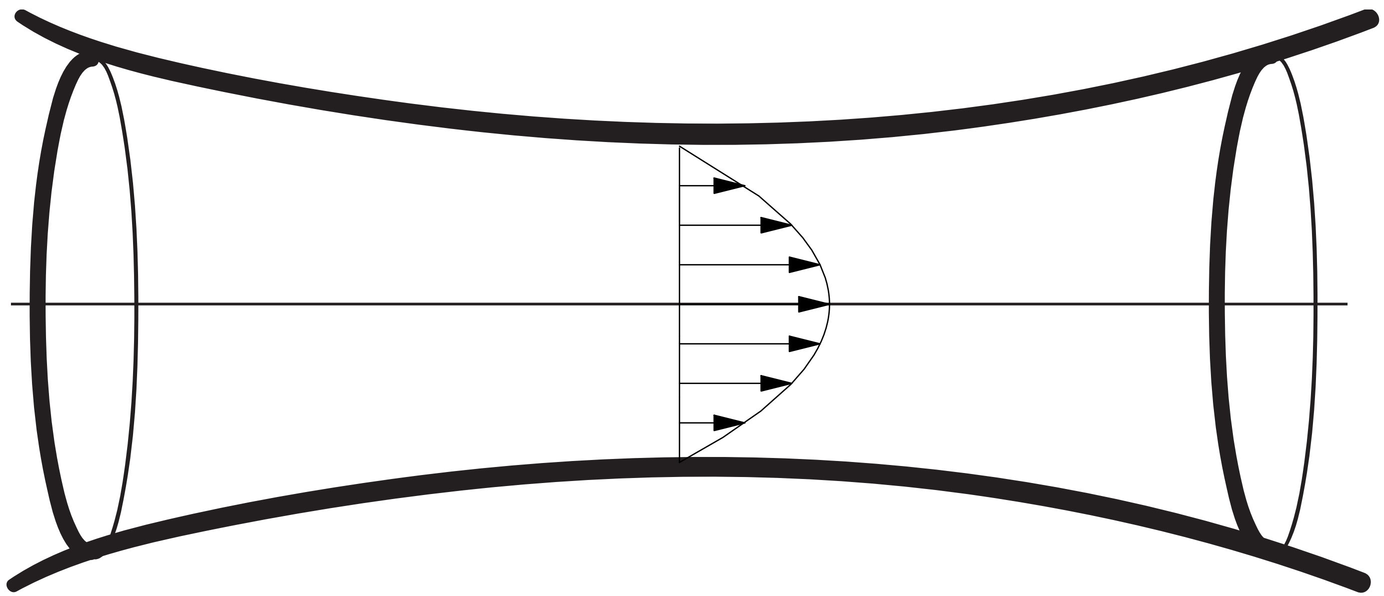
$$0 = -\frac{\partial p}{\rho \partial r}$$

Conditions aux limites



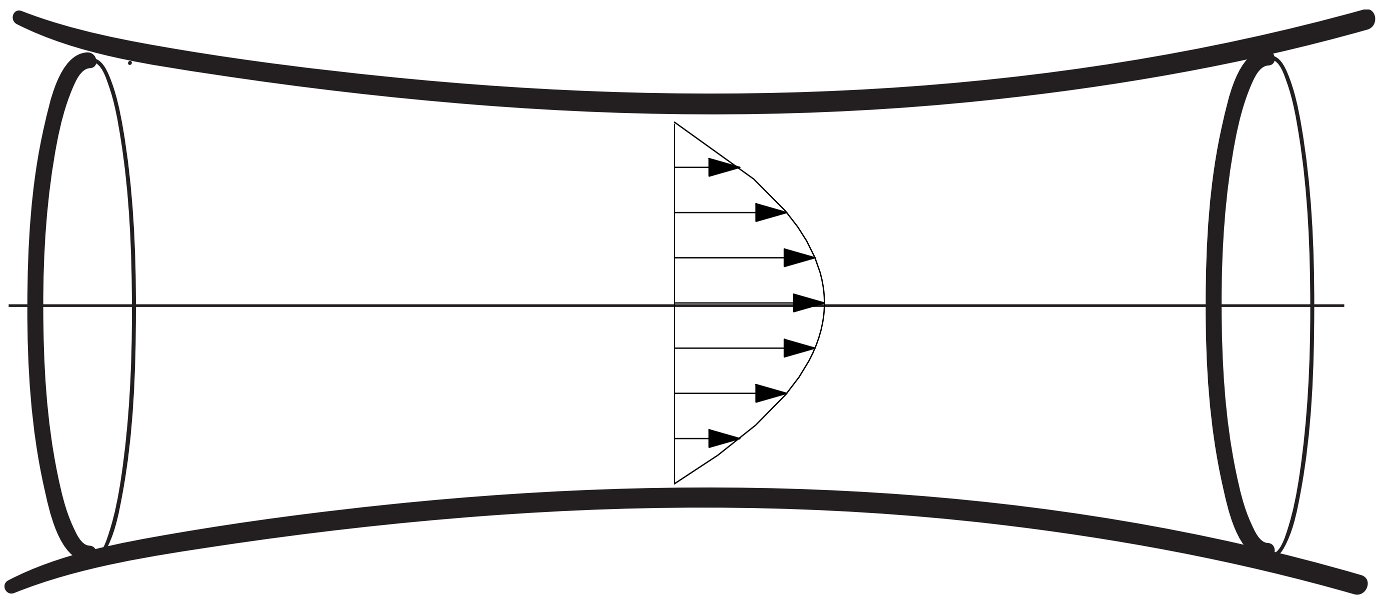
Paroi Rigide: $u = v = 0$

Conditions aux limites



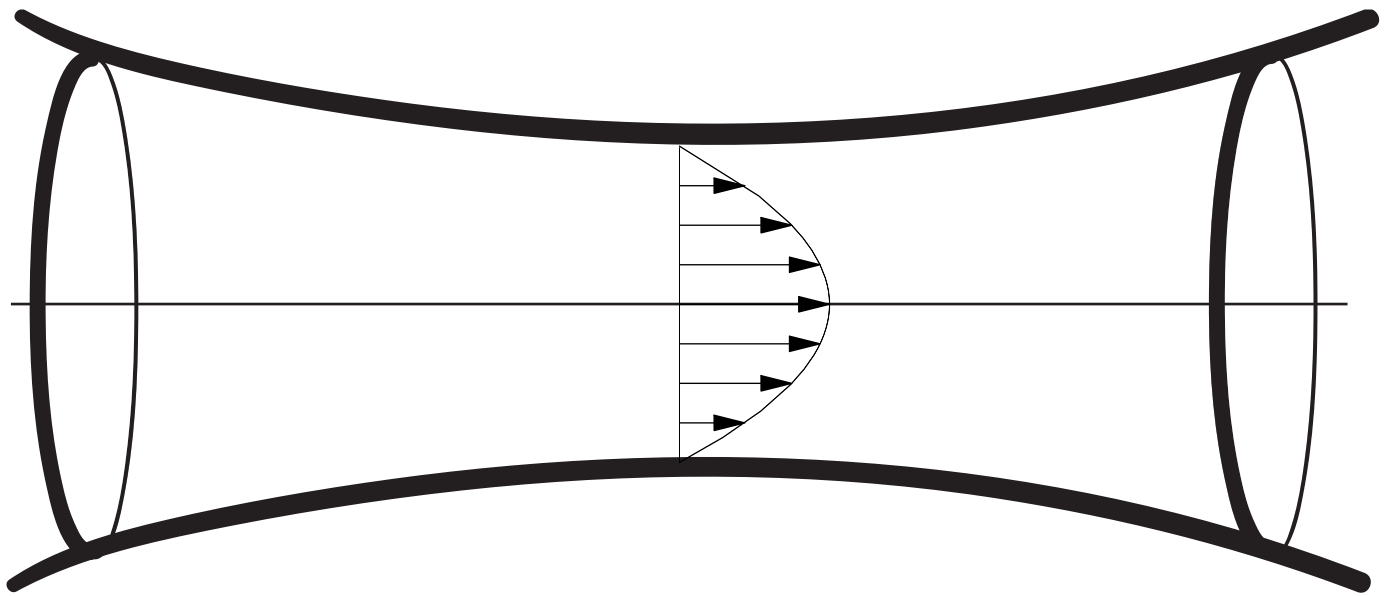
Paroi mobile

Conditions aux limites



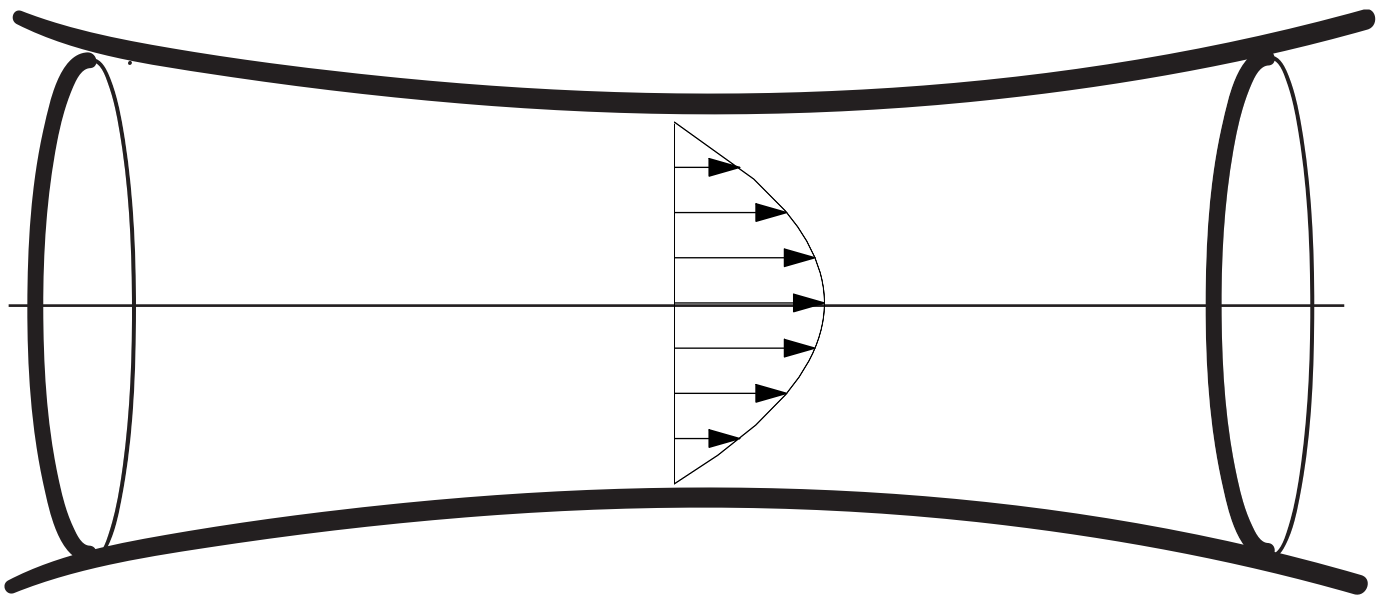
Paroi mobile

Conditions aux limites



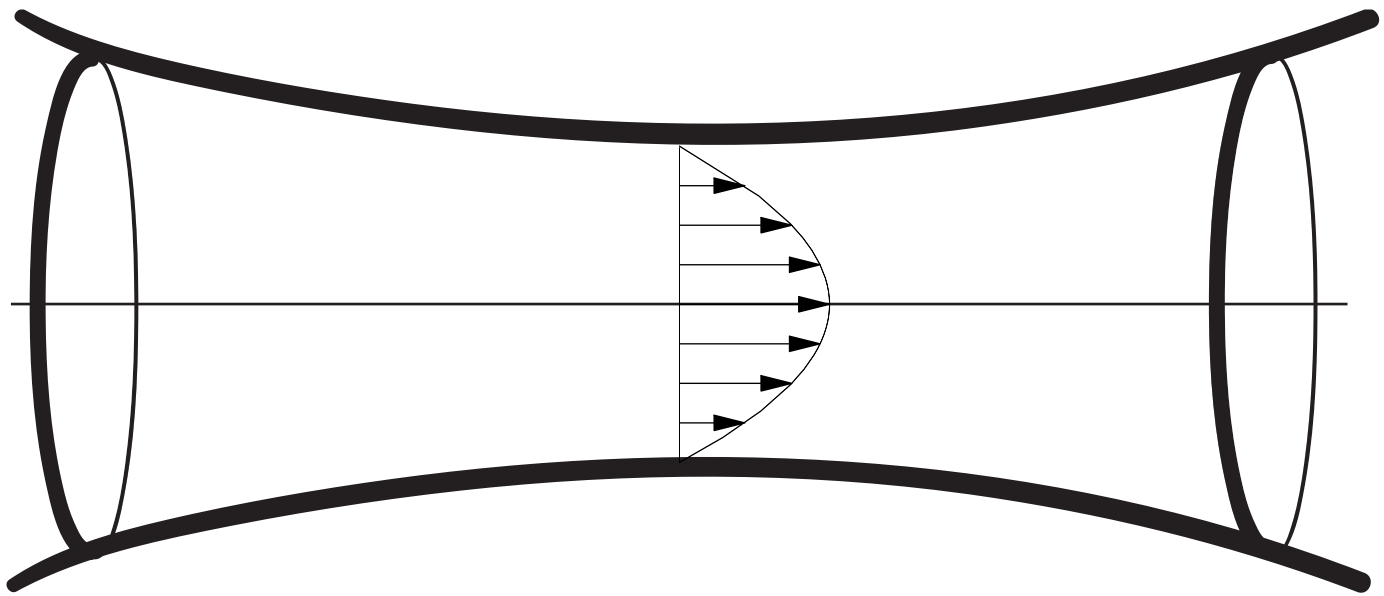
Paroi mobile

Conditions aux limites



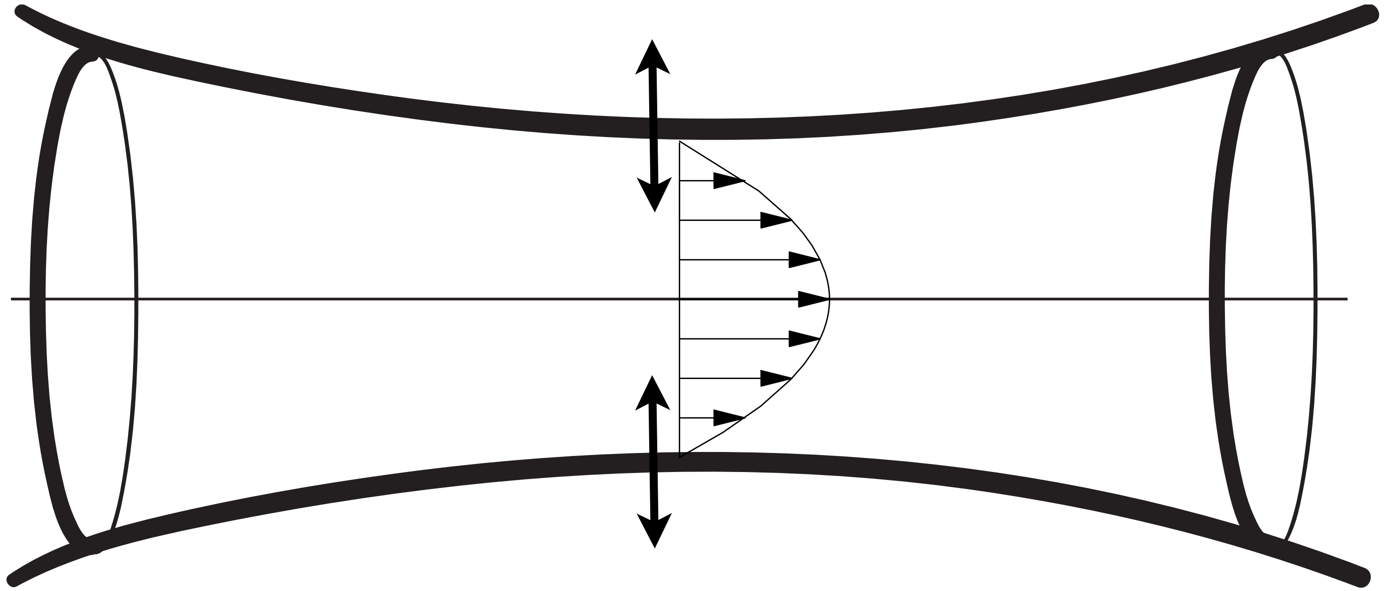
Paroi mobile

Conditions aux limites



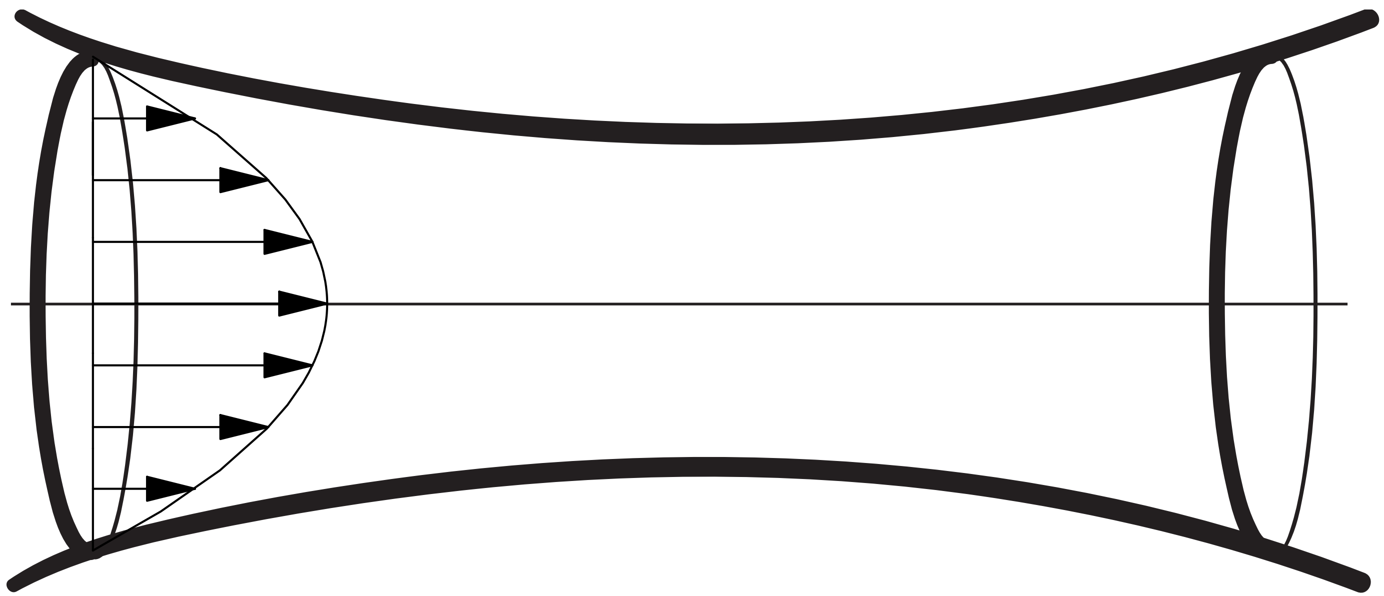
Paroi mobile

Conditions aux limites



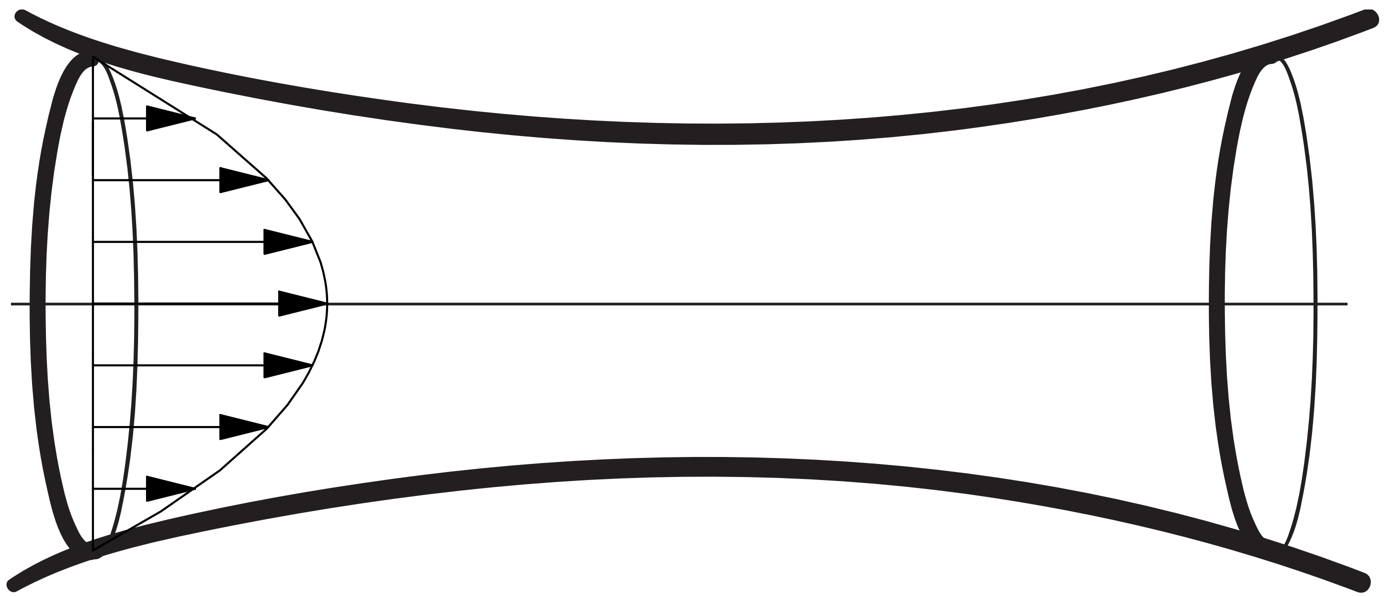
Paroi mobile $$v = \frac{\partial R}{\partial t}$$

Conditions aux limites



Profil initial donné:

Conditions aux limites

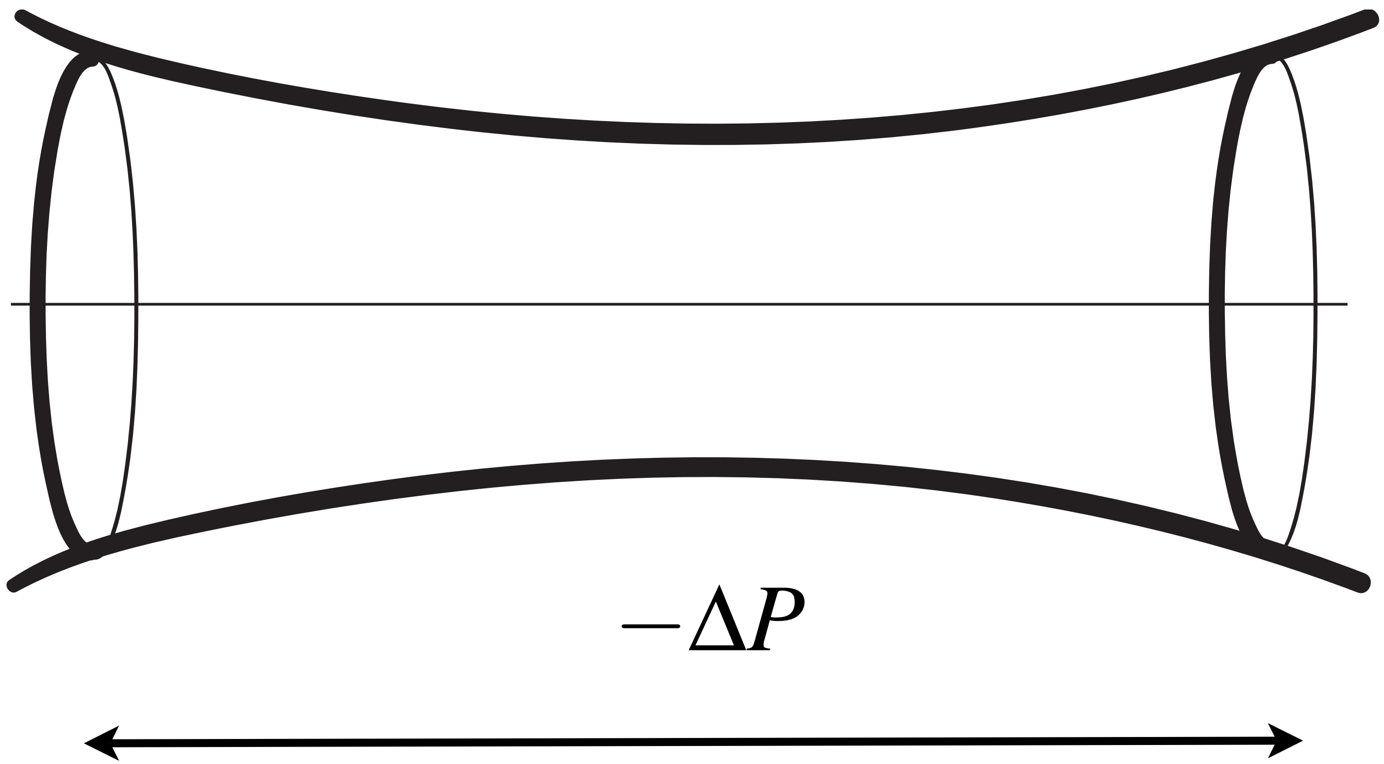


Profil initial donné:

procédure de marche

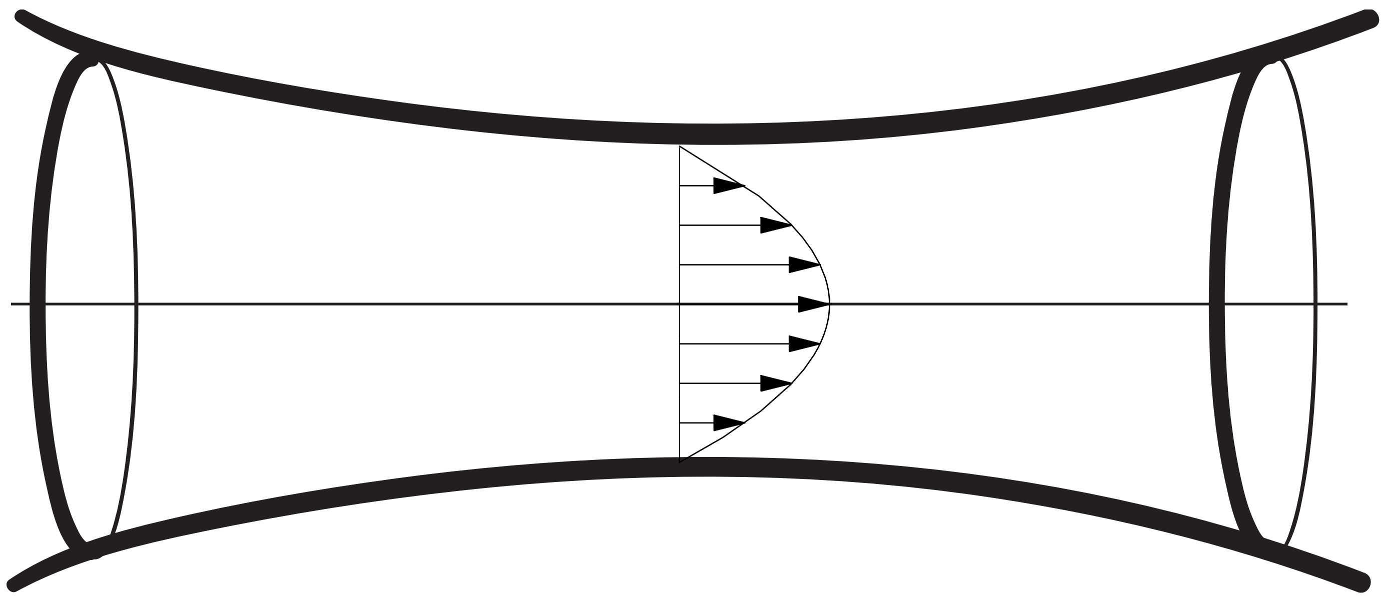
la distribution de pression est un résultat

Conditions aux limites

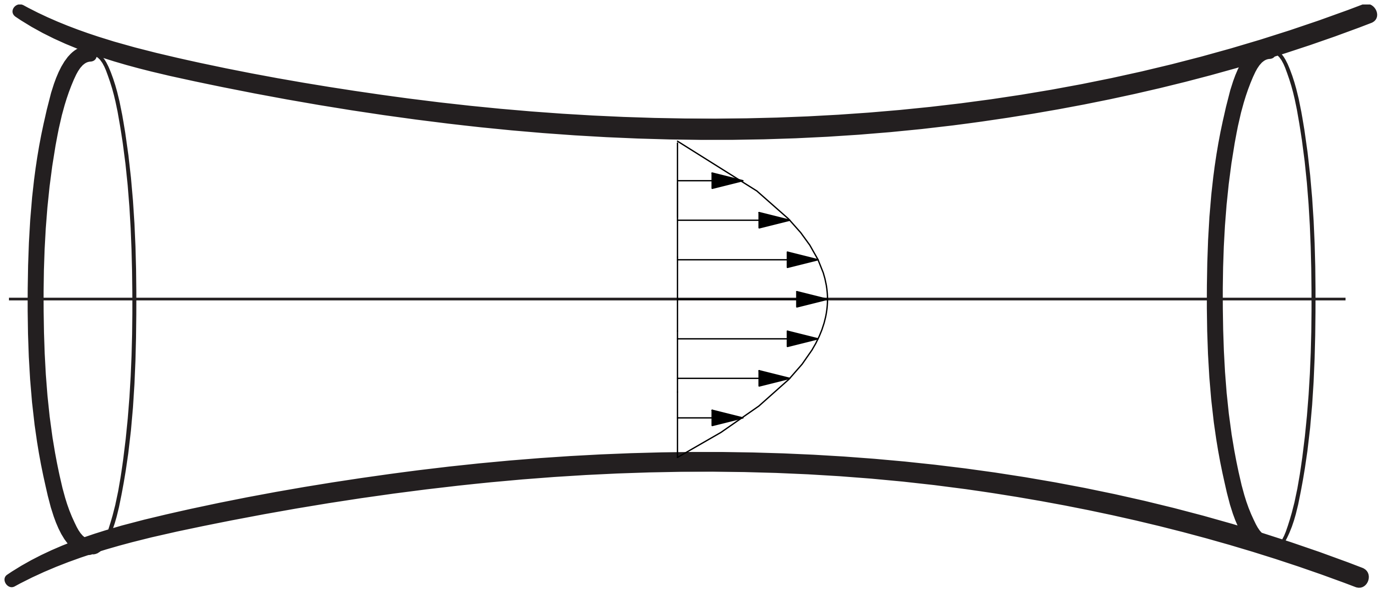


ou à chute de pression donnée
par une itération de Newton sur le flux d'entrée

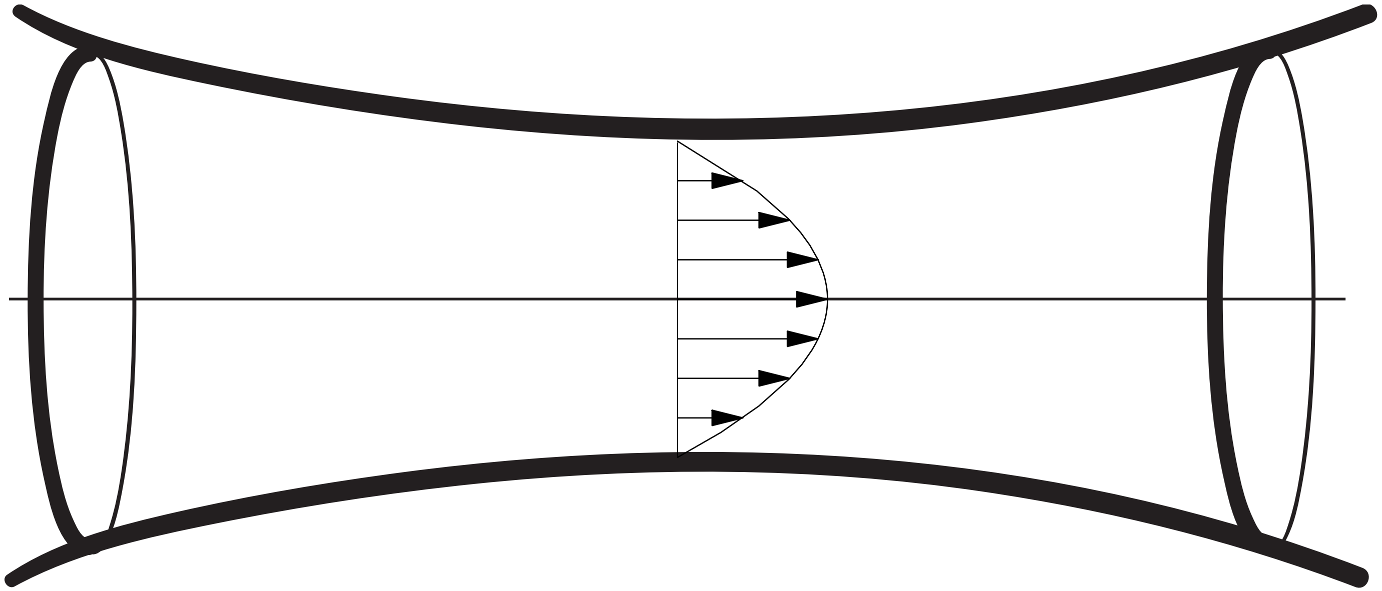
Résolution Numerique



différences finies,
implicite en temps

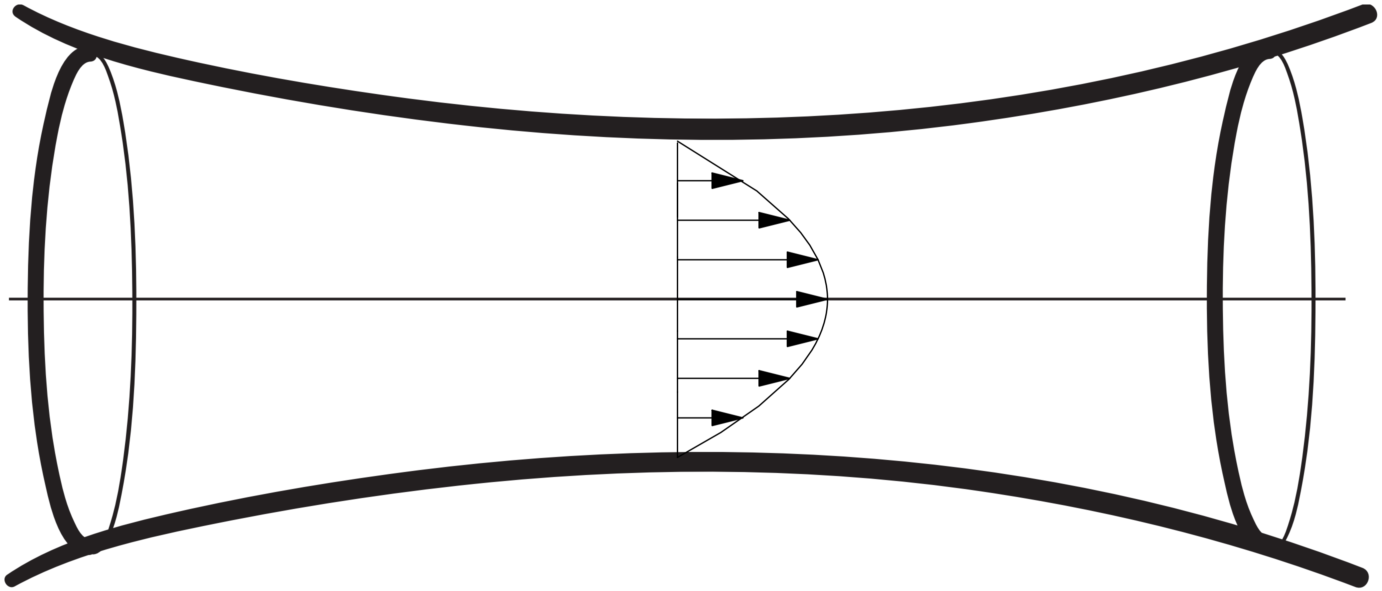


$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$



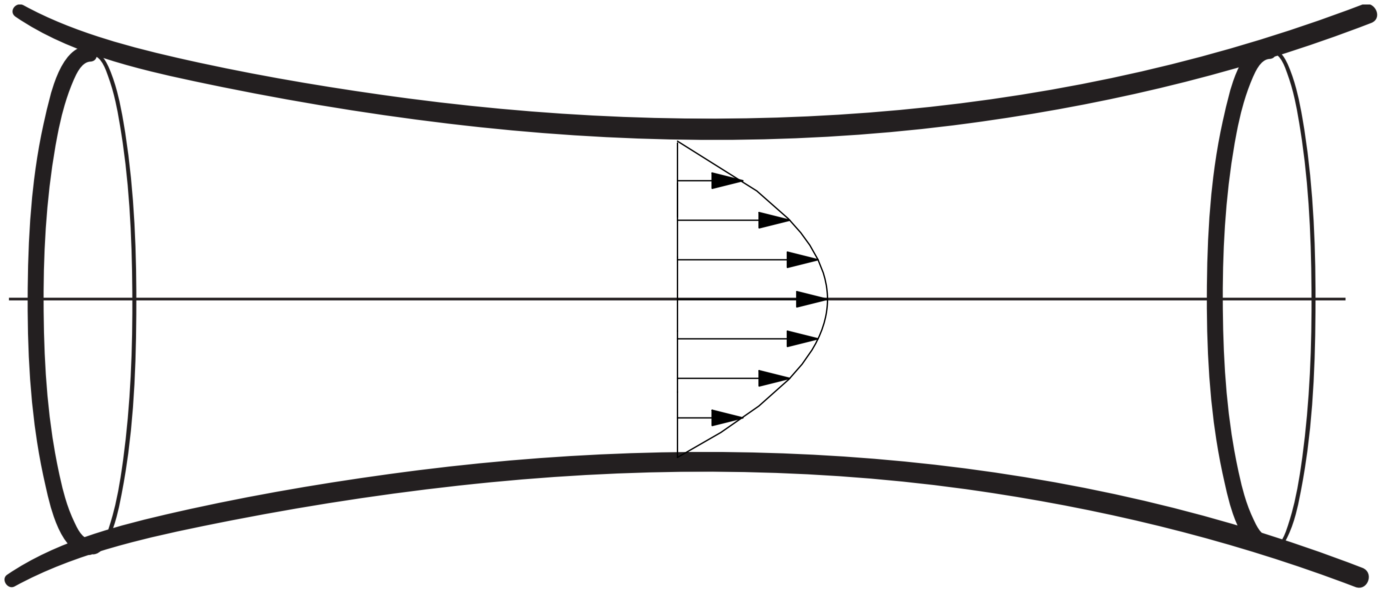
$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

$$p^{given} \rightarrow u^*$$



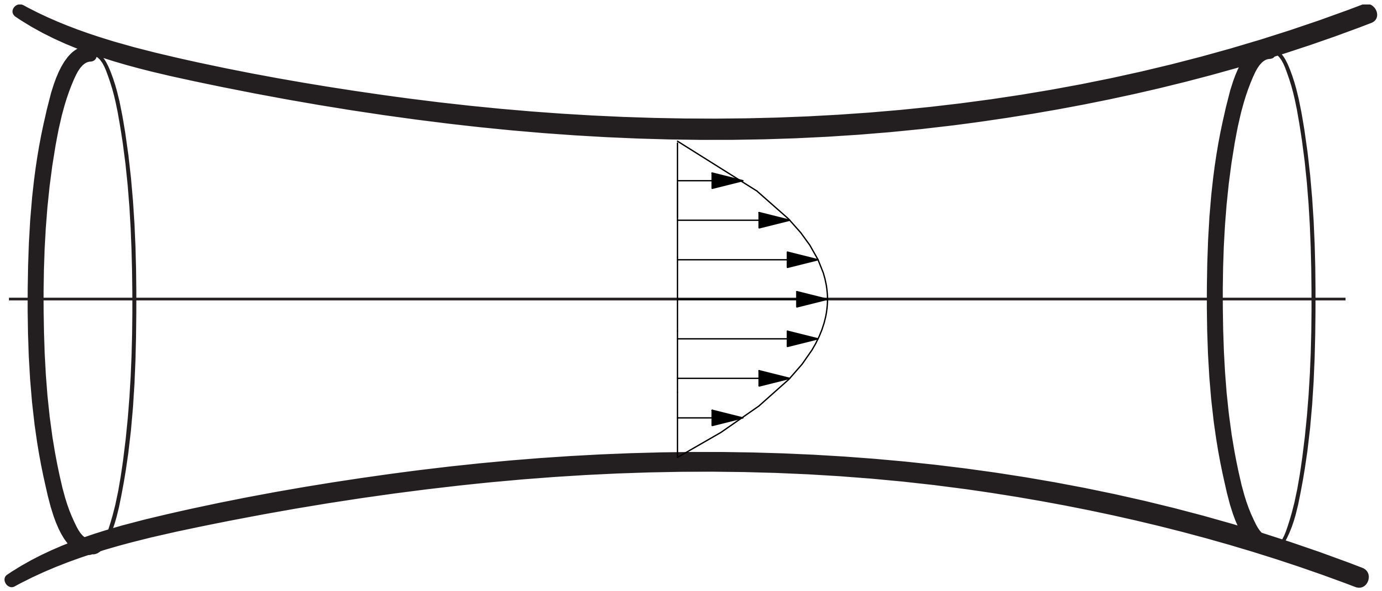
$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

$$p^{given} \rightarrow u^* \qquad rv^*(R) = -\int_0^R r \frac{\partial u^*}{\partial x} dr$$



$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

$$p^{given} \rightarrow u^* \qquad rv^*(R) = -\int_0^R r\frac{\partial u^*}{\partial x} dr \,\Bigg|\, \begin{matrix} \frac{\partial R}{\partial t} \\ 0? \end{matrix} \;?$$

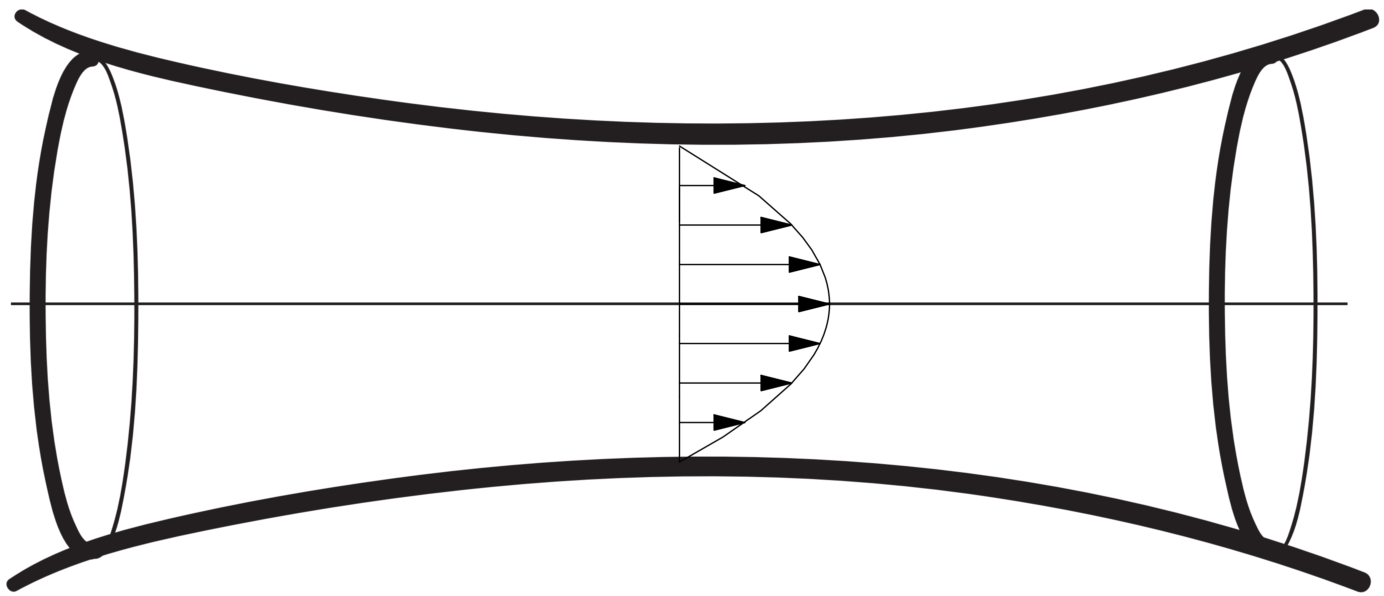


Newton sur la pression pour trouver la condition à la limite

$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

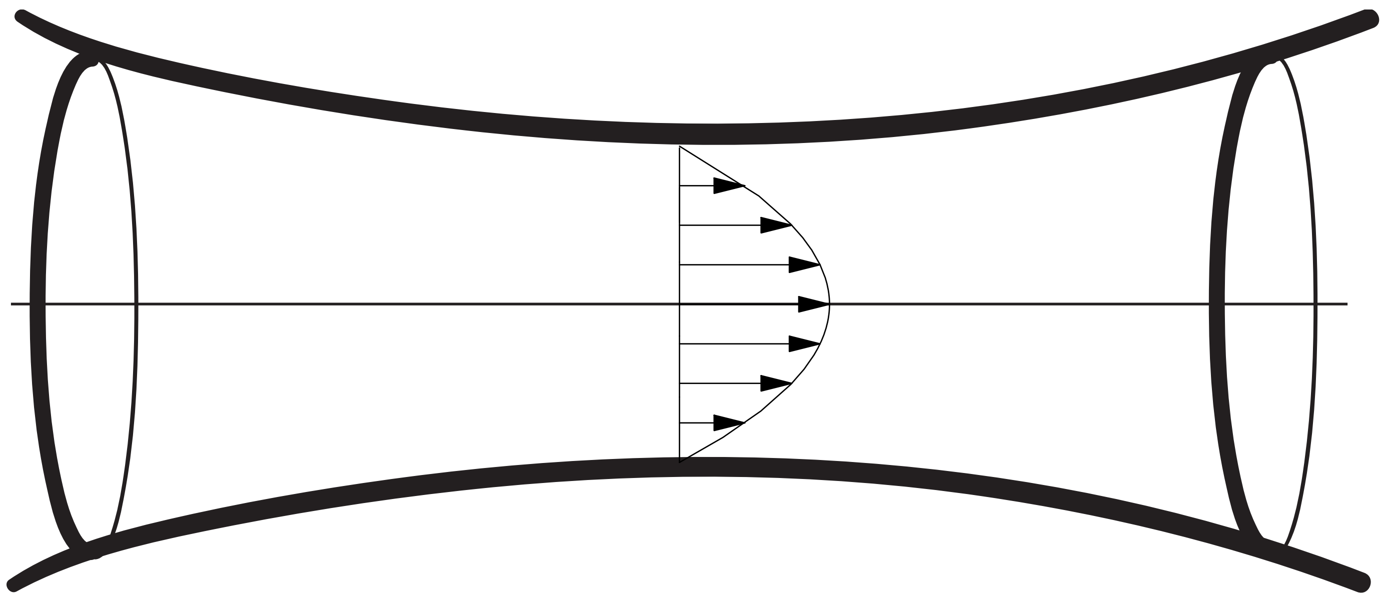
$$p^{given} \rightarrow u^* \longrightarrow rv^*(R) = -\int_0^R r\frac{\partial u^*}{\partial x} dr \Big| \frac{\partial R}{\partial t} \; ?$$

0?

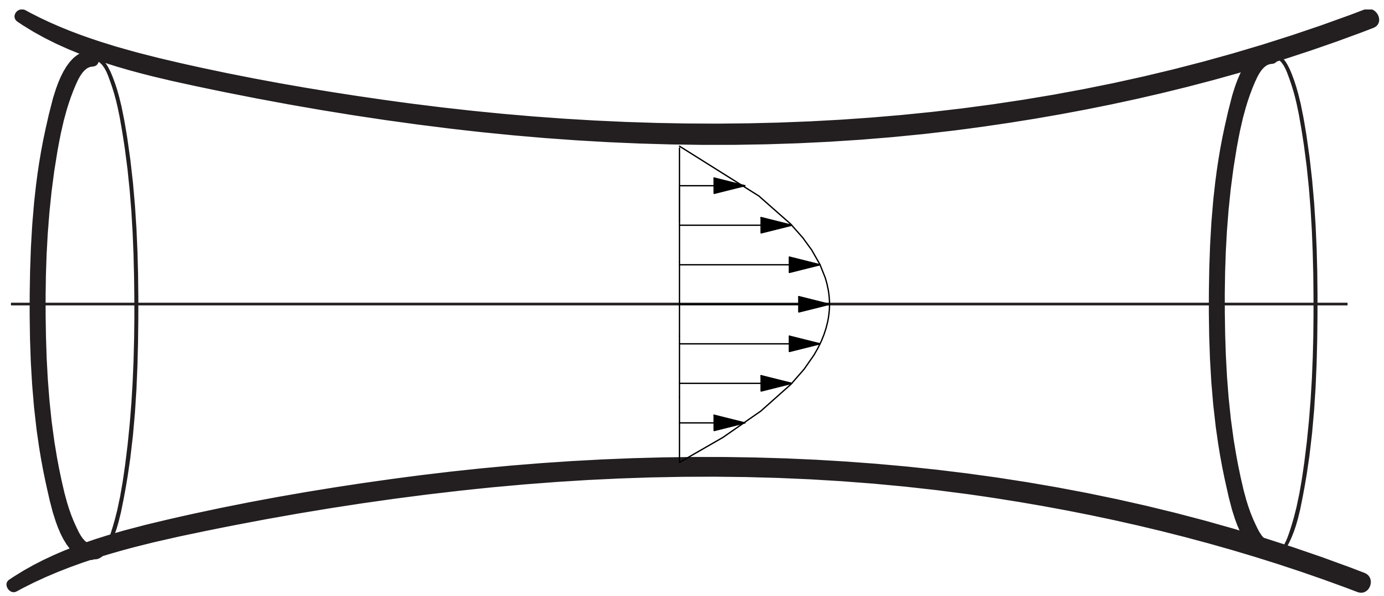


la Pression est un résultat du calcul

Résolution Intégrale



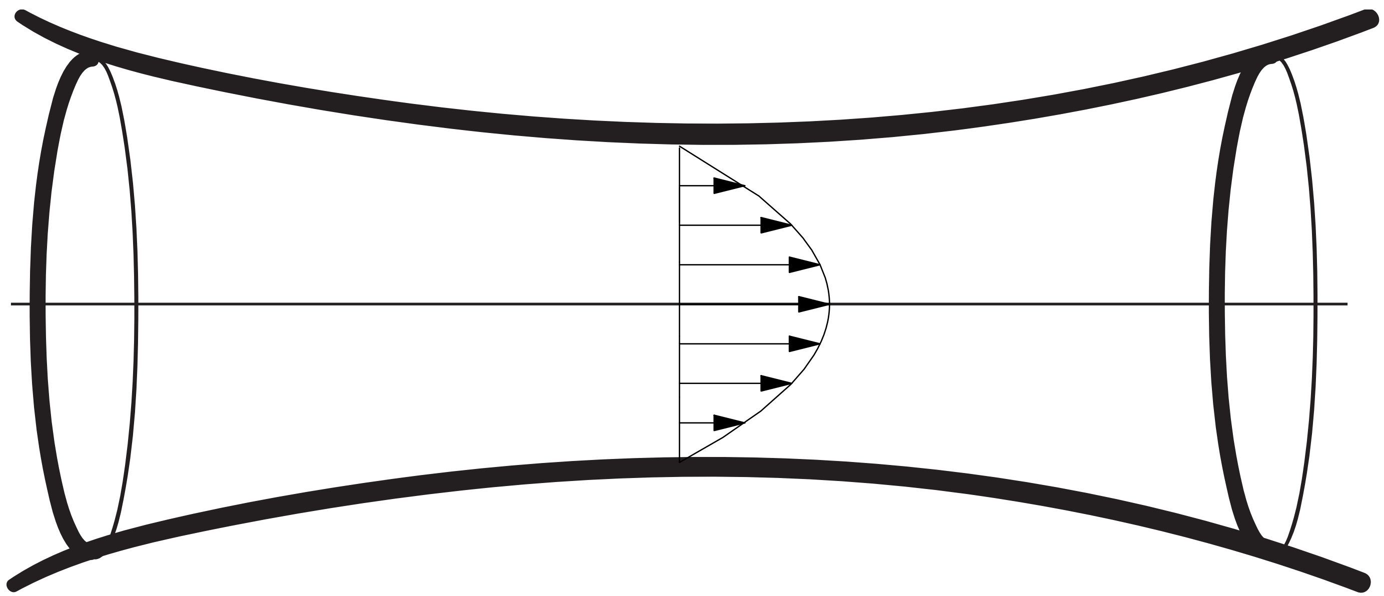
Résolution Intégrale



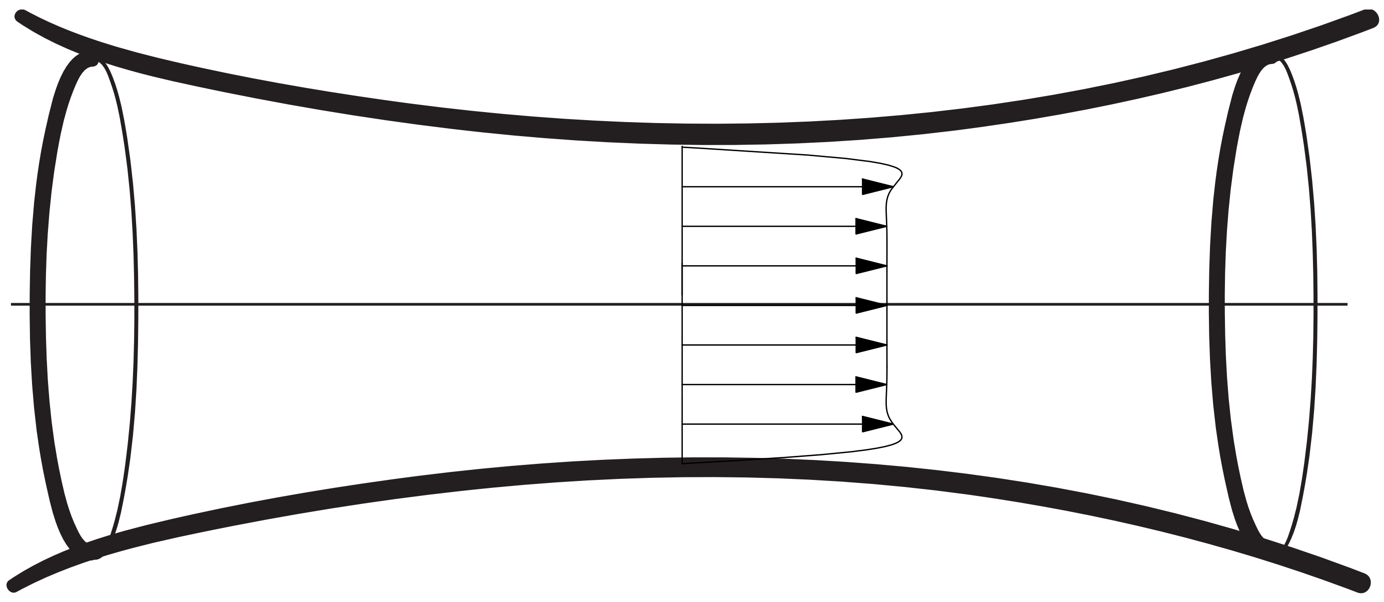
- système integral (1D) est inclus dans RNSP

- on calcule des profils plus réalistes

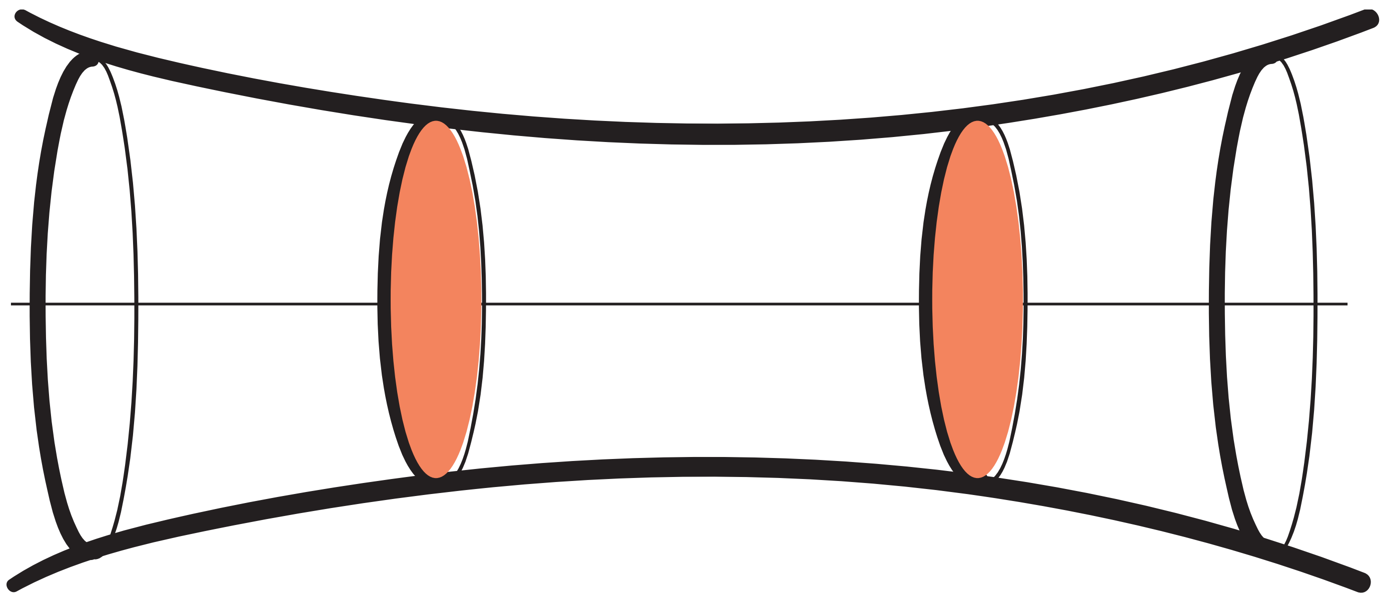
Résolution Intégrale



Résolution Intégrale

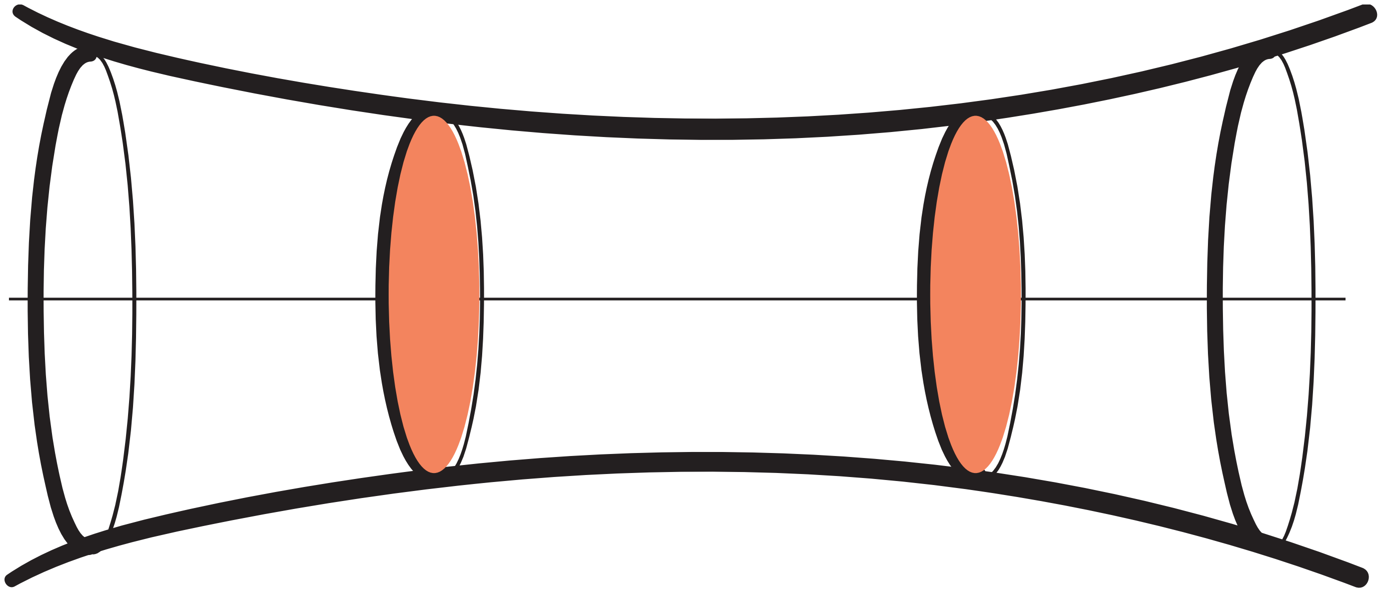


Résolution Intégrale



$$Q = \int_0^R 2\pi r u dr$$

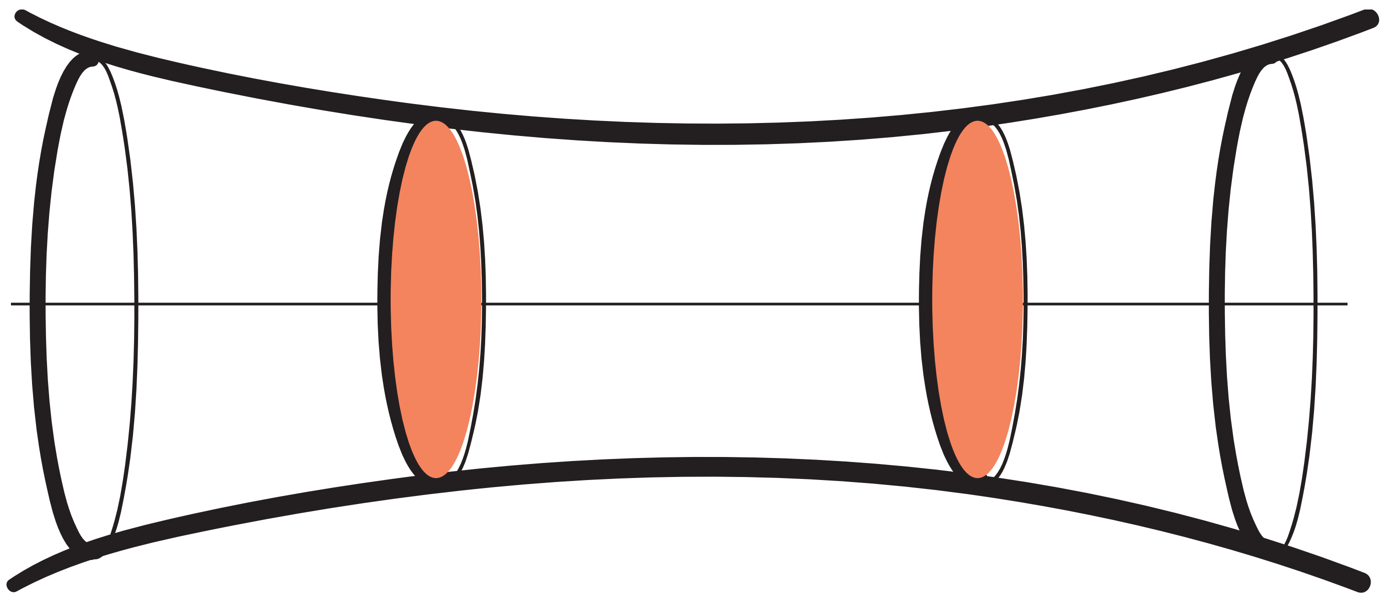
Résolution Intégrale



$$Q = \int_0^R 2\pi r u dr$$

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

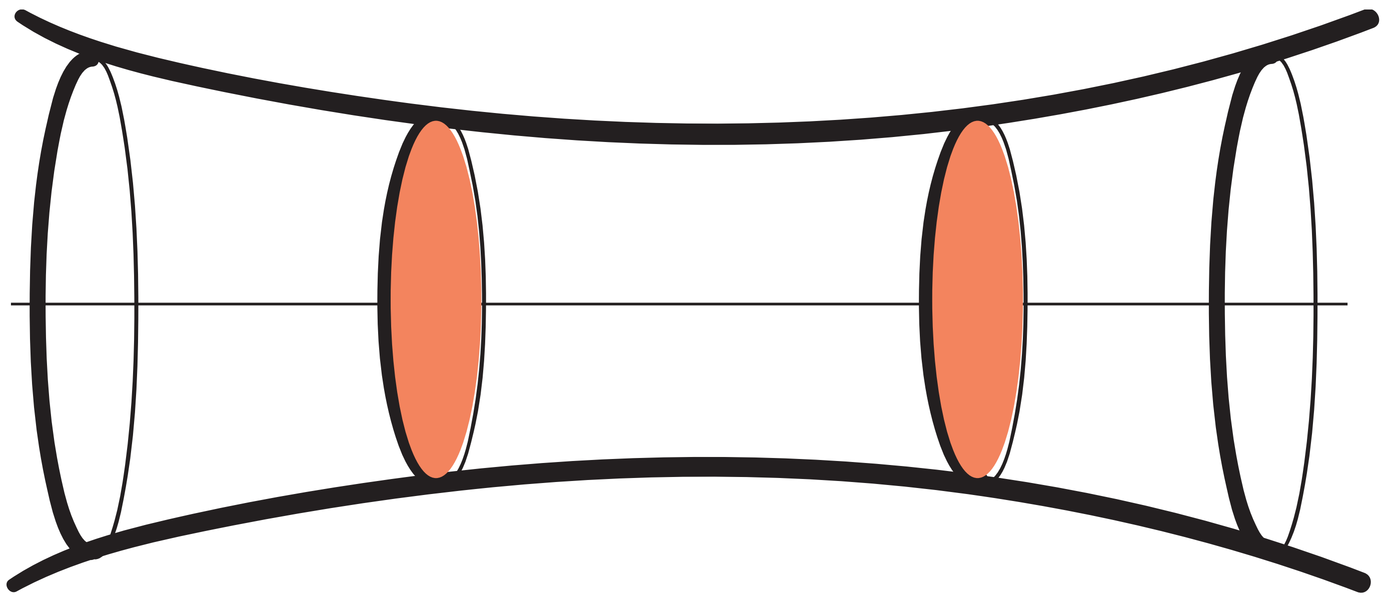
Résolution Intégrale



$$Q = \int_0^R 2\pi r u dr$$

$$\int_0^R 2\pi r dr \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} \right) = 0$$

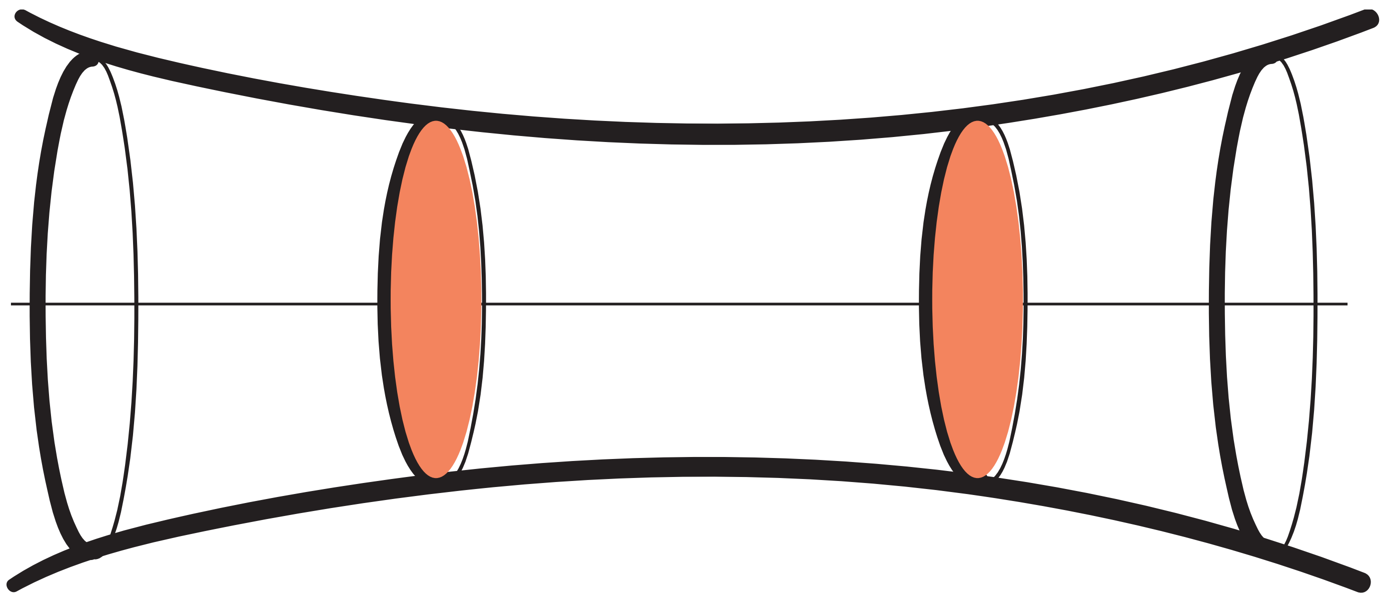
Résolution Intégrale



$$Q = \int_0^R 2\pi r u dr$$

$$\int_0^R 2\pi r dr \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} \right) = 0 \longrightarrow \frac{\partial (2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

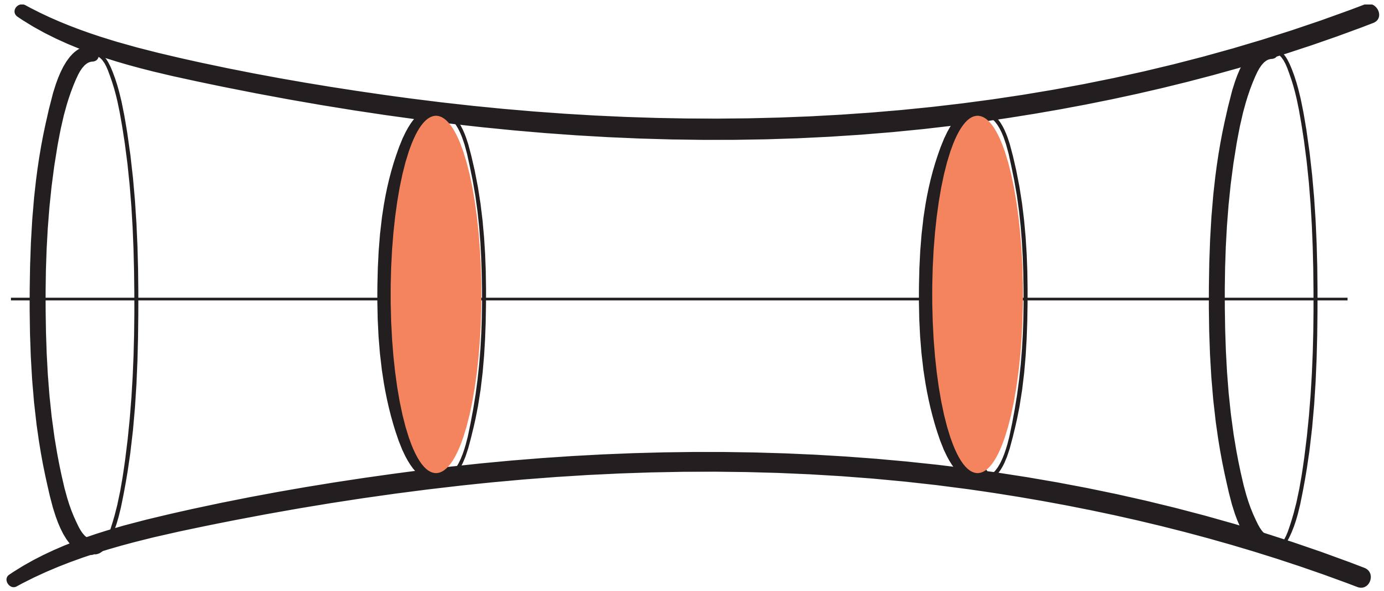
Résolution Intégrale



$$Q = \int_0^R 2\pi r u dr$$

$$\frac{\partial(2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

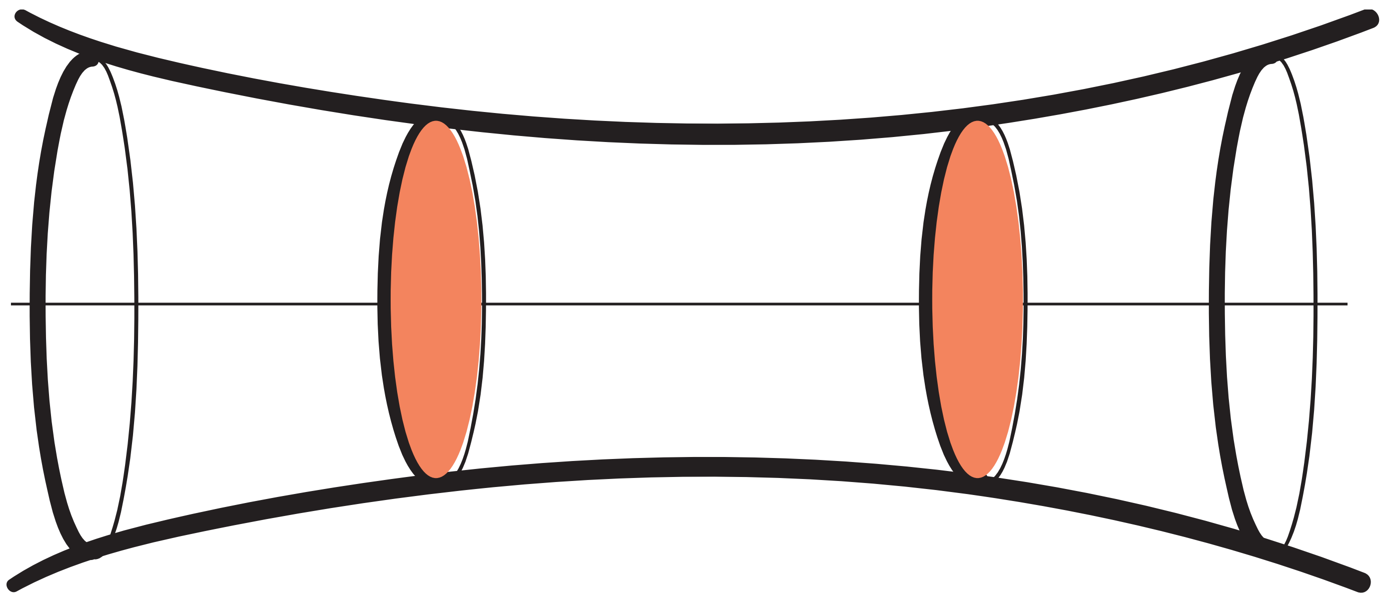
Résolution Intégrale



$$Q = \int_0^R 2\pi r u dr$$

$$\tau = \frac{\partial u}{\partial r}$$

Résolution Intégrale

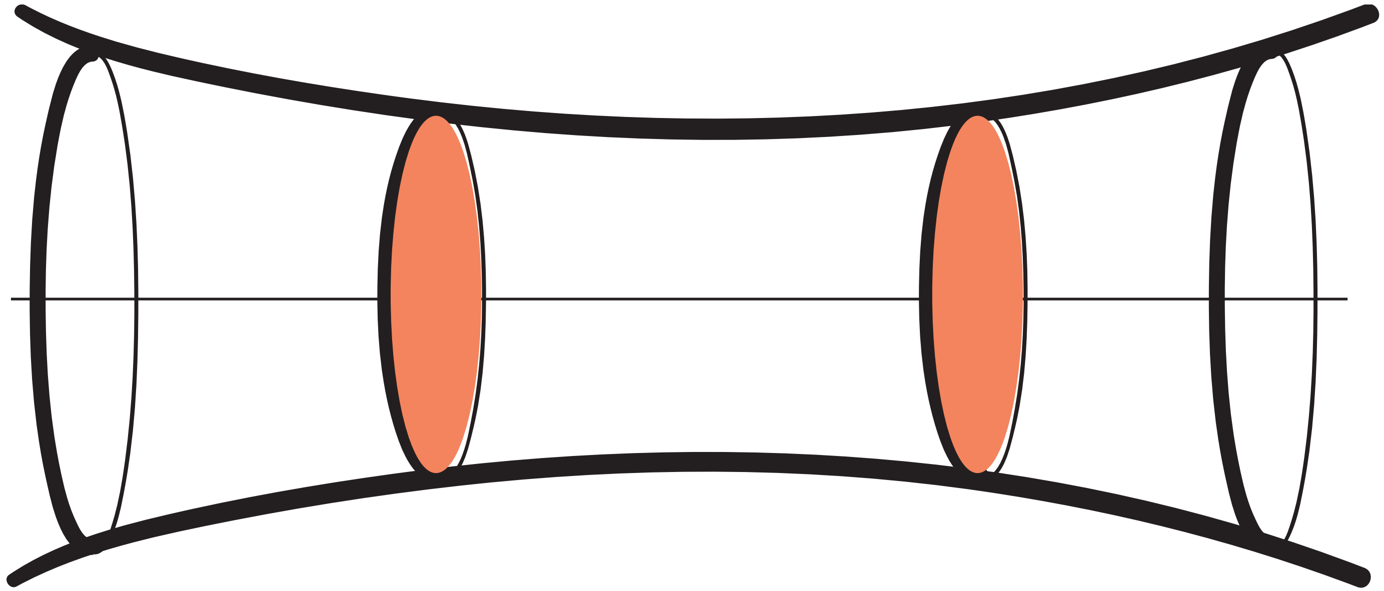


$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\int\left(\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r} \right)$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

Résolution Intégrale

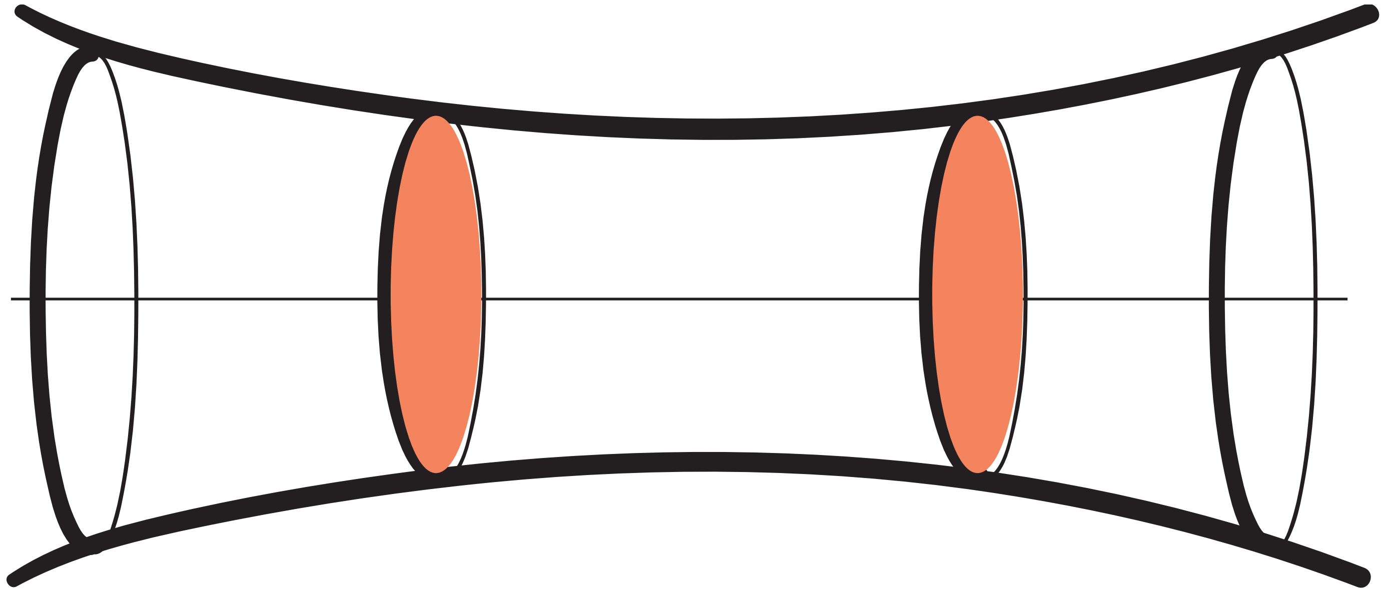


$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\int \left(\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r} \right) \quad \frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2)\frac{\partial p}{\partial x} - \tau$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

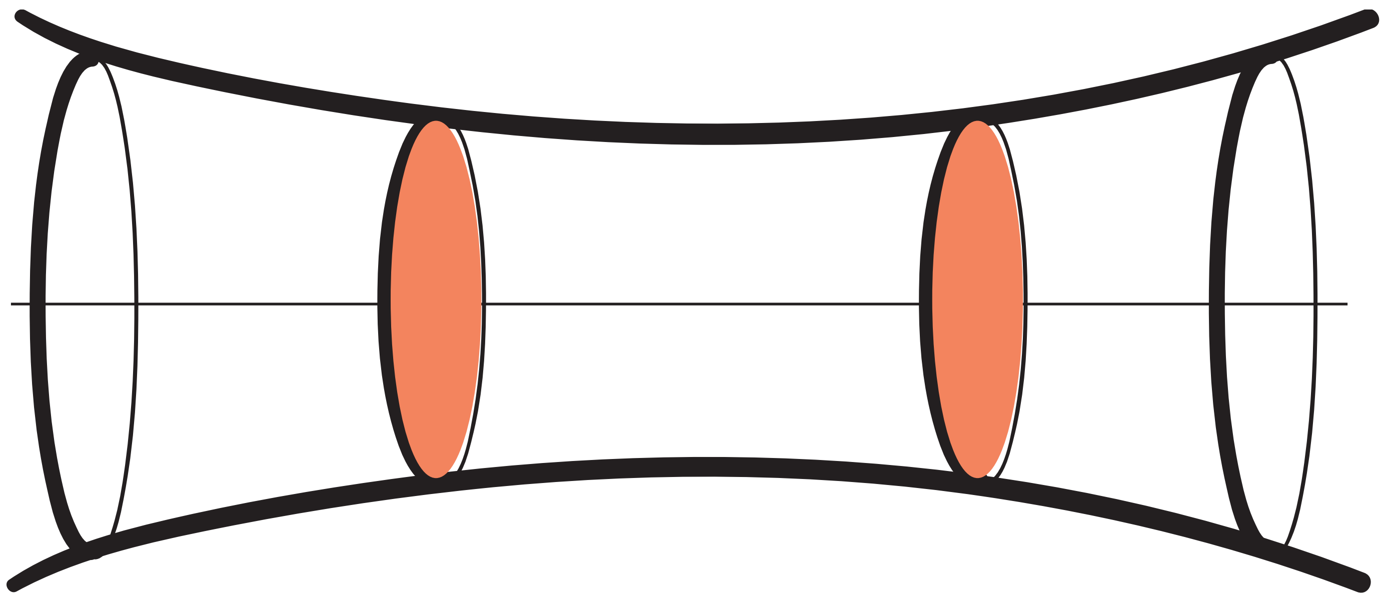
Résolution Intégrale



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2)\frac{\partial p}{\partial x} - \tau$$

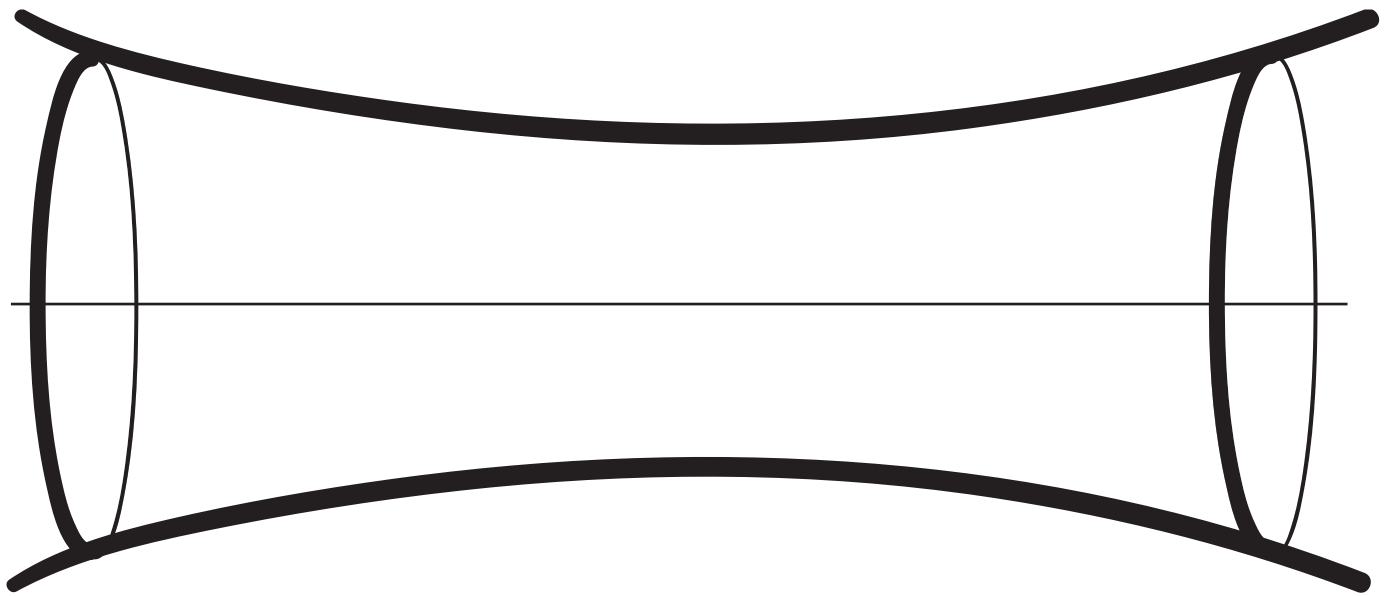
Résolution Intégrale équations 1D



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\frac{\partial(2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2)\frac{\partial p}{\partial x} - \tau$$

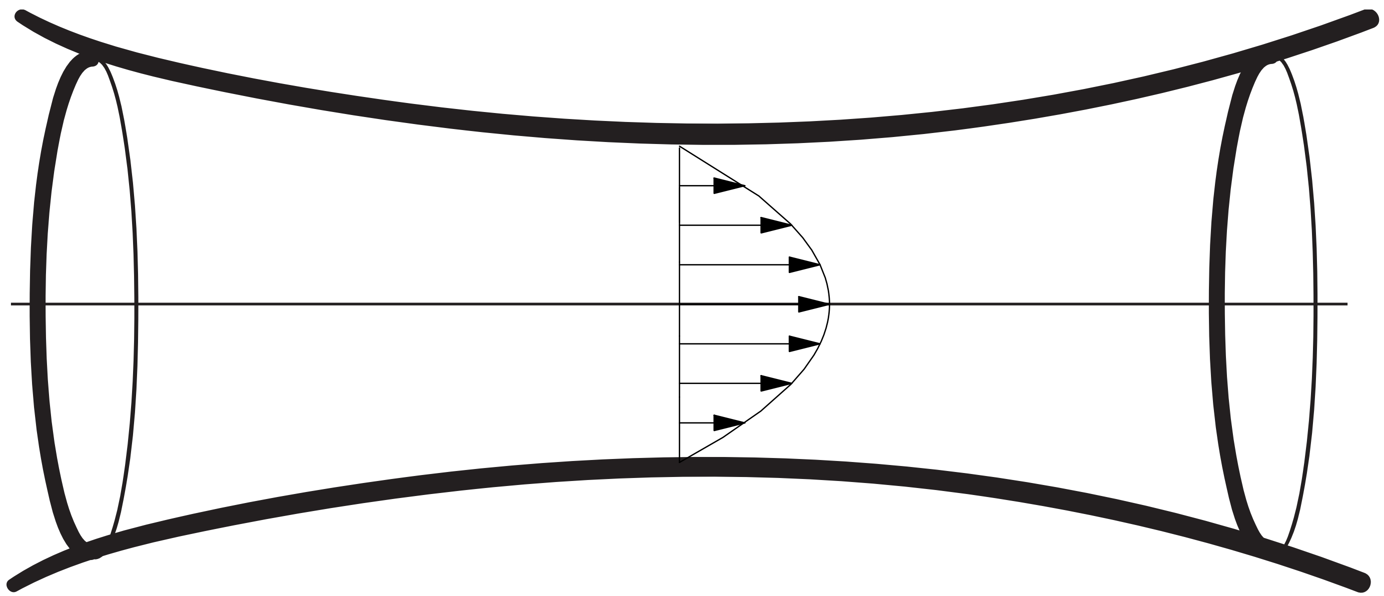
Résolution Intégrale équations 1D



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

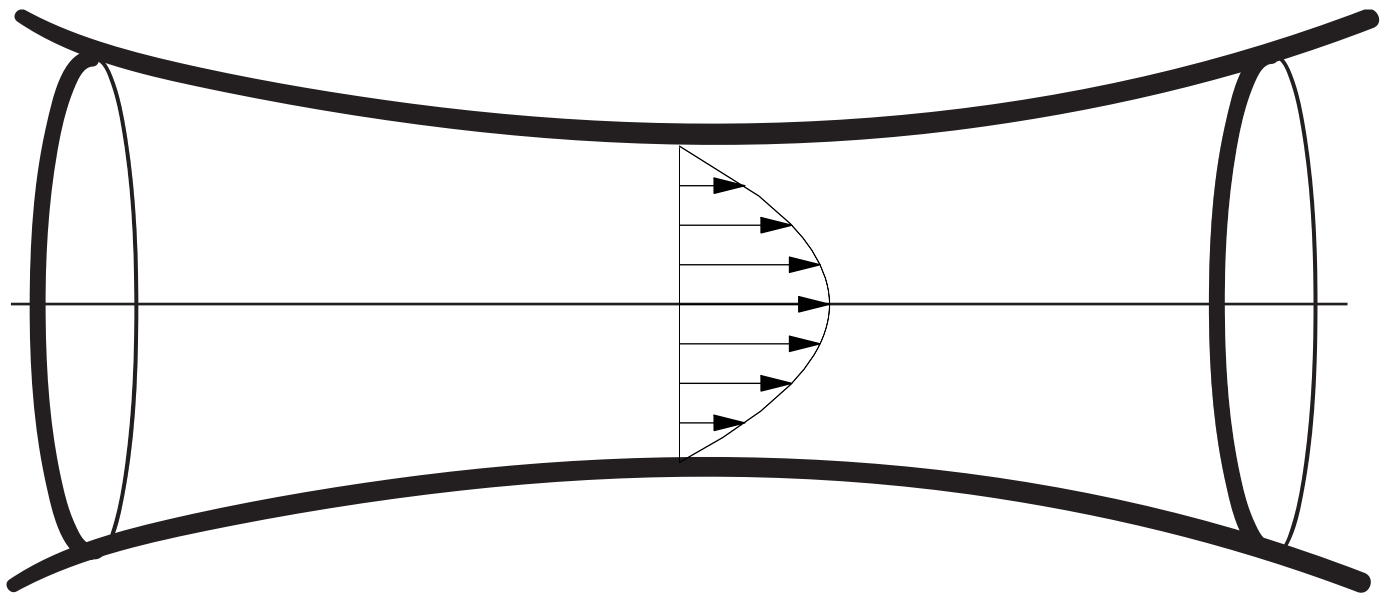
donne Q_2 fonction de Q et τ fonction de Q

Résolution Intégrale équations 1D



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

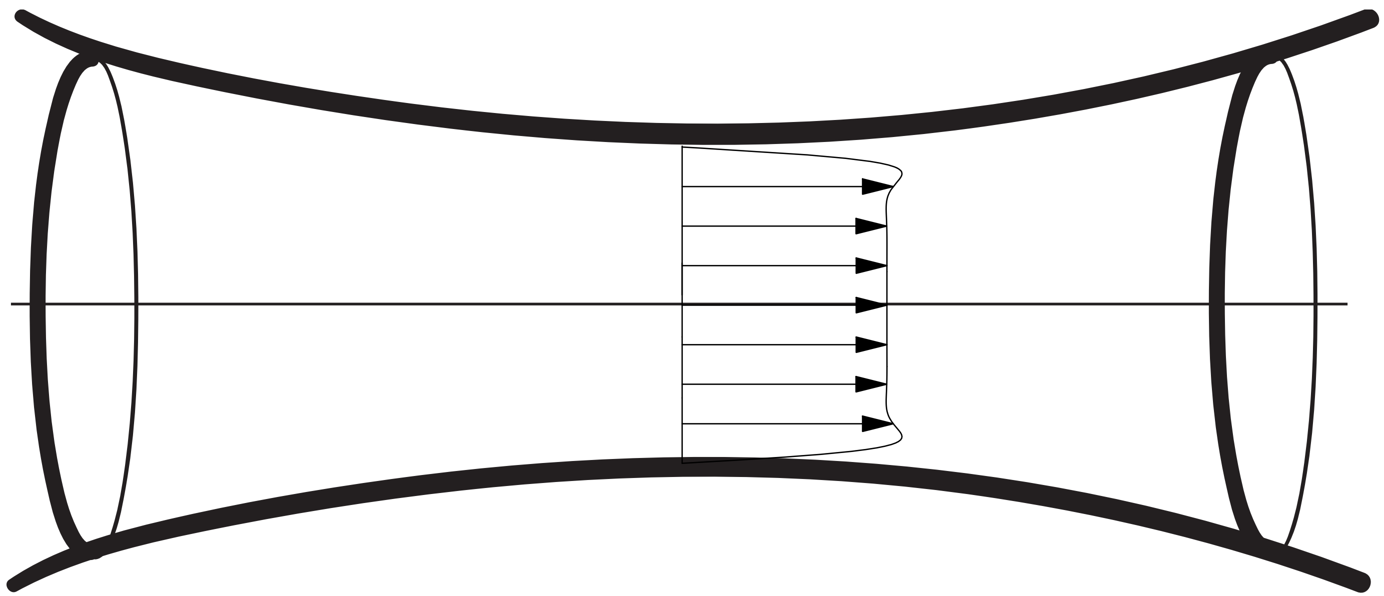
Résolution Intégrale équations 1D



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

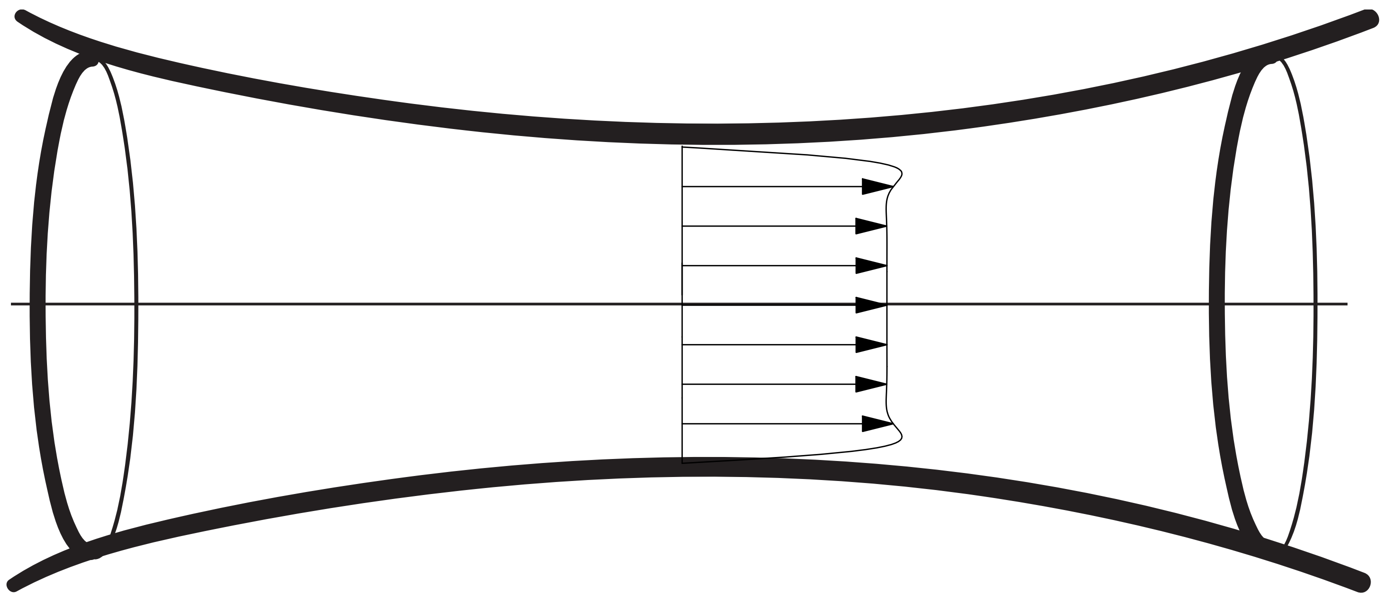
$$Q_2 = (\frac{4}{3})\frac{Q^2}{\pi R^2} \qquad \tau = (8\pi)\frac{Q}{\pi R^2}$$

Résolution Intégrale équations 1D



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

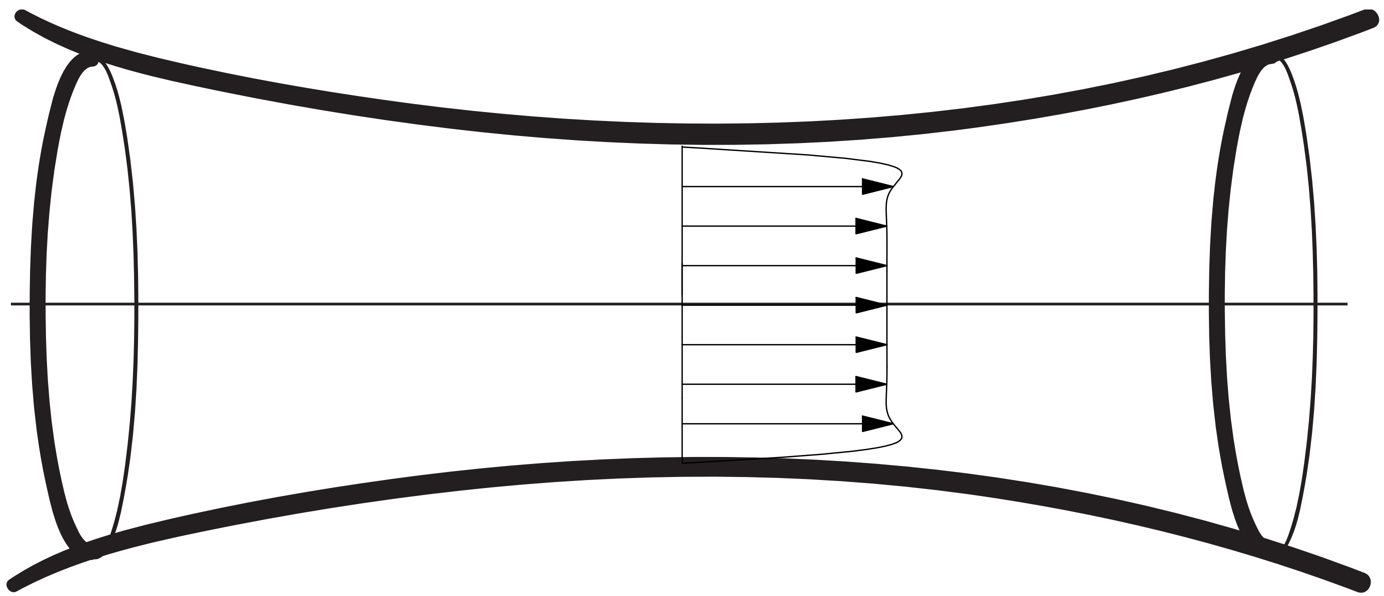
Résolution Intégrale équations 1D



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

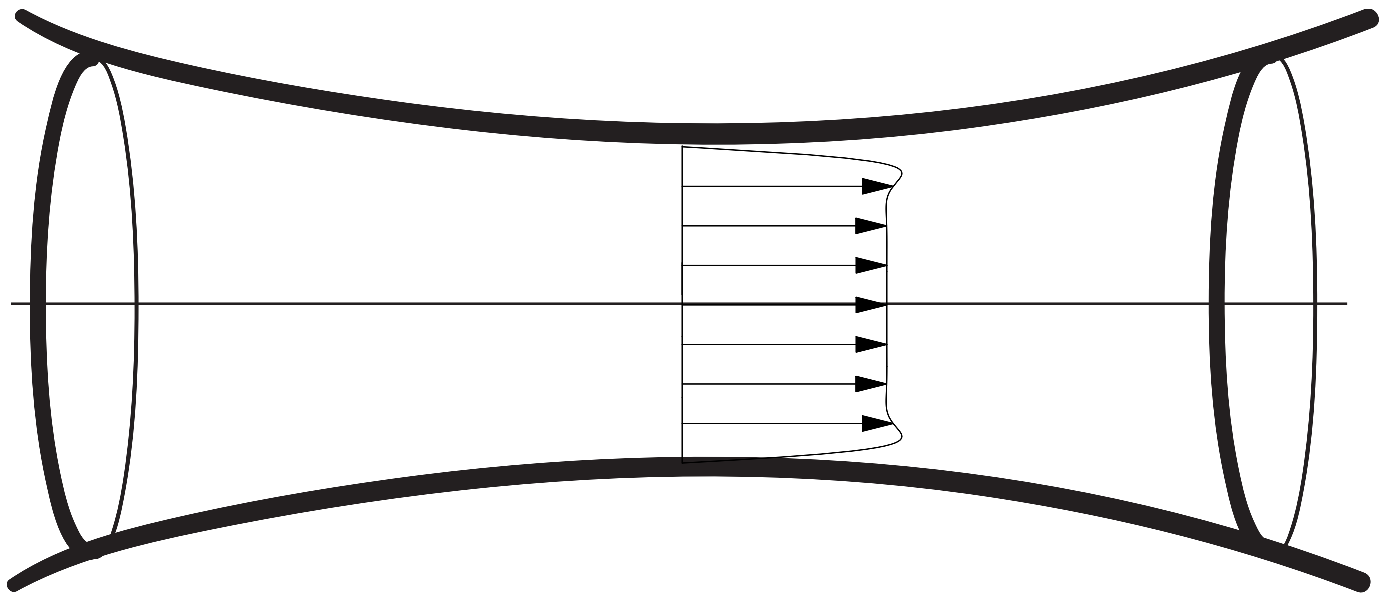
$$Q_2 = \frac{Q^2}{\pi R^2} \qquad \tau = F(Q)$$

Résolution Intégrale équations 1D



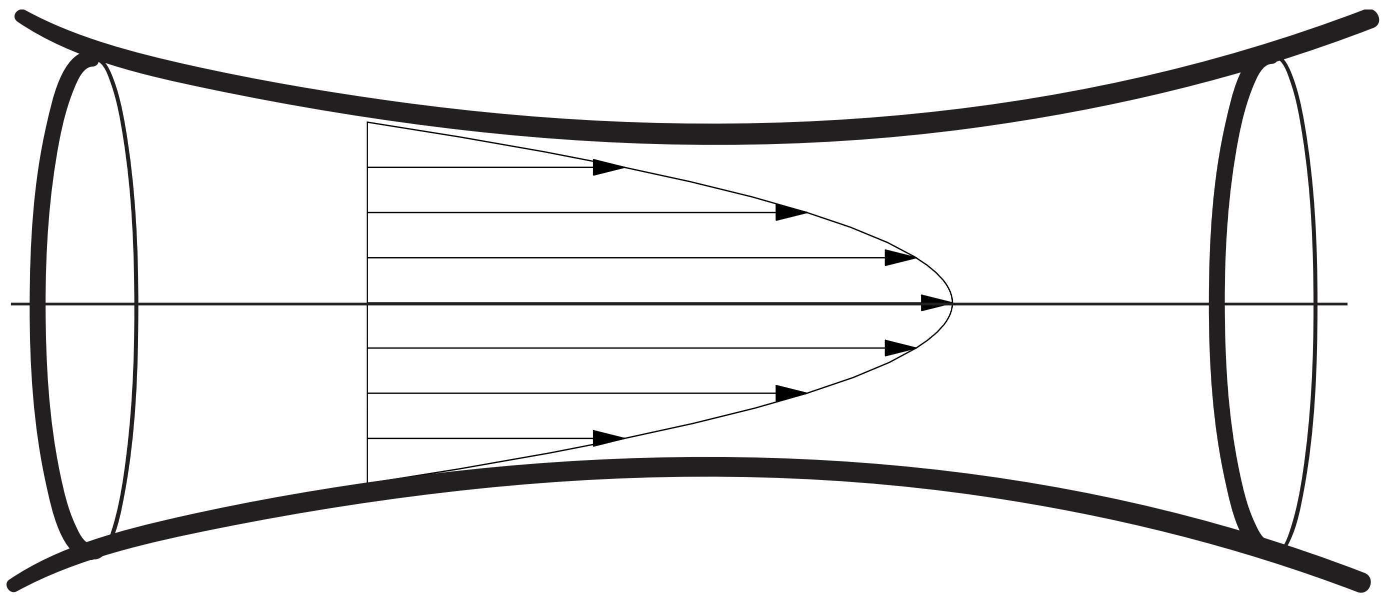
besoin d'un profil

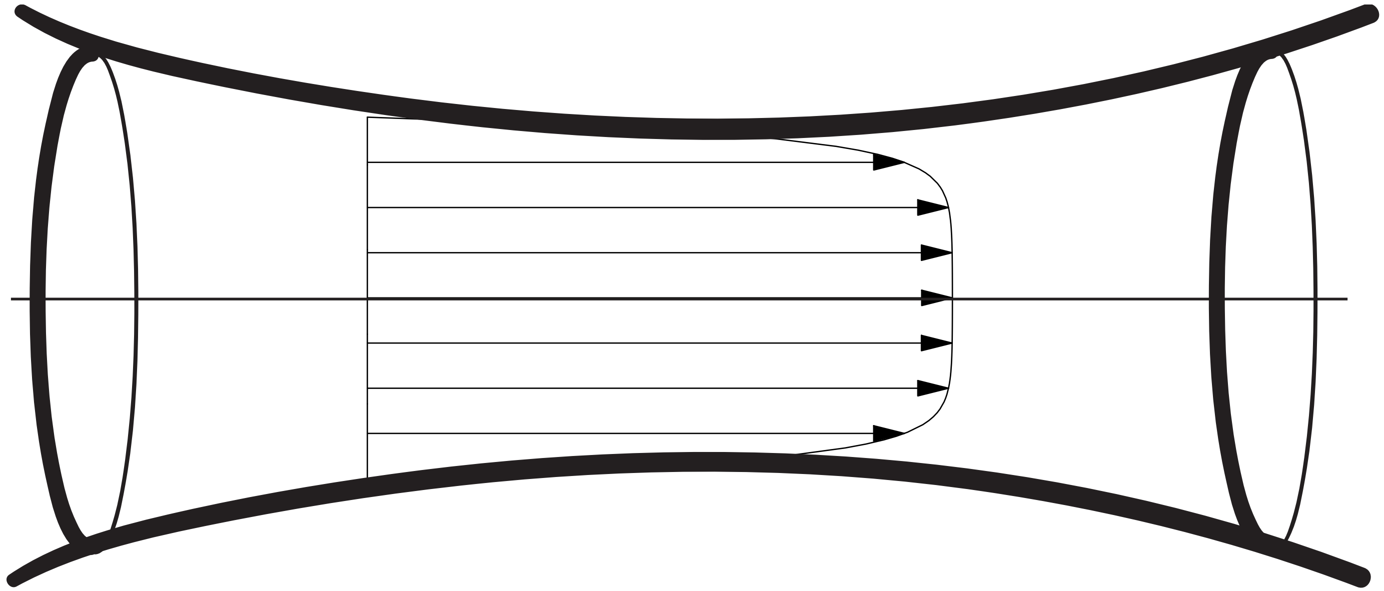
Résolution Intégrale équations 1D

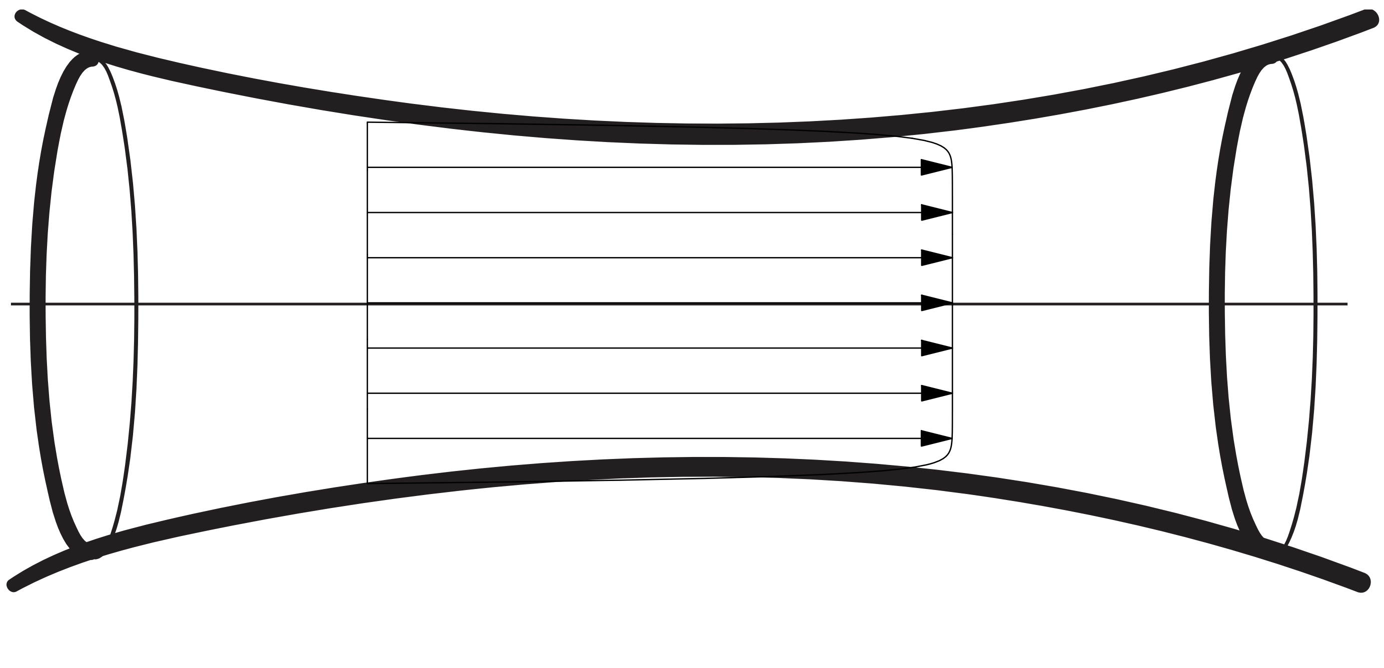


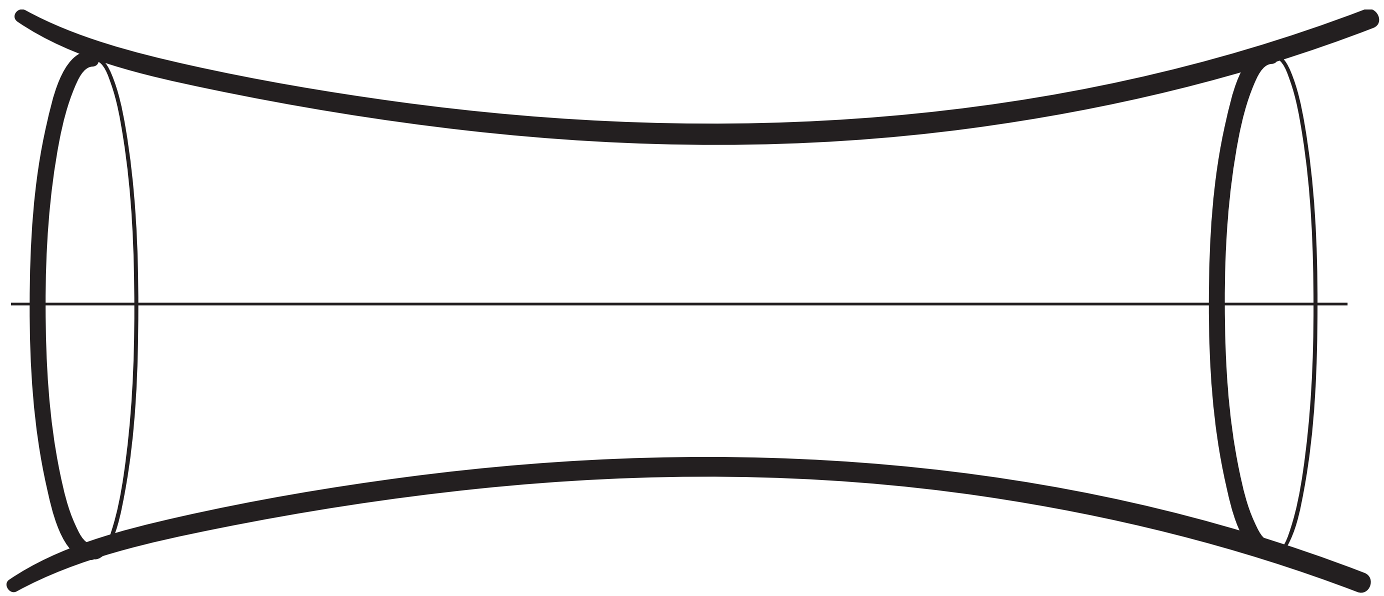
Les équations 1D “habituelles” sont
une simplification de RNSP

Choix des profils

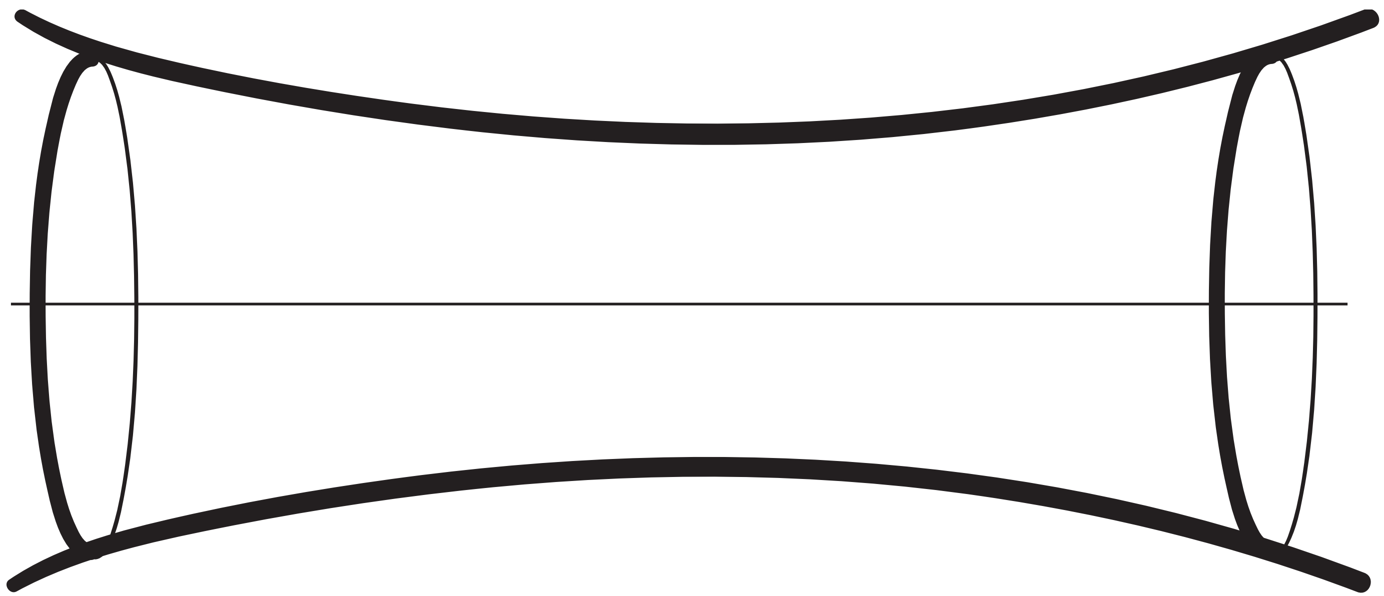








Choix d’une famille de profils simples

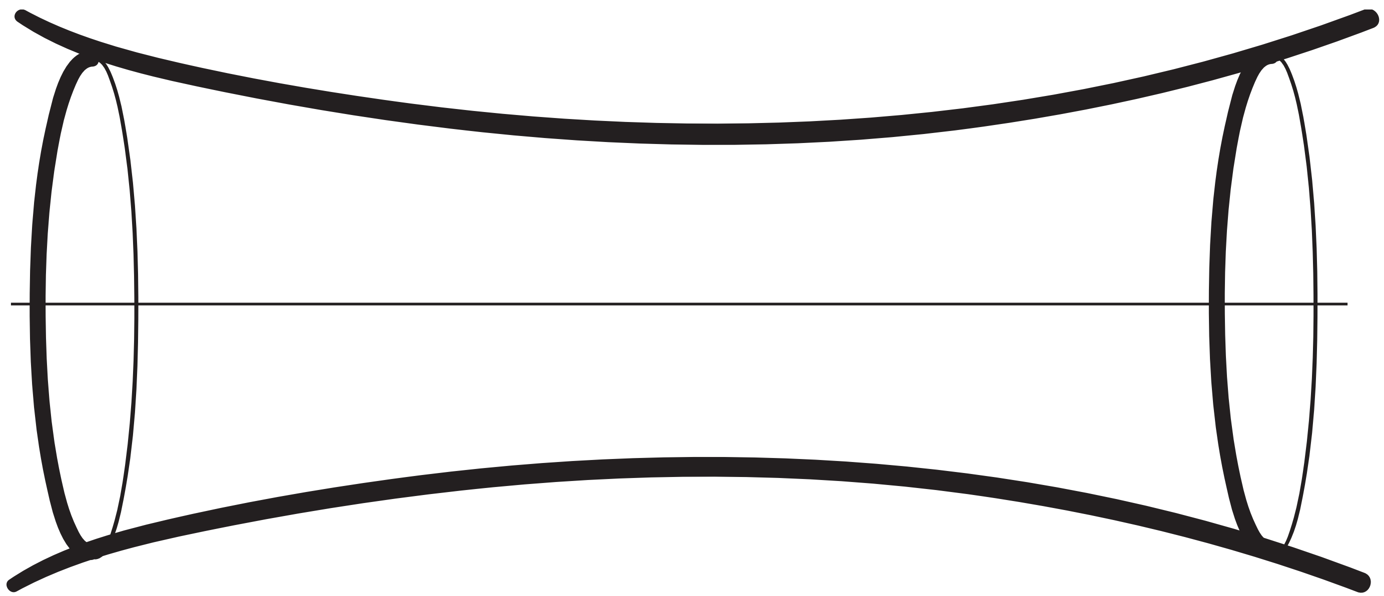


Choix d'une famille de profils simples

Dans un écoulement instationnaire, il est naturel de prendre les profils de Womersley

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r}$$

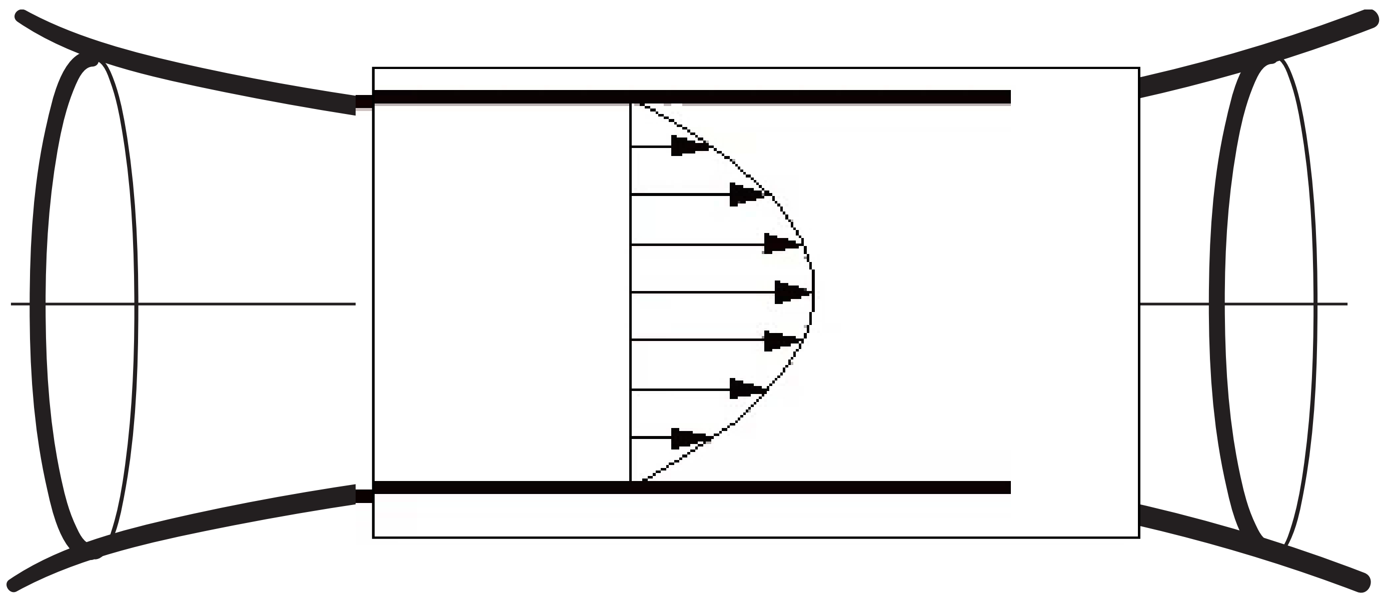
$$0 = -\frac{\partial p}{\rho \partial r}$$



Choix d’une famille de profils simples

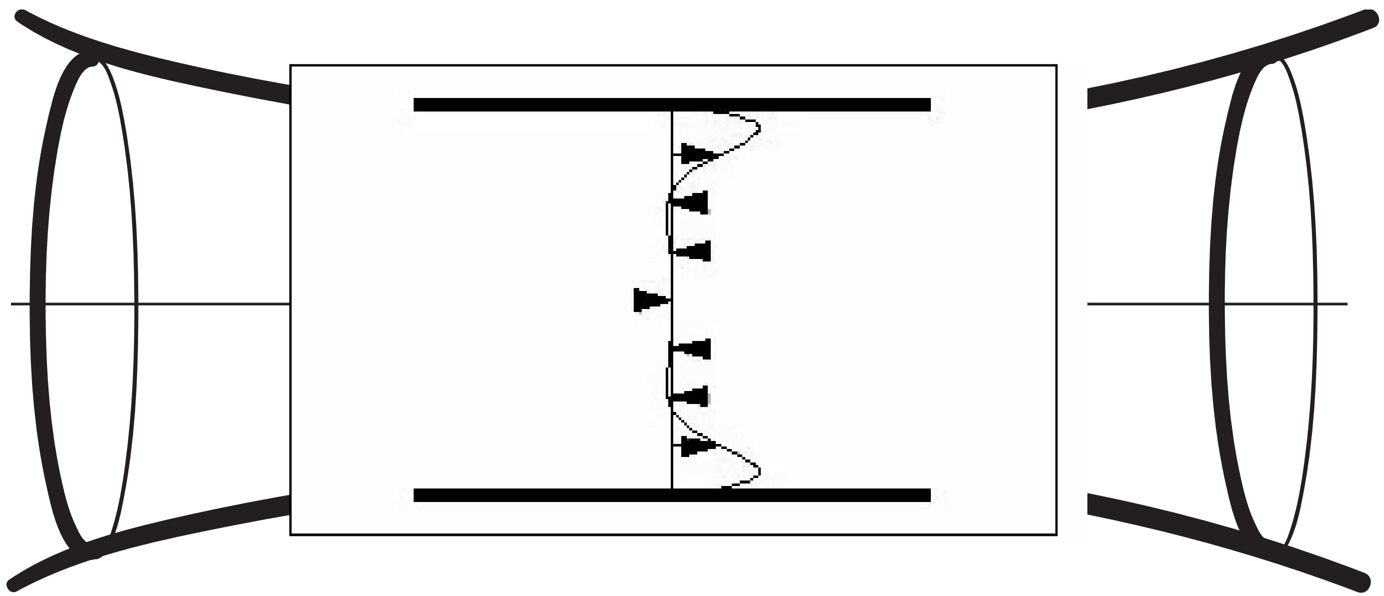
Dans un écoulement instationnaire, il est naturel de prendre les profils de Womersley

les profils de Womersley sont solution de RNSP



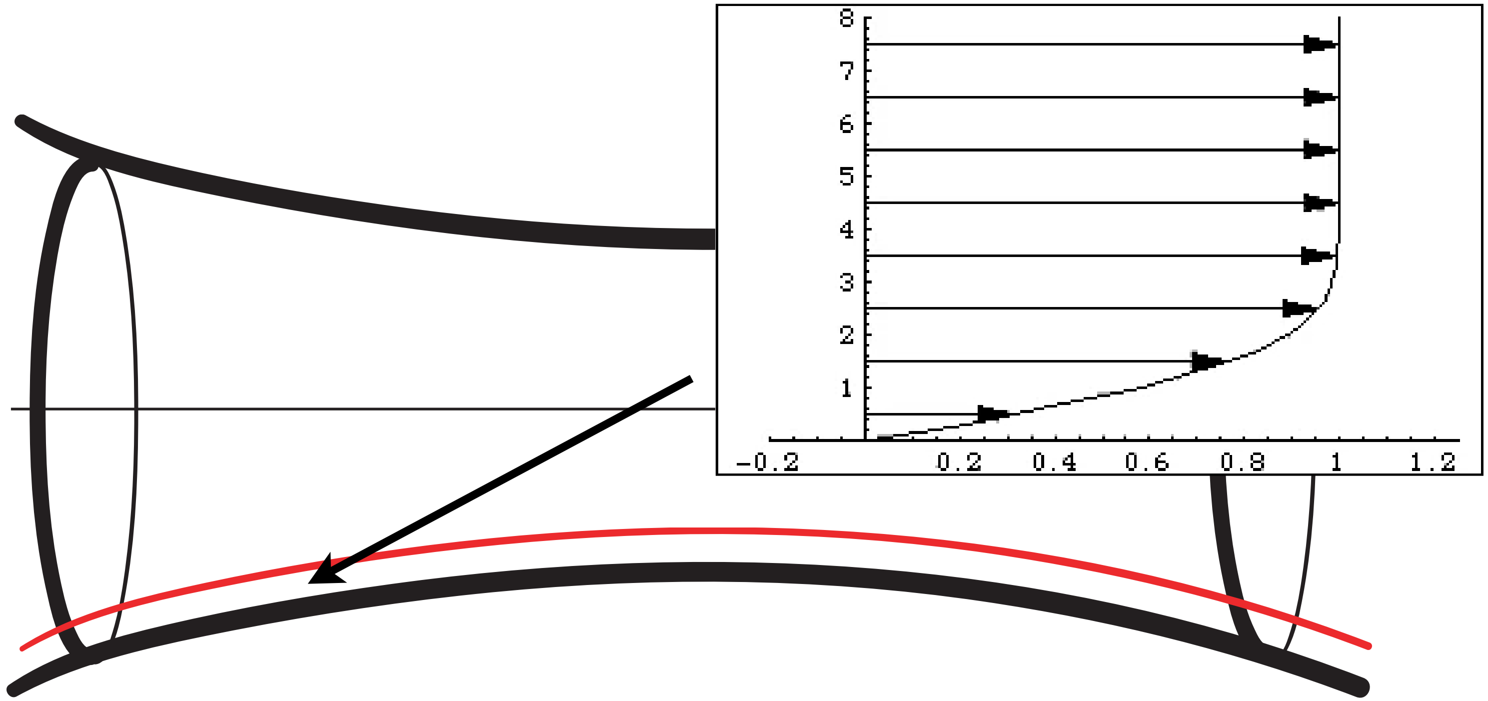
Choix d’une famille de profils simples

Dans un écoulement instationnaire, il est naturel de prendre les profils de Womersley



Choix d’une famille de profils simples

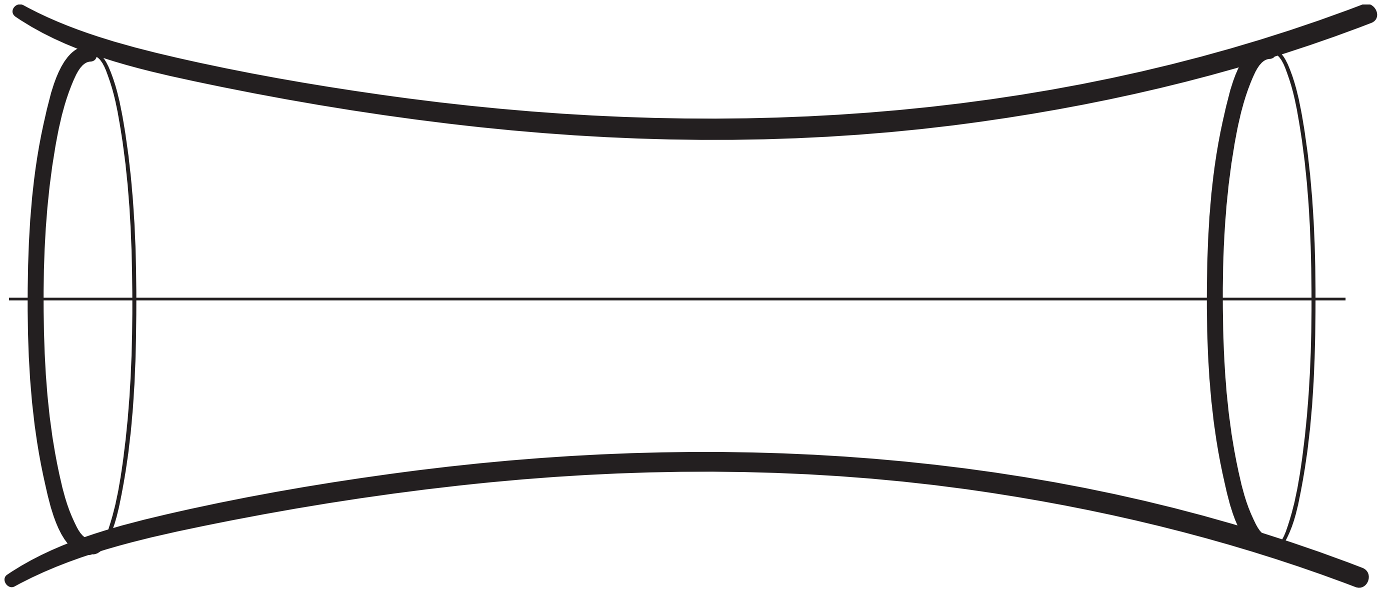
Dans un écoulement instationnaire, il est naturel de prendre les profils de Womersley



Choix d'une famille de profils simples

Dans un écoulement stationnaire, il est naturel de prendre les profils de Falkner Skan

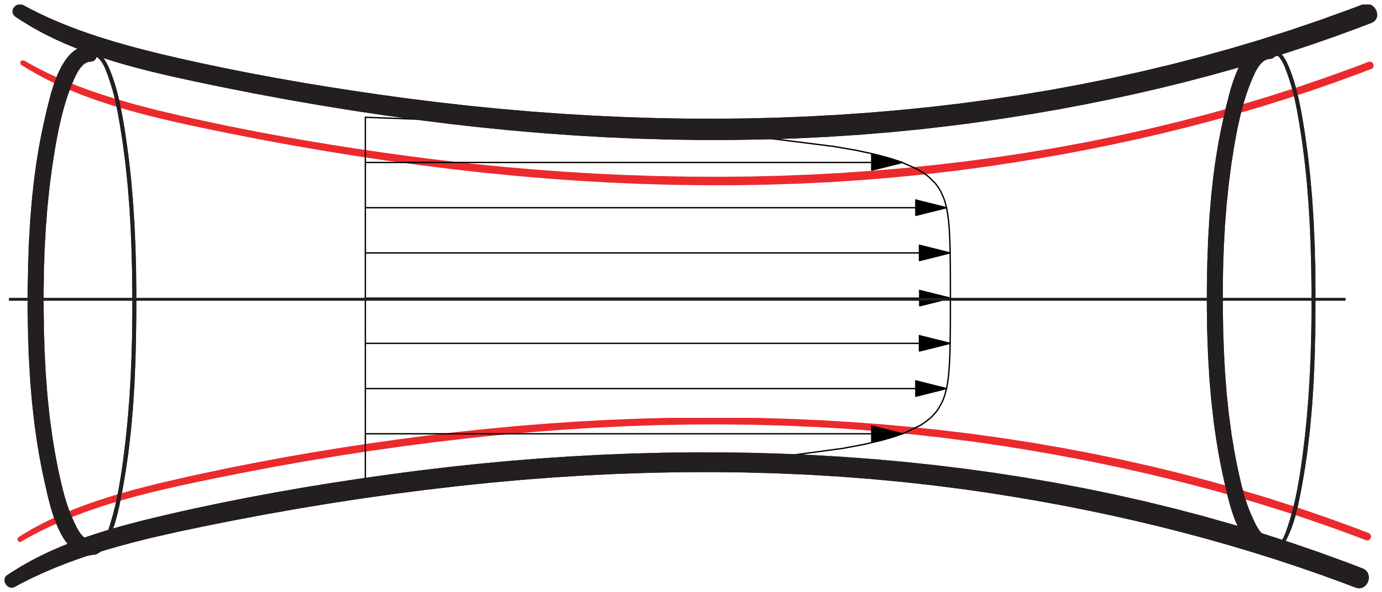
Résolution Intégrale



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

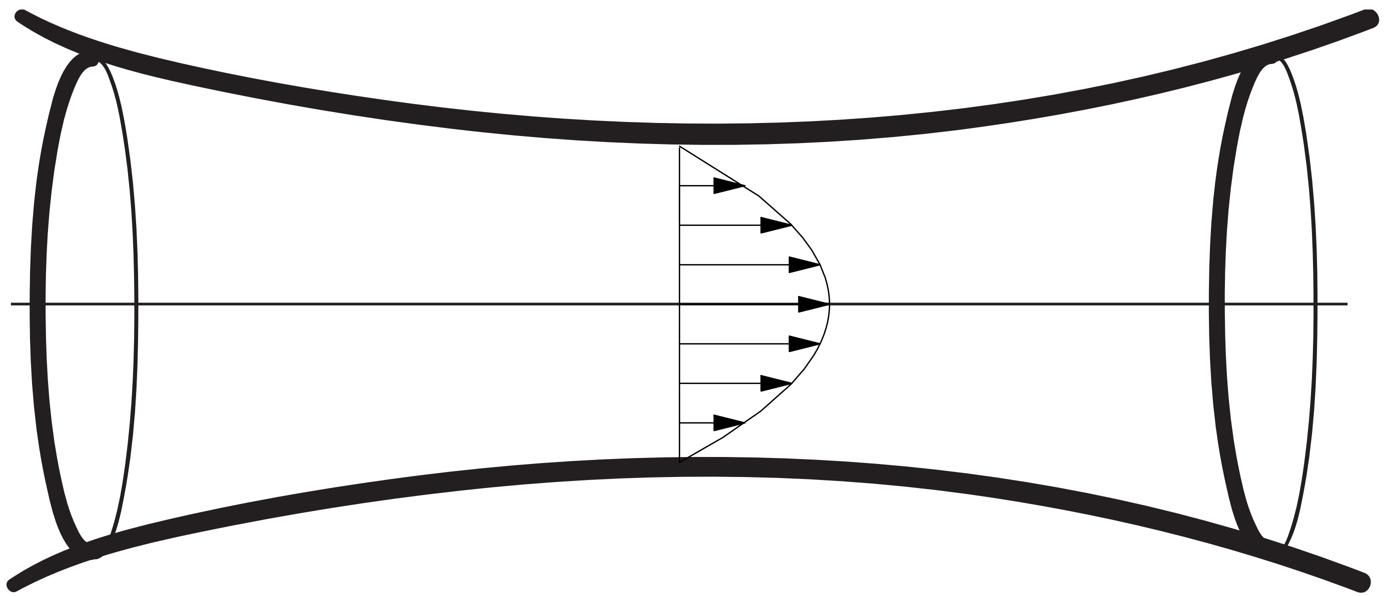
donne Q_2 fonction de Q et τ fonction de Q

Résolution Intégrale

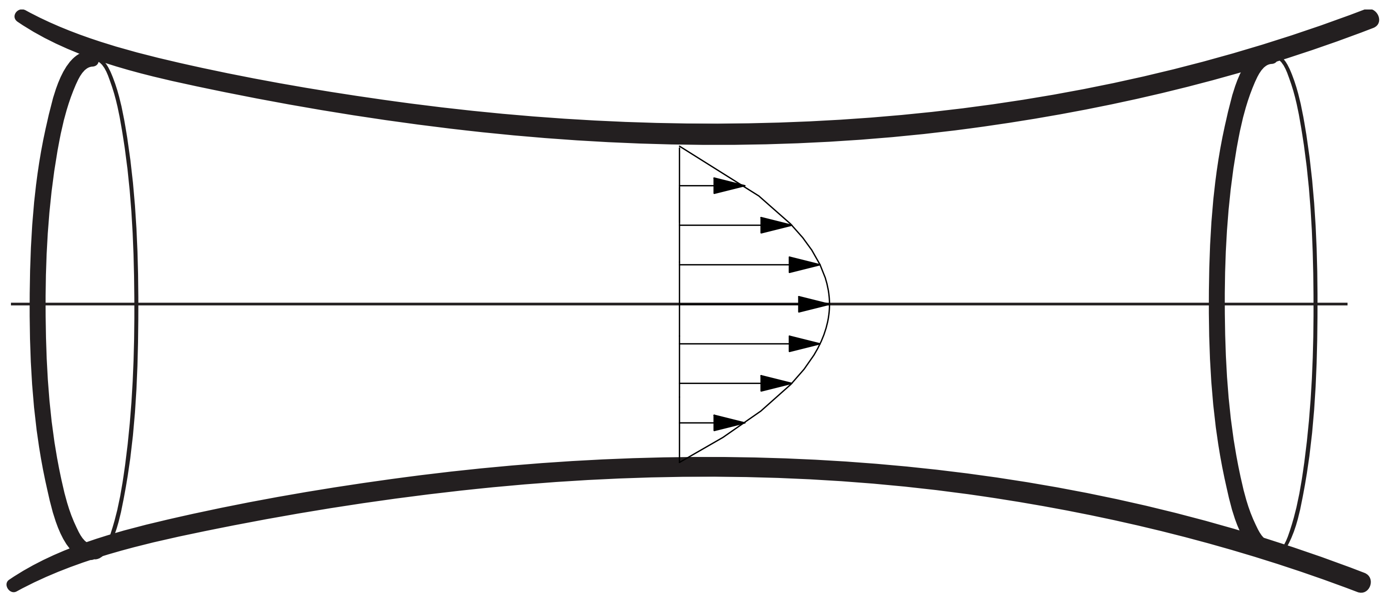


Résolution numérique:
différences finies

Interactive Boundary Layer/ Couche limite interactive

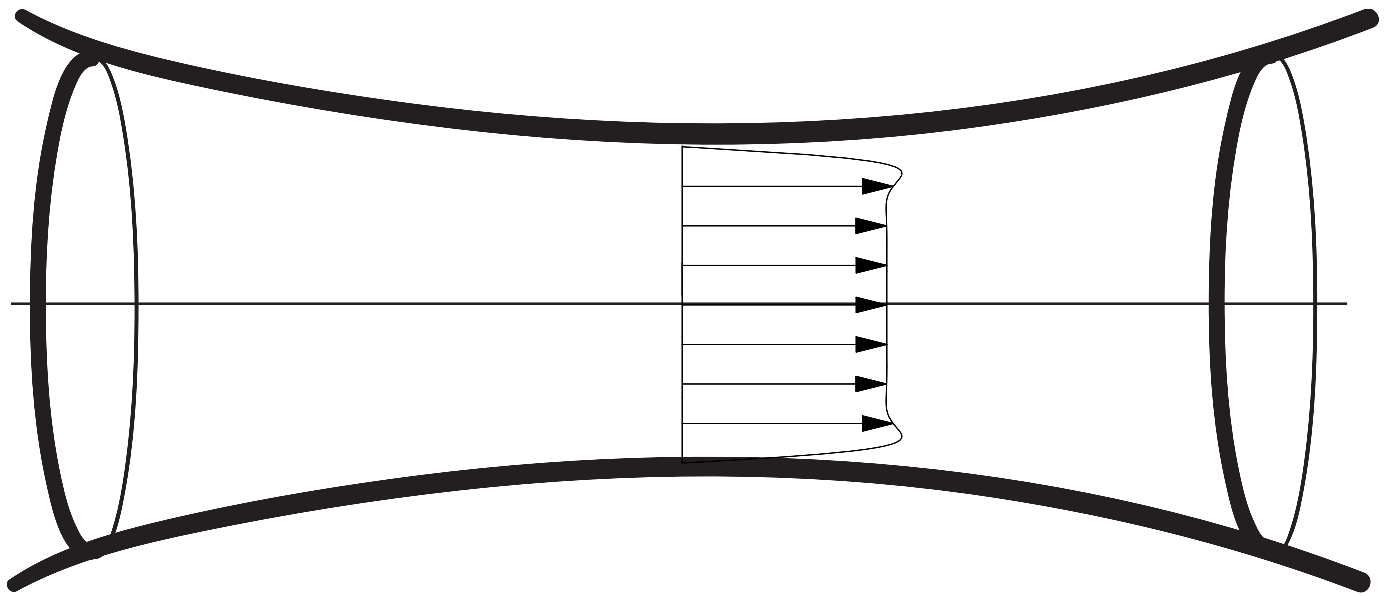


Interactive Boundary Layer/ Couche limite interactive

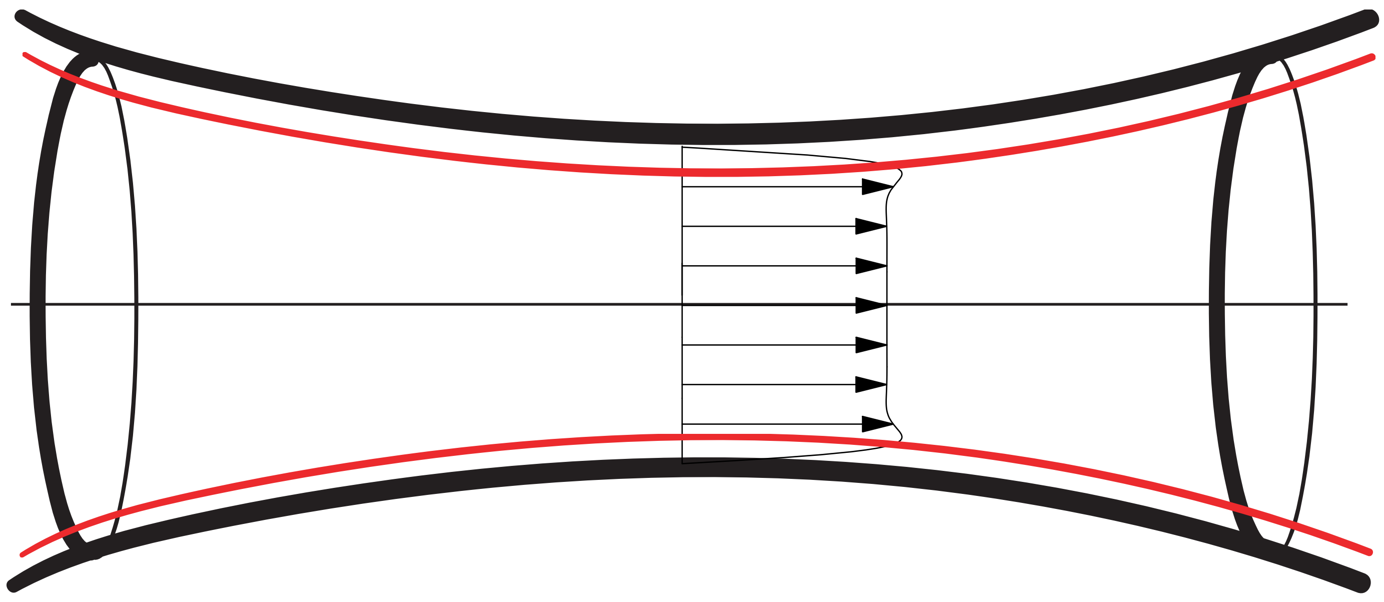


IBL est inclus dans RNSP

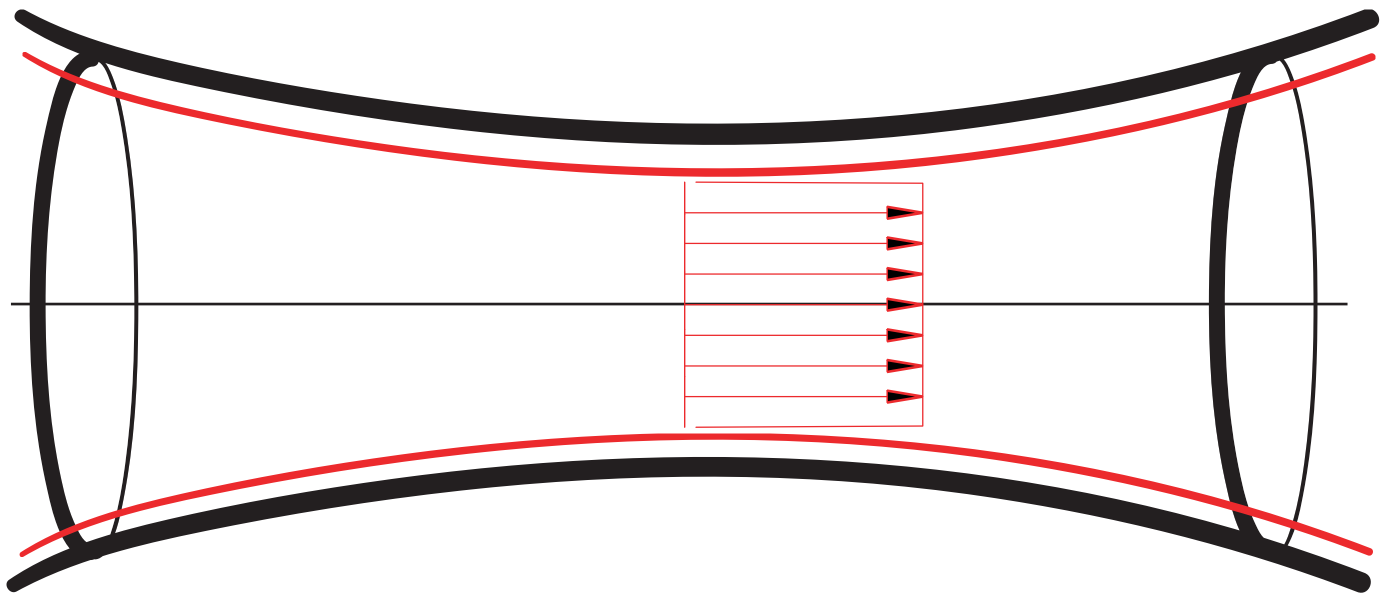
Interactive Boundary Layer/ Couche limite interactive



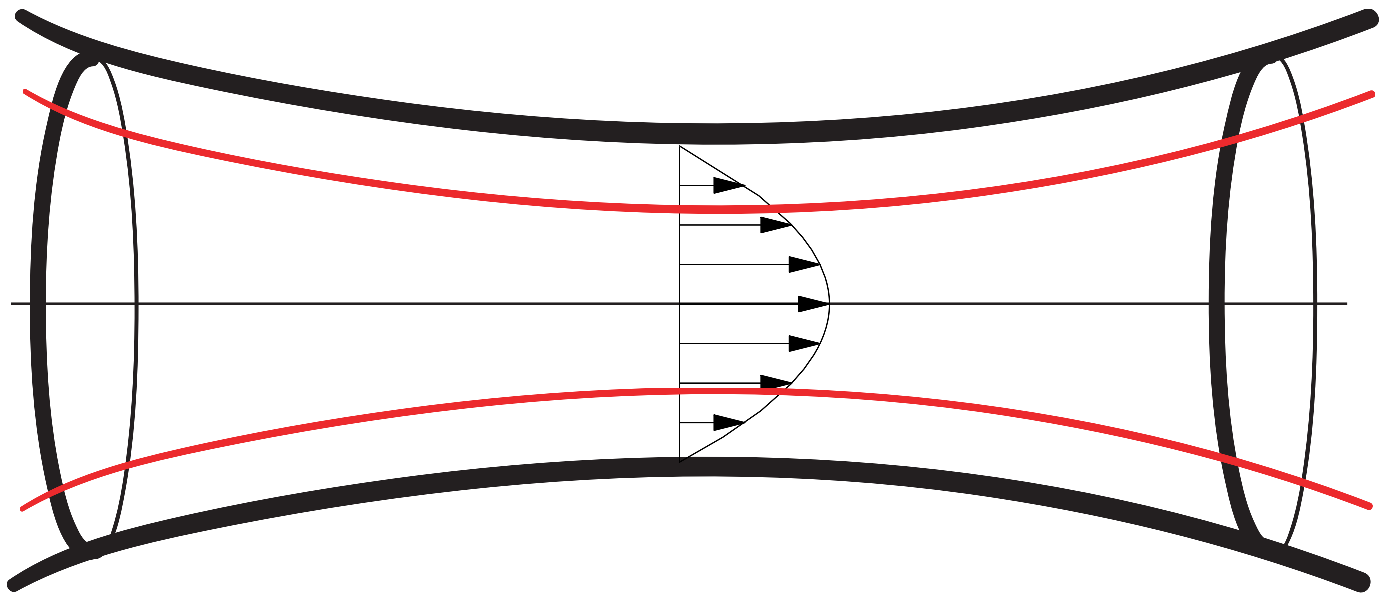
Interactive Boundary Layer/ Couche limite interactive



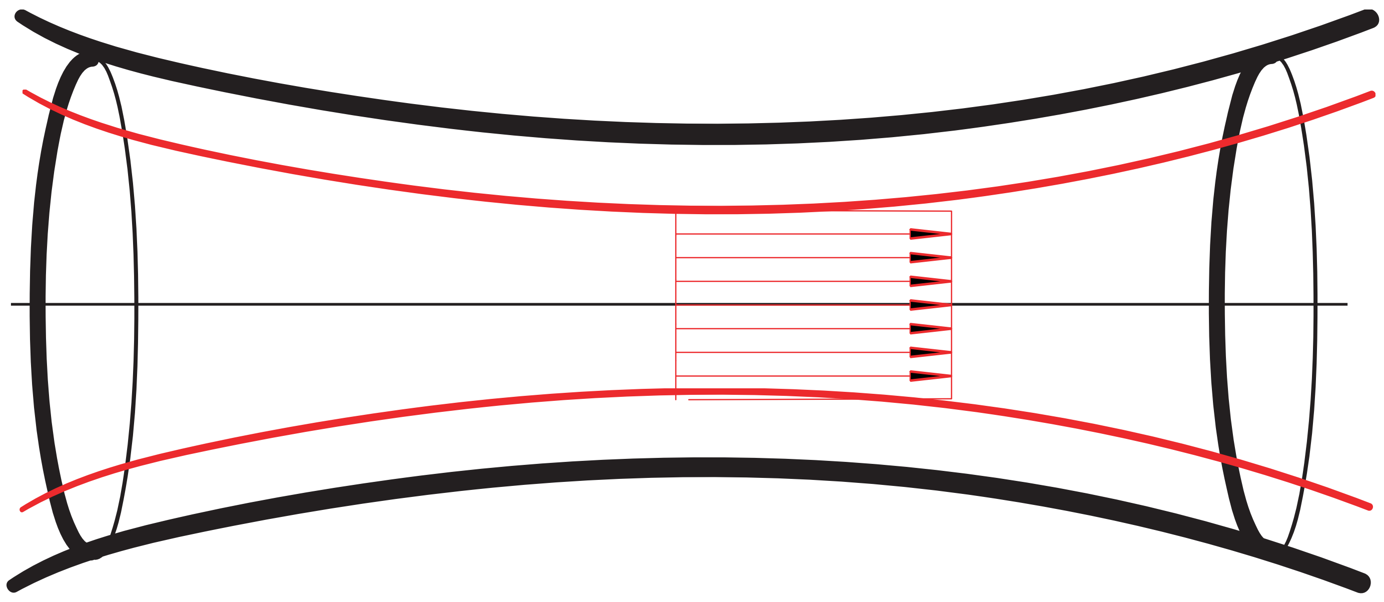
Interactive Boundary Layer/ Couche limite interactive



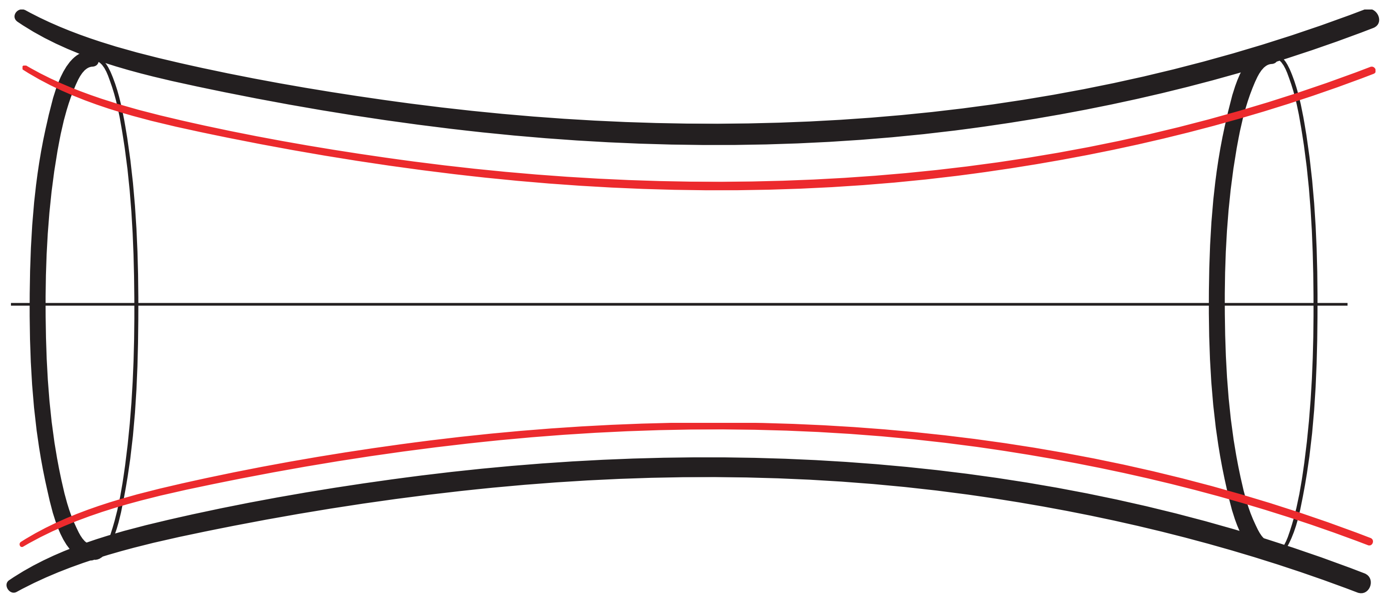
Interactive Boundary Layer/ Couche limite interactive



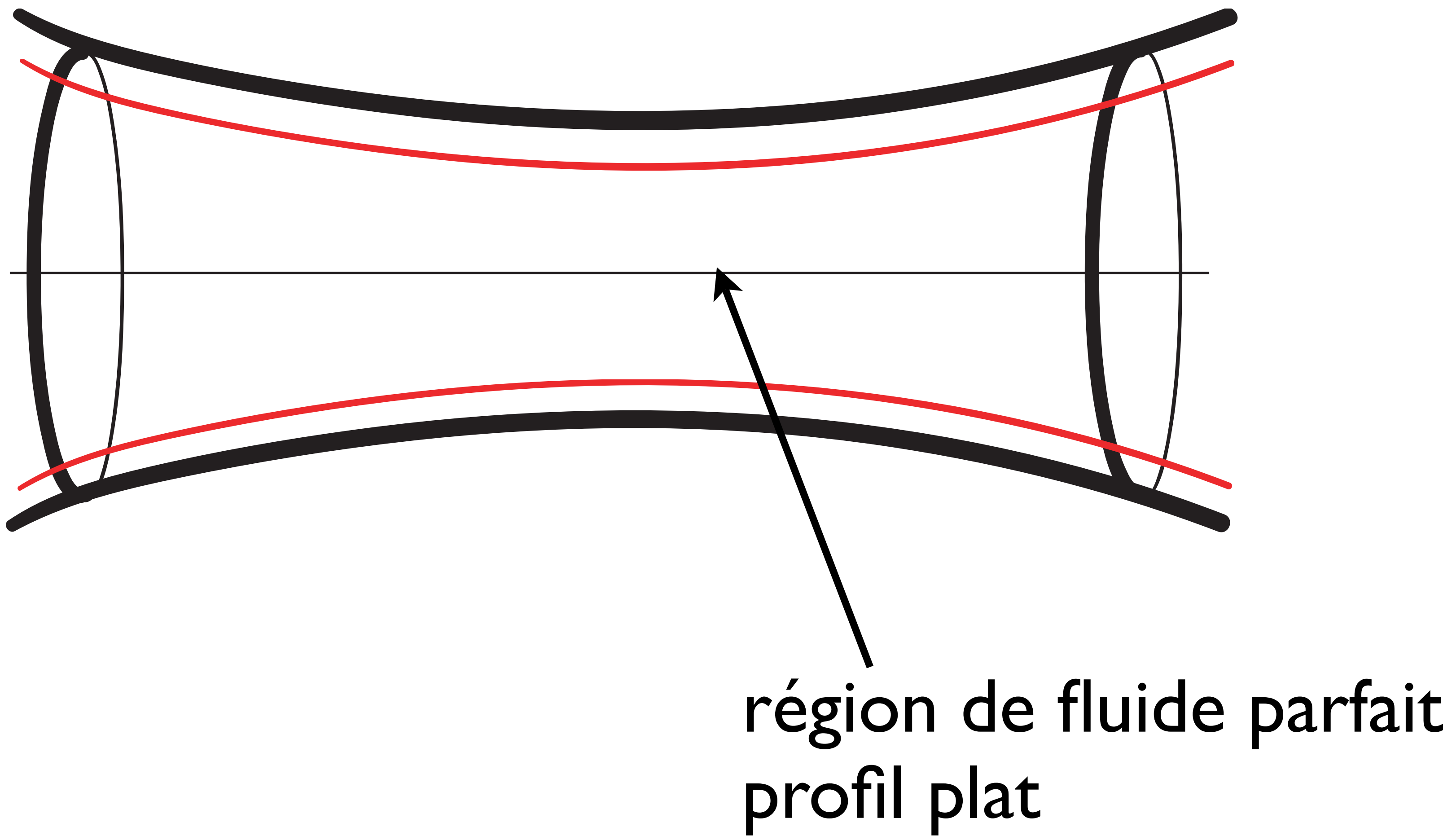
Interactive Boundary Layer/ Couche limite interactive



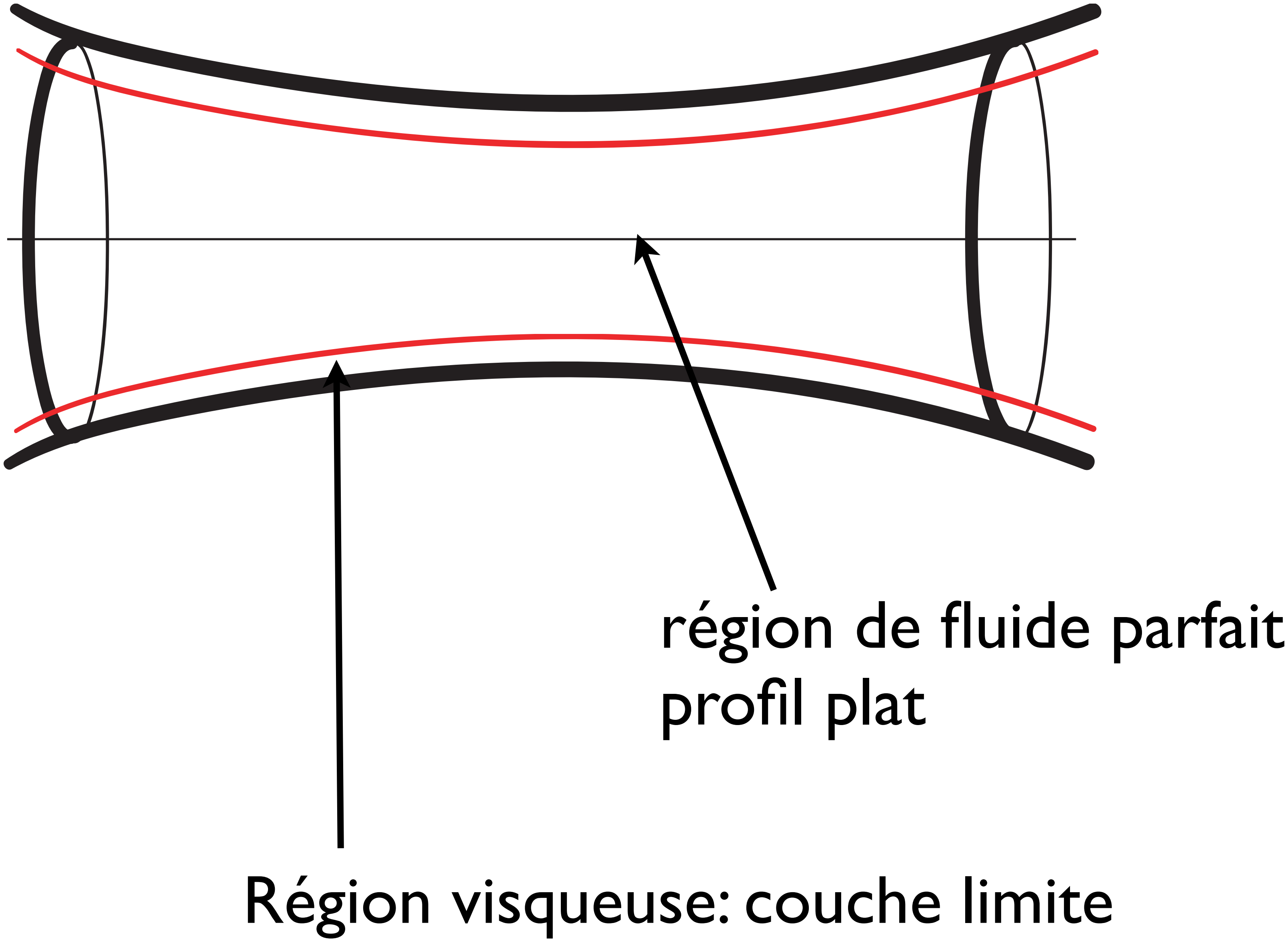
Interactive Boundary Layer/ Couche limite interactive



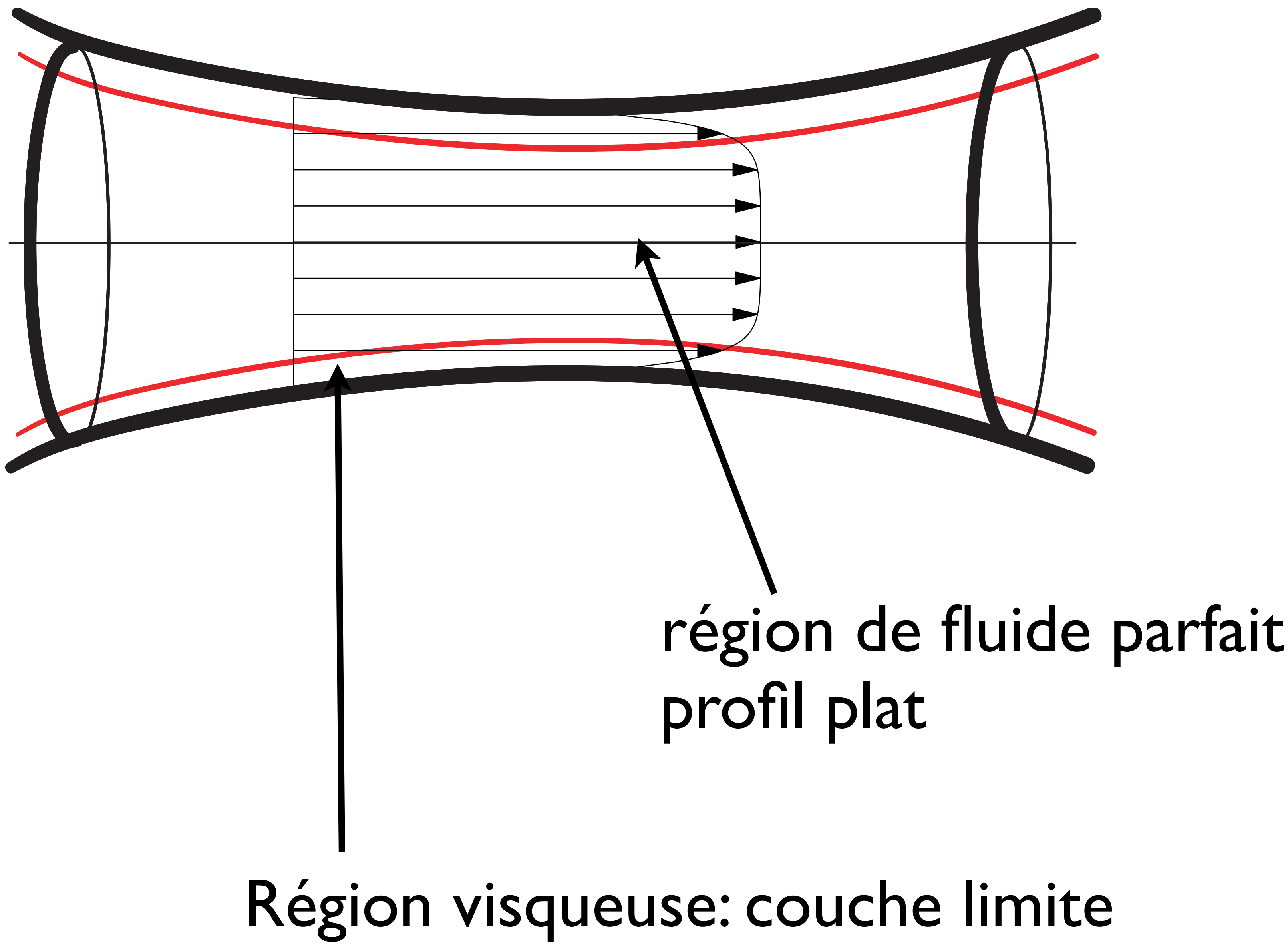
Interactive Boundary Layer/ Couche limite interactive



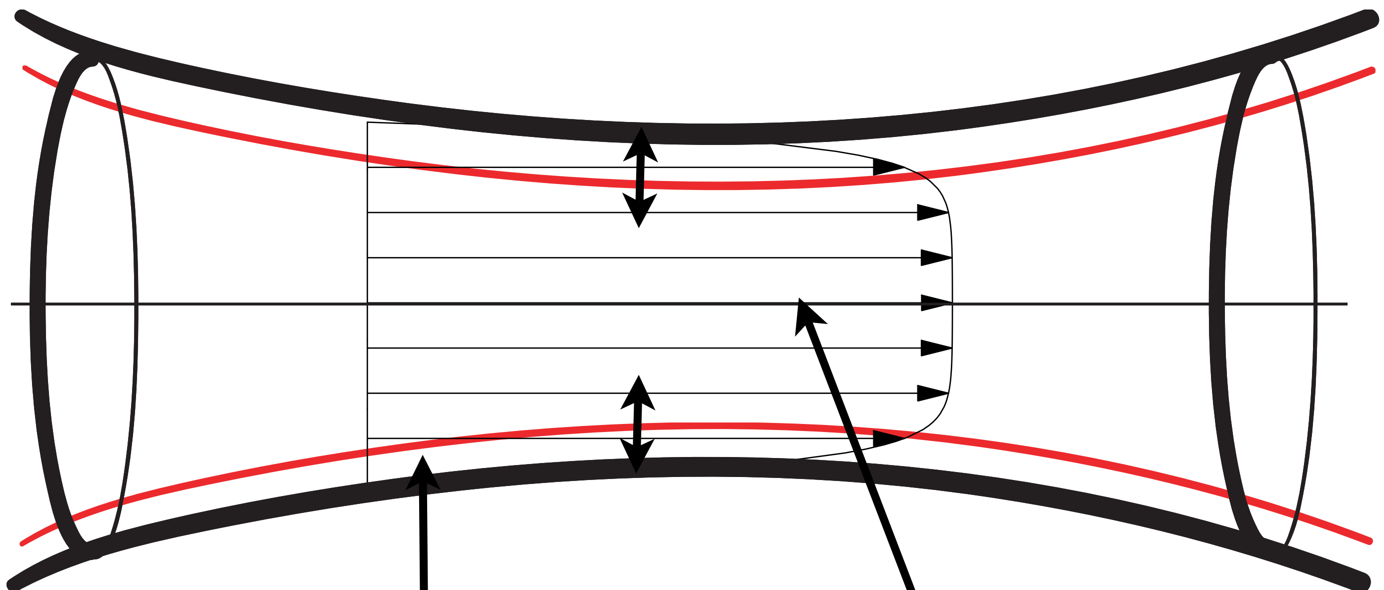
Interactive Boundary Layer/ Couche limite interactive



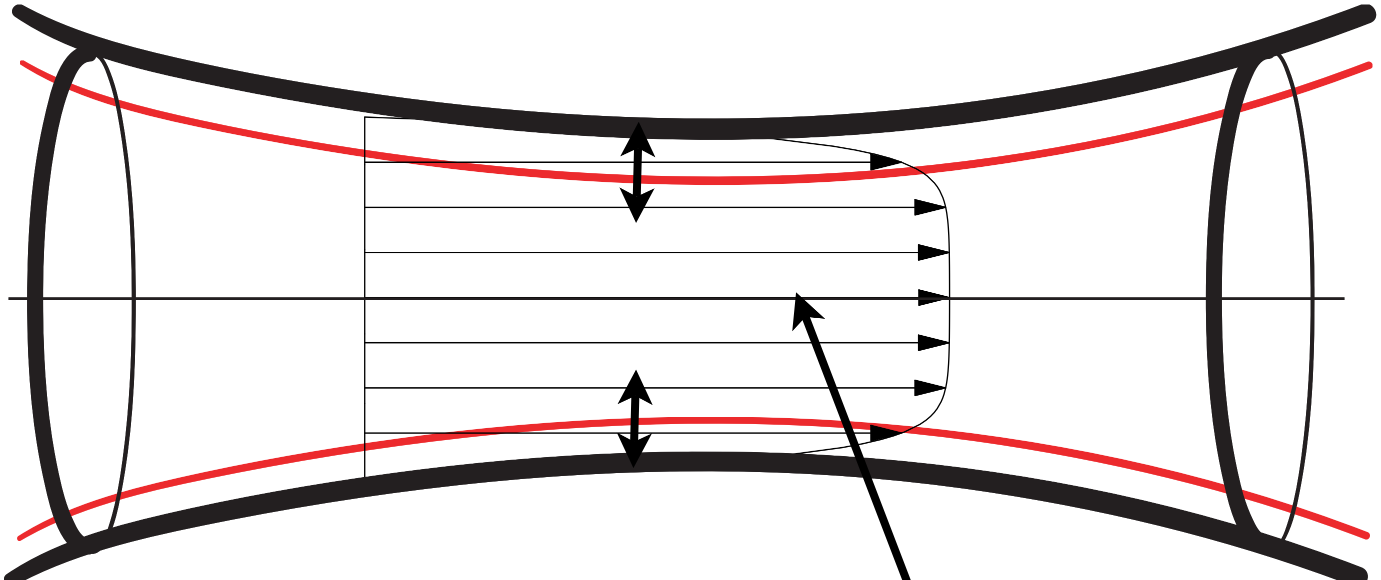
Interactive Boundary Layer/ Couche limite interactive



Interactive Boundary Layer/ Couche limite interactive



Interactive Boundary Layer/ Couche limite interactive

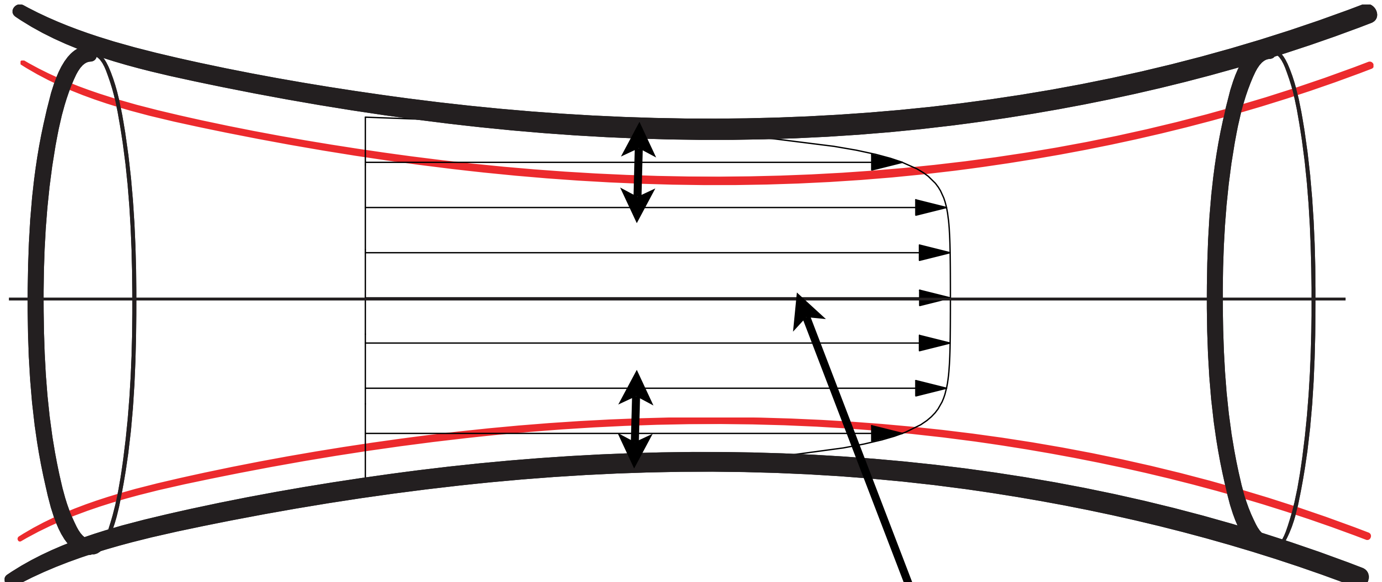


$$\frac{\partial u}{\partial x} + \frac{\partial rv}{r\partial r} = 0$$

région de fluide parfait
profil plat

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho\partial x} + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r} \qquad 0 = -\frac{\partial p}{\rho\partial r}$$

Interactive Boundary Layer/ Couche limite interactive

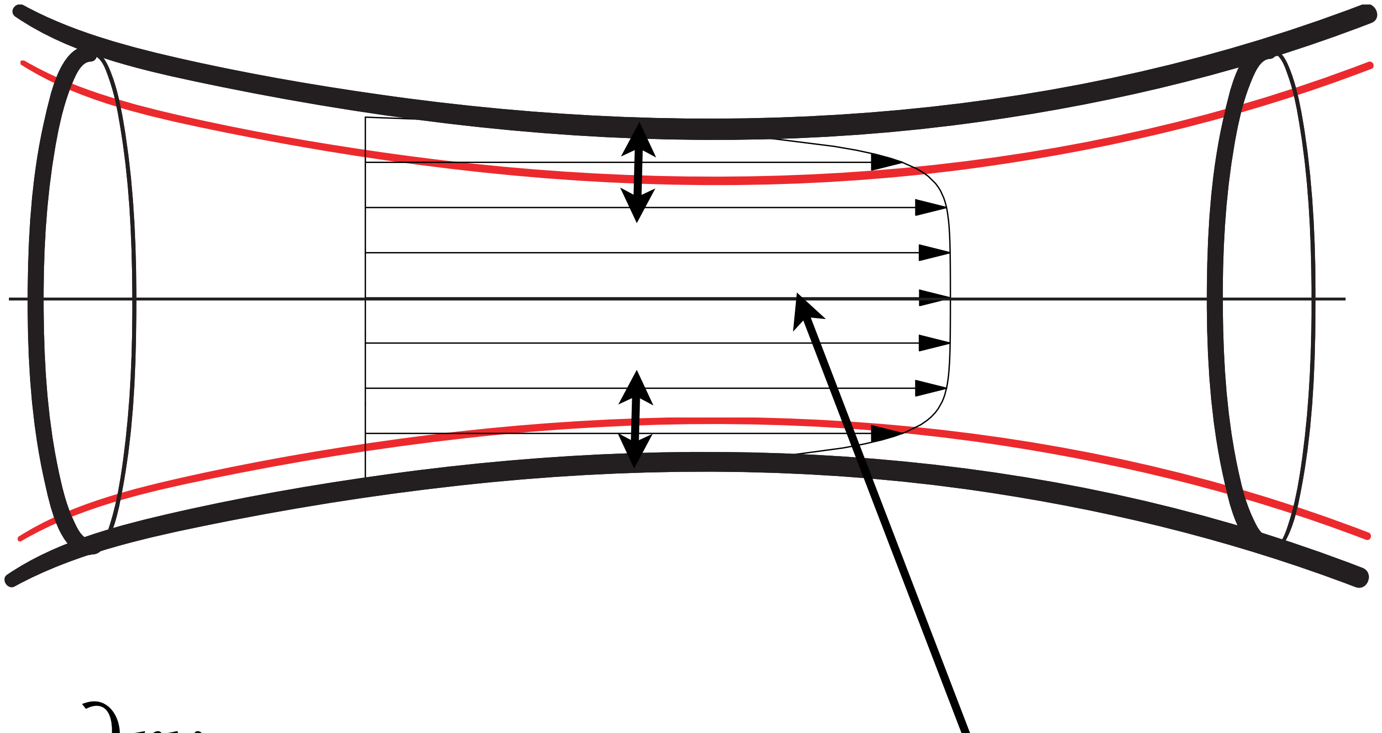


région de fluide parfait
profil plat

$$\frac{\partial u}{\partial x}+\frac{\partial rv}{r\partial r}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial r}u=-\frac{\partial p}{\rho\partial x}+\nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r} \qquad 0=-\frac{\partial p}{\rho\partial r}$$

Interactive Boundary Layer/ Couche limite interactive



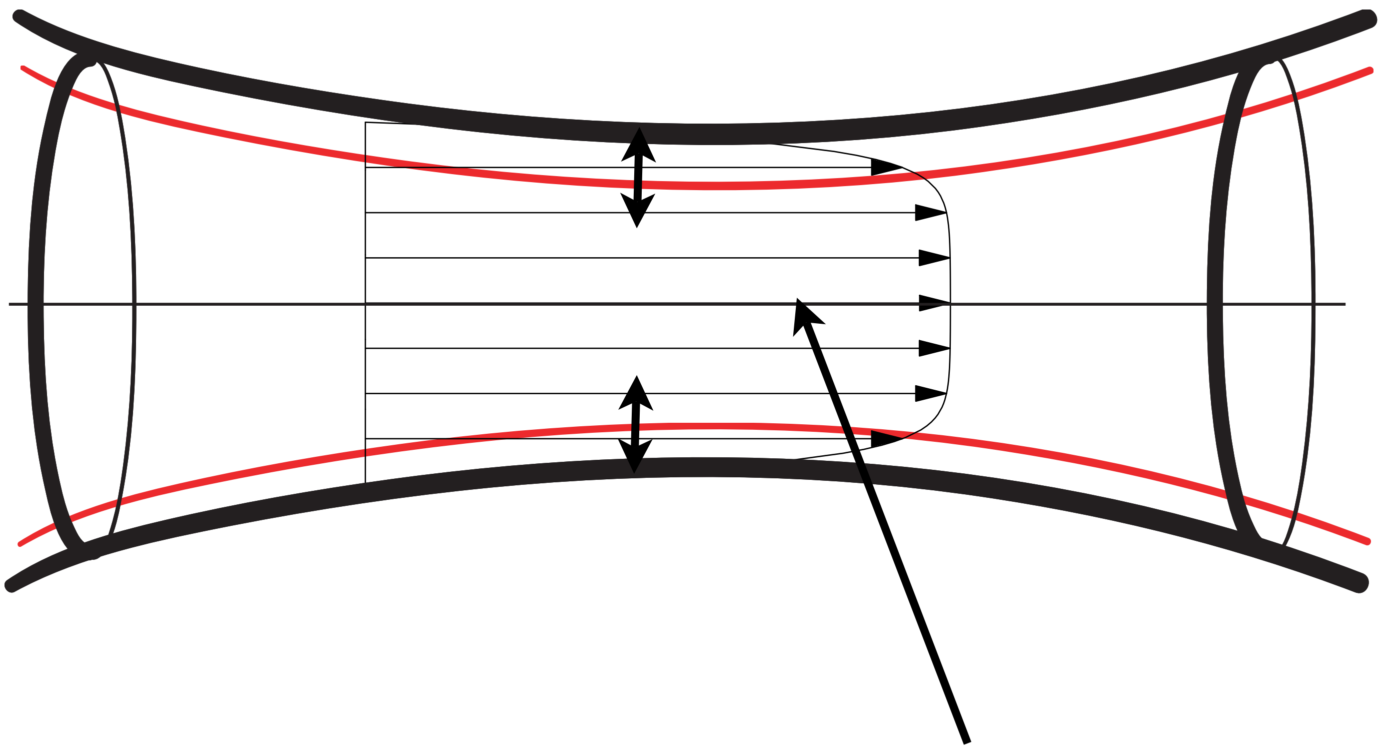
région de fluide parfait
profil plat

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \cancel{\frac{\partial}{r\partial r}} r\frac{\partial u}{\partial r} \qquad 0 = -\frac{\partial p}{\rho \partial r}$$

steady/ or large convective acceleration

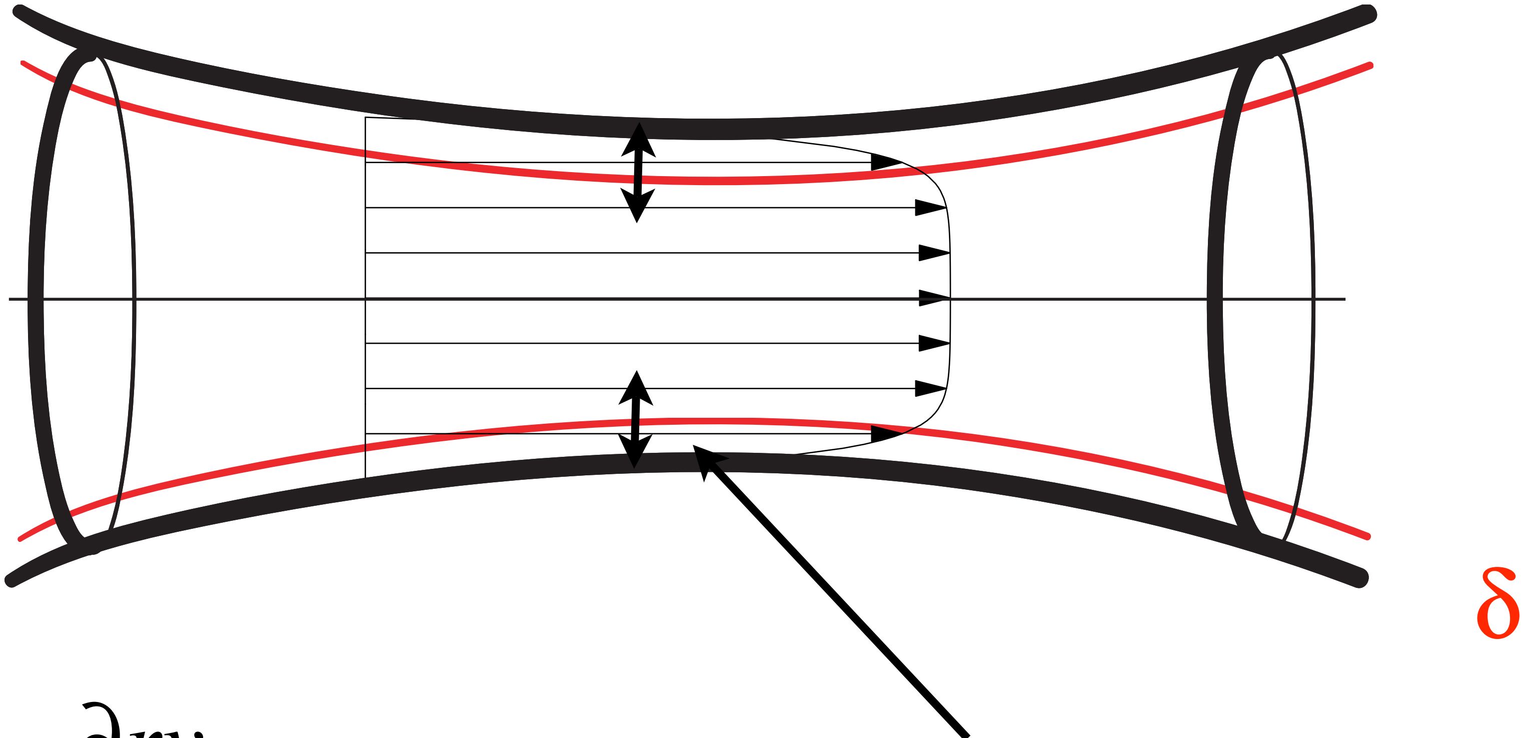
Interactive Boundary Layer/ Couche limite interactive



région de fluide parfait
profil plat

$$U_e S = cst$$

Interactive Boundary Layer/ Couche limite interactive



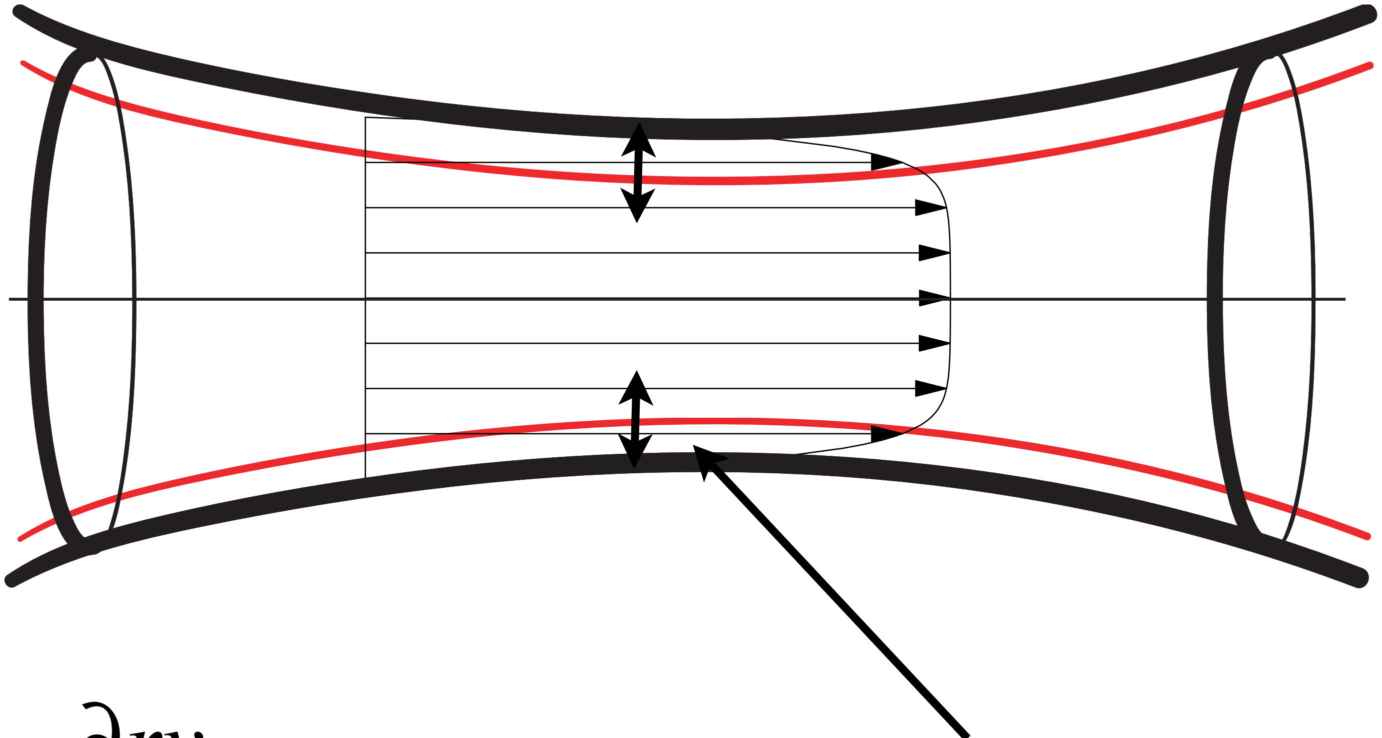
Région visqueuse: couche limite

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r} \qquad 0 = -\frac{\partial p}{\rho \partial r}$$

steady/ or large convective acceleration

Interactive Boundary Layer/ Couche limite interactive

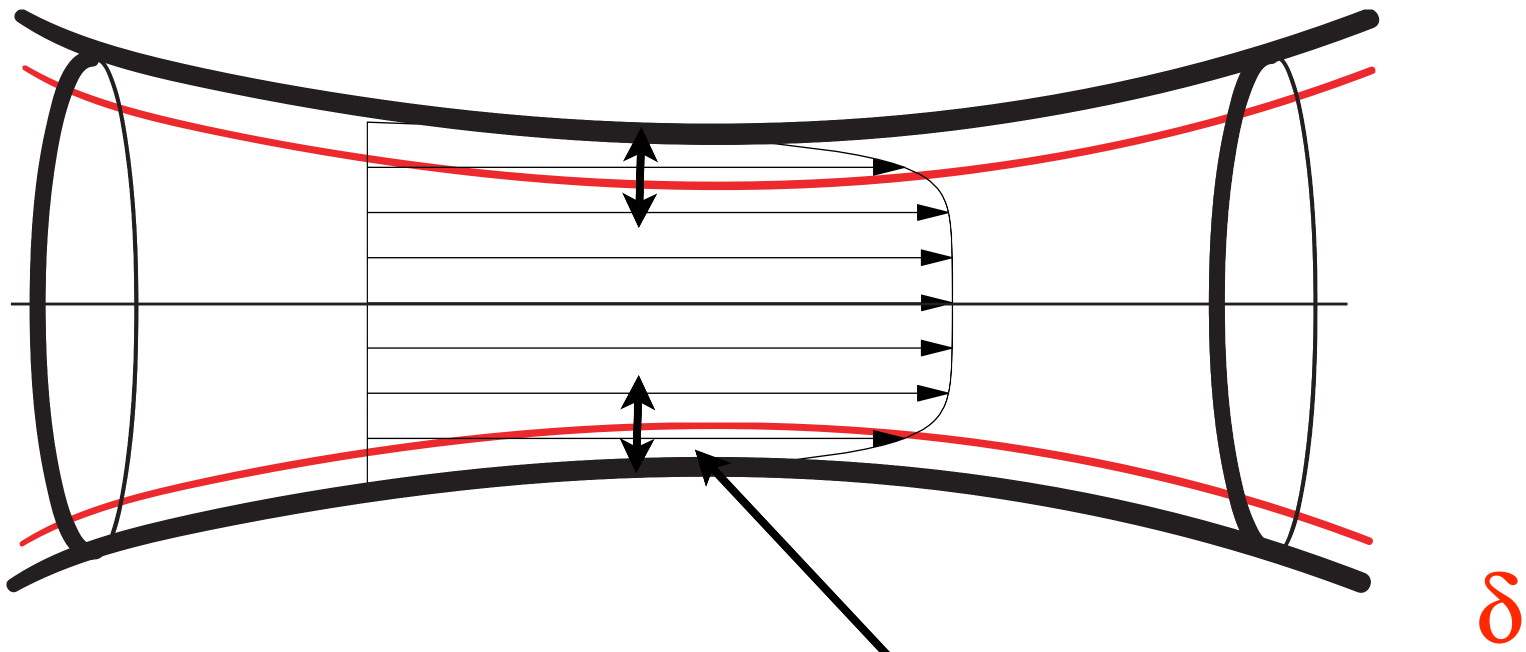


$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\boxed{\frac{U_0^2}{\lambda}} = -\frac{\partial p}{\rho \partial x} + \boxed{\frac{\nu}{U_0 \lambda} \frac{\lambda^2}{\delta^2} \frac{U_0^2}{\lambda}} \quad 0 = -\frac{\partial p}{\rho \partial r}$$

steady/ or large convective acceleration

Interactive Boundary Layer/ Couche limite interactive



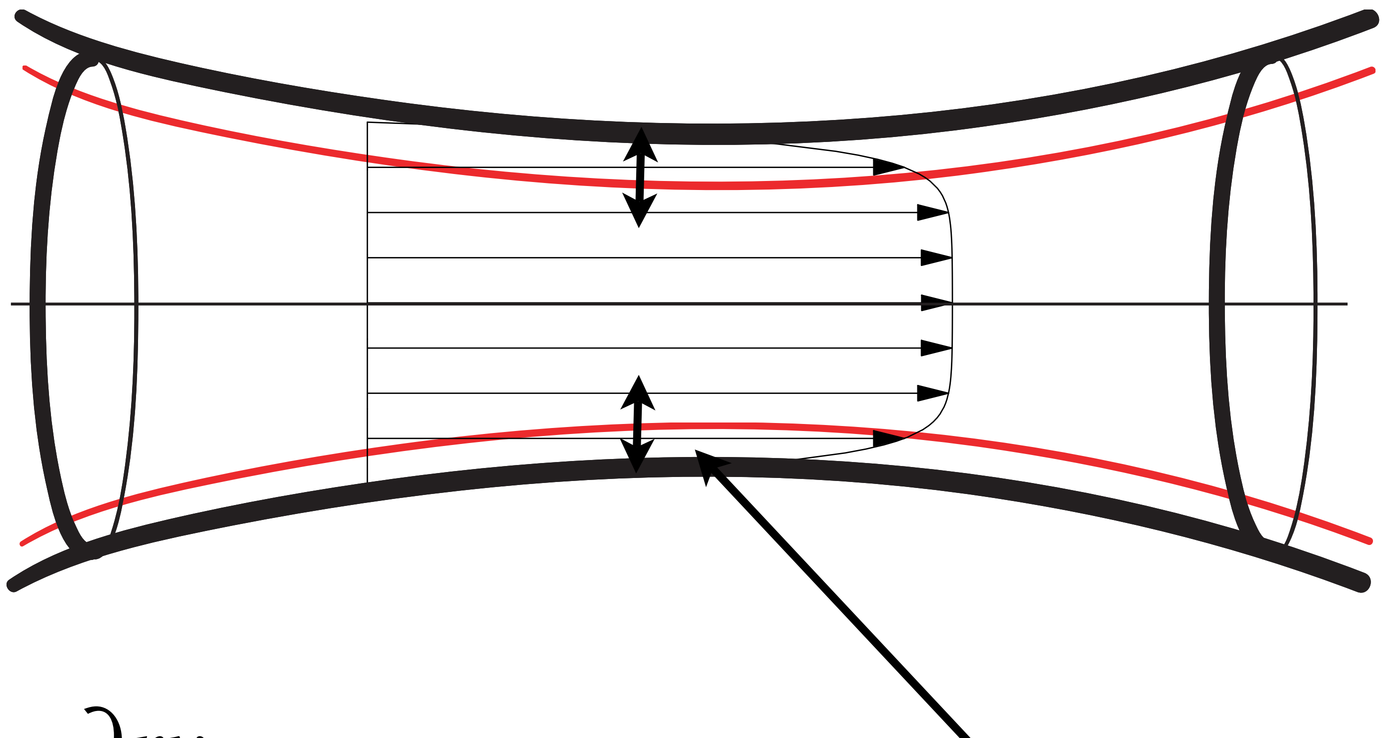
$$\frac{\partial u}{\partial x} + \frac{\partial rv}{r\partial r} = 0$$

Région visqueuse: couche limite

$$\boxed{\frac{U_0^2}{\lambda}} = -\frac{\partial p}{\rho \partial x} + \boxed{\frac{1}{Re}\frac{\lambda^2}{\delta^2}\frac{U_0^2}{\lambda}} \quad 0 = -\frac{\partial p}{\rho \partial r}$$

steady/ or large convective acceleration

Interactive Boundary Layer/ Couche limite interactive

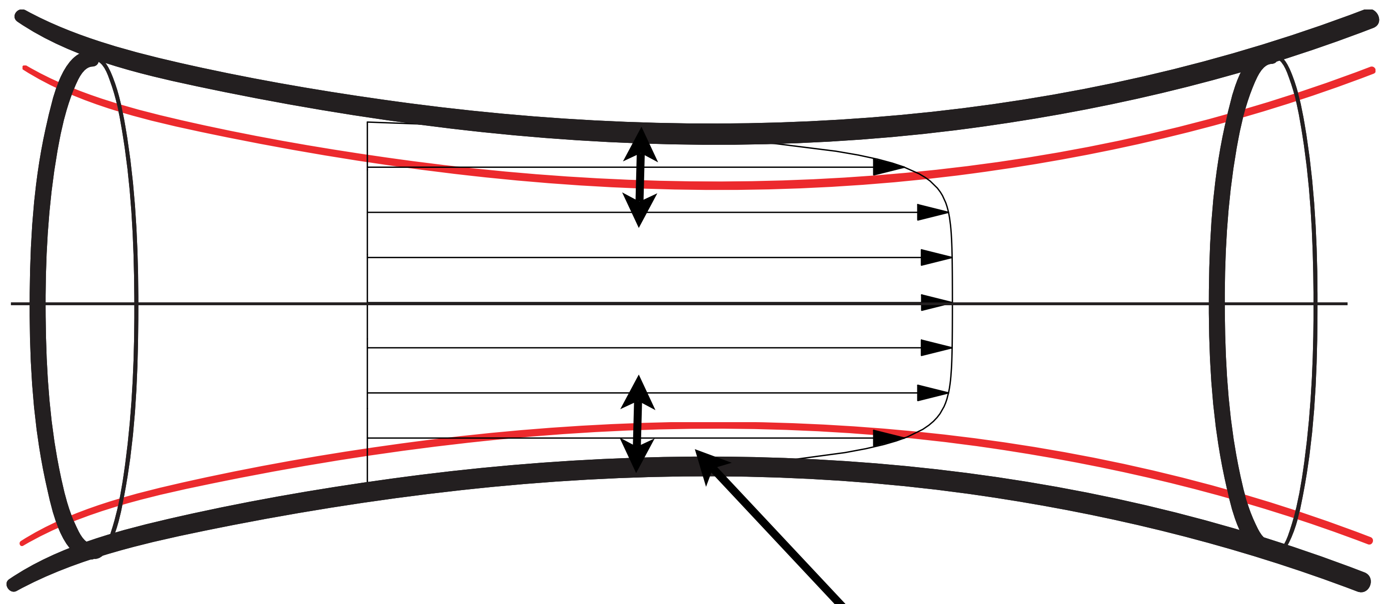


$$\delta \sim \frac{\lambda}{Re^{1/2}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\boxed{\frac{U_0^2}{\lambda}} = -\frac{\partial p}{\rho \partial x} + \boxed{\frac{1}{Re}\frac{\lambda^2}{\delta^2}\frac{U_0^2}{\lambda}} \quad 0 = -\frac{\partial p}{\rho \partial r}$$

Interactive Boundary Layer/ Couche limite interactive



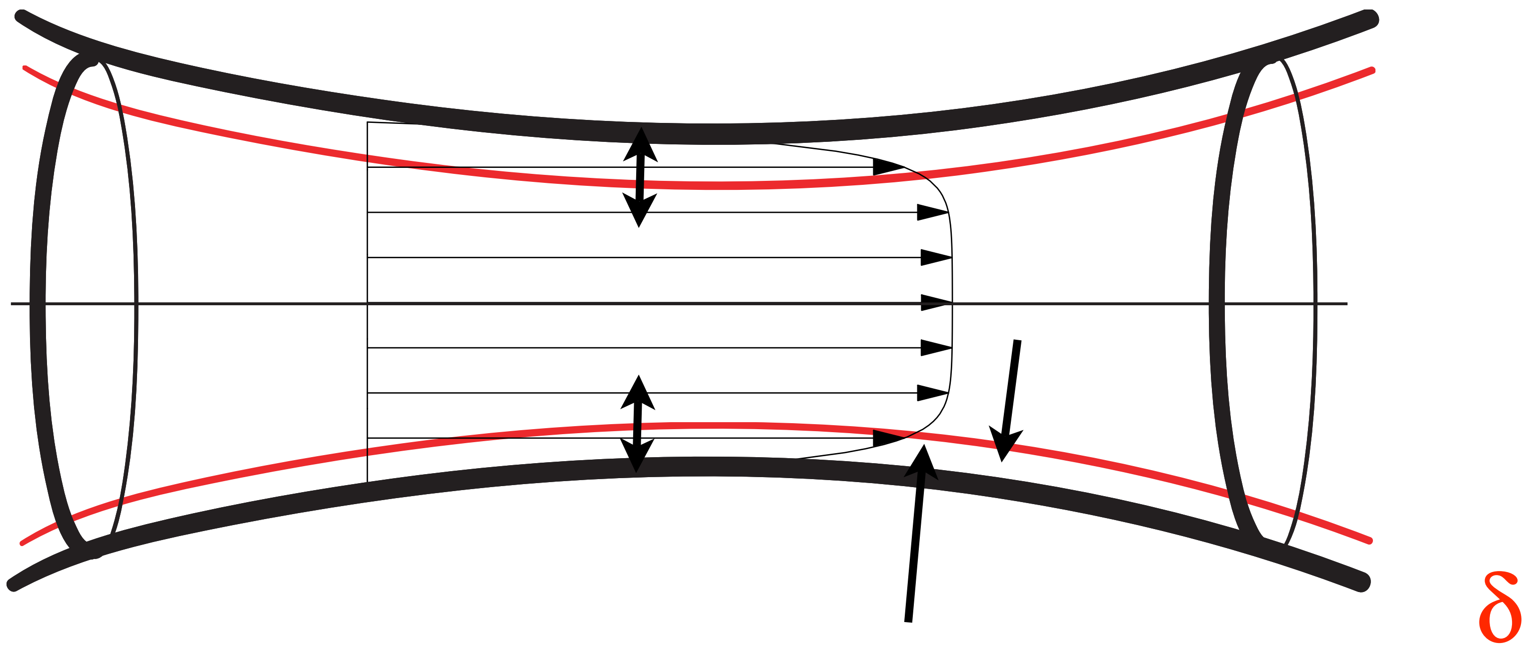
$$\delta \sim \frac{\lambda}{Re^{1/2}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0$$

Région visqueuse: couche limite

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = -\frac{\partial p}{\partial x} + \frac{\partial^2}{\partial n^2}u \qquad 0 = -\frac{\partial p}{\partial n}$$

Interactive Boundary Layer/ Couche limite interactive

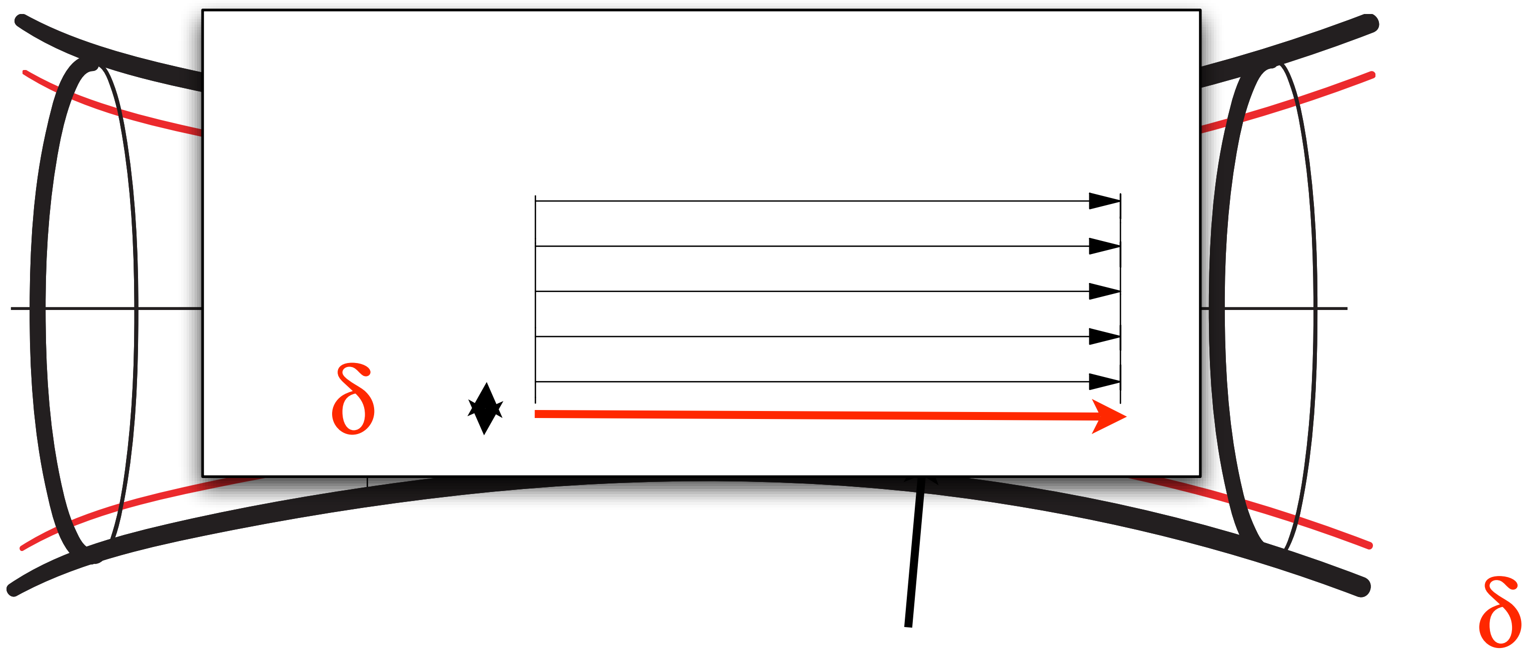


Raccord des vitesses de la couche limite au fluide parfait

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = -\frac{\partial p}{\partial x} + \frac{\partial^2}{\partial n^2}u \qquad 0 = -\frac{\partial p}{\partial n}$$

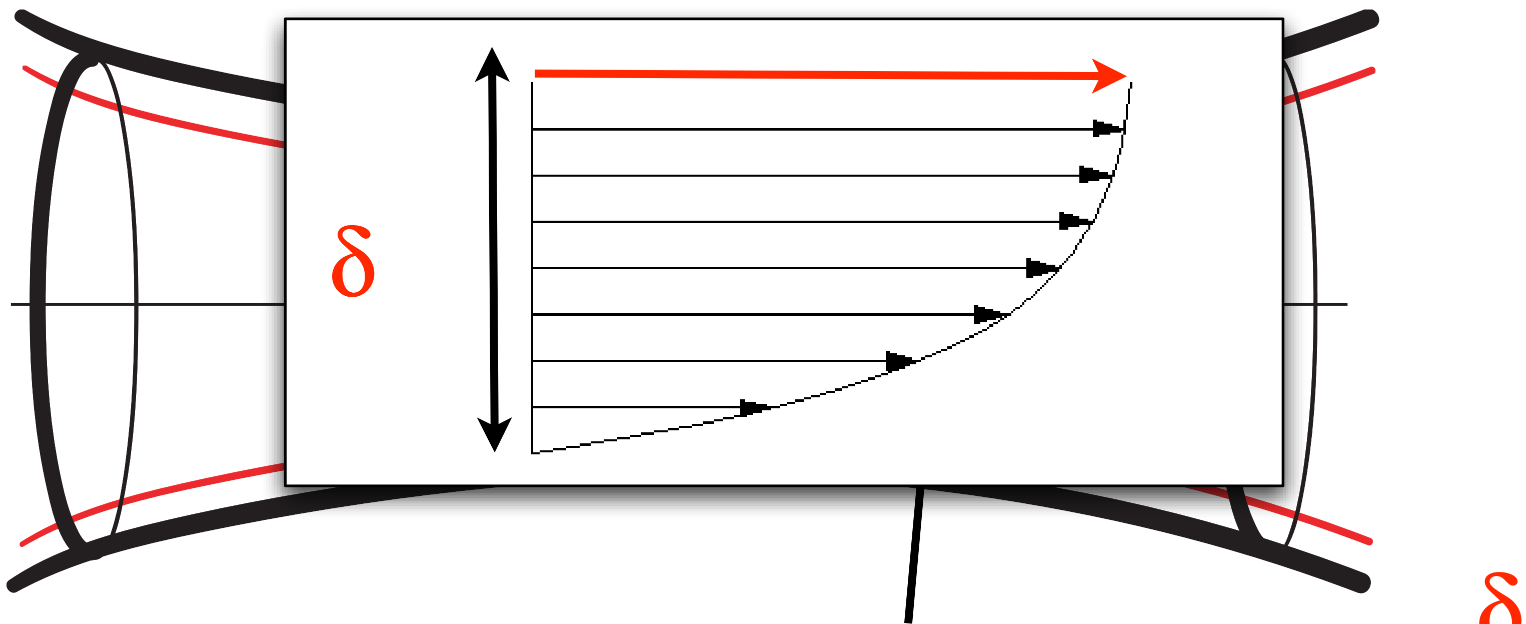
Interactive Boundary Layer/ Couche limite interactive



Raccord des vitesses de la couche limite au fluide parfait

U_e à la paroi

Interactive Boundary Layer/ Couche limite interactive



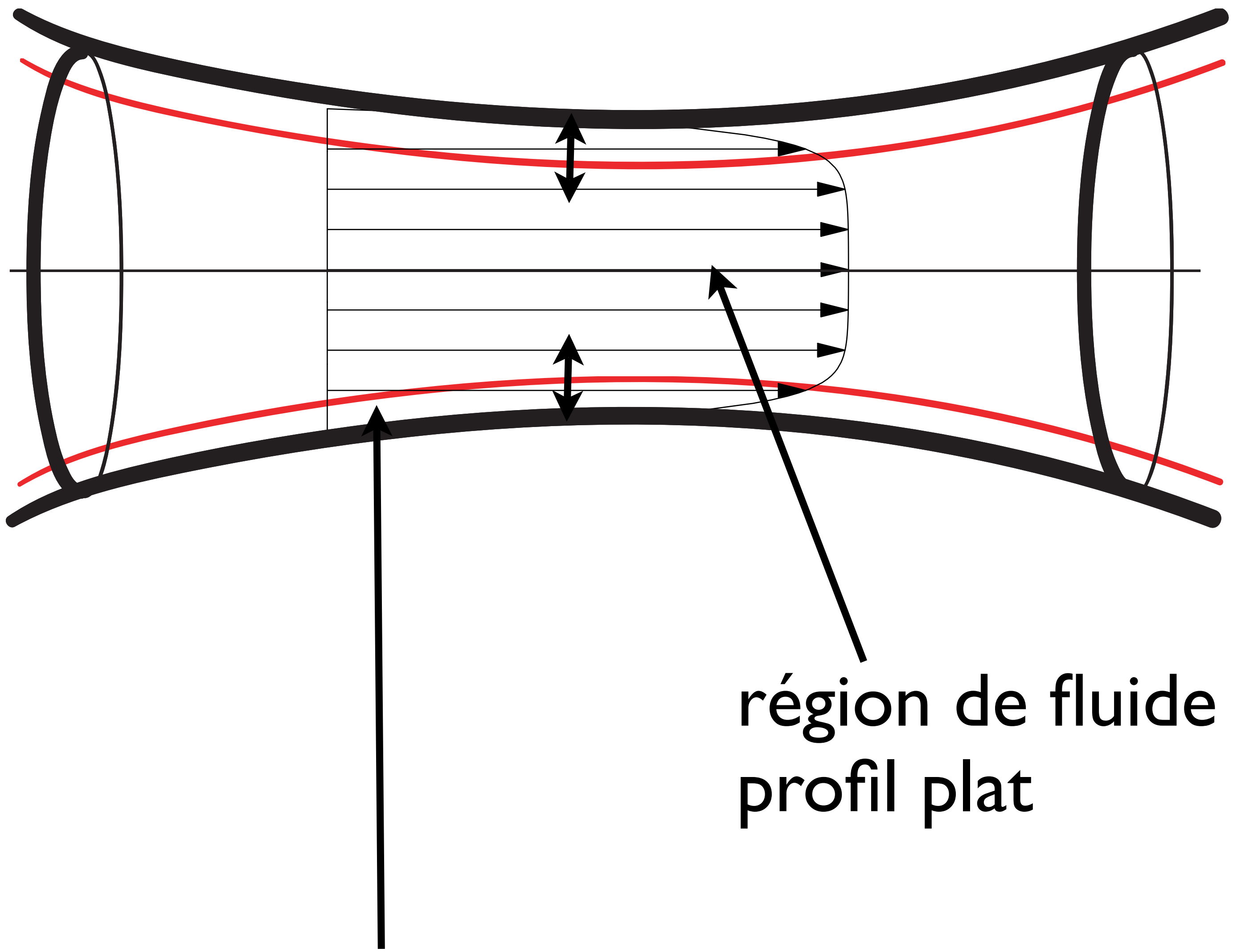
Raccord des vitesses de la couche limite au fluide parfait

U_e à la paroi

est la vitesse à la lisière de la couche limite $u(x,\infty)$

à l'"infini"

Interactive Boundary Layer/ Couche limite interactive

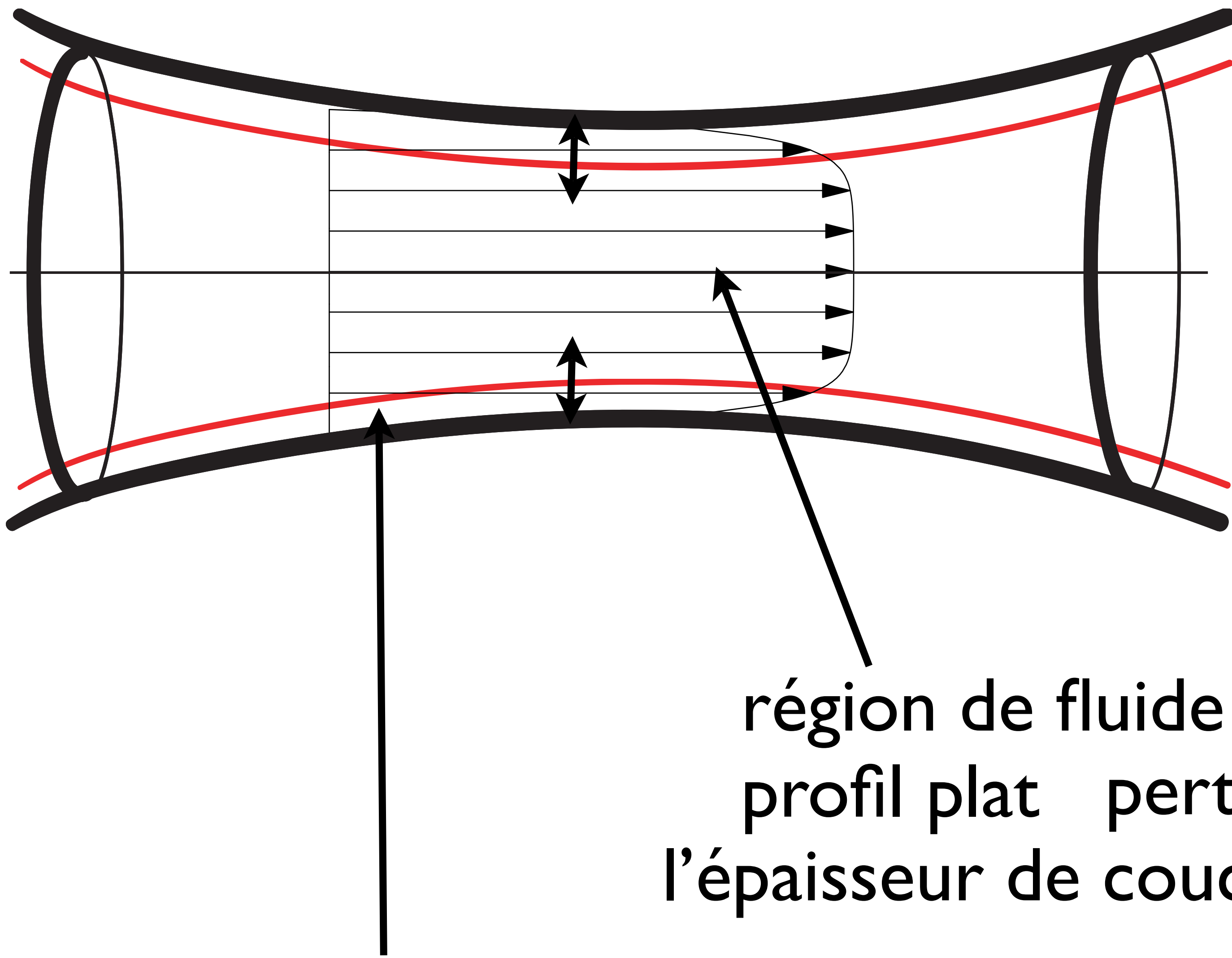


région de fluide parfait
profil plat

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn$$

déplacement des lignes de courant

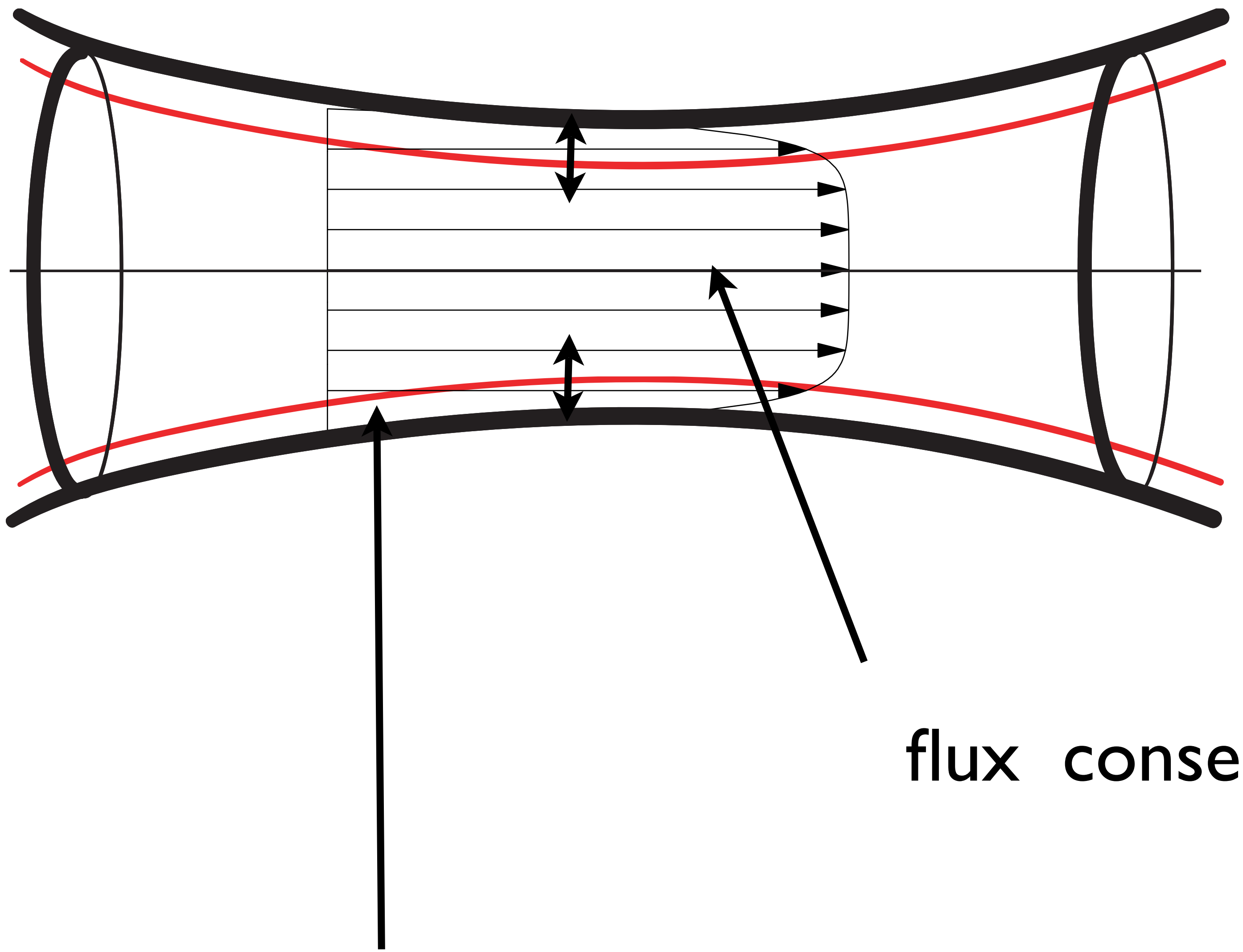
Interactive Boundary Layer/ Couche limite interactive



région de fluide parfait
profil plat perturbé par
l'épaisseur de couche limite

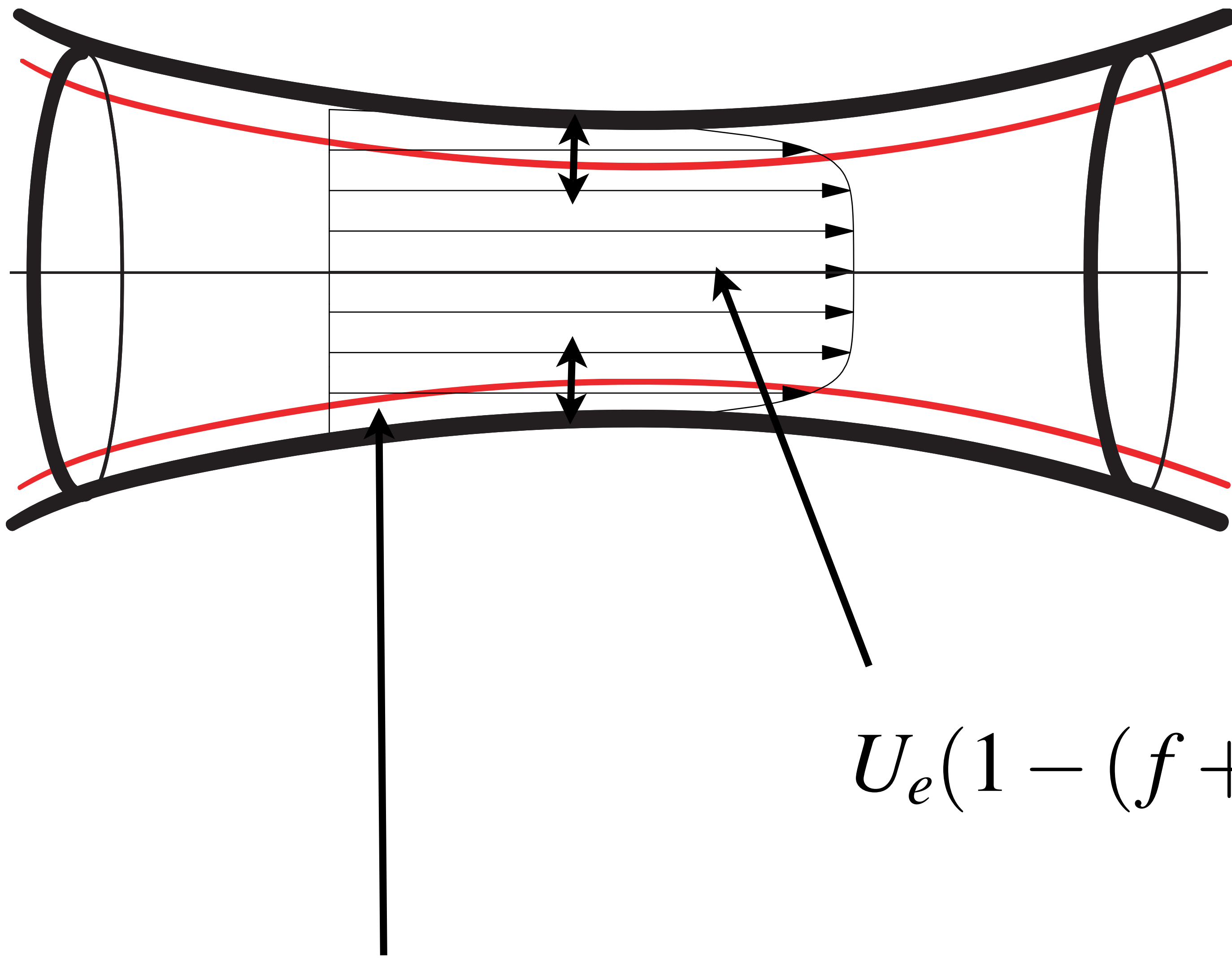
$$\delta_1 = \int_0^{\infty} (1 - \frac{u}{U_e}) dn$$

Interactive Boundary Layer/ Couche limite interactive



$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn$$

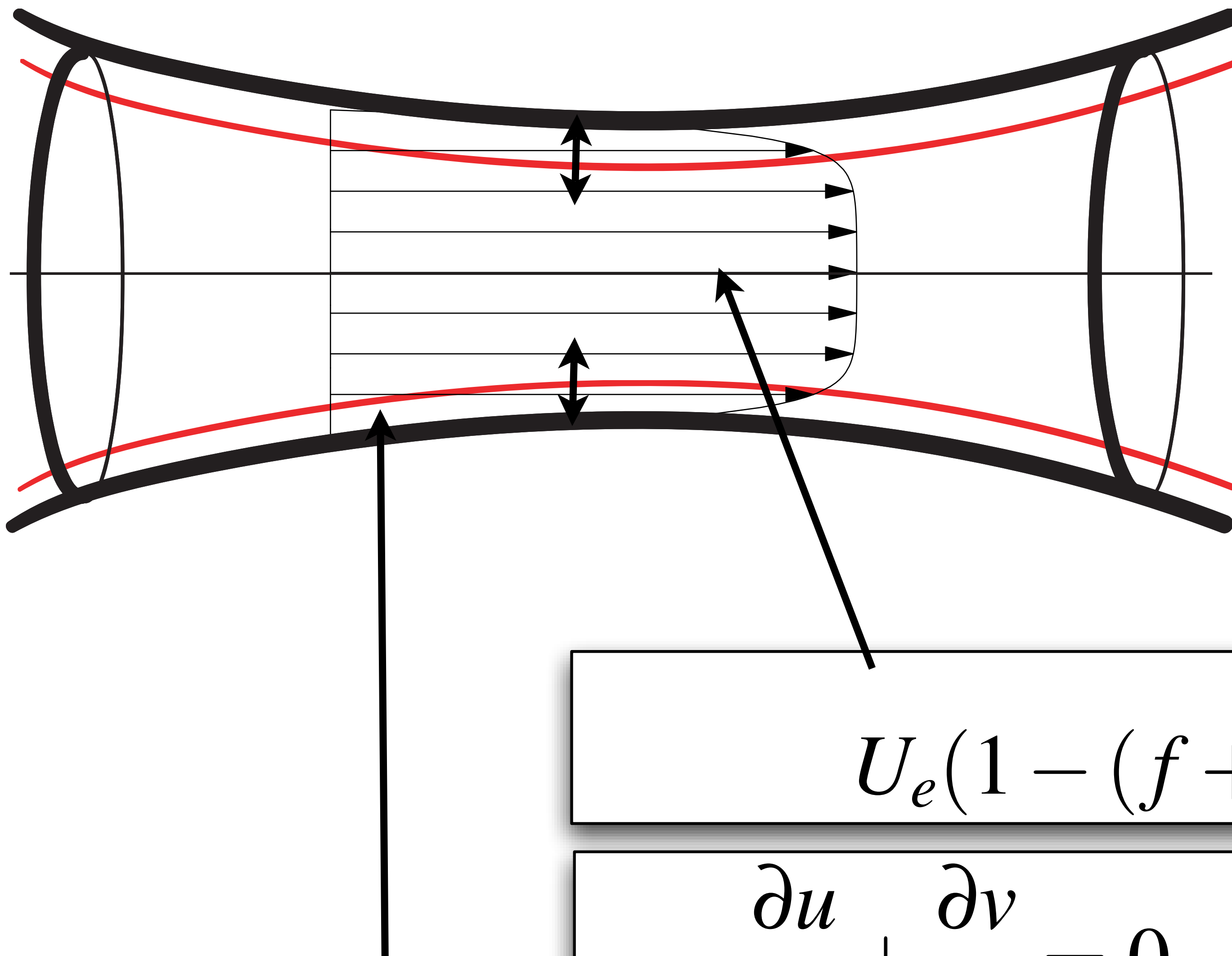
Interactive Boundary Layer/ Couche limite interactive



$$U_e(1-(f+\delta_1))^2 = 1$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn$$

Interactive Boundary Layer/ Couche limite interactive



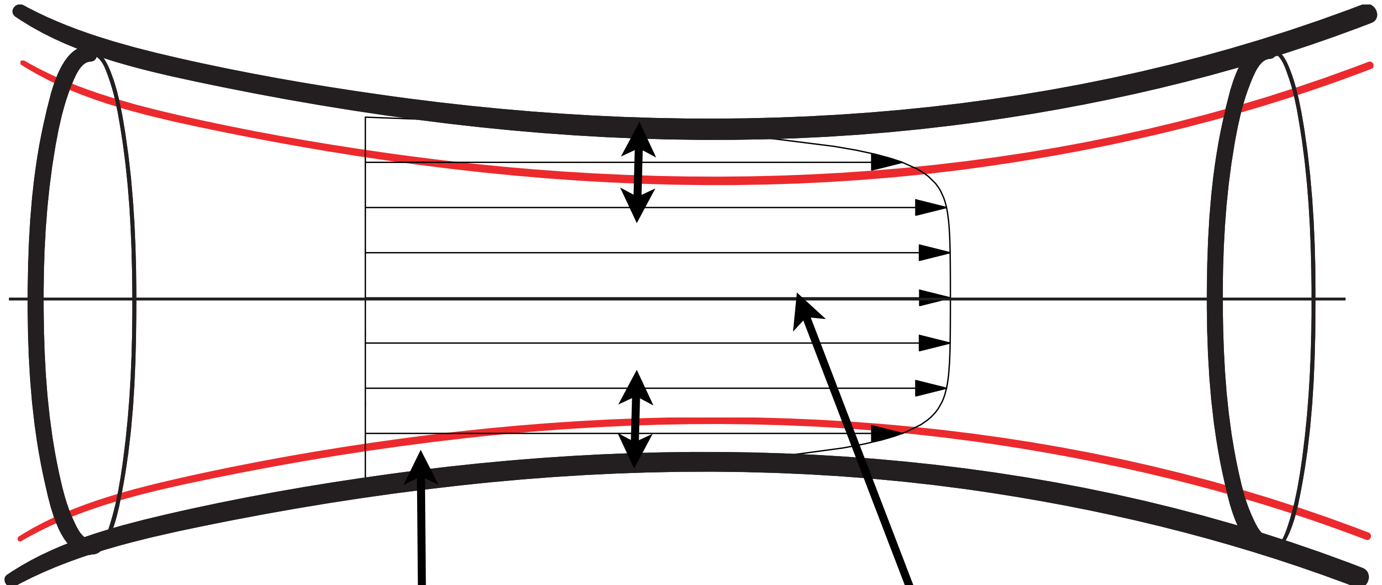
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\delta_1 = \int_0^\infty (1-\frac{u}{U_e})dn$$

Interactive Boundary Layer/ Couche limite interactive



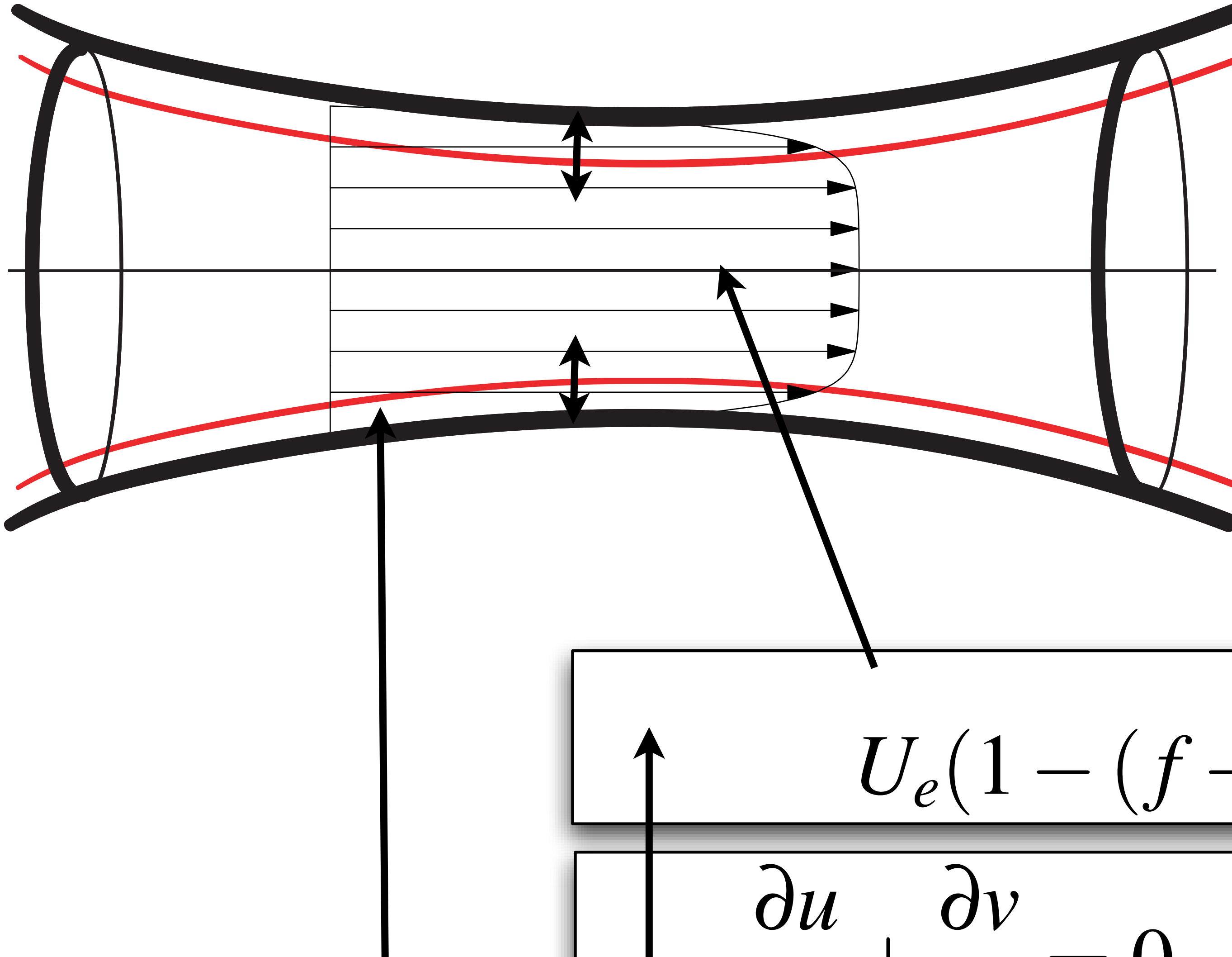
$$U_e(1-(f+\delta_1))^2=1$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial n}=0 \qquad u(x,\infty)=U_e$$

$$u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial n}u=\frac{dU_e}{dx}+\frac{\partial^2}{\partial n^2}u$$

$$\delta_1=\int_0^\infty(1-\frac{u}{U_e})dn$$

Interactive Boundary Layer/ Couche limite interactive



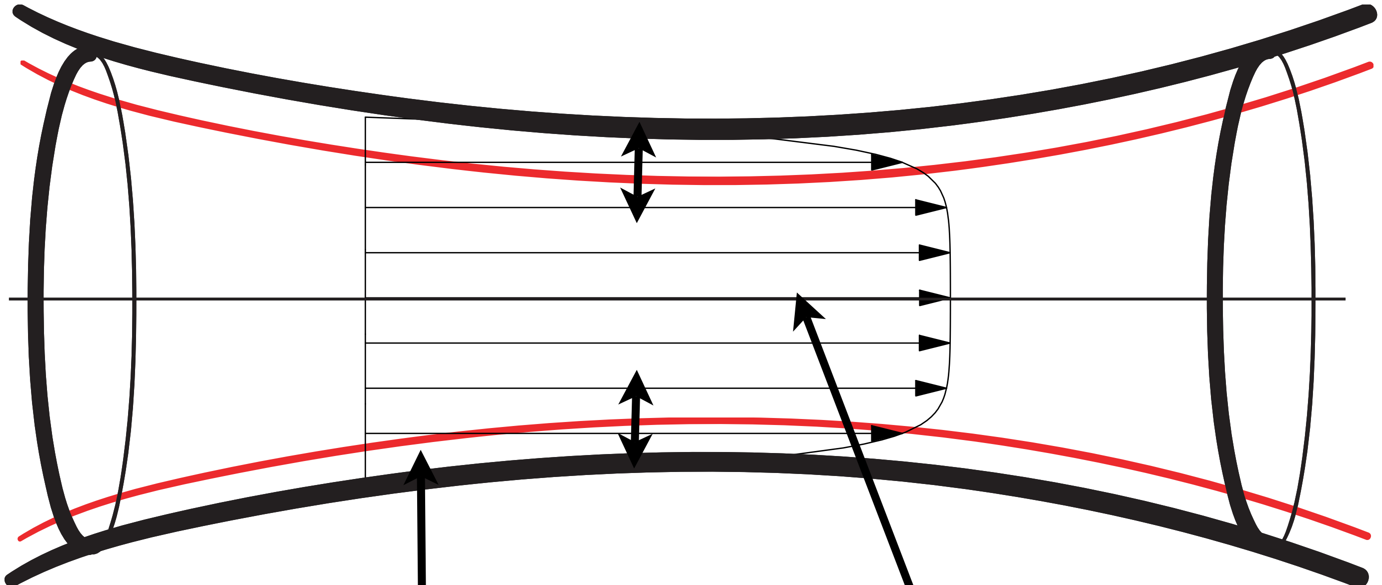
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \qquad u(x,\infty) = U_e$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e})dn$$

Interactive Boundary Layer/ Couche limite interactive



$$U_e(1-(f+\delta_1))^2 = 1$$

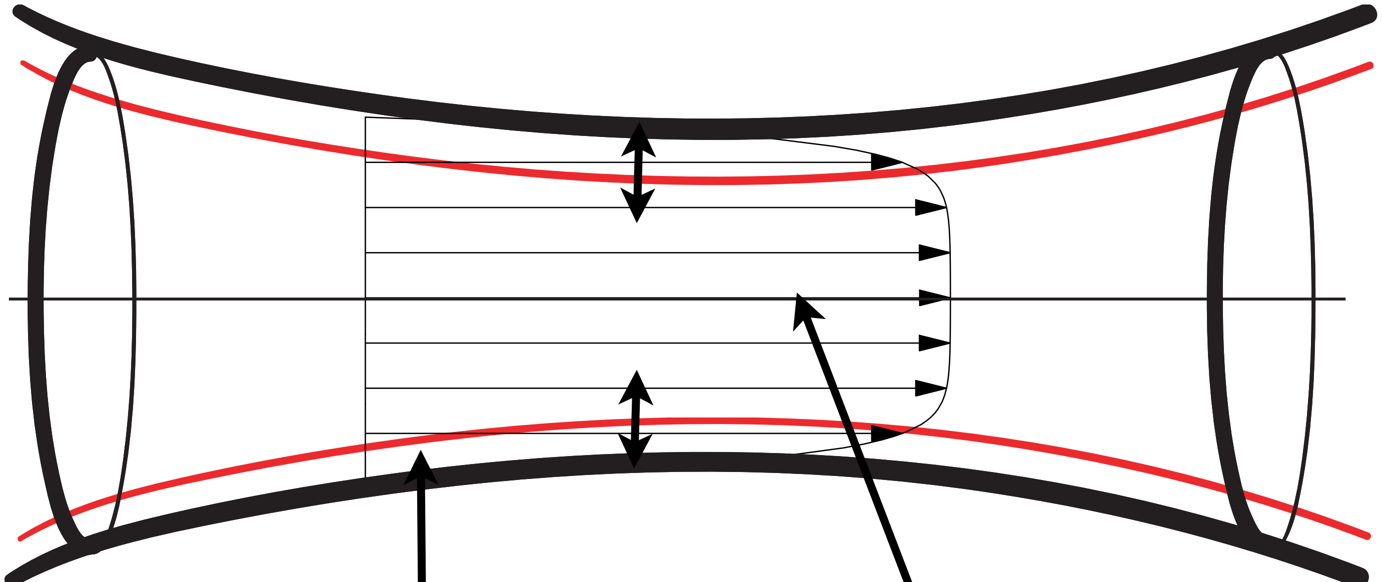
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial n}=0$$

$$u(x,\infty)=U_e$$

$$u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial n}u=\frac{dU_e}{dx}+\frac{\partial^2}{\partial n^2}u$$

$$\delta_1=\int_0^\infty(1-\frac{u}{U_e})dn$$

Interactive Boundary Layer/ Couche limite interactive



Problème couplé à résoudre

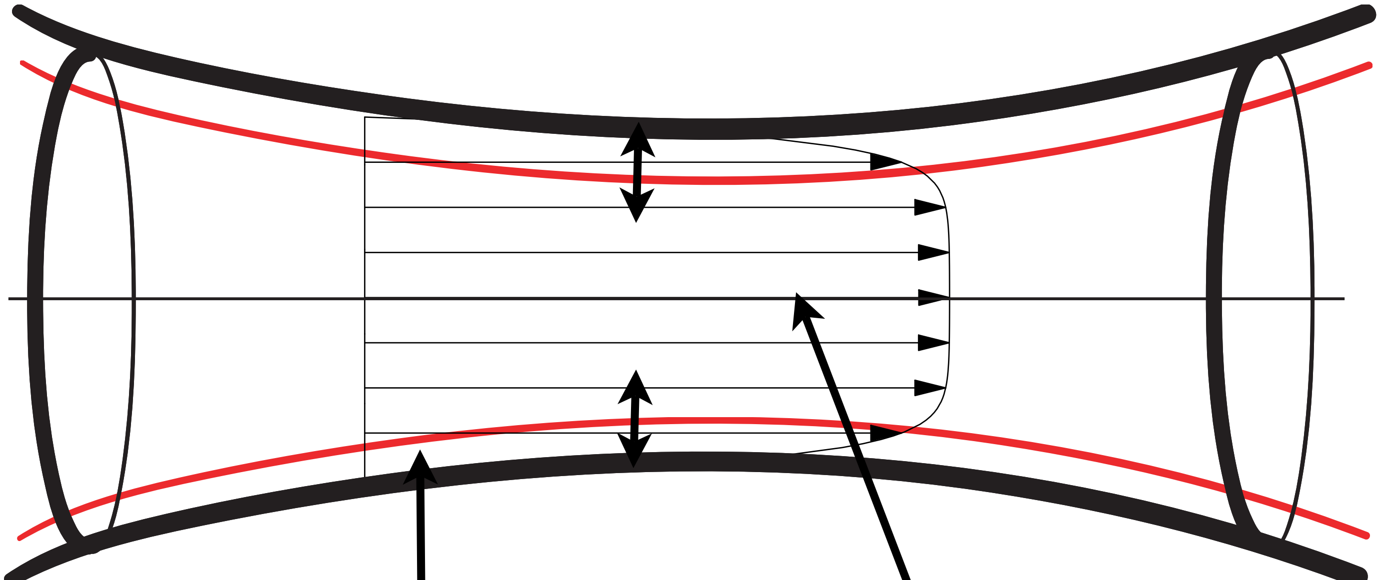
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \qquad u(x,\infty) = U_e$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e})dn$$

Interactive Boundary Layer/ Couche limite interactive



Problème couplé à résoudre

$$U_e(1-(f+\delta_1))^2 = 1$$

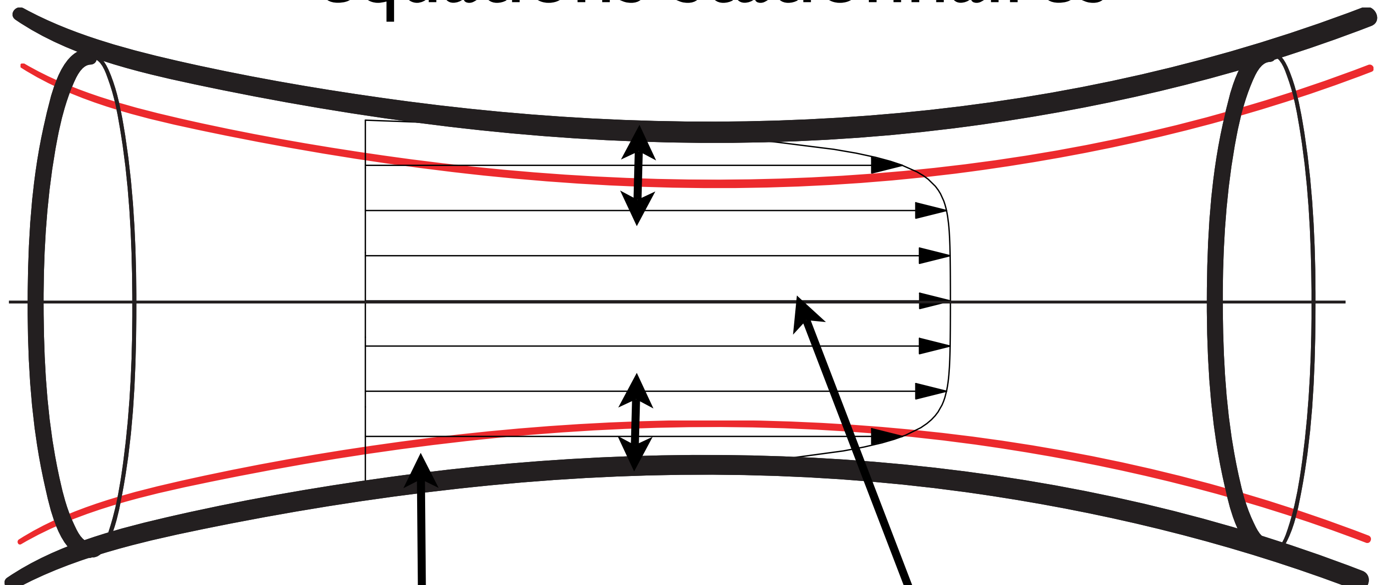
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \qquad u(x,\infty) = U_e$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\int_0^\infty dn$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e})dn$$

Résolution Intégrale équations stationnaires



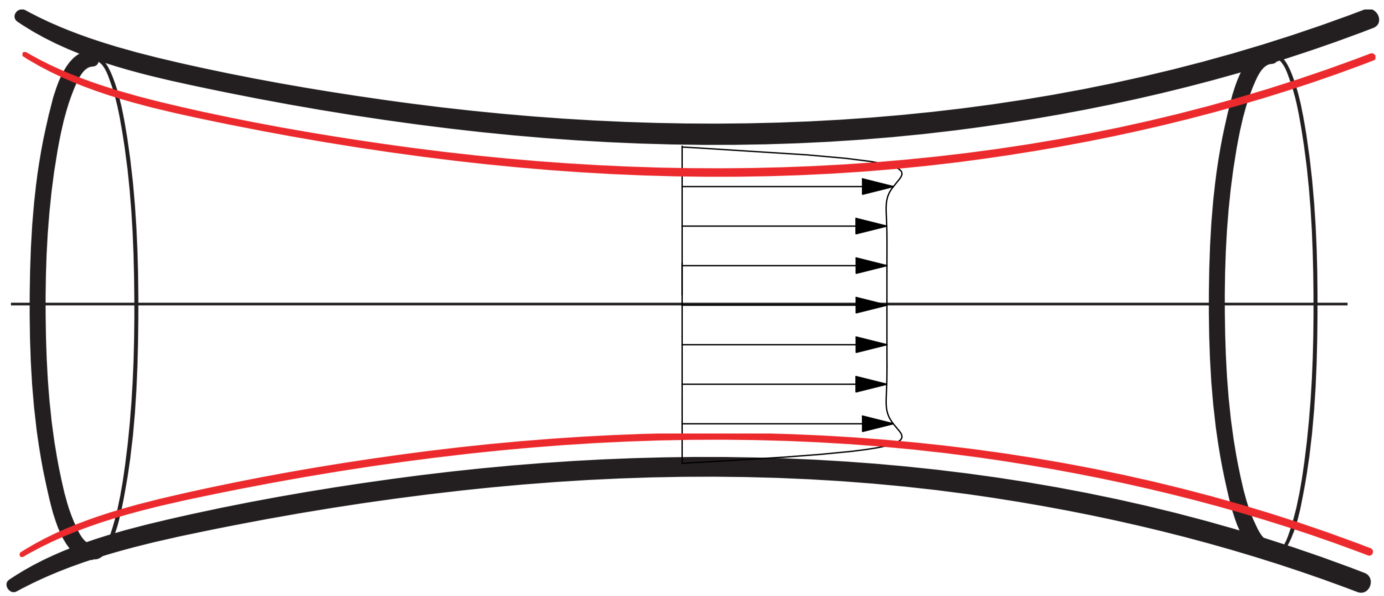
Problème couplé à résoudre

$$U_e(1-(f+\delta_1))^2 = 1$$

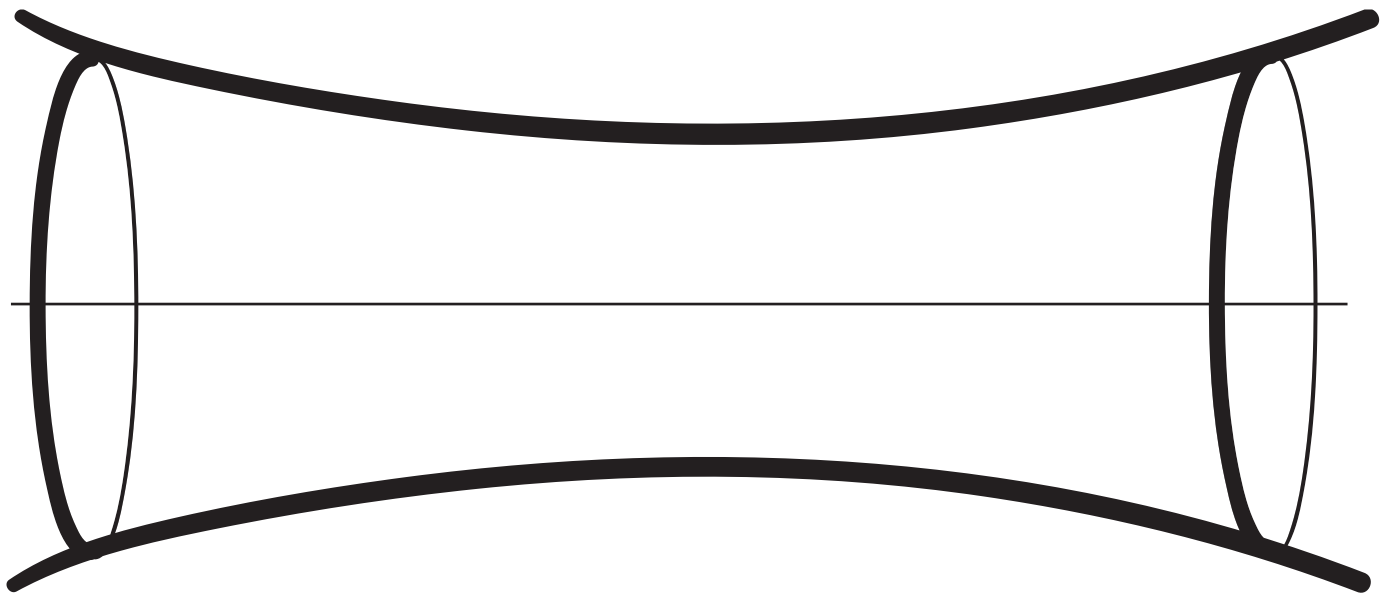
$$\delta_1 = \int_0^\infty (1-\frac{u}{U_e})dn$$

$$\frac{d}{dx}(\frac{\delta_1}{H}) + \frac{\delta_1}{U_e}(1+\frac{2}{H})\frac{dU_e}{dx} = \frac{f_2 H}{\delta_1 U_e}$$

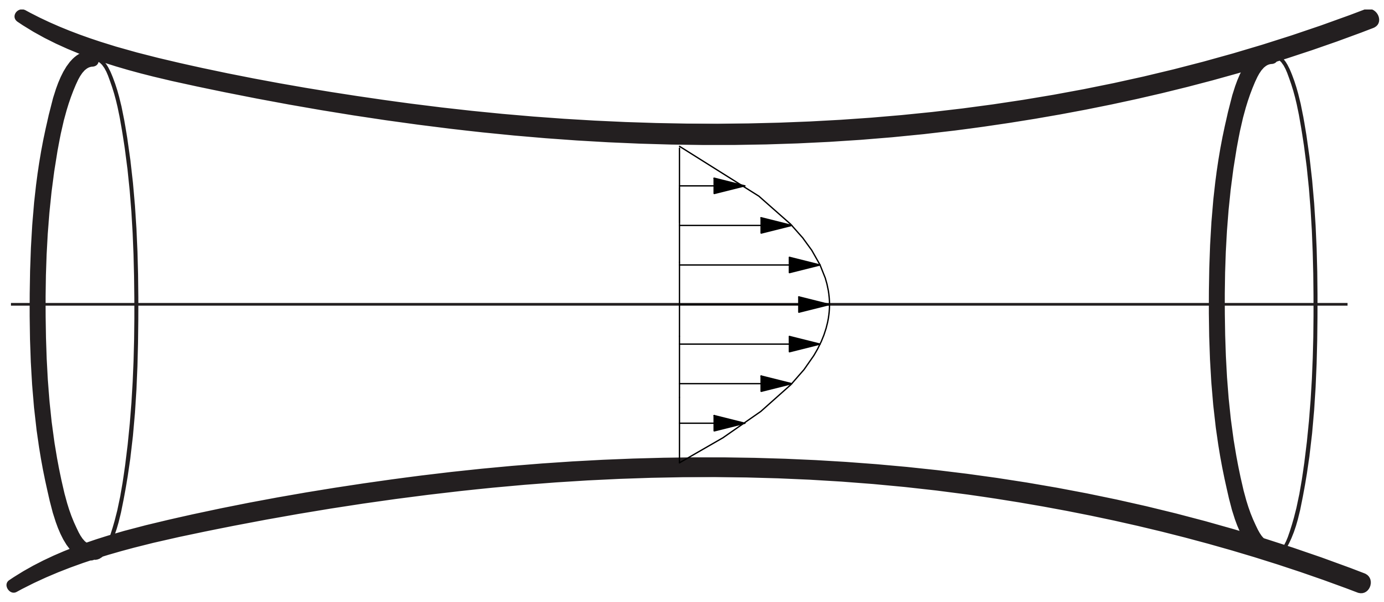
Interactive Boundary Layer/ Couche limite interactive

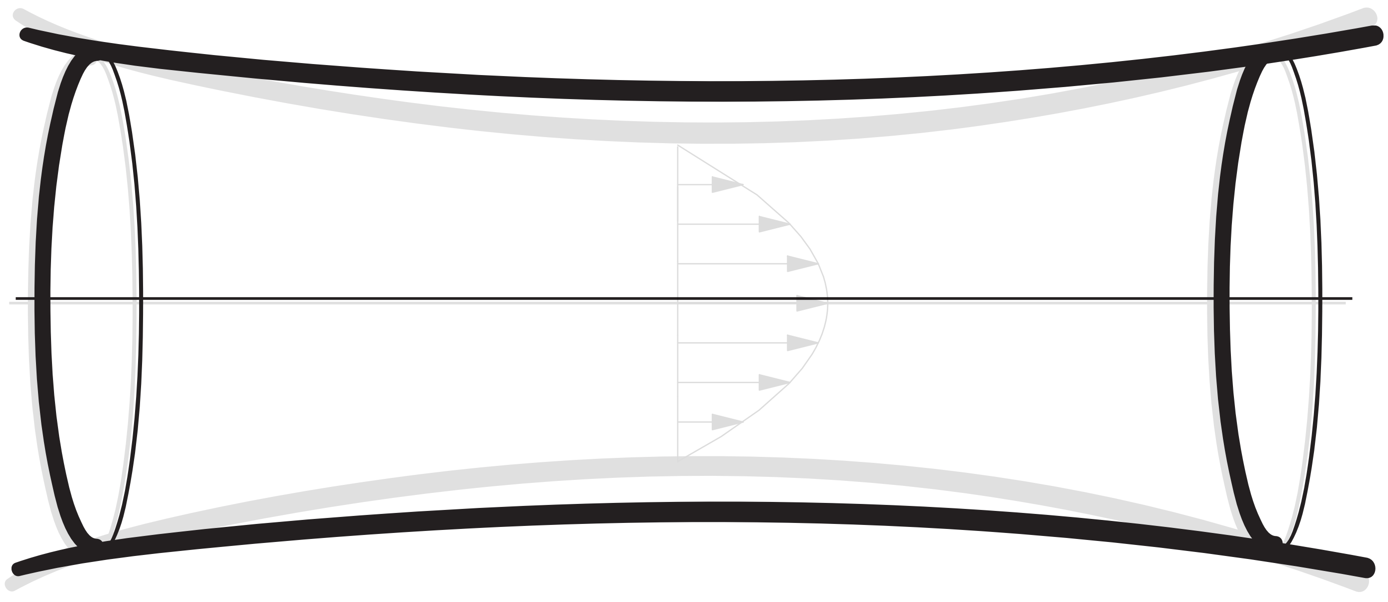


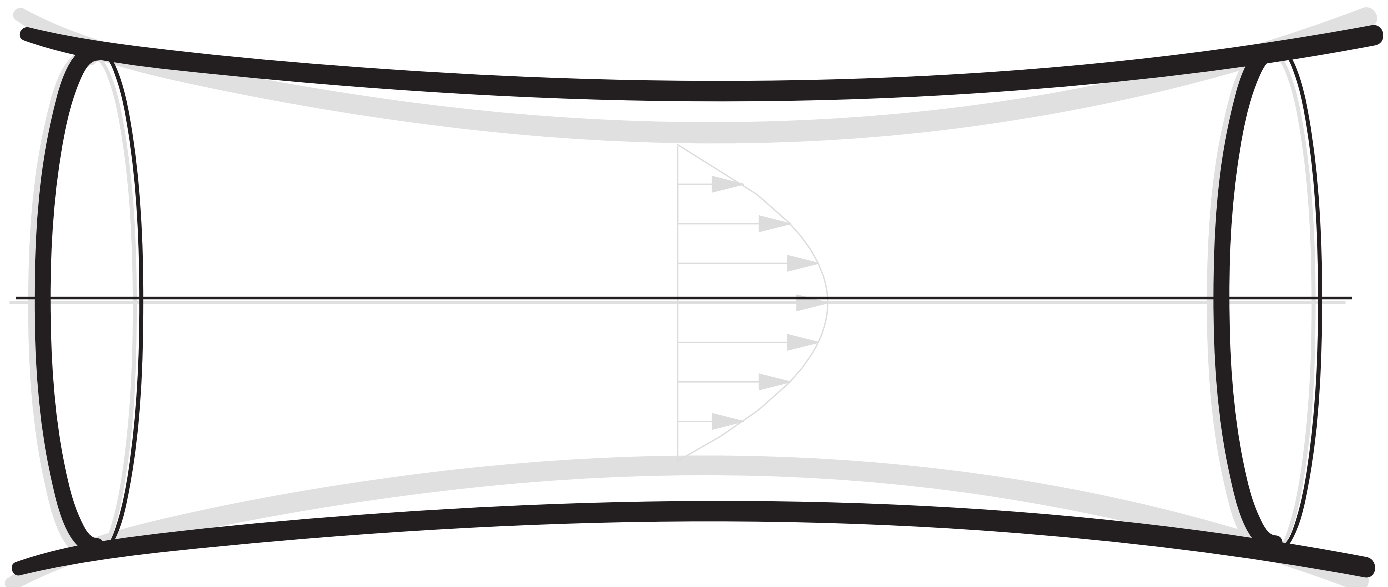
IBL est inclus dans RNSP

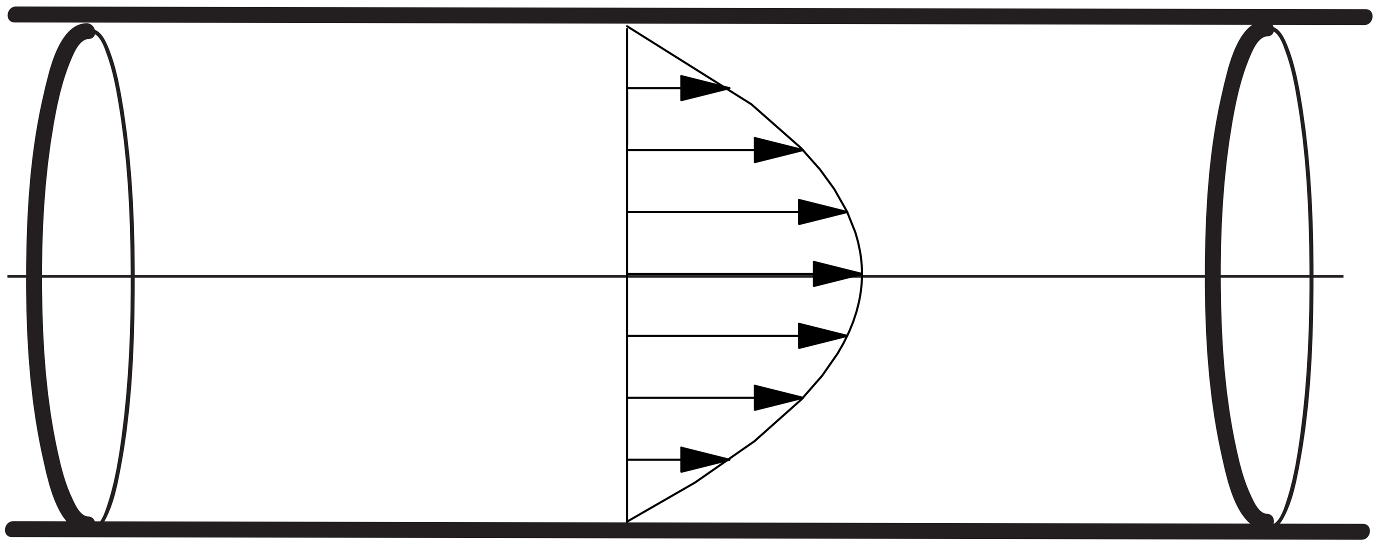


RNSP comptient les équations 1D habituelles
RNSP comptient les profils de Womersley
RNSP comptient la couche limite interactive(IBL)

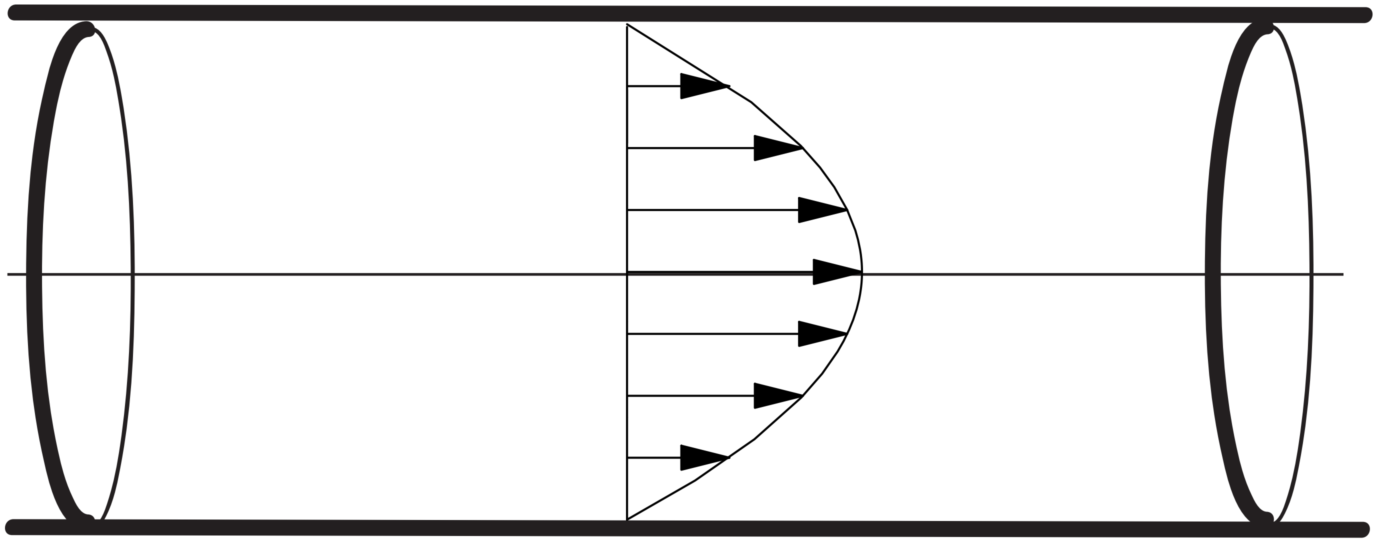


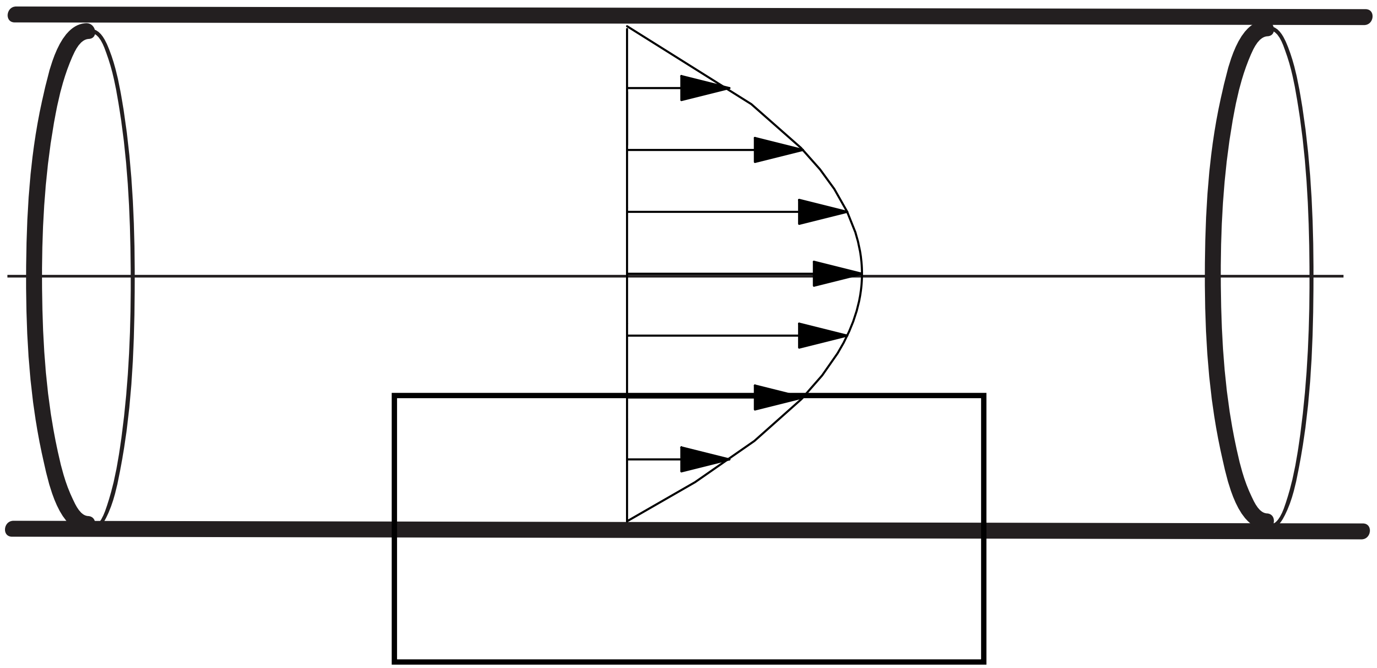


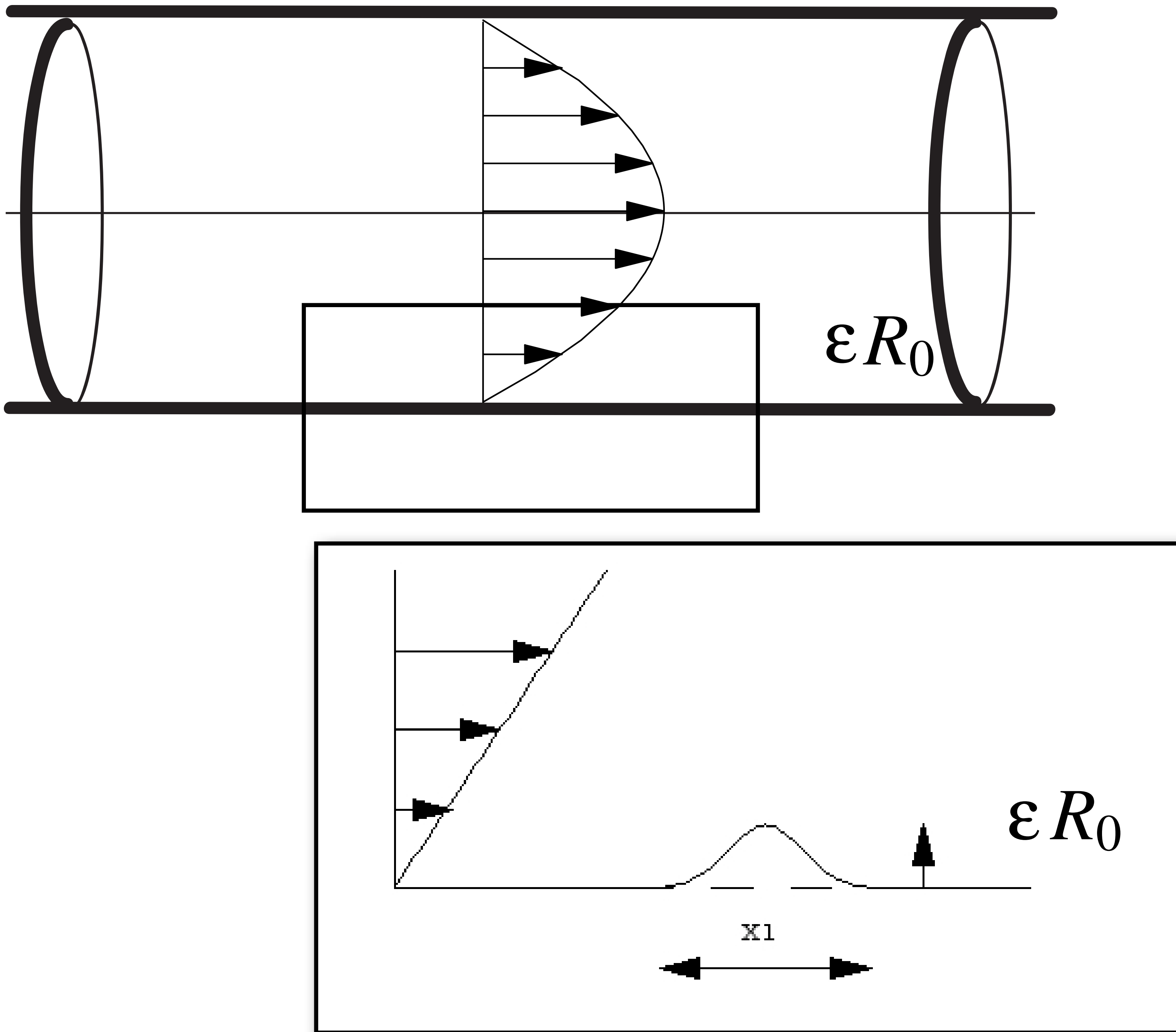




La double/triple couche?



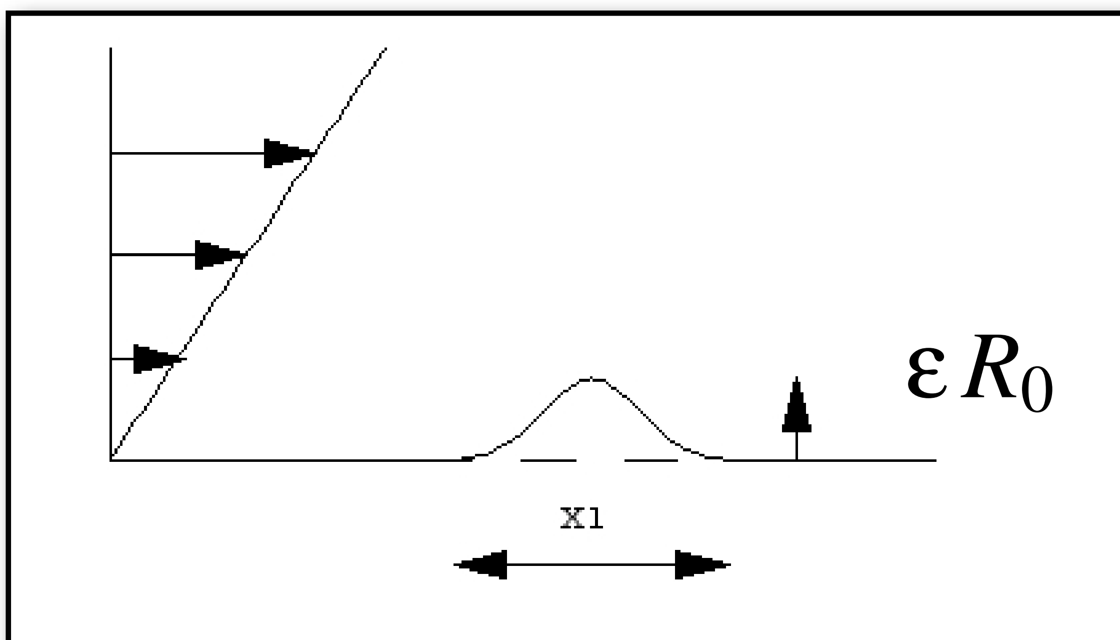




εR0
εR0
x1

vitesse linéaire
en y en amont

$U_0\varepsilon$

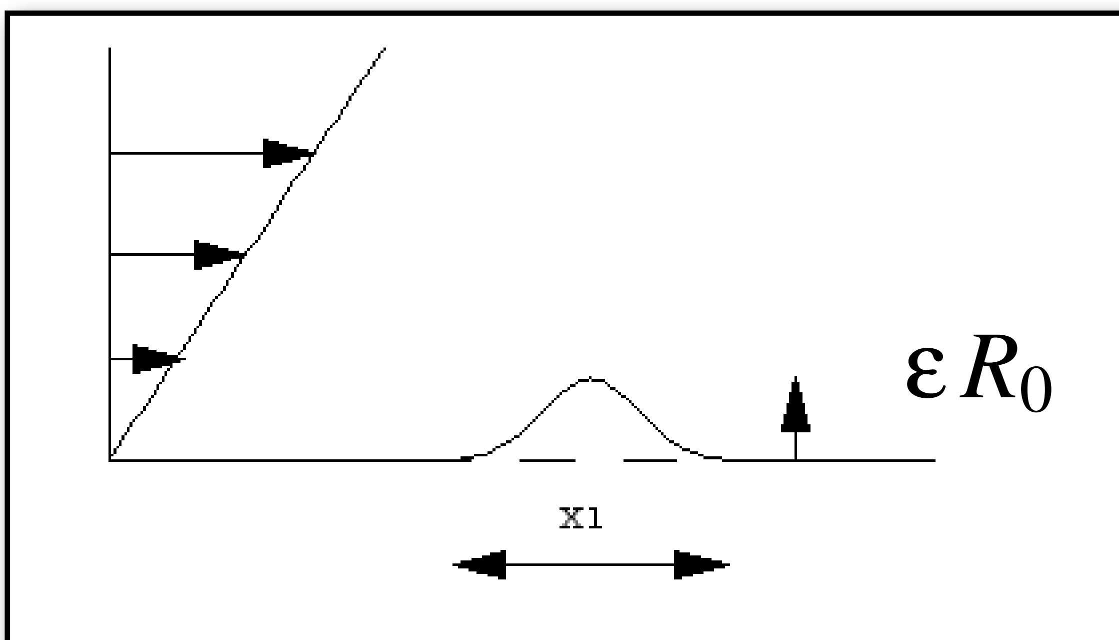


convection diffusion

$$u\frac{\partial u}{\partial x} \propto \nu \frac{\partial^2 u}{\partial y^2}$$

vitesse linéaire
en y en amont

$U_0 \varepsilon$



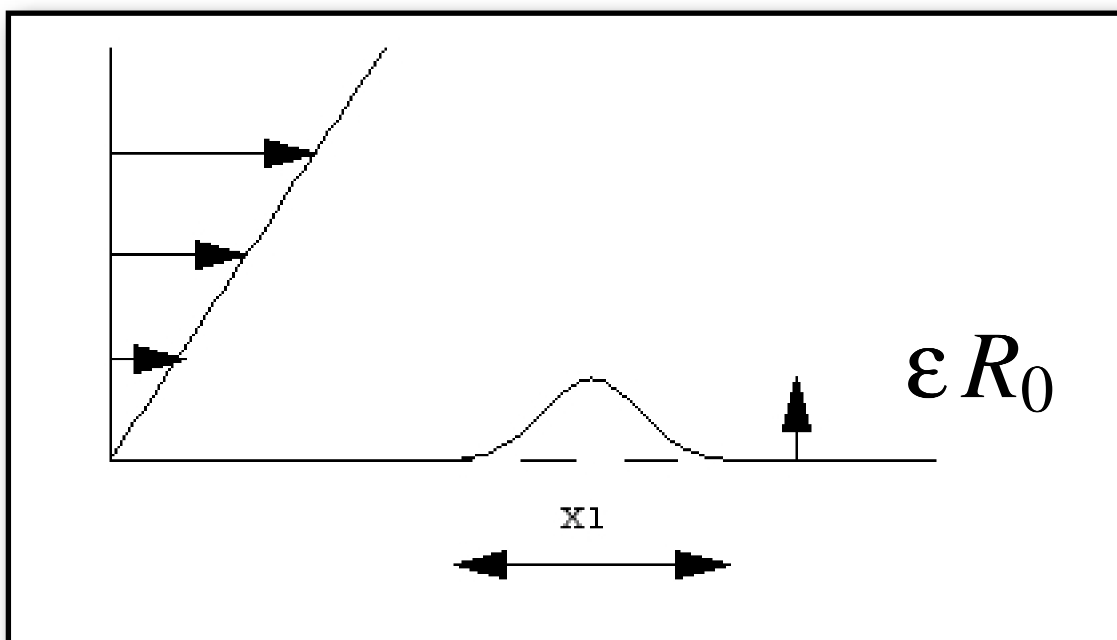
εR_0

convection diffusion

$$\frac{(U_0\varepsilon)^2}{x_l} \propto \nu \frac{U_0\varepsilon}{(\varepsilon R_0)^2}$$

vitesse linéaire
en y en amont

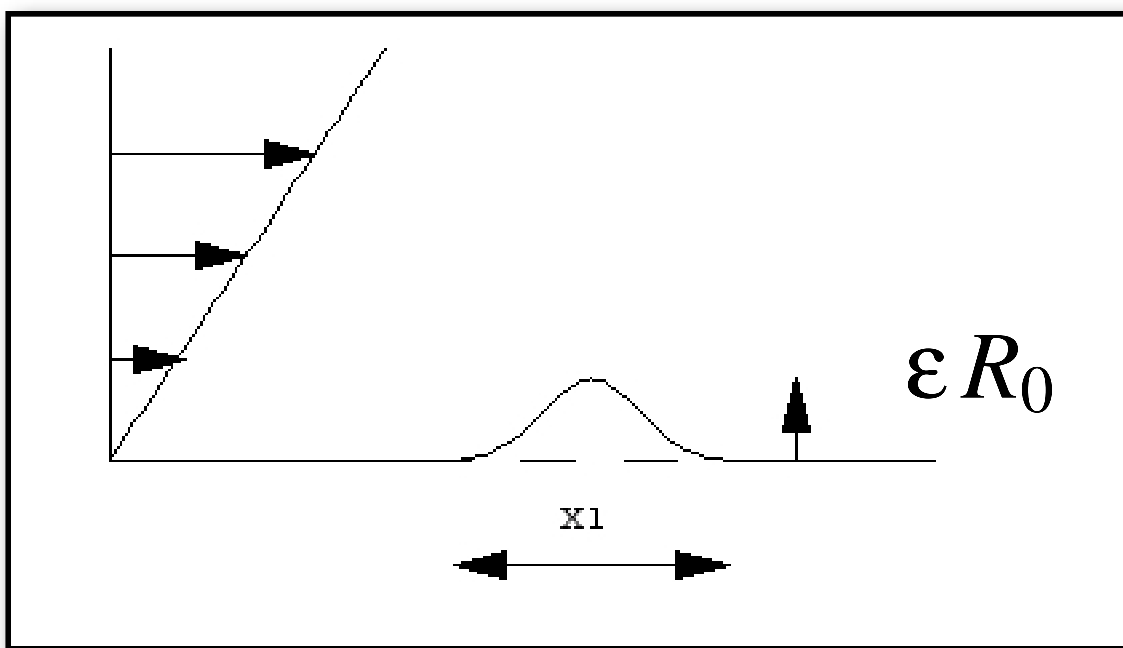
$U_0\varepsilon$



convection diffusion

$$\frac{x_l}{R_0} = \frac{U_0 R_0}{\nu} \varepsilon^3$$

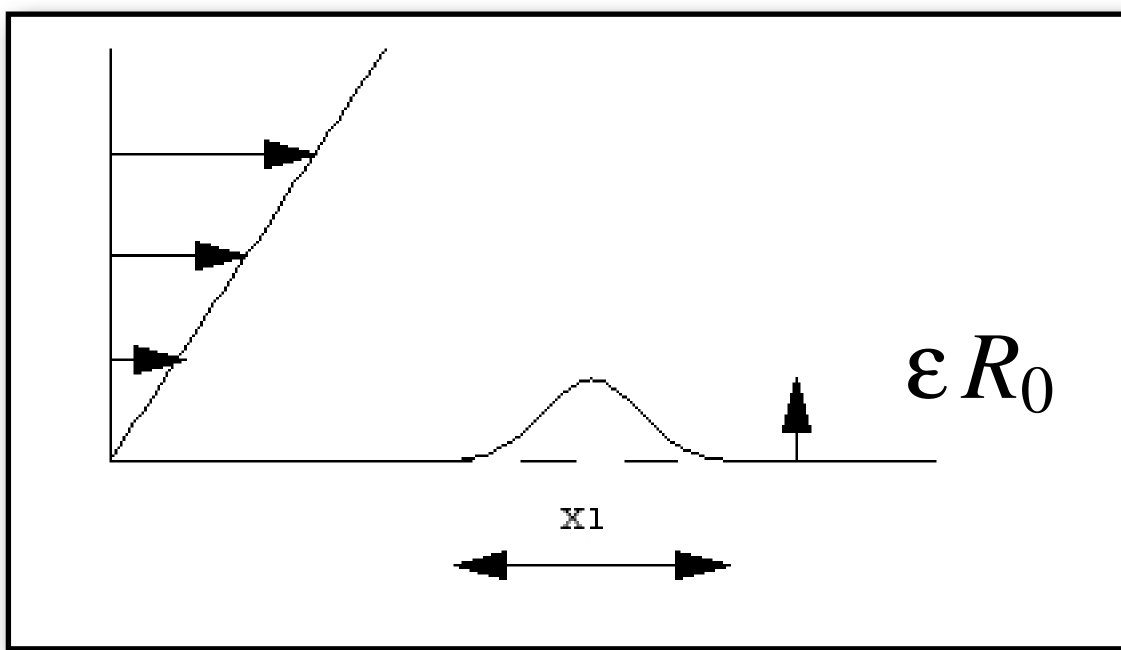
$U_0 \varepsilon$



εR_0

$$\varepsilon = Re^{-1/3}$$

$U_0\varepsilon$



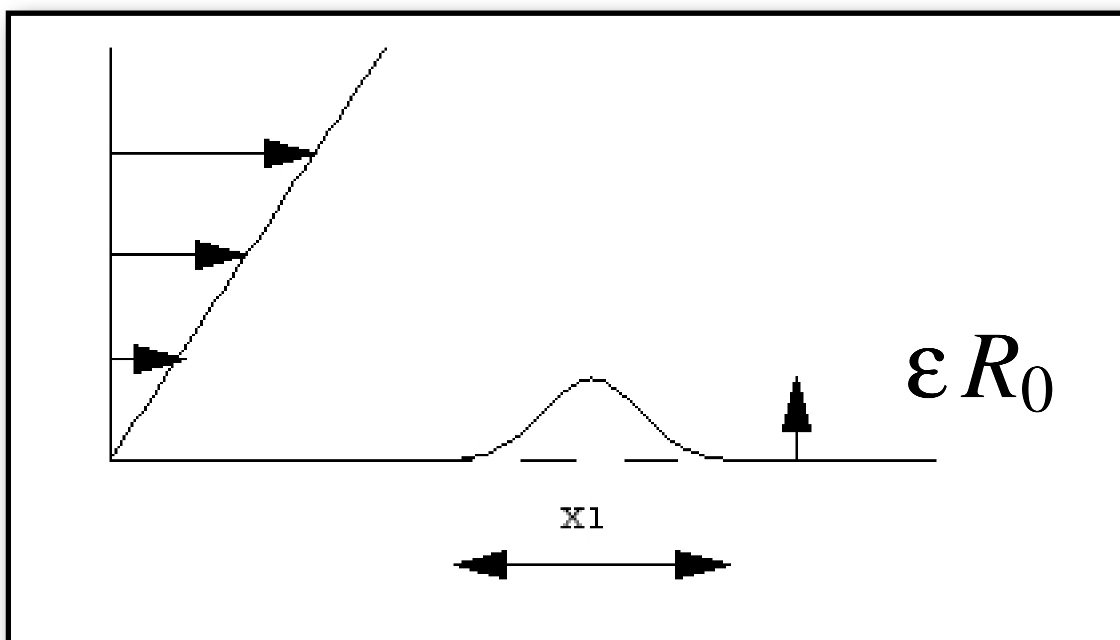
εR_0

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}p + \frac{\partial^2}{\partial y^2}u$$

$$0 = -\frac{\partial}{\partial y}p$$

$$u \rightarrow y$$



εR_0

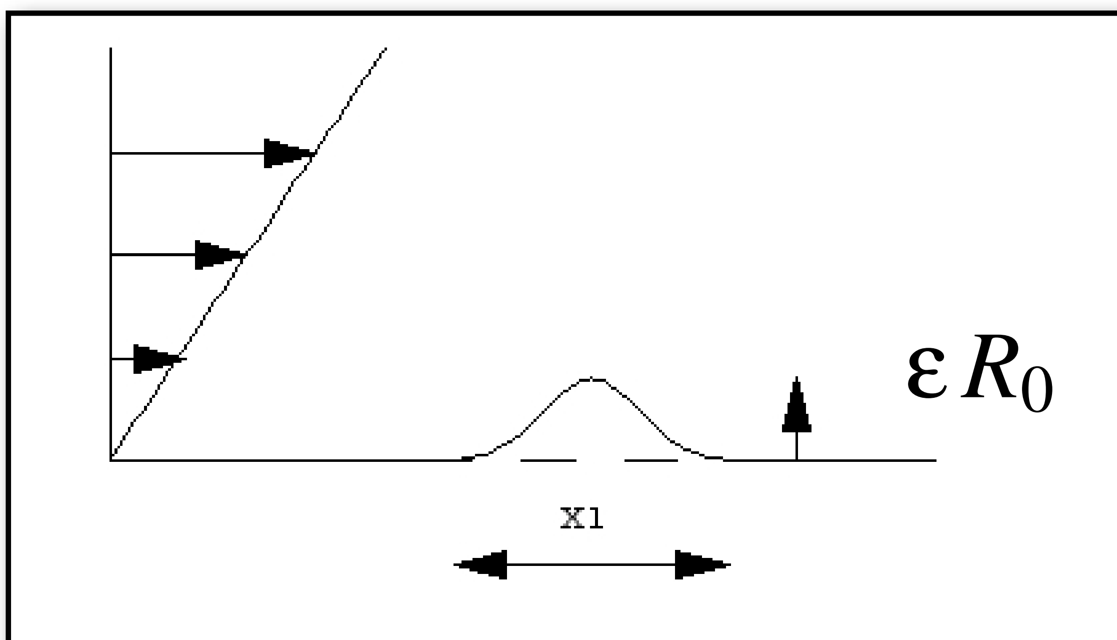
$$\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v=0$$

$$u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial y}u=-\frac{\partial}{\partial x}p+\frac{\partial^2}{\partial y^2}u$$

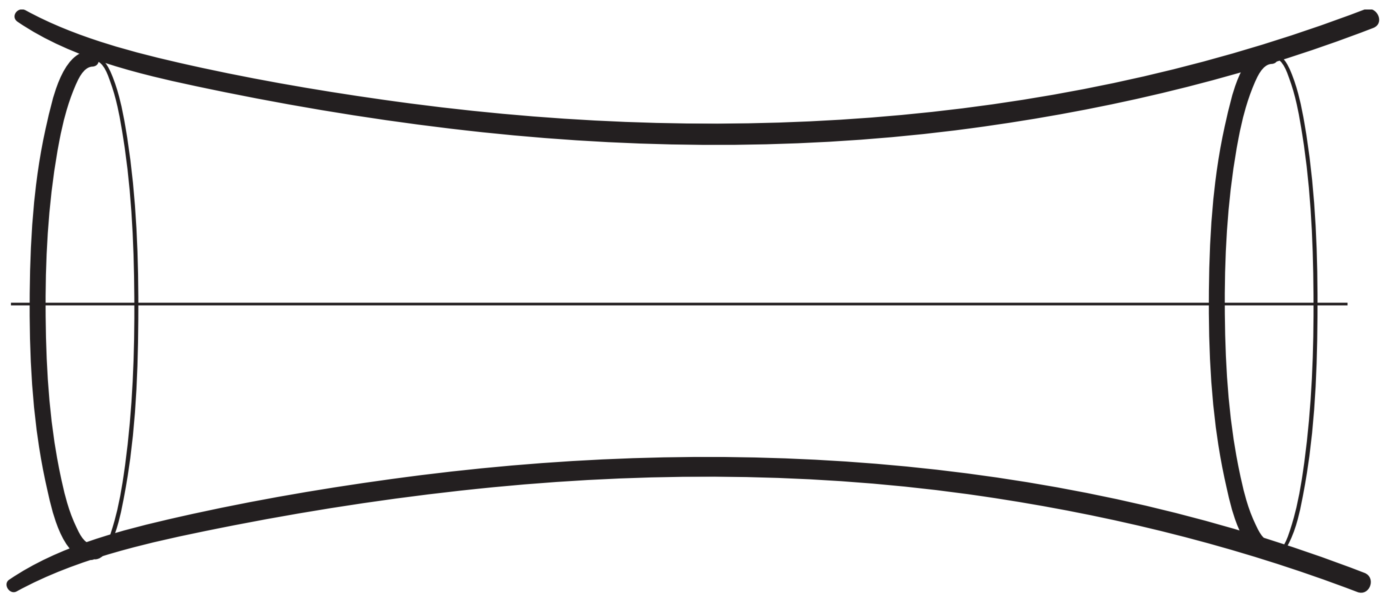
$$0=-\frac{\partial}{\partial y}p$$

$$u\longrightarrow y$$

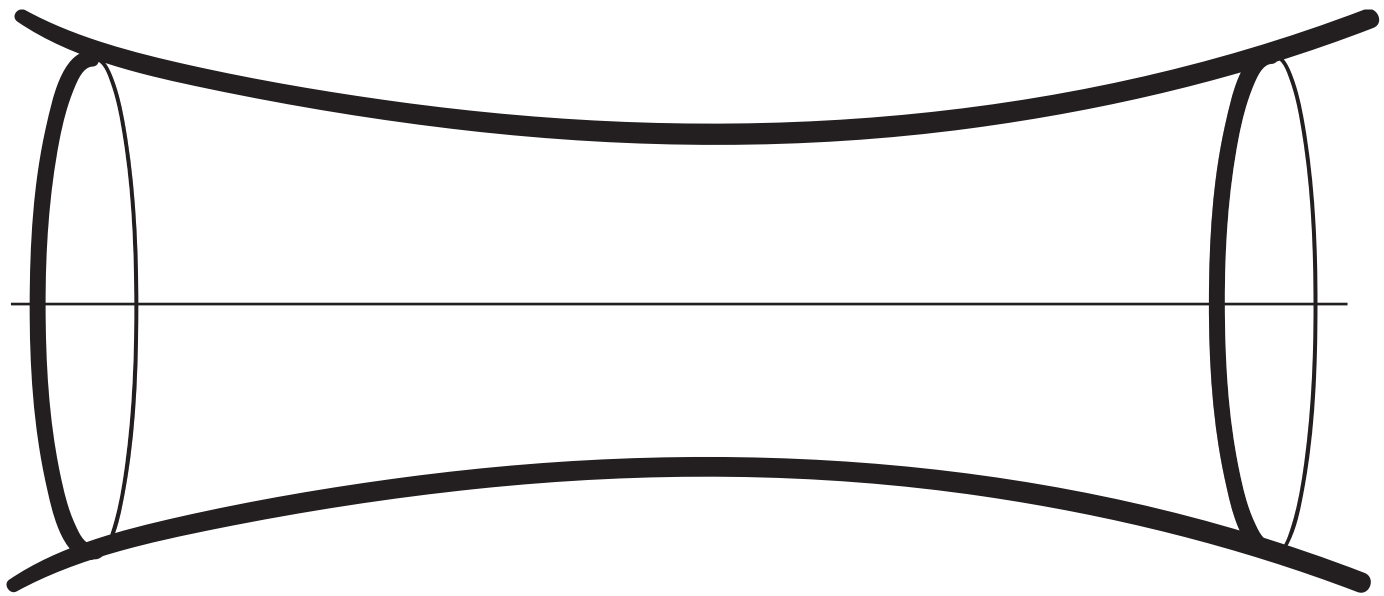
on retrouve les équations MAIS à des échelles différentes
ET avec des conditions limites différentes



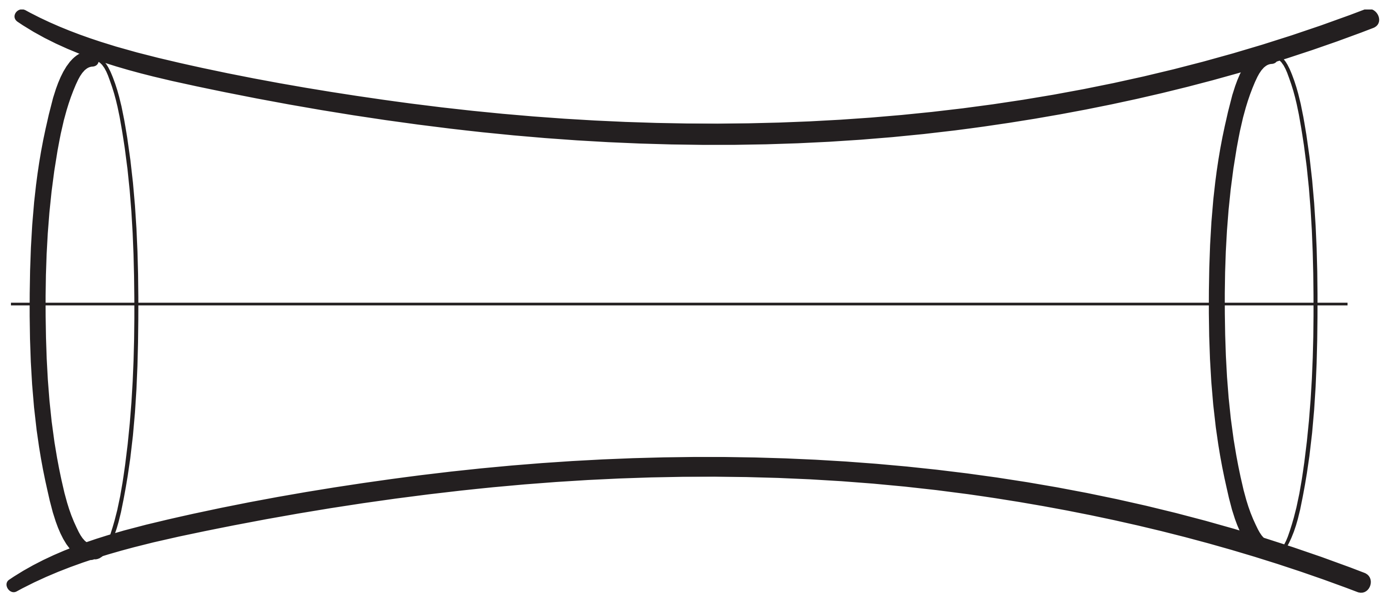
εR_0



RNSP contient les équations 1D habituelles
RNSP contient les profils de Womersley
RNSP contient la couche limite interactive(IBL)
RNSP contient la double/triple couche
RNSP contient les équations de jet



Comparaisons

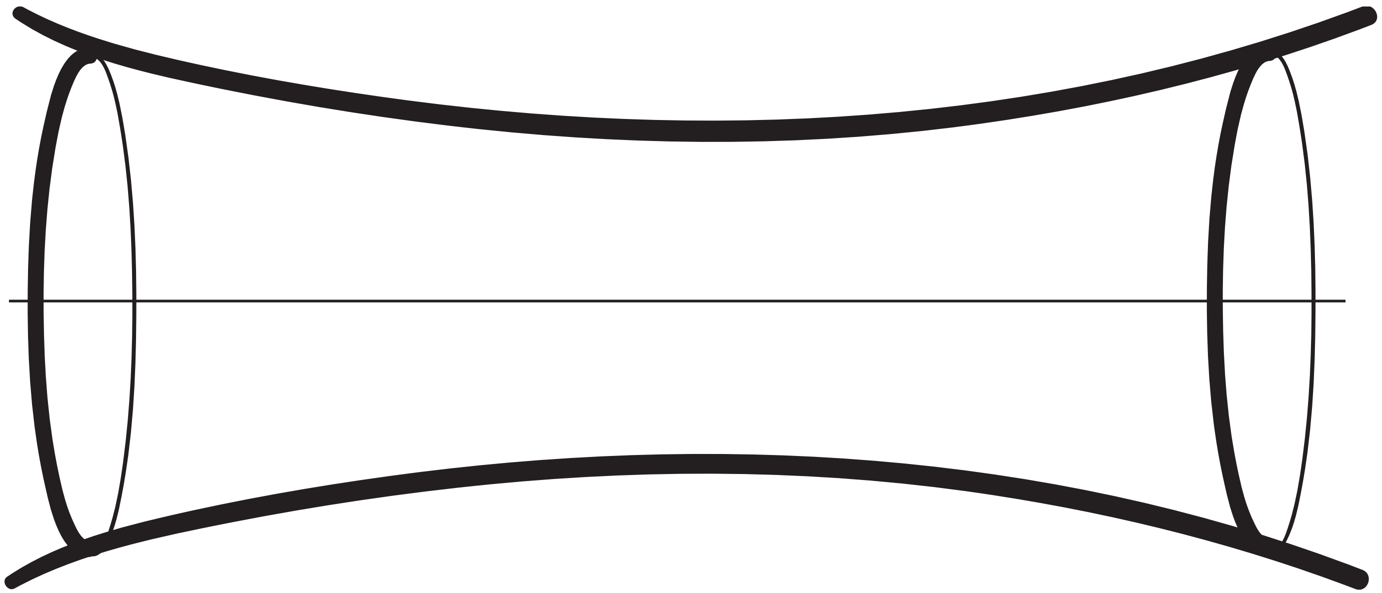


Comparaisons

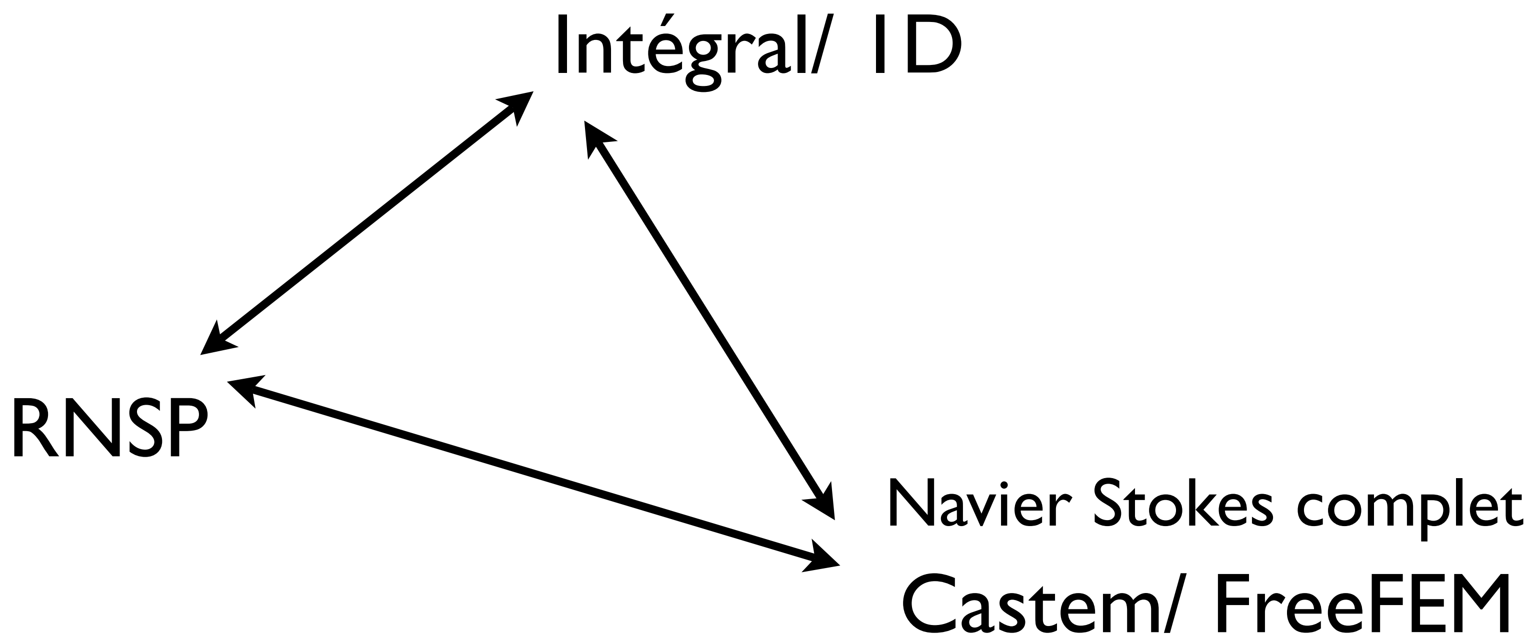
Intégral/ 1D

Navier Stokes complet

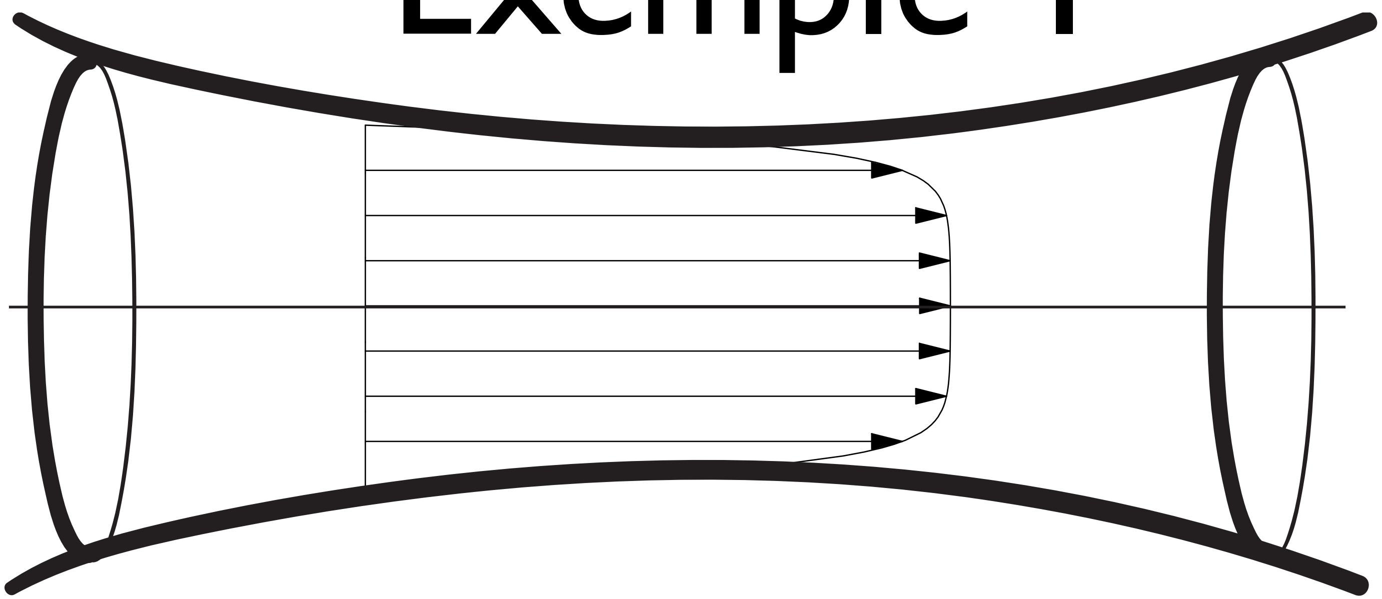
Castem/ FreeFEM



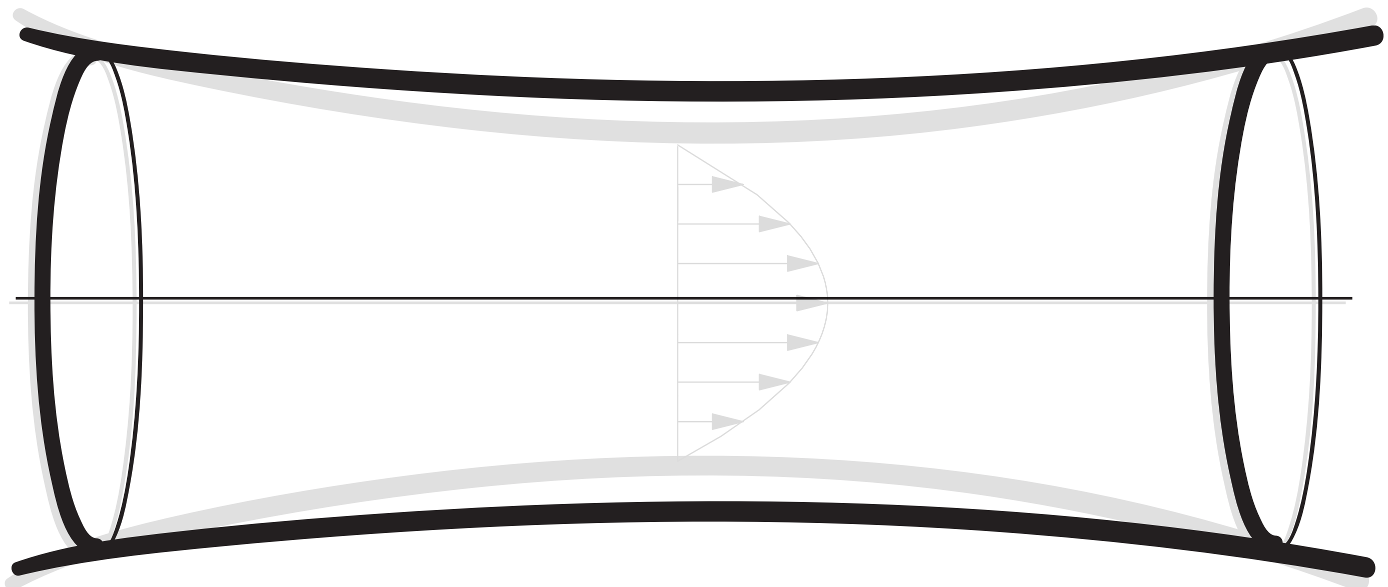
Comparaisons

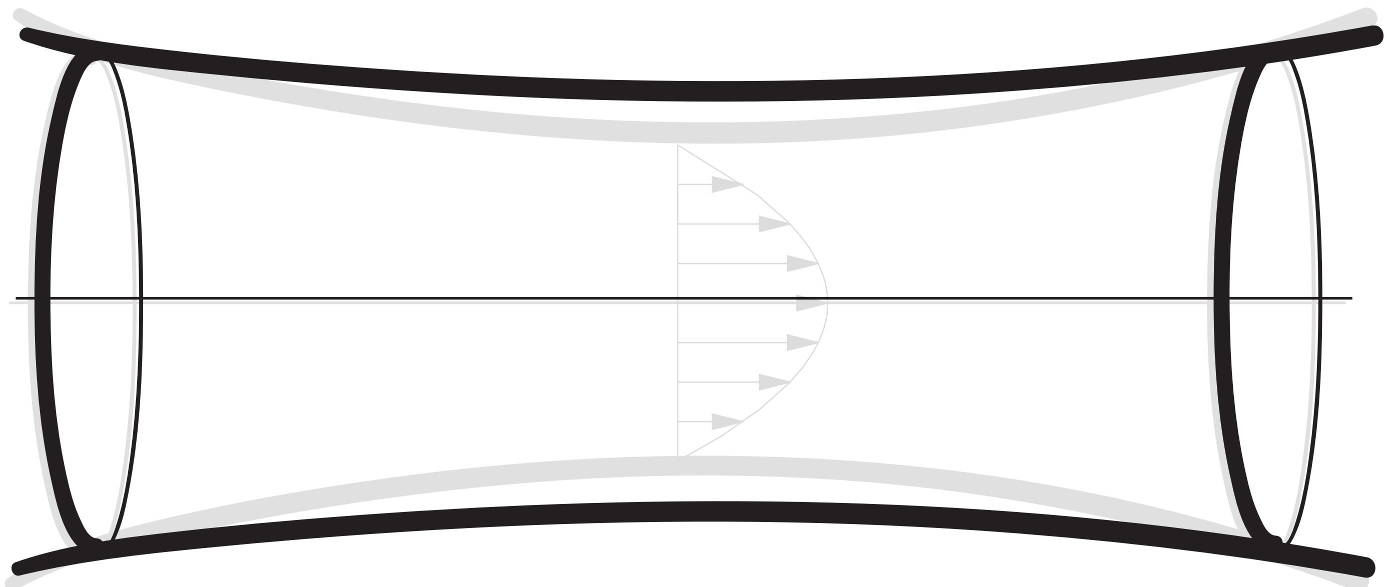


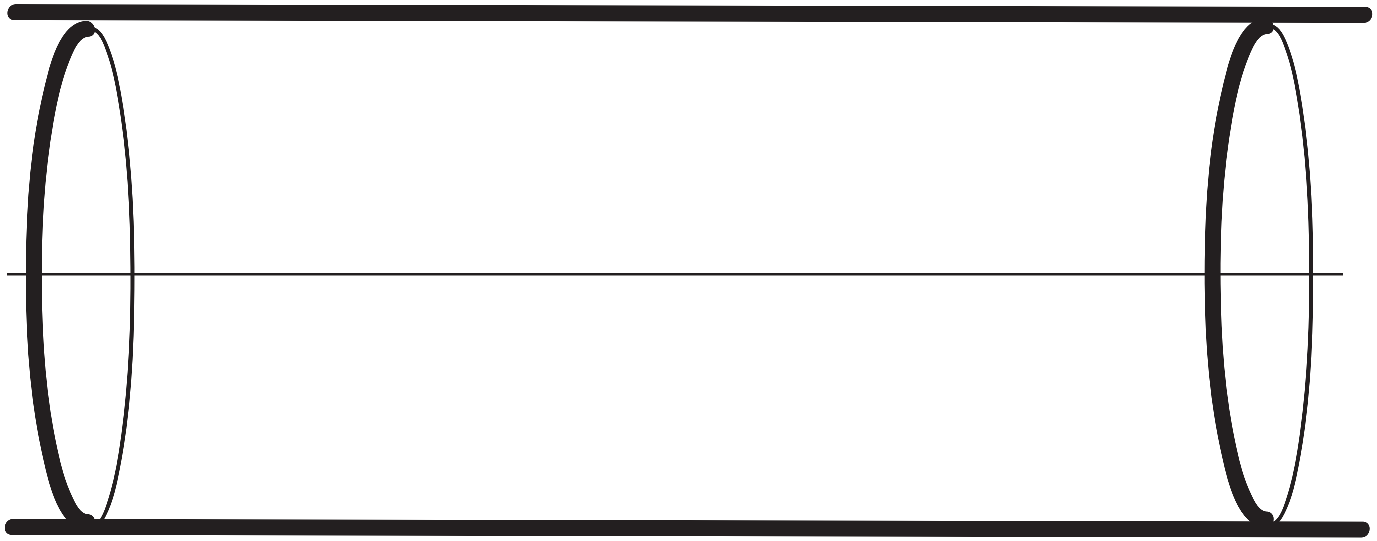
Exemple 1

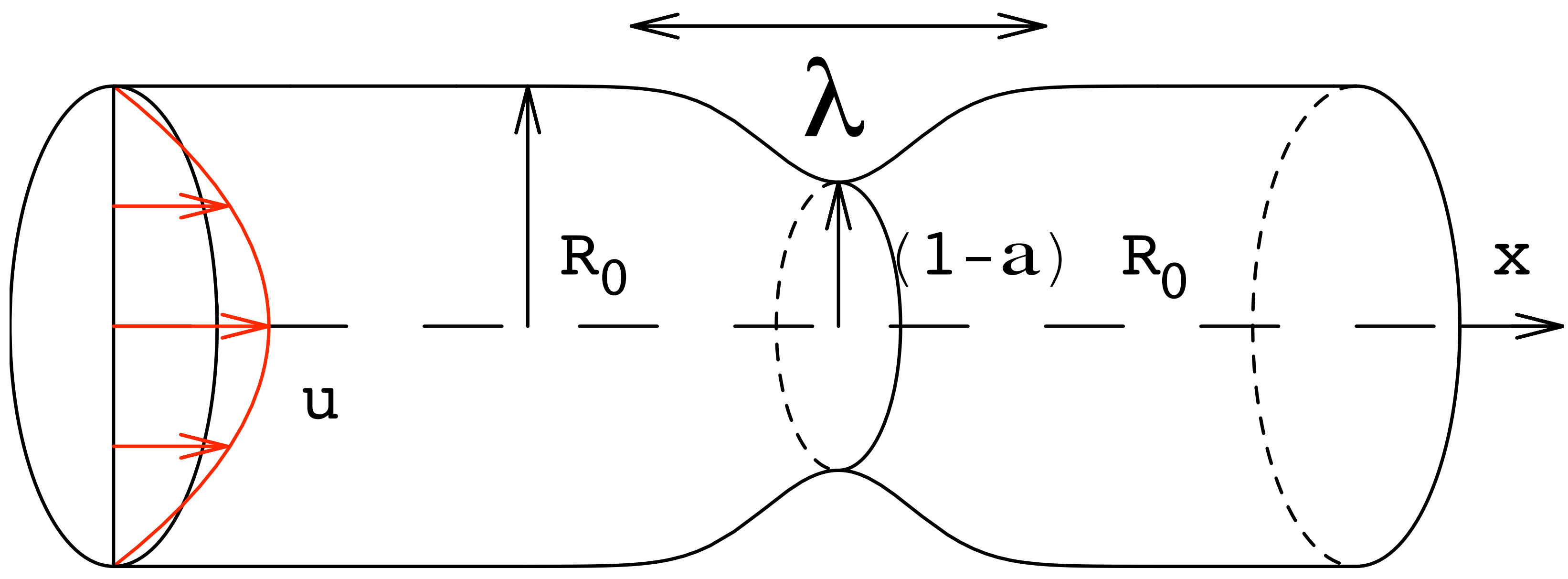


- Ecoulement dans les vaisseaux sténosés
- stationnaire, paroi rigide

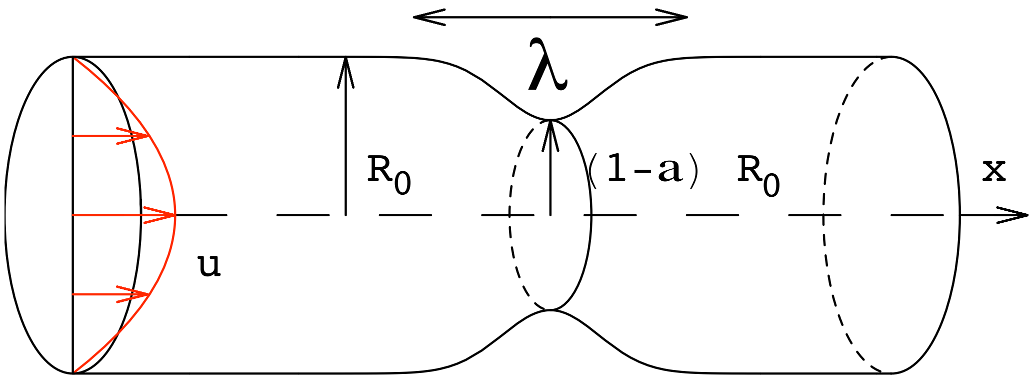







λ
R_0
(1-a) R_0
x
u

RNSP Scales

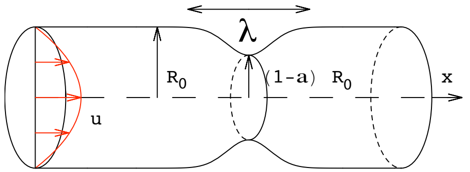


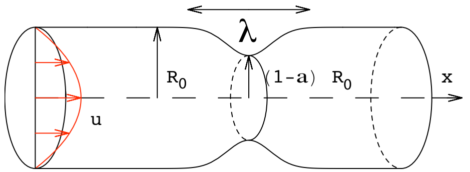
En utilisant:

$x^* = xR_0Re$, $r^* = rR_0$, $u^* = U_0u$, $v^* = \frac{U_0}{Re}v$, $t^* = t\frac{R_0}{U_0}Re$,
$p^* = p_0^* + \rho_0 U_0^2 p$ and $\tau^* = \frac{\rho U_0^2}{Re}\tau$

le système suivant d'équations différentielles est obtenu à partir de Navier Stokes, lorsque $Re \to \infty$:

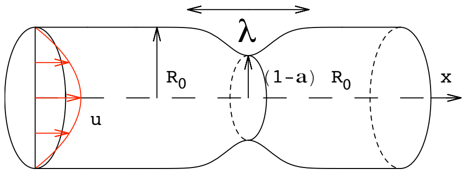
RNSP: Reduced Navier Stokes/ Prandtl System





RNSP: Reduced Navier Stokes/ Prandtl System

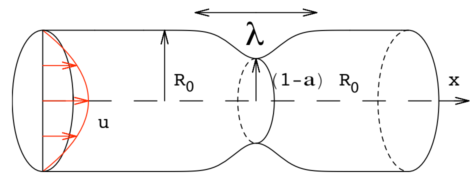
$$\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0,\\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u),\\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}$$



RNSP: Reduced Navier Stokes/ Prandtl System

$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0,\\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u),\\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$

+ Les conditions aux limites.



RNSP: Reduced Navier Stokes/ Prandtl System

$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0,\\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u),\\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$

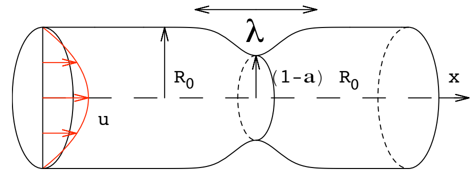
- symétrie axiale ($\partial_r u = 0$ et $v = 0$ en $r = 0$),

- adhérence à la paroi ($u = v = 0$ en $r = 1 - f(x)$),

- profils d'entrée ($u(0, r)$ et $v(0, r)$) donnés

- *pas* de condition de sortie en $x_{out} = \frac{x^*_{out}}{R_0 Re}$

- streamwise marching (résolution en suivant l'écoulement) même lorsqu'il y a séparation .



RNSP: Reduced Navier Stokes/ Prandtl System

$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0, \\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), \\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$

Problème parabolique- Marching Problem

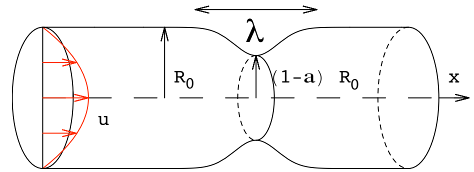
- symétrie axiale ($\partial_r u = 0$ et $v = 0$ en $r = 0$),

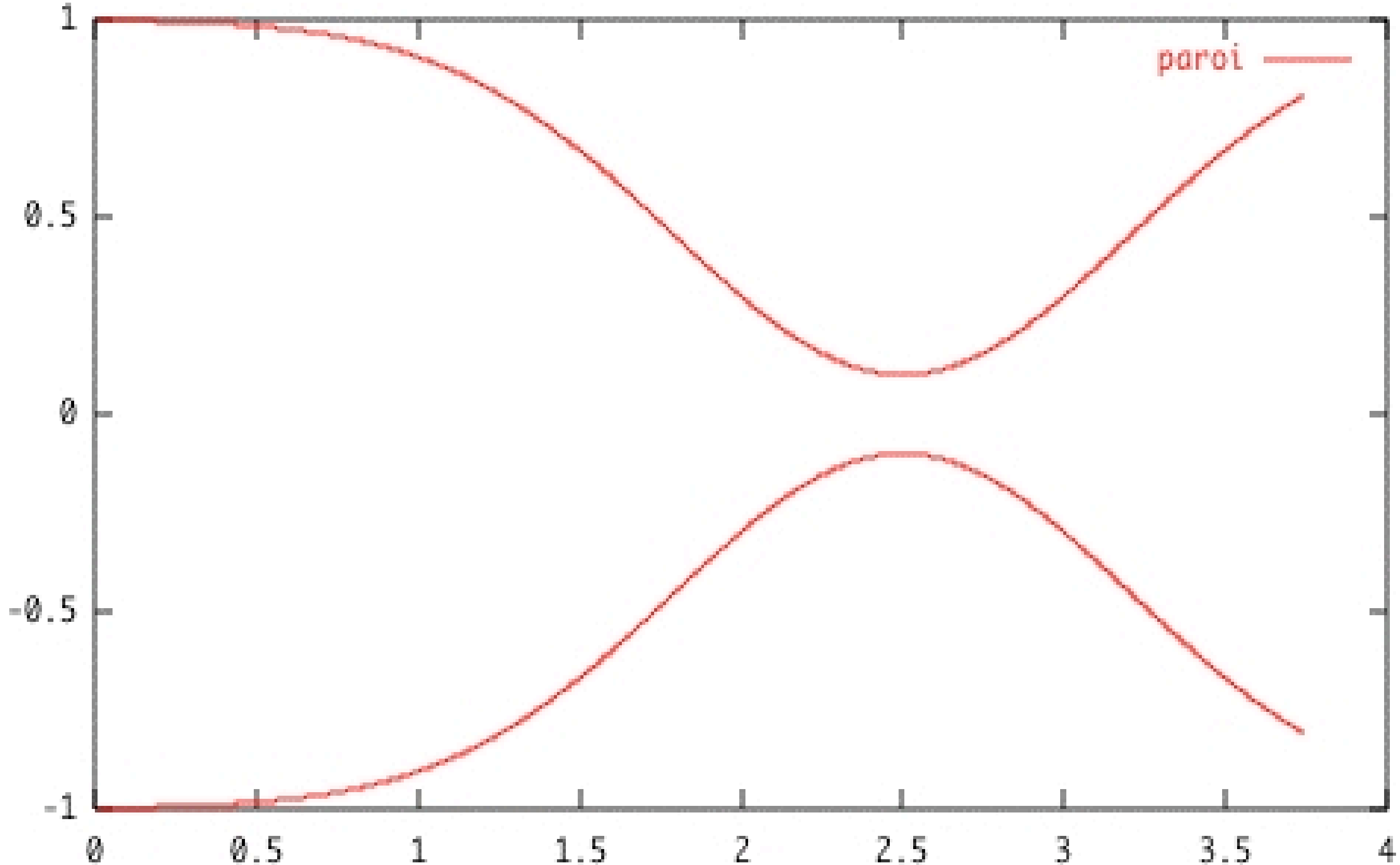
- adhérence à la paroi ($u = v = 0$ en $r = 1 - f(x)$),

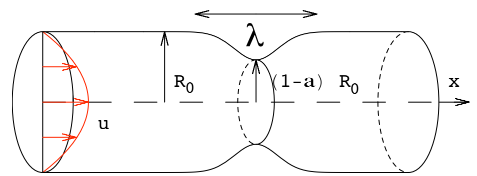
- profils d'entrée ($u(0, r)$ et $v(0, r)$) donnés

- *pas* de condition de sortie en $x_{out} = \frac{x^*_{out}}{R_0 Re}$

- streamwise marching (résolution en suivant l'écoulement) même lorsqu'il y a séparation .


λ
R0
(1-a) R0
x
u

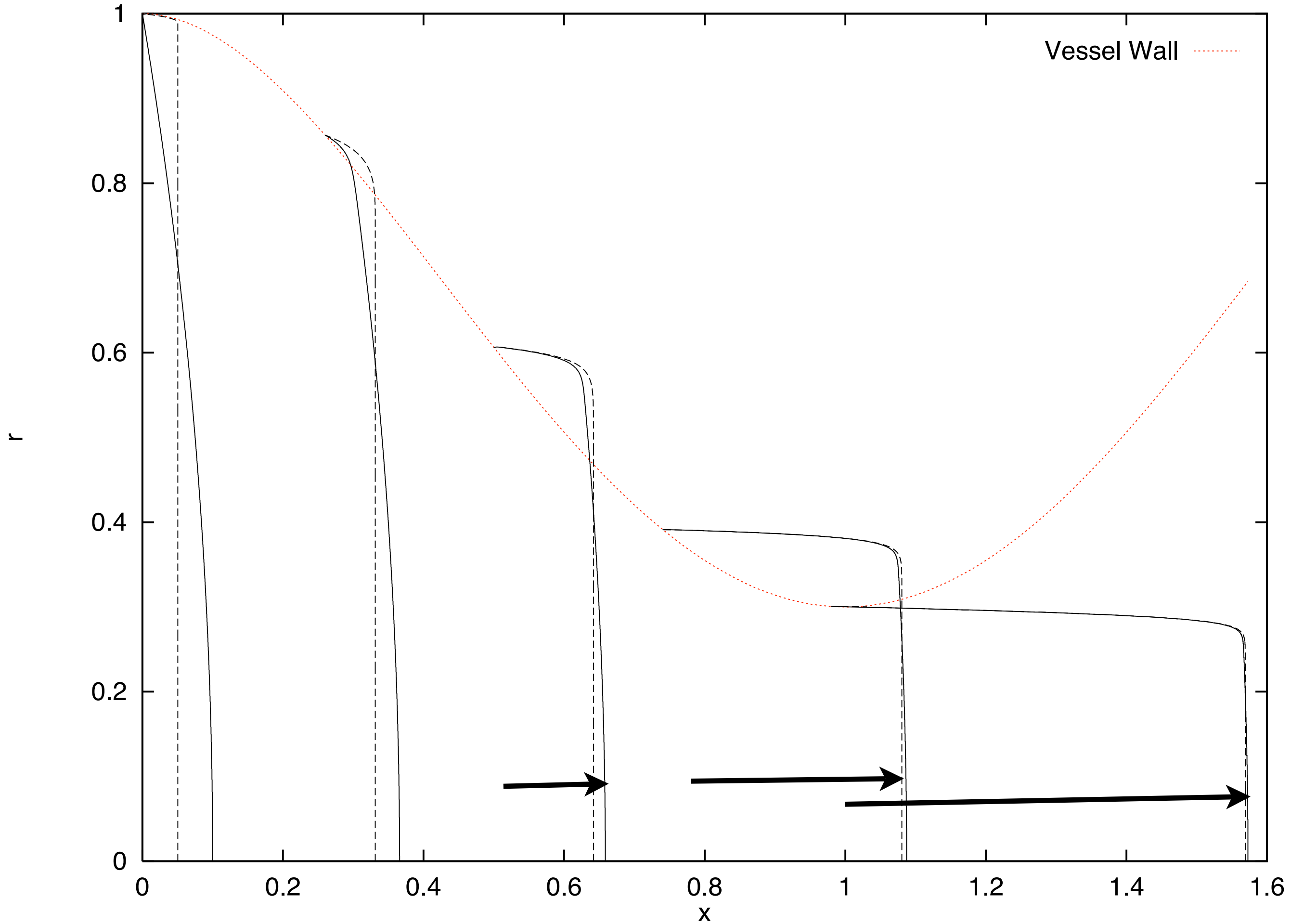

paroi
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
3
3.5
4

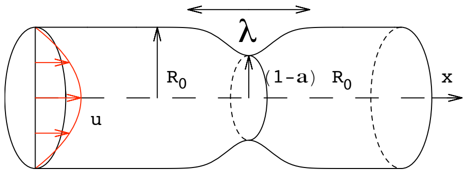


Evolution du profil de vitesse le long du convergent dans une sténose à 70% $(Re = 500)$;

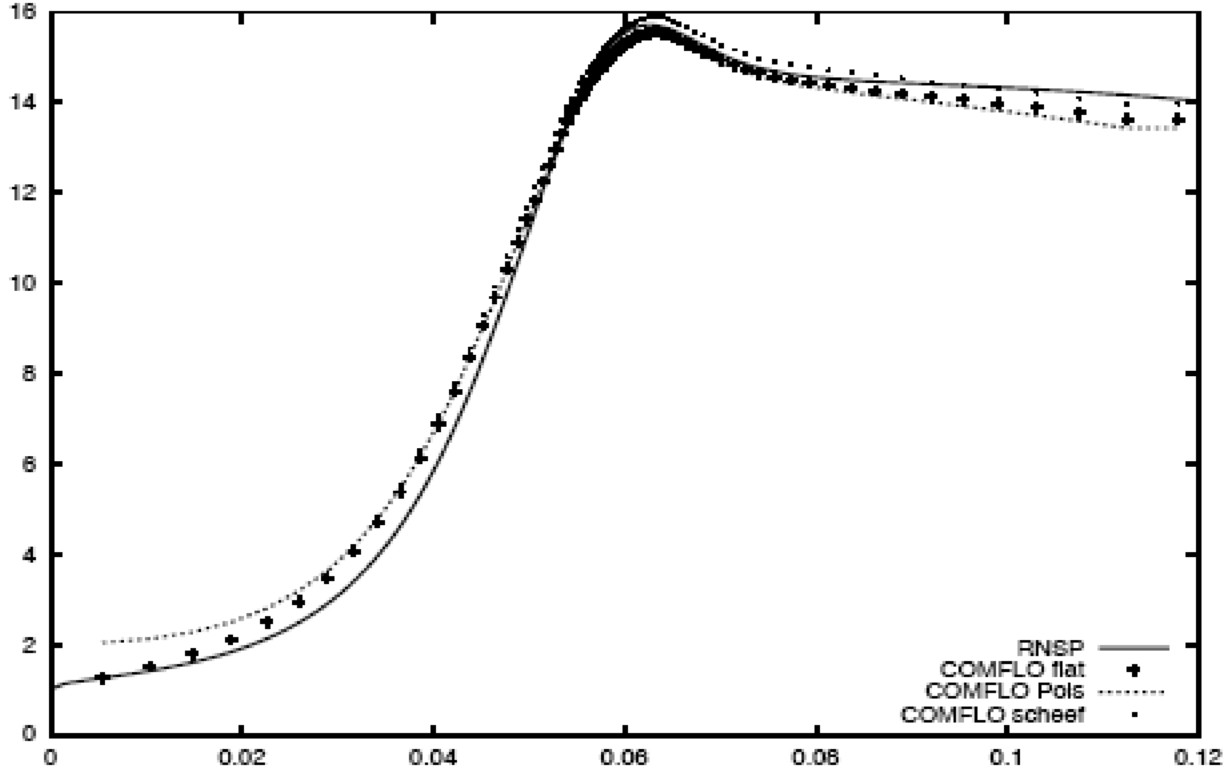
—— trait plein:
Poiseuille en entrée

- - - - trait pointillé:
profil plat en entrée





Testing asymmetry in the entry profile

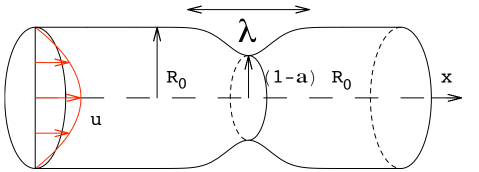


The velocities in the middle for Comflo and RNS.
Comflo uses here 50X50X100 points. Dimensionless scales!

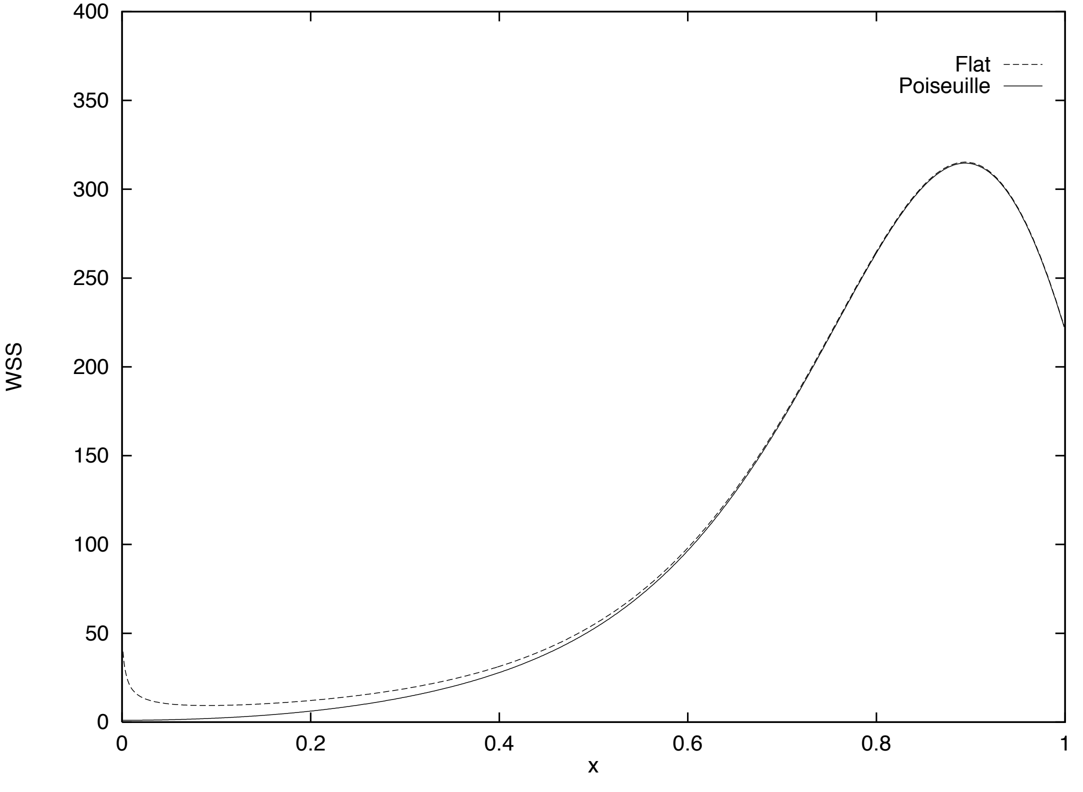
* Using COMFLO (Veldman, de Bruin R*u*G)

Q = 240 ml/min, $\nu = 0.03$, $R_0 = 0.2$cm, $R_{col} = 0.3*R_0$, L=5cm

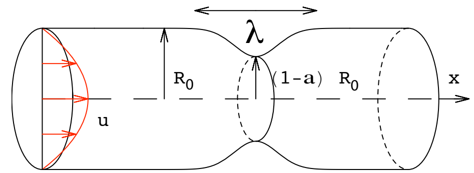
$U_0 = 31.8$cm/s, Re=212

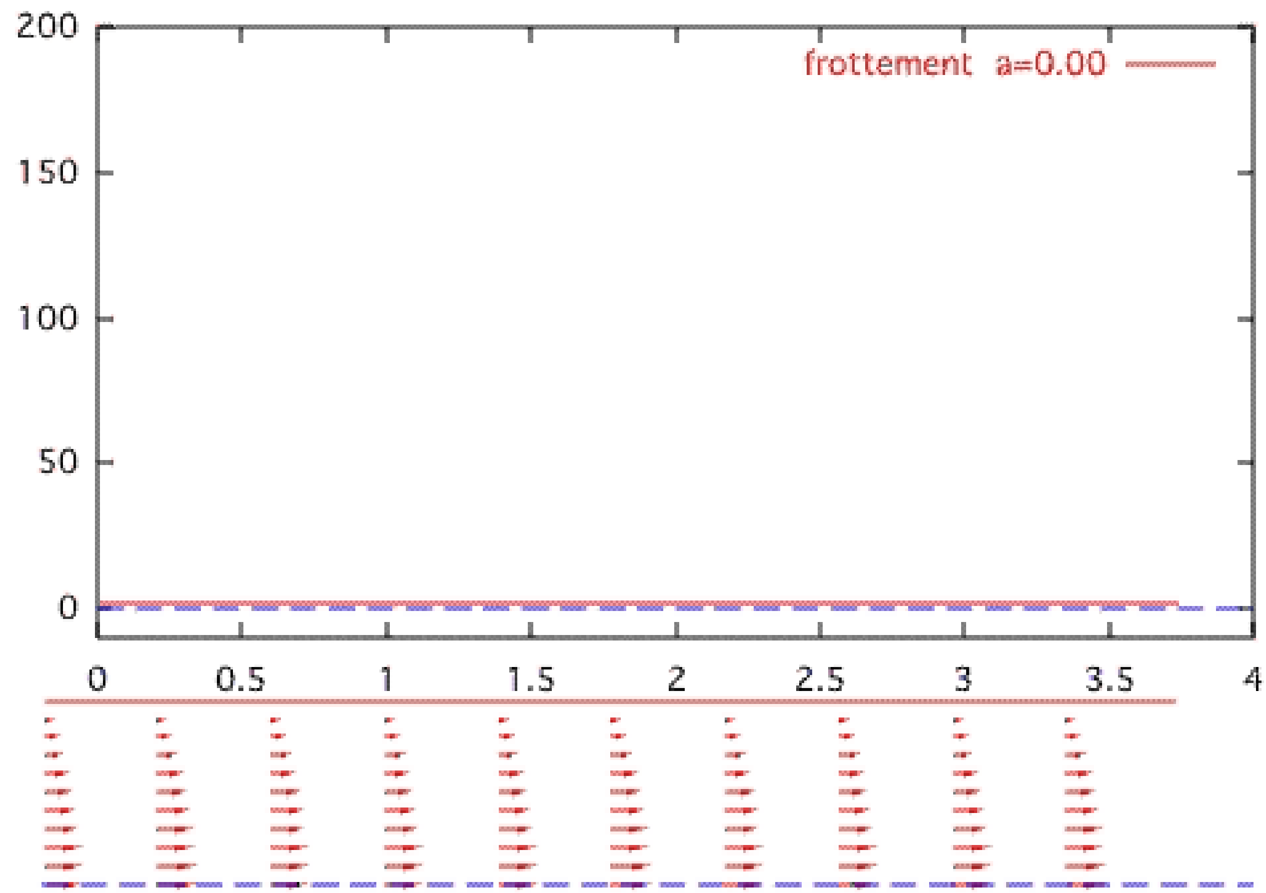


Frottement pariétal – Wall Shear Stress

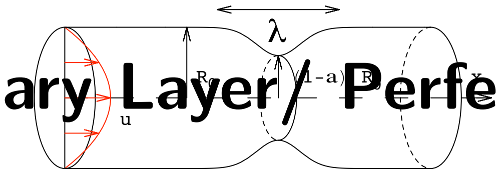


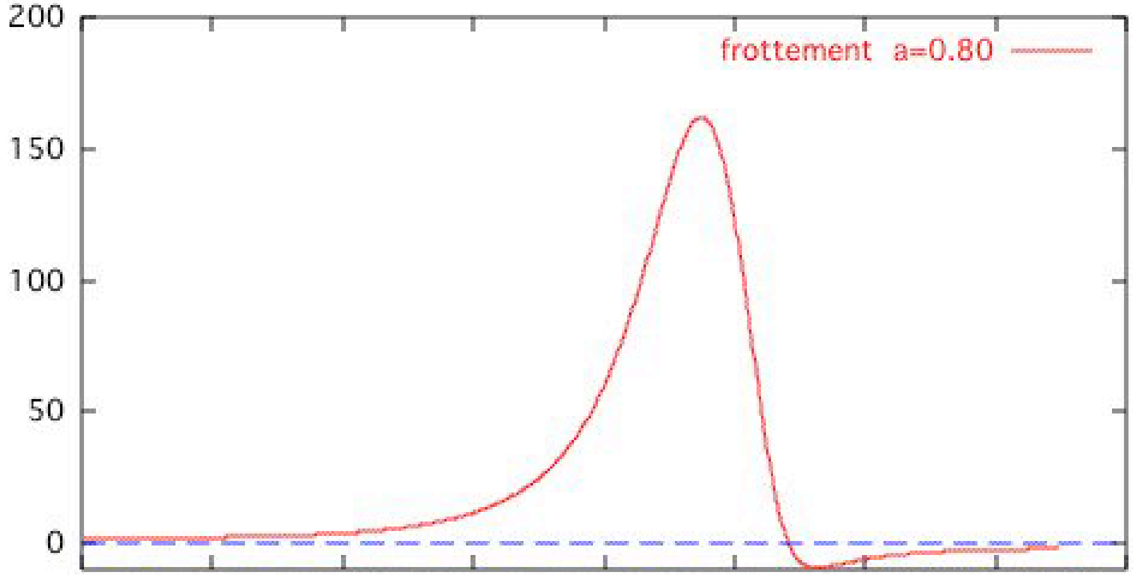
Evolution de la distribution de WSS le long du convergent (sténose de 70% $Re = 500$);
—— trait plein: Poiseuille en entrée, - - - - trait pointillé: profil plat

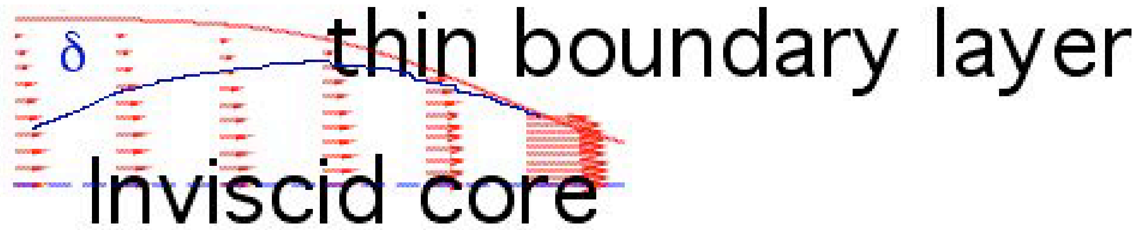

λ
R0
(1-a) R0
x
u

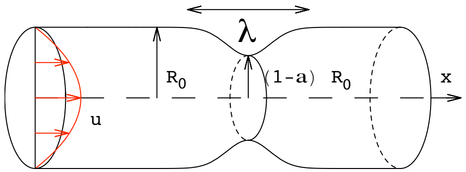

200
150
100
50
0
frottement a=0.00
0
0.5
1
1.5
2
2.5
3
3.5
4

Boundary Layer / Perfect Fluid

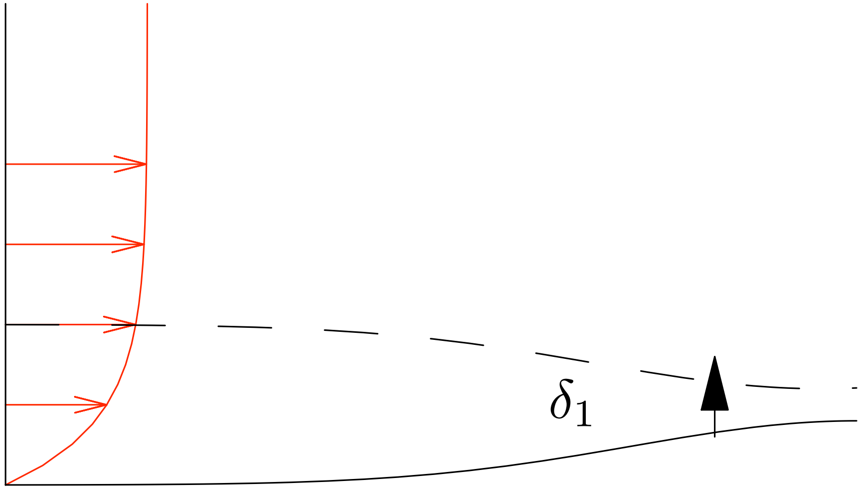




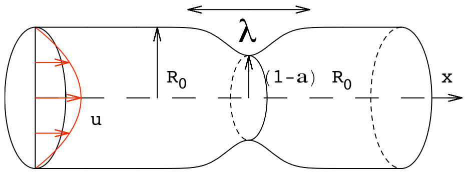




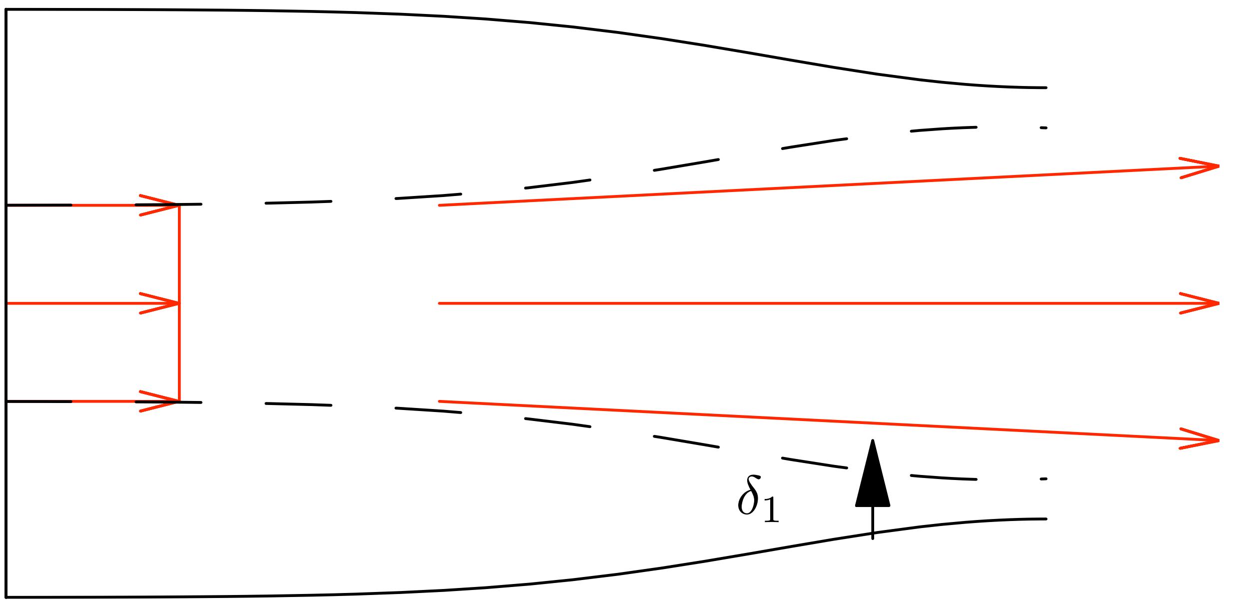
Boundary Layer/ Perfect Fluid



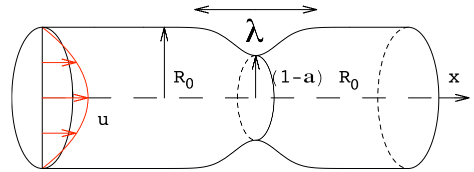
La Couche Limite est générée près de la paroi
δ_1 l'épaisseur de déplacement.



Couche Limite / Fluide Parfait



δ_1 l'épaisseur de déplacement se comporte comme une nouvelle paroi!!!!
$\rightarrow$ Interacting Boundary Layer (IBL) [Couche limite Interactive]



RNSP/ IBL

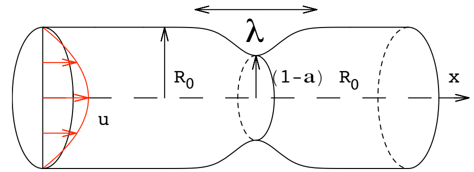
Après adimensionnement:
$r = R(\bar{x}) - (\lambda/Re)^{-1/2}\bar{y}$, $u = \bar{u}$, $v = (\lambda/Re)^{1/2}\bar{v}$ et $x - x_b = (\lambda/Re)\bar{x}$, $p = \bar{p}$, où x_b est la position de la sténose, les équations RNSP(x) donnent le problème final IBL (interacting Boundary Layer):

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{n}} = 0$$

$$(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{n}}) = \bar{u}_e\frac{d\bar{u}_e}{d\bar{s}} + \frac{\partial}{\partial \bar{n}}\frac{\partial \bar{u}}{\partial \bar{n}}$$

avec: $\bar{u}(\bar{x},0) = 0$, $\bar{v}(\bar{x},0) = 0$ $\bar{u}(\bar{x},\infty) = u_e$, où $\bar{\delta}_1 = \int_0^\infty (1 - \frac{\bar{u}}{\bar{u}_e})d\bar{n}$, et

$$\bar{u}_e = \frac{1}{(R^2 - 2((\lambda/Re)^{-1/2})\bar{\delta}_1)}.$$



IBL intégral: équation 1D

$$\frac{d}{d\bar{x}}(\frac{\bar{\delta}_1}{H}) = \bar{\delta}_1(1+\frac{2}{H})\frac{d\bar{u}_e}{d\bar{x}} + \frac{f_2 H}{\bar{\delta}_1 \bar{u}_e},$$
$$\bar{u}_e = \frac{1}{(R^2 - 2(\lambda/Re)^{-1/2}\bar{\delta}_1)}.$$

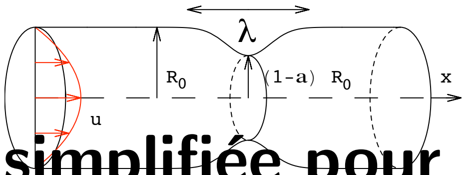
Pour résoudre ce problème une relation de fermeture liant H et f_2 à la vitesse et à l'épaisseur de déplacement doit être trouvée:

On définit $\Lambda_1 = \bar{\delta}_1^2 \frac{d\bar{u}_e}{d\bar{x}}$,

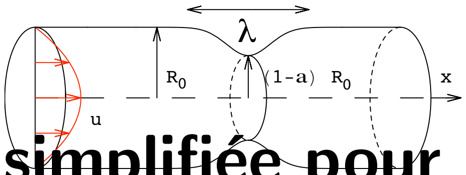
le système est fermé par la résolution des profils de Falkner Skan:

si $\Lambda_1 < 0.6$ alors $H = 2.5905 exp(-0.37098\Lambda_1)$, sinon $H = 2.074$.

De H,f_2 est calculé par $f_2 = 1.05(-H^{-1} + 4H^{-2})$.
exemples de $\underline{profils}$

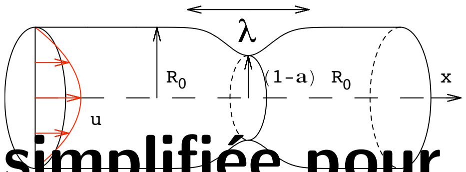


IBL intégral: Equation simplifiée pour le frottement (Shear Stress)



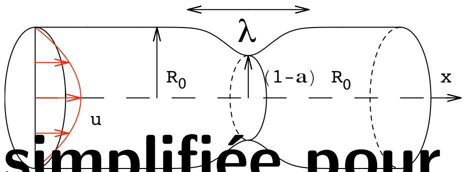
IBL intégral: Equation simplifiée pour le frottement (Shear Stress)

- variation de la vitesse (conservation du flux)



IBL intégral: Equation simplifiée pour le frottement (Shear Stress)

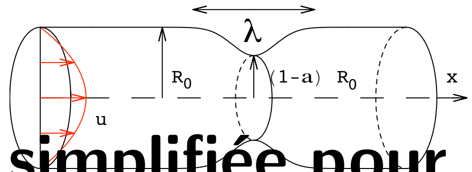
- variation de la vitesse (conservation du flux) $U_0 \rightarrow U_0/(1-\alpha-\delta_1)^2$



IBL intégral: Equation simplifiée pour le frottement (Shear Stress)

- variation de la vitesse (conservation du flux) $U_0 \rightarrow U_0/(1-\alpha-\delta_1)^2$

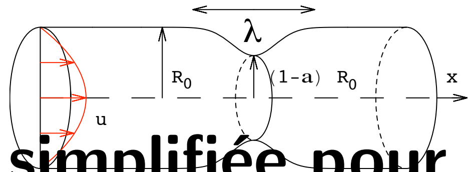
- accélération: couche limite $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$,



IBL intégral: Equation simplifiée pour le frottement (Shear Stress)

- variation de la vitesse (conservation du flux) $U_0 \rightarrow U_0/(1-\alpha-\delta_1)^2$

- accélération: couche limite $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$, avec $Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2\nu} = \frac{Re\lambda}{(1-\alpha)^2}$

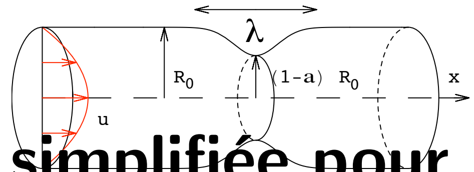


IBL intégral: Equation simplifiée pour le frottement (Shear Stress)

- variation de la vitesse (conservation du flux) $U_0 \to U_0/(1-\alpha-\delta_1)^2$

- accélération: couche limite $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$, avec $Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2\nu} = \frac{Re\lambda}{(1-\alpha)^2}$

- WSS = (variation de vitesse)/(épaisseur de couche limite)

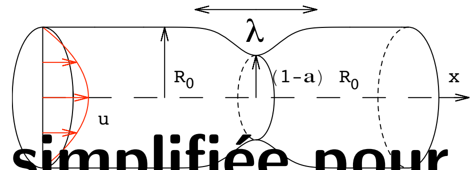


IBL intégral: Equation simplifiée pour le frottement (Shear Stress)

- variation de la vitesse (conservation du flux) $U_0 \rightarrow U_0/(1-\alpha-\delta_1)^2$

- accélération: couche limite $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$, avec $Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2\nu} = \frac{Re\lambda}{(1-\alpha)^2}$

- WSS $=$ (variation de vitesse)/(épaisseur de couche limite) $= \frac{(Re/\lambda)^{1/2}}{(1-\alpha)^3}$



IBL intégral: Equation simplifiée pour le frottement (Shear Stress)

- variation de la vitesse (conservation du flux) $U_0 \rightarrow U_0/(1-\alpha-\delta_1)^2$

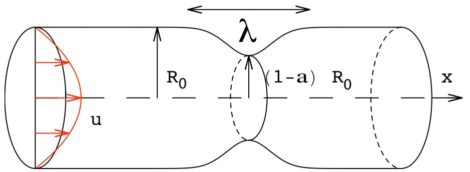
- accélération: couche limite $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$, avec $Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2\nu} = \frac{Re\lambda}{(1-\alpha)^2}$

- WSS = (variation de vitesse)/(épaisseur de couche limite) $= \frac{(Re/\lambda)^{1/2}}{(1-\alpha)^3}$

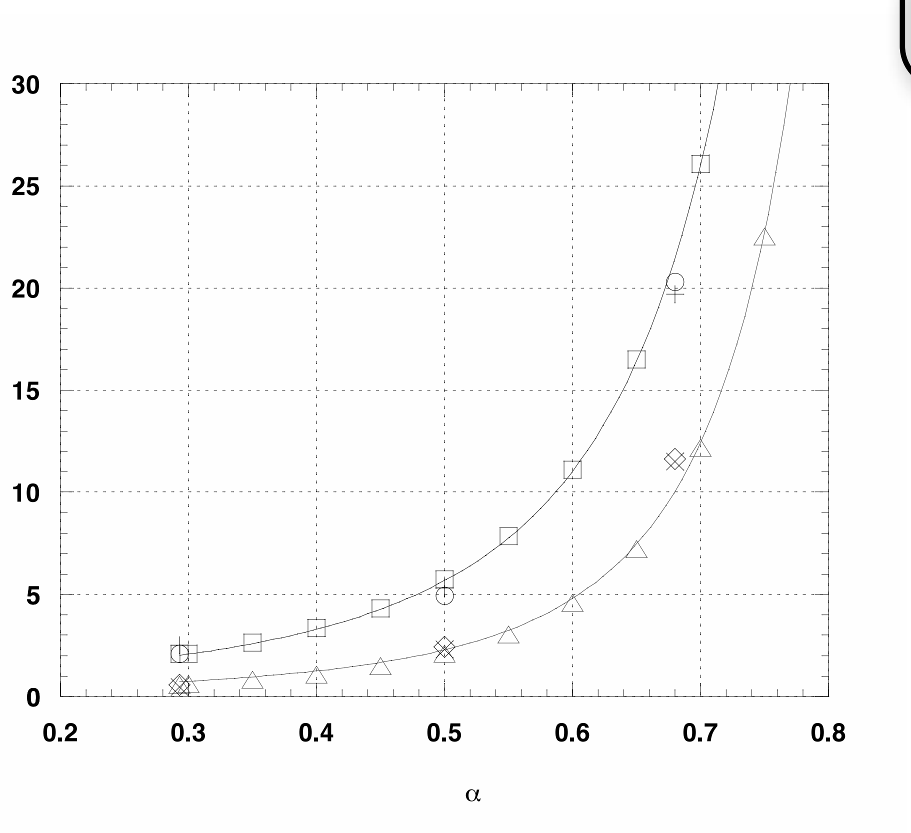
Une formule simple peut être déduite:

$$WSS = (\mu\frac{\partial u^*}{\partial y^*})/((\mu\frac{4U_0}{R})) \sim .22\frac{((Re/\lambda)^{1/2}+3)}{(1-\alpha)^3}$$

Le nombre de Reynolds pertinent n'est plus $Re = U_0R_0/\nu$ mais $Re\lambda$ et $(Re/\lambda)^{1/2}$ est l'inverse de l'épaisseur relative de couche limite



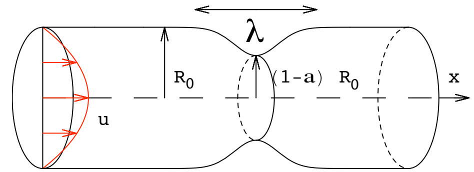
IBL intégral: Comparaison avec Navier Stokes (Siegel et al. 1994)

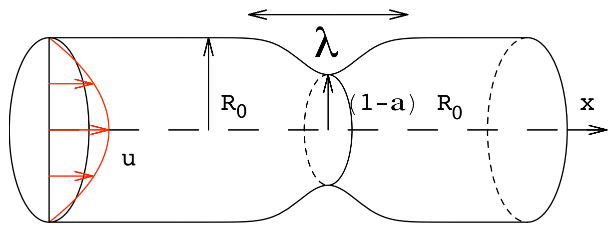


$$WSS = aRe^{1/2} + b$$

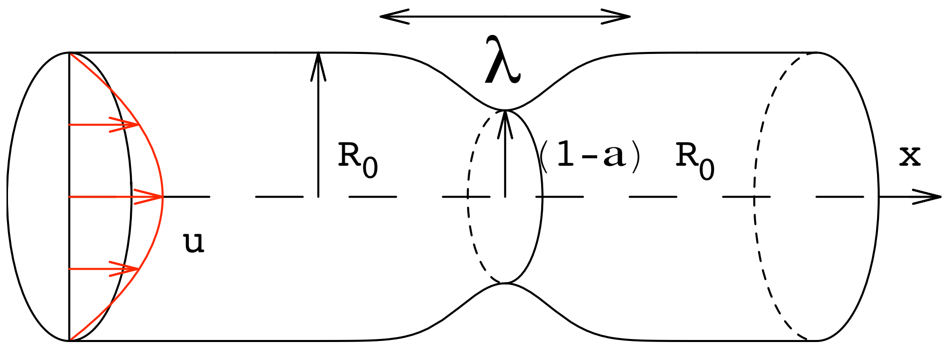
Coefficient a et b du maximum de WSS.
lignes avec triangle $\triangle$ et "carré" : coefficient a et b obtenus en utilisant la méthode IBL;

$\diamond$: coefficient a obtenu par Siegel pour $\lambda = 3$;
$\times$: coefficient a obtenu par Siegel pour $\lambda = 6$;
$\bigcirc$: coefficient b obtenu par Siegel pour $\lambda = 3$;
$+$: coefficient b obtenu par Siegel pour $\lambda = 6$.

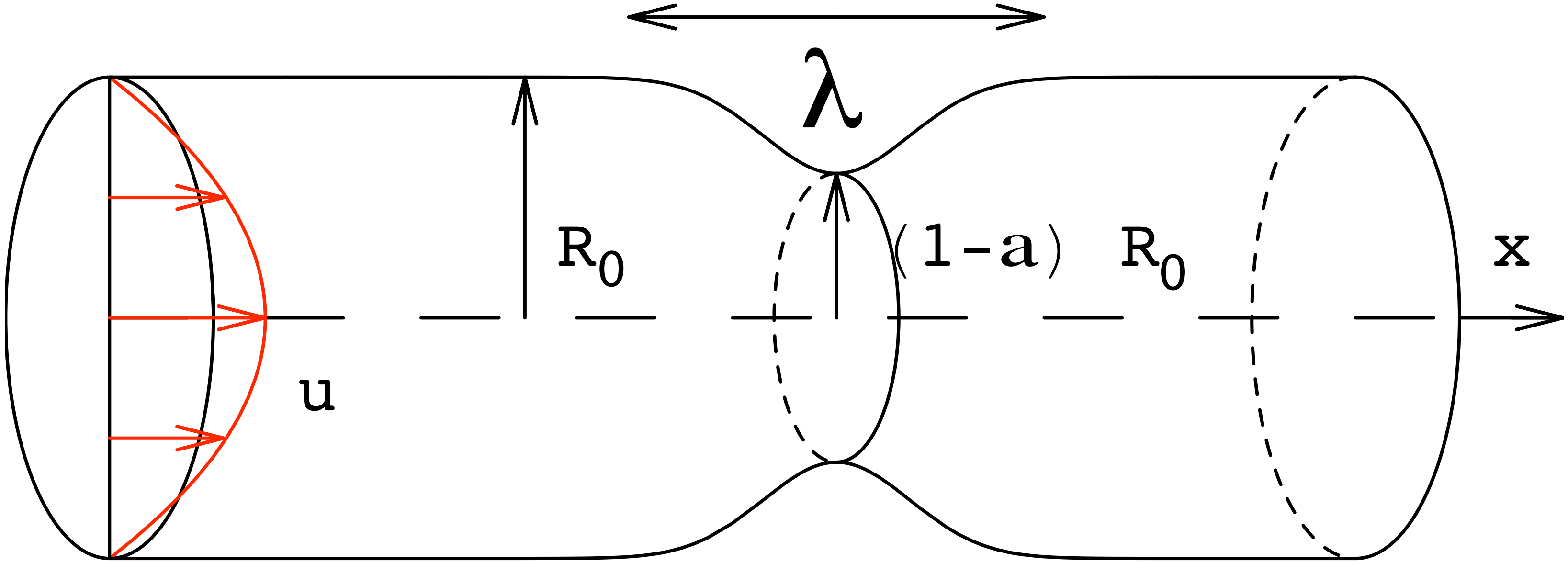

λ
R0
(1-a) R0
x
u



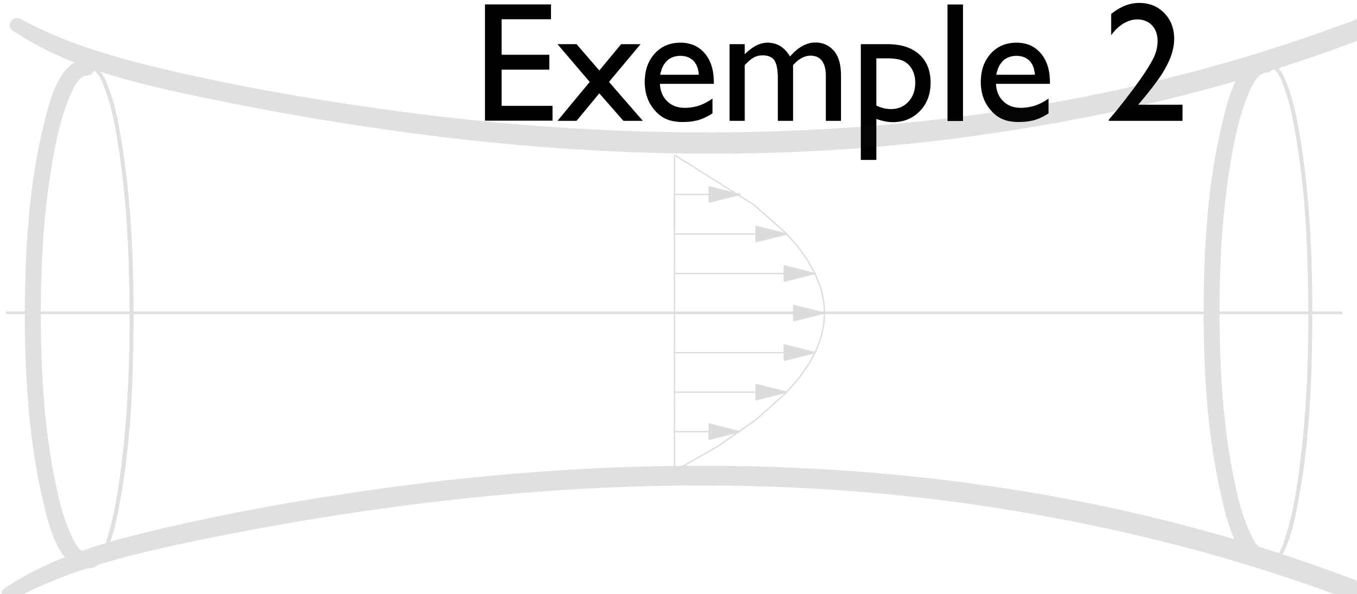
λ
R0
(1-a) R0
x
u



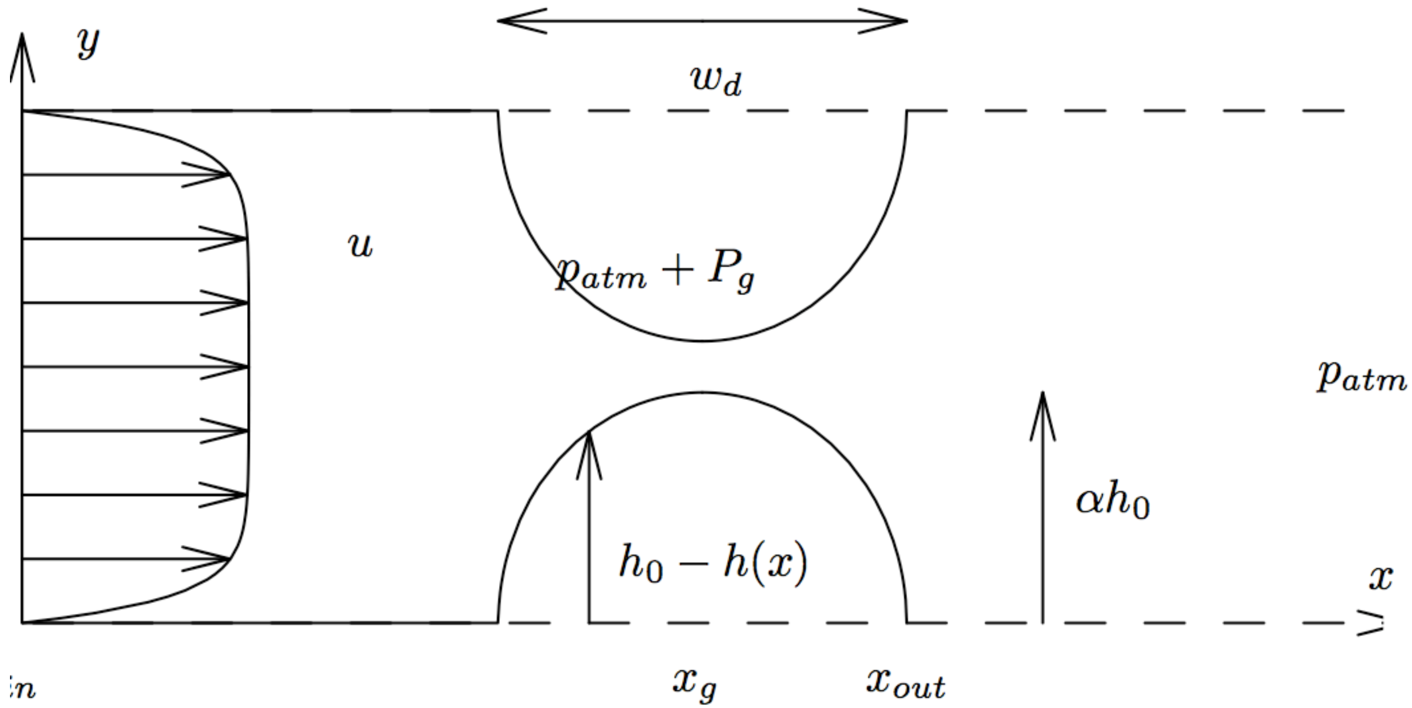
λ
R0
(1-a) R0
x
u


λ
R0
(1-a) R0
x
u

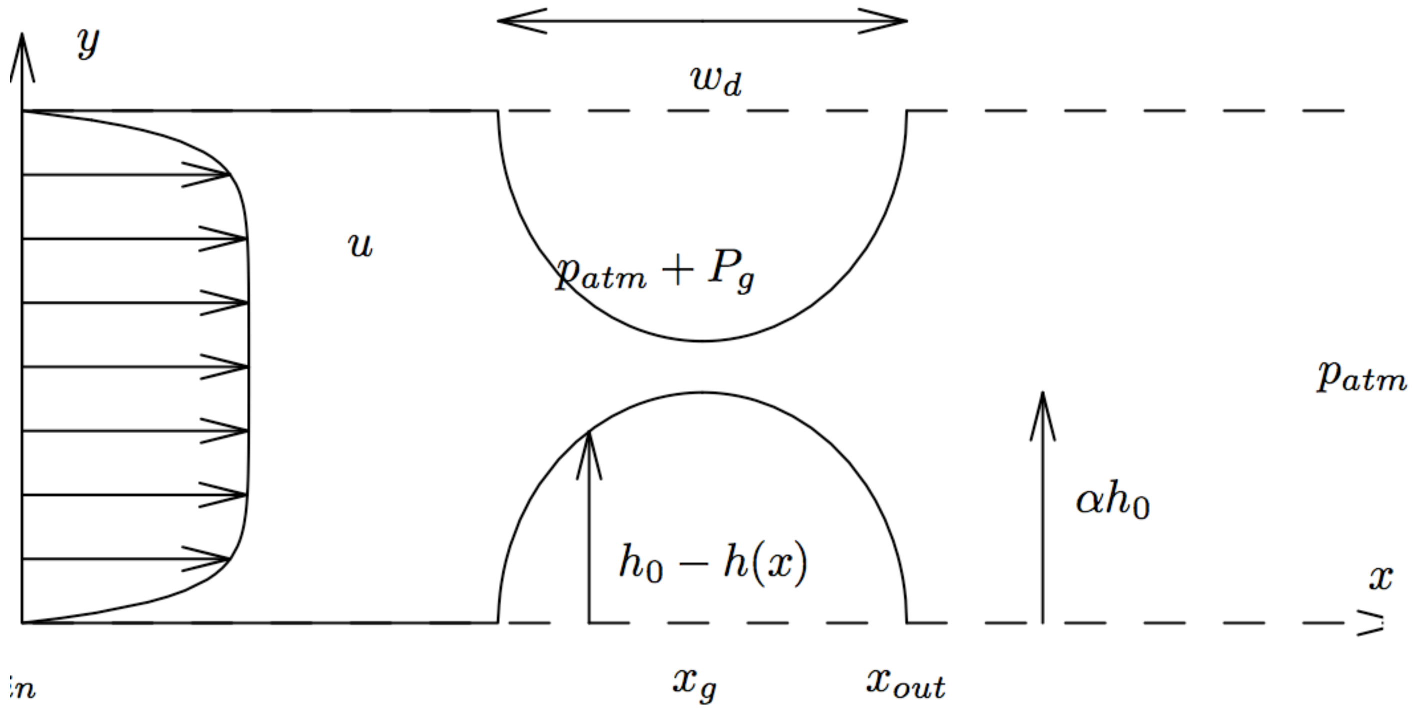
Exemple 2



- écoulement dans un tuyau
- stationnaire, parois rigides



- écoulement dans un tuyau sténosé
- stationnaire, parois rigides

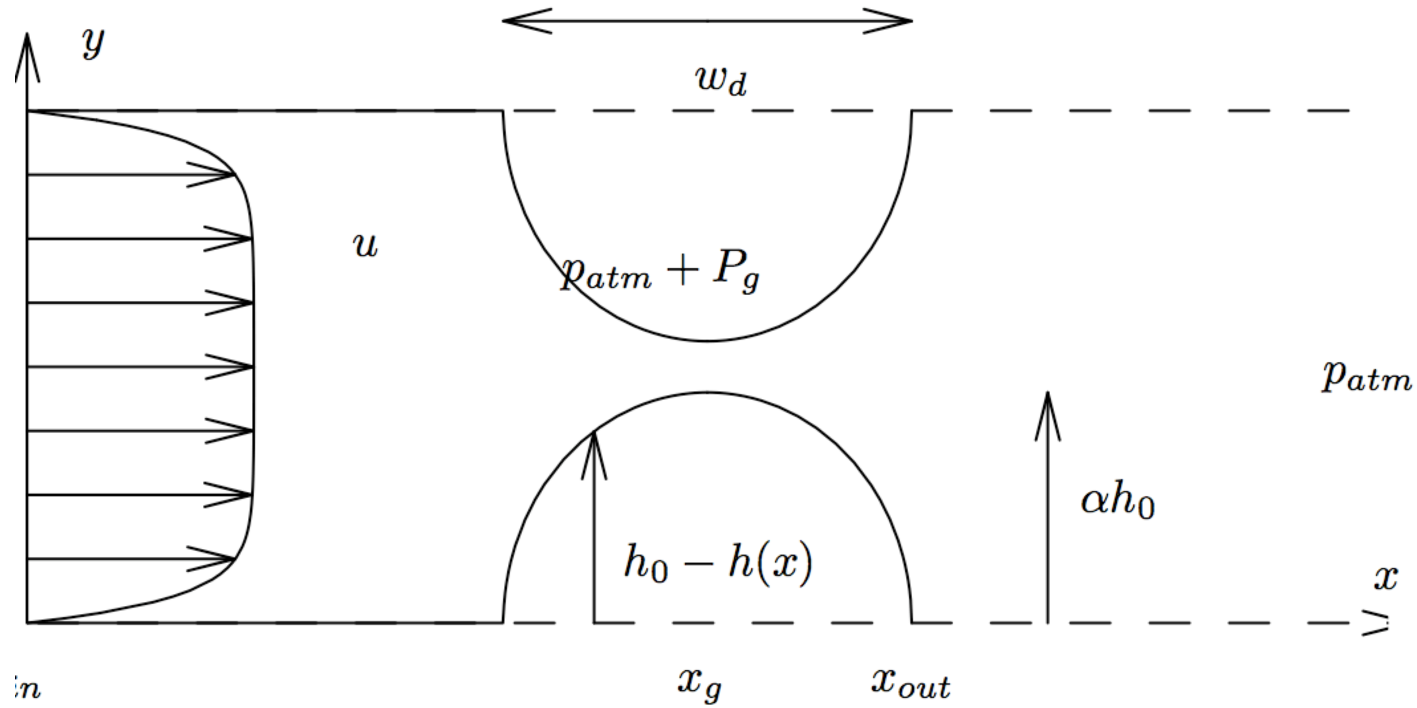


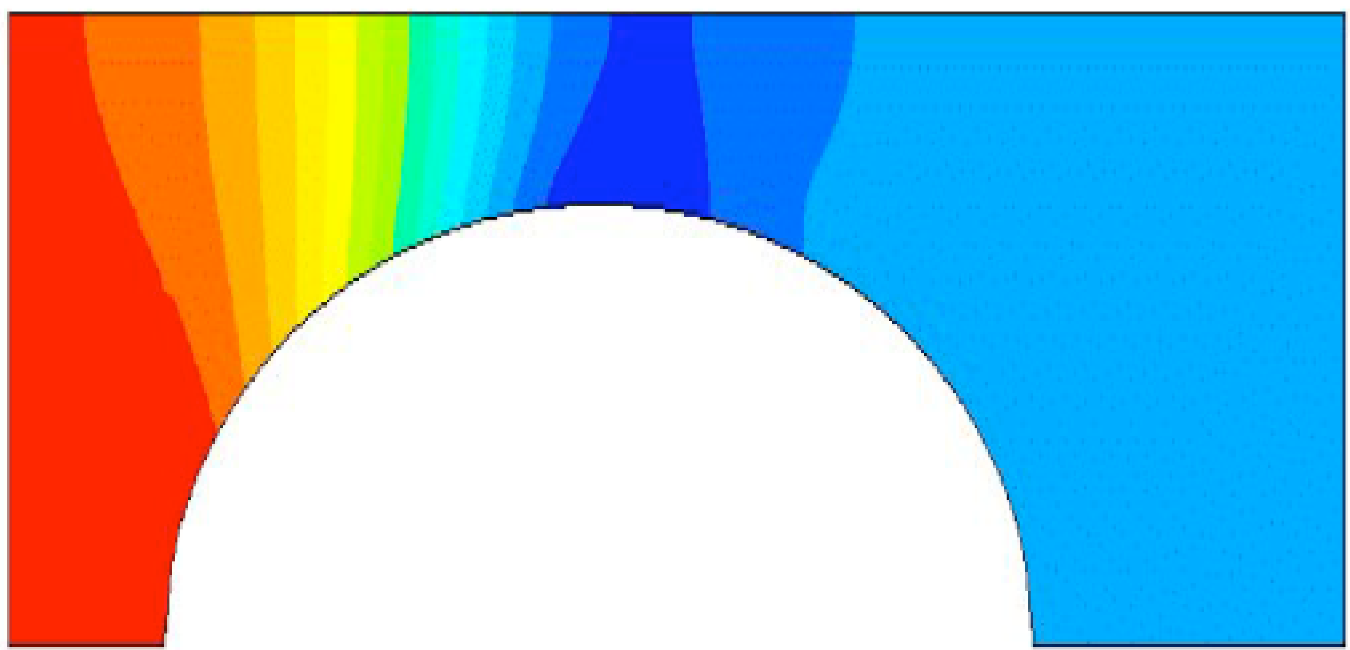
$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

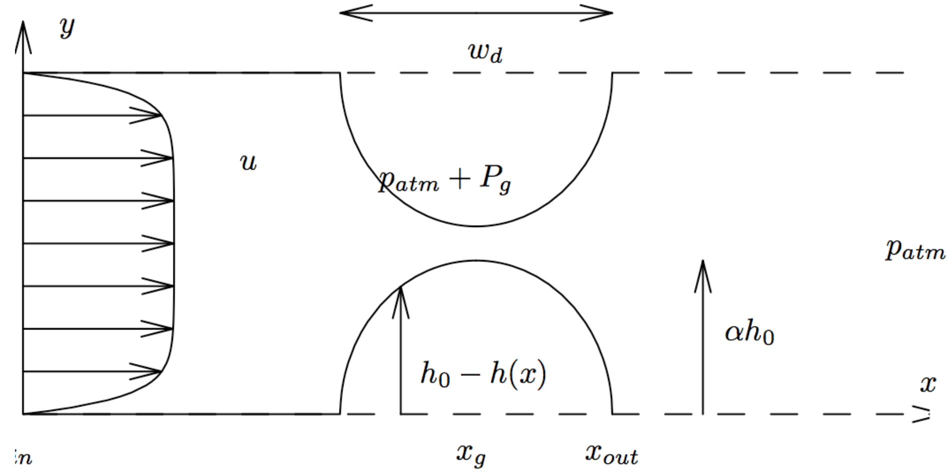
$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}p + \frac{\partial^2}{\partial y^2}u$$

$$0 = -\frac{\partial}{\partial y}p$$

RNSP sans dimension


y
w_d
u
$p_{atm} + P_g$
p_{atm}
αh_0
$h_0 - h(x)$
x
x_g
x_{out}





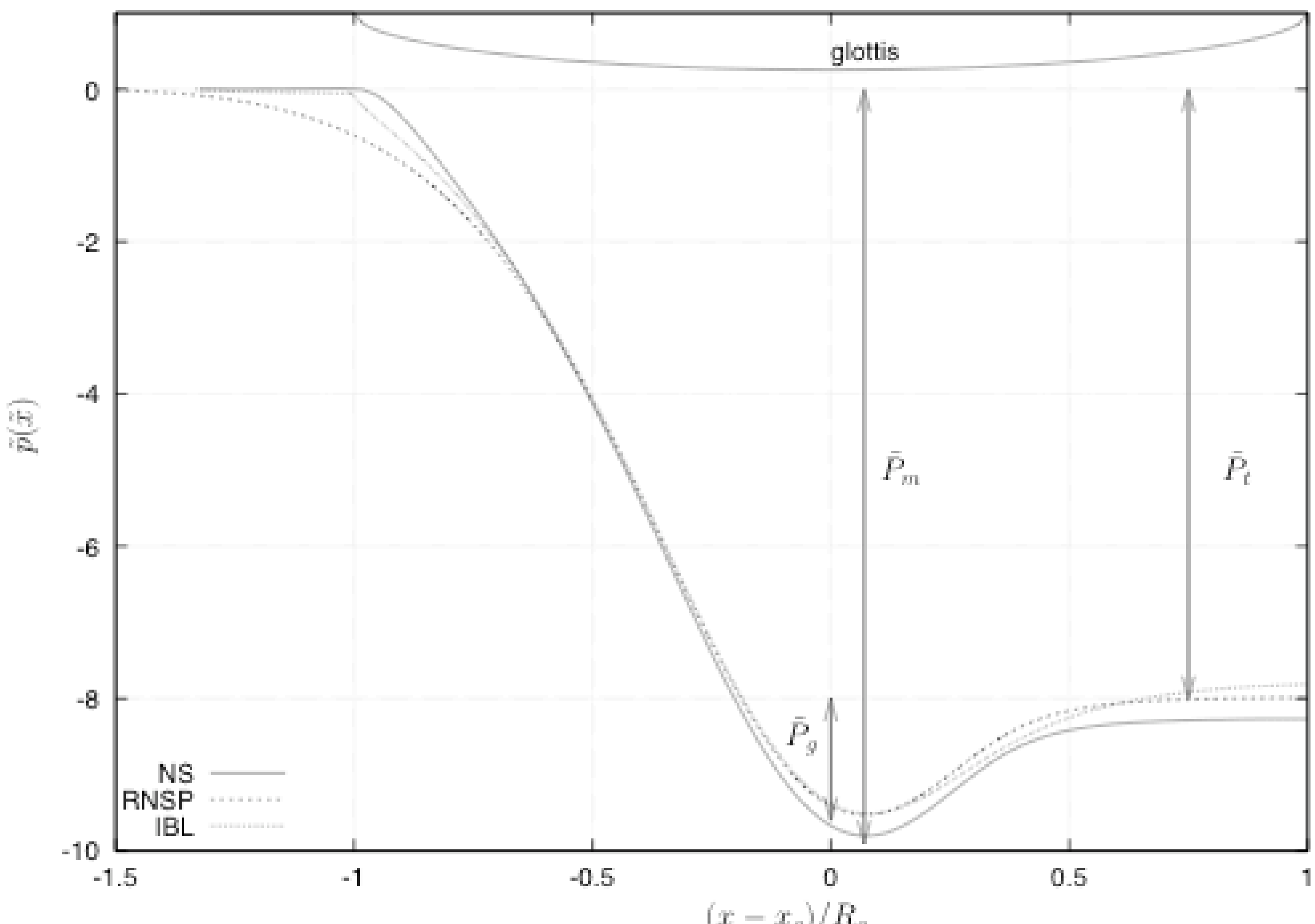
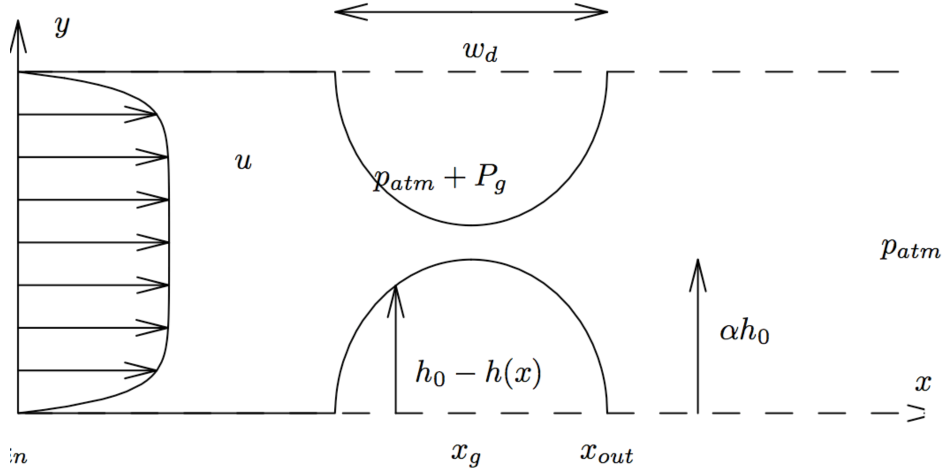
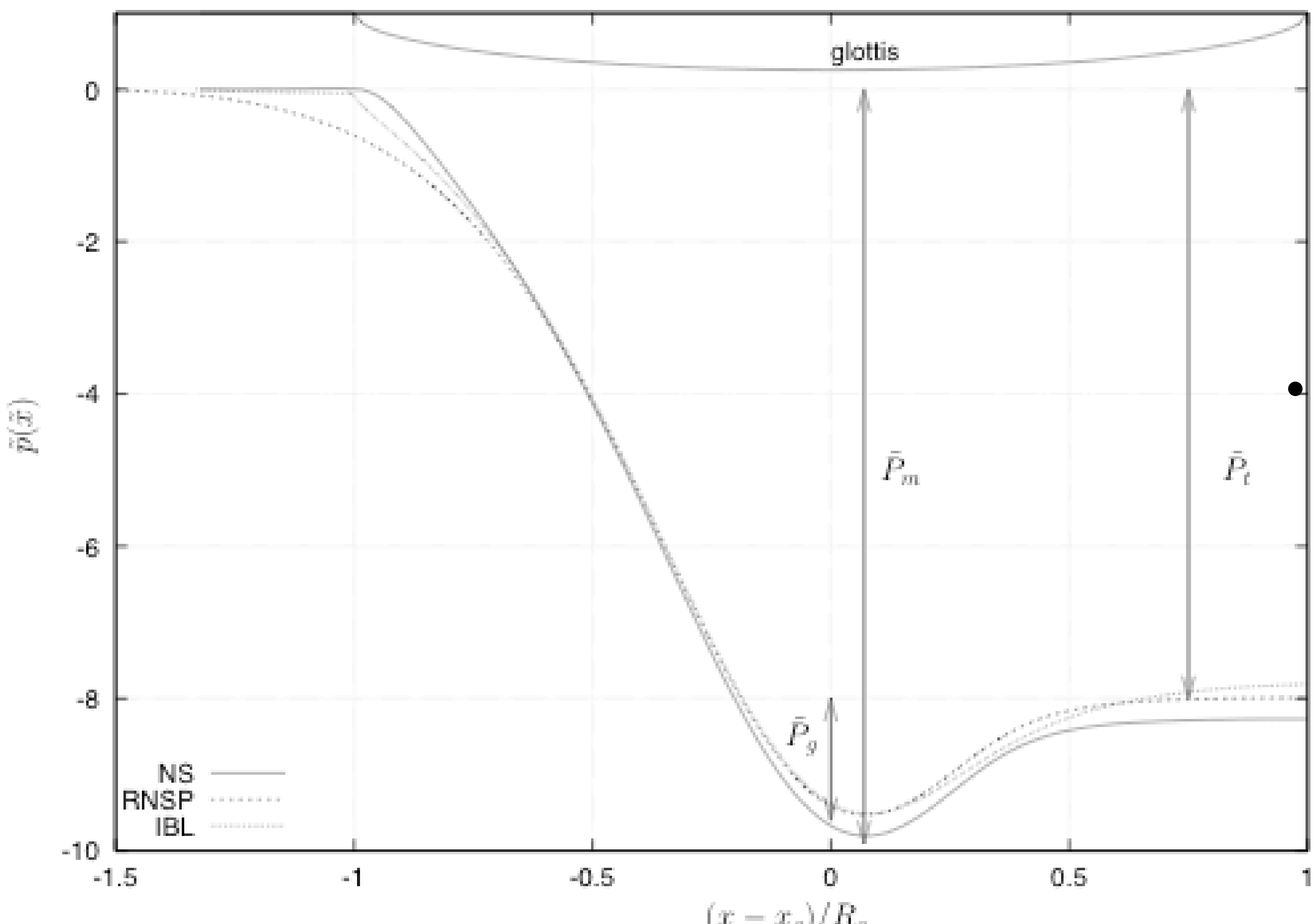


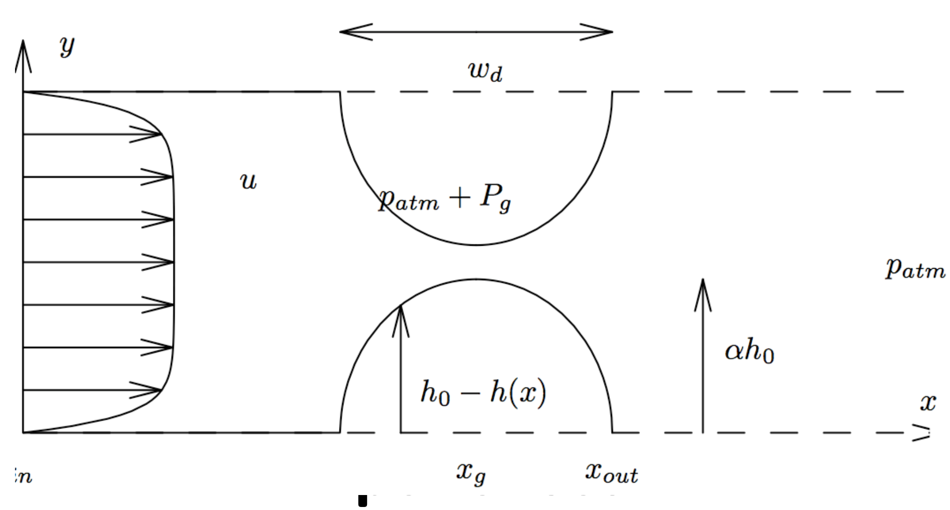
Fig. 2 A comparison between computed non-dimensional pressure for the three models (NS, IBL and RNSP, in this last case the wall has



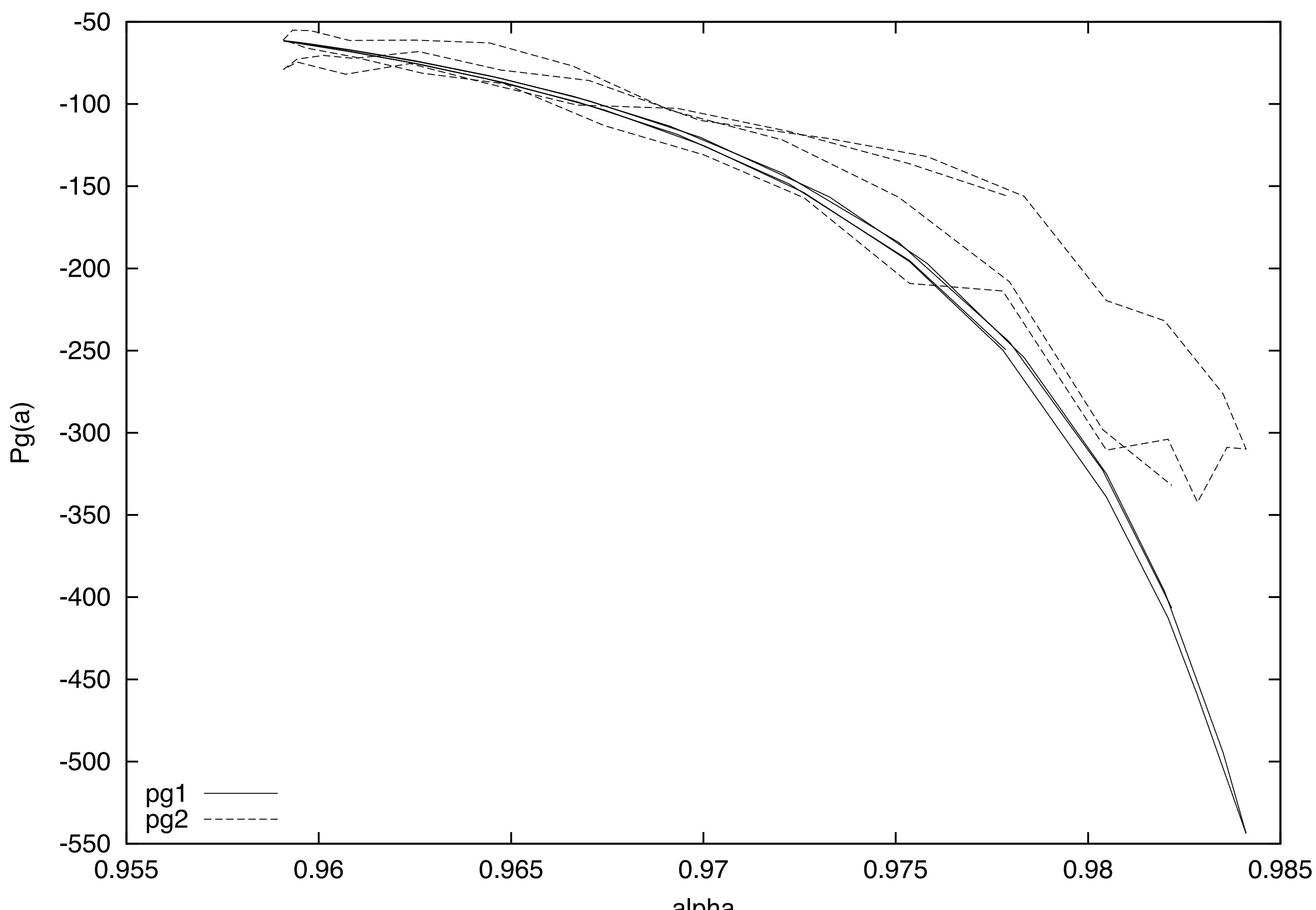


- "distribution" de pression quasi invariante:
  $K_e = P_t/P_m$ à peu près constant $K_e \simeq 0.82$
  $K_g = P_g/P_m$ à peu près constant $K_g \simeq 0.97$

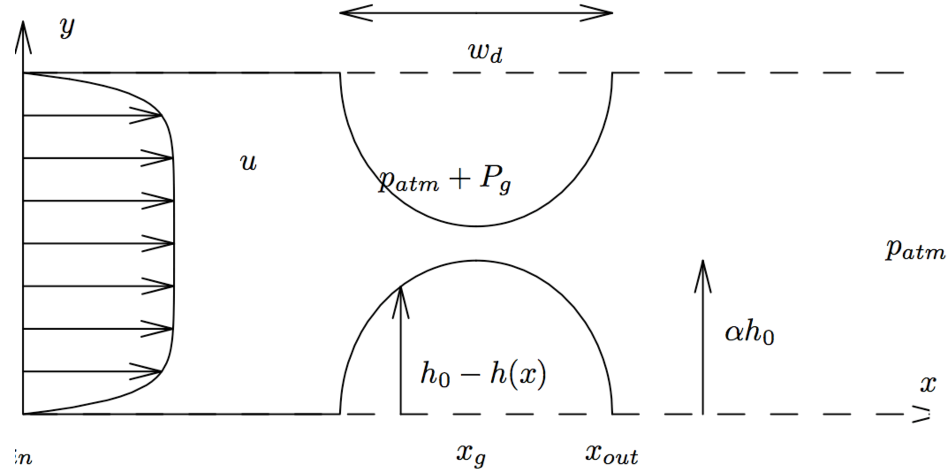
Fig. 2 A comparison between computed non-dimensional pressure for the three models (NS, IBL and RNSP, in this last case the wall has



y
w_d
u
$p_{atm} + P_g$
p_{atm}
αh_0
$h_0 - h(x)$
x
x_g
x_{out}



Pg(a)
alpha
-50
-100
-150
-200
-250
-300
-350
-400
-450
-500
-550
0.955
0.96
0.965
0.97
0.975
0.98
0.985
pg1
pg2



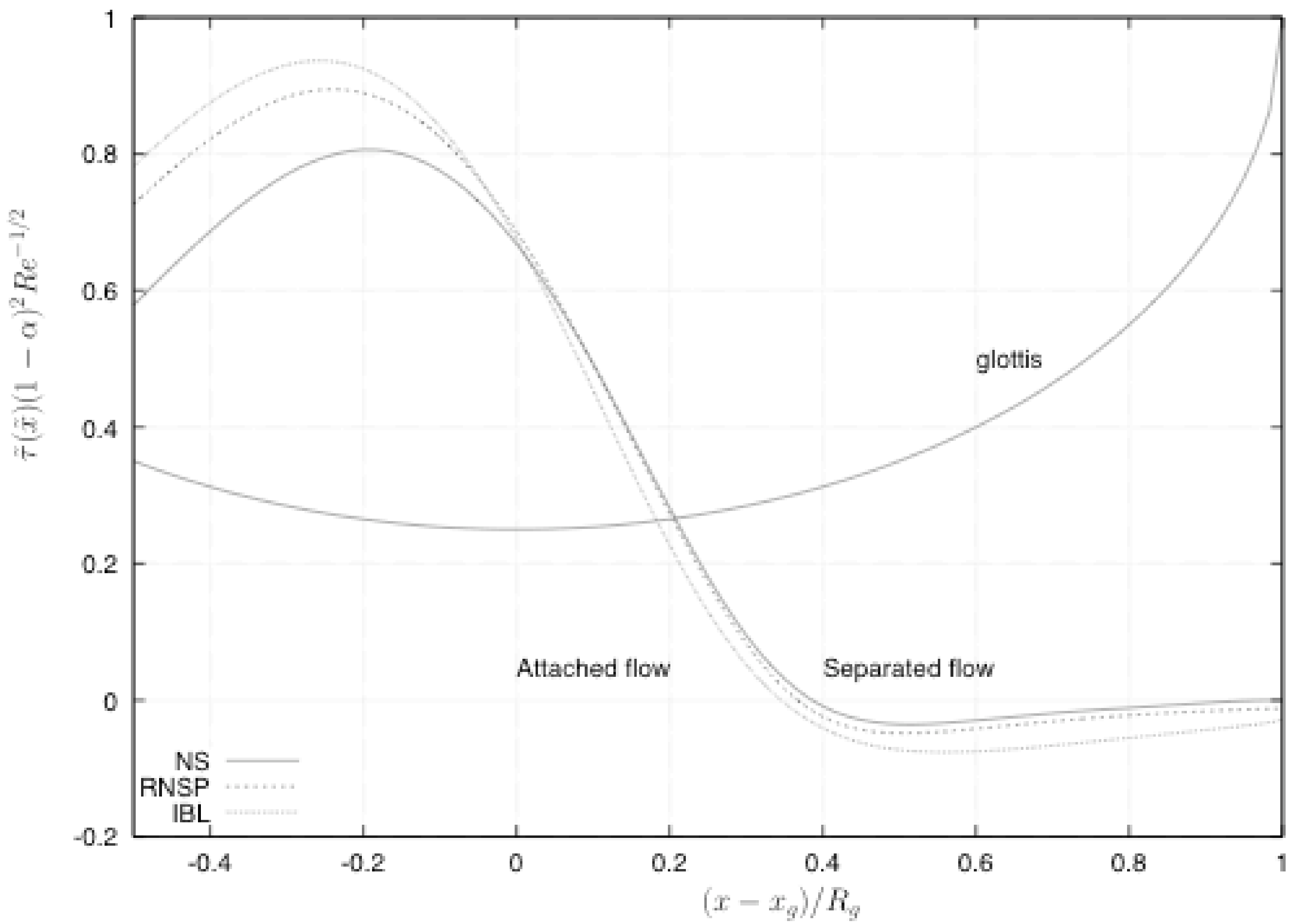
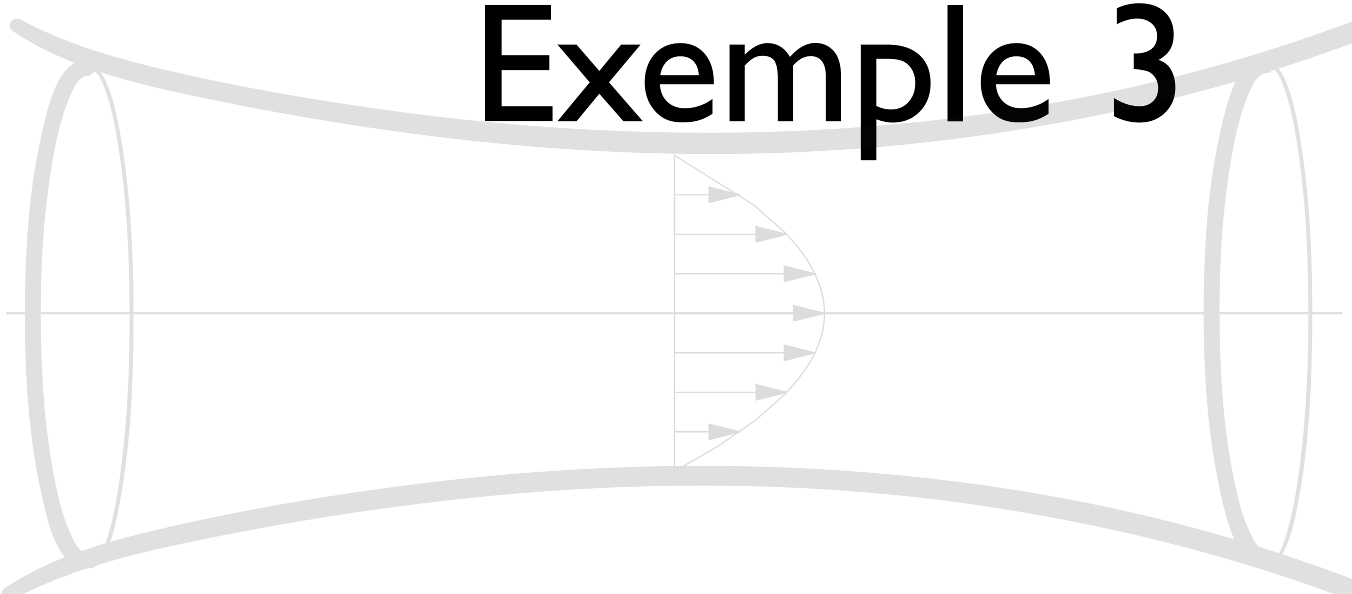


Fig. 4. A comparison between computed skin friction divided by $(0.47 + 2.07)(1-\alpha)^{-1}/\bar{\delta}_{...} \propto (1-\alpha)^{-2} Re^{1/2}$ for the three models

Exemple 3



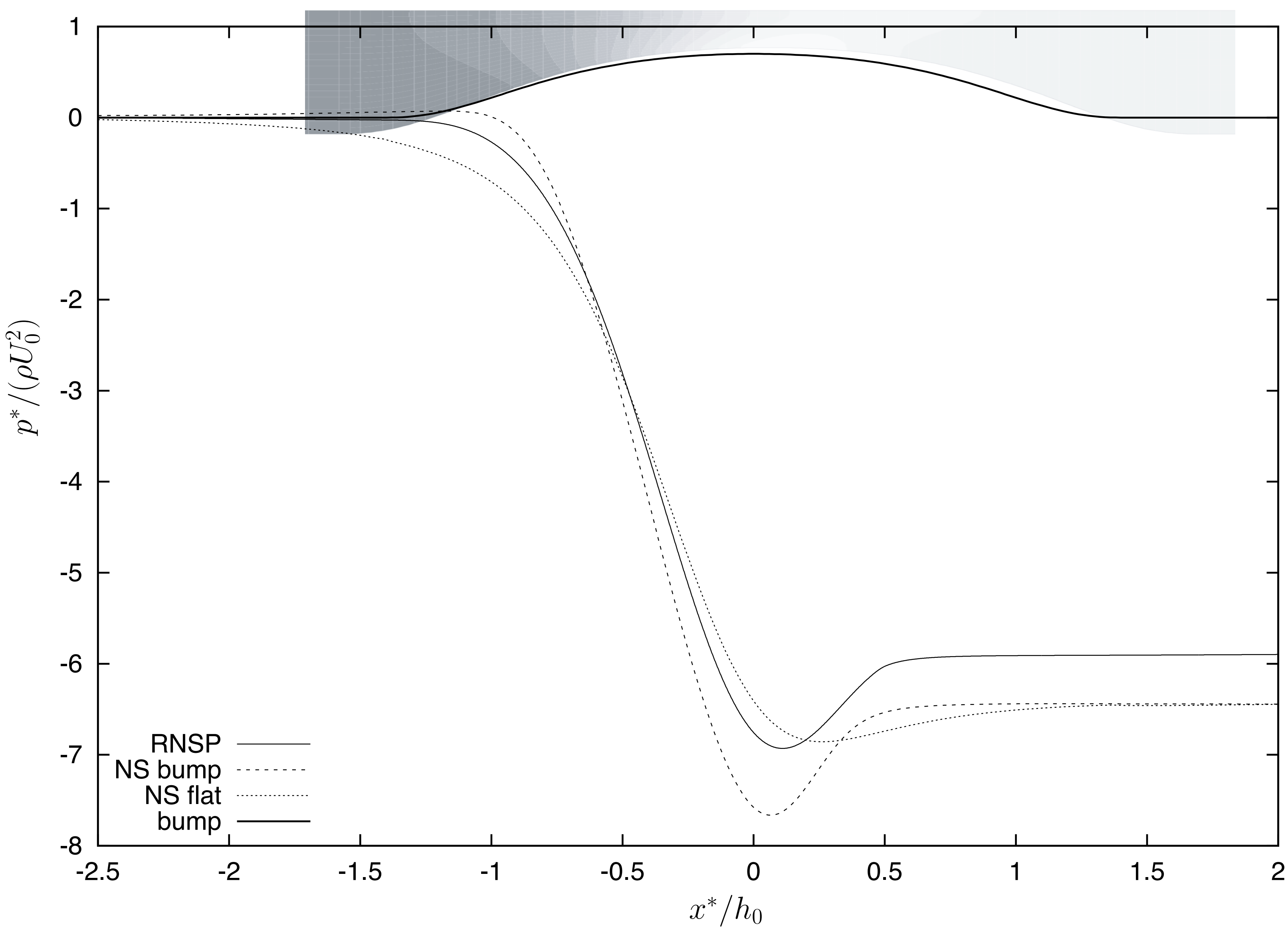
- écoulement dans un tuyau sténosé
- stationnaire, parois rigides
- cas non symétrique

cas non symétrique

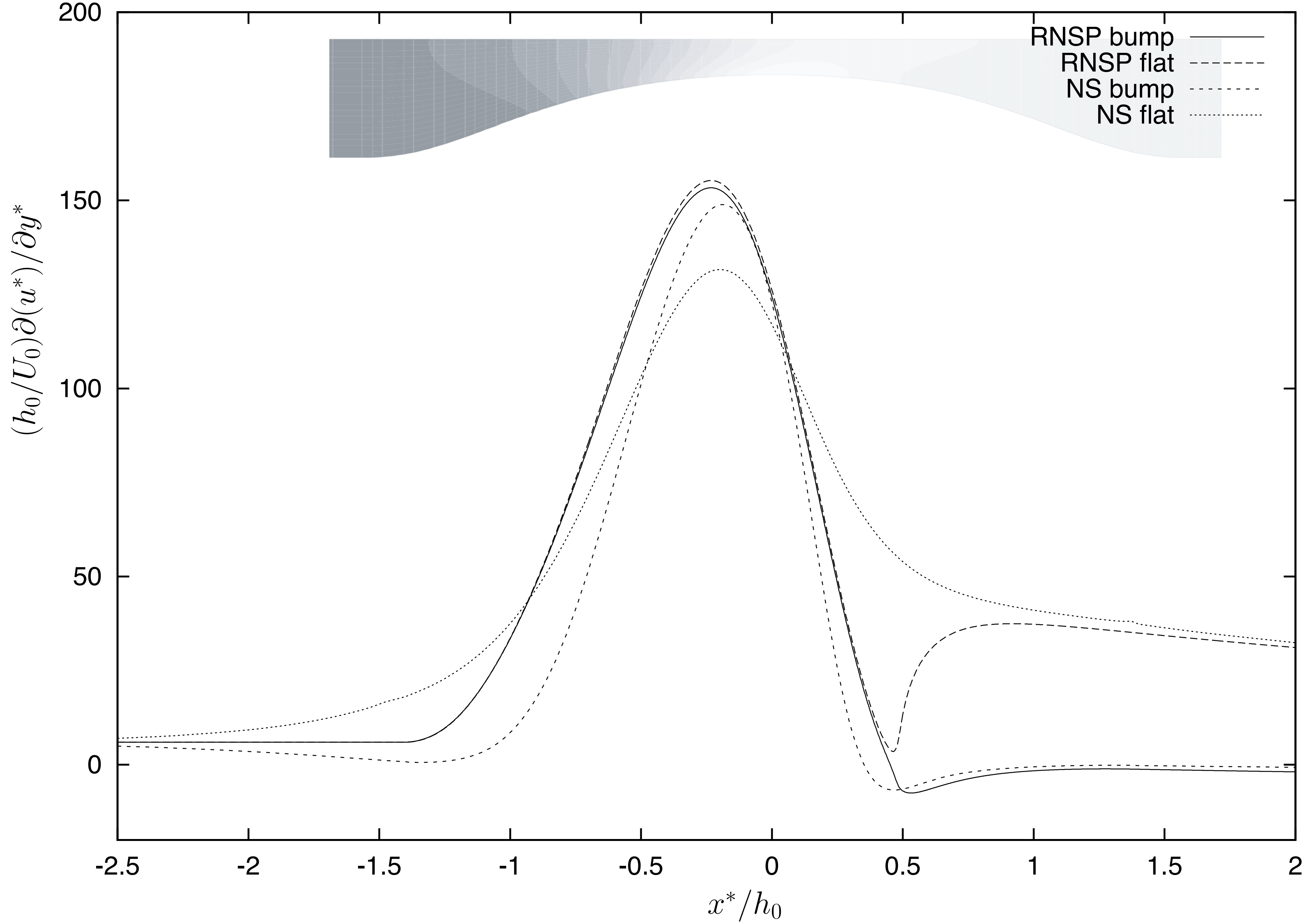
- RNSP
- méthode intégrale modifiée pour tenir compte de la variation transverse de pression
- NS

cas non symétrique

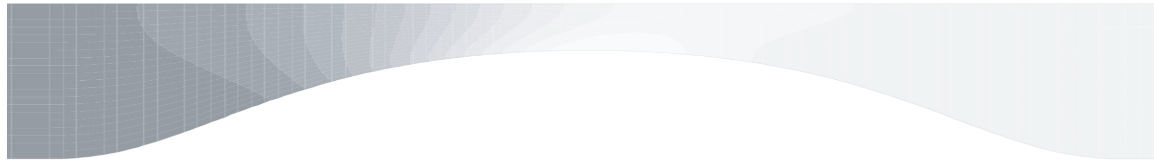
- RNSP
- méthode intégrale modifiée pour tenir compte de la variation transverse de pression
- NS



pression RNSP et NS sur les deux parois



frottement pariétal RNSP et NS sur les deux parois

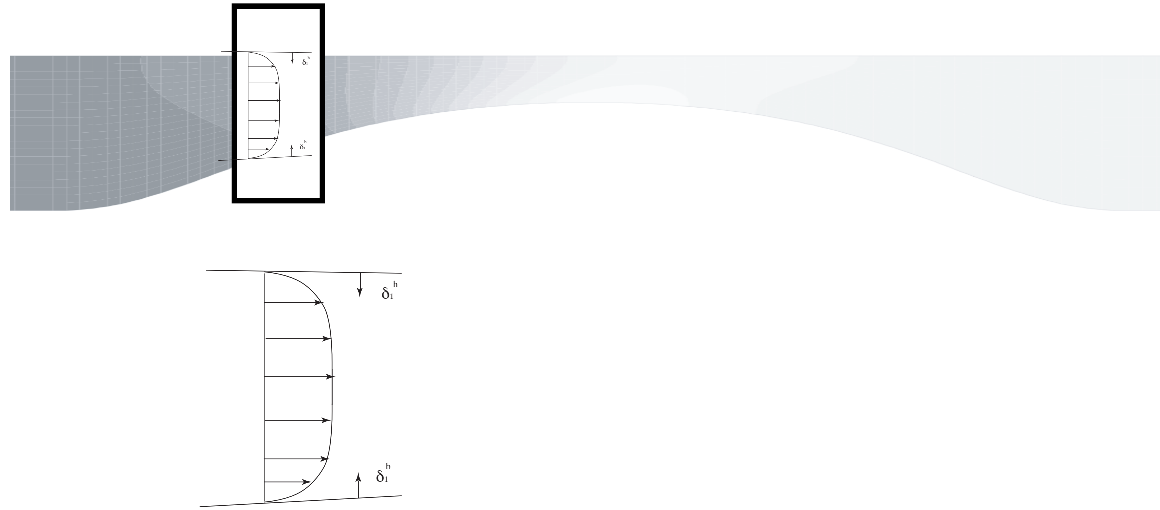


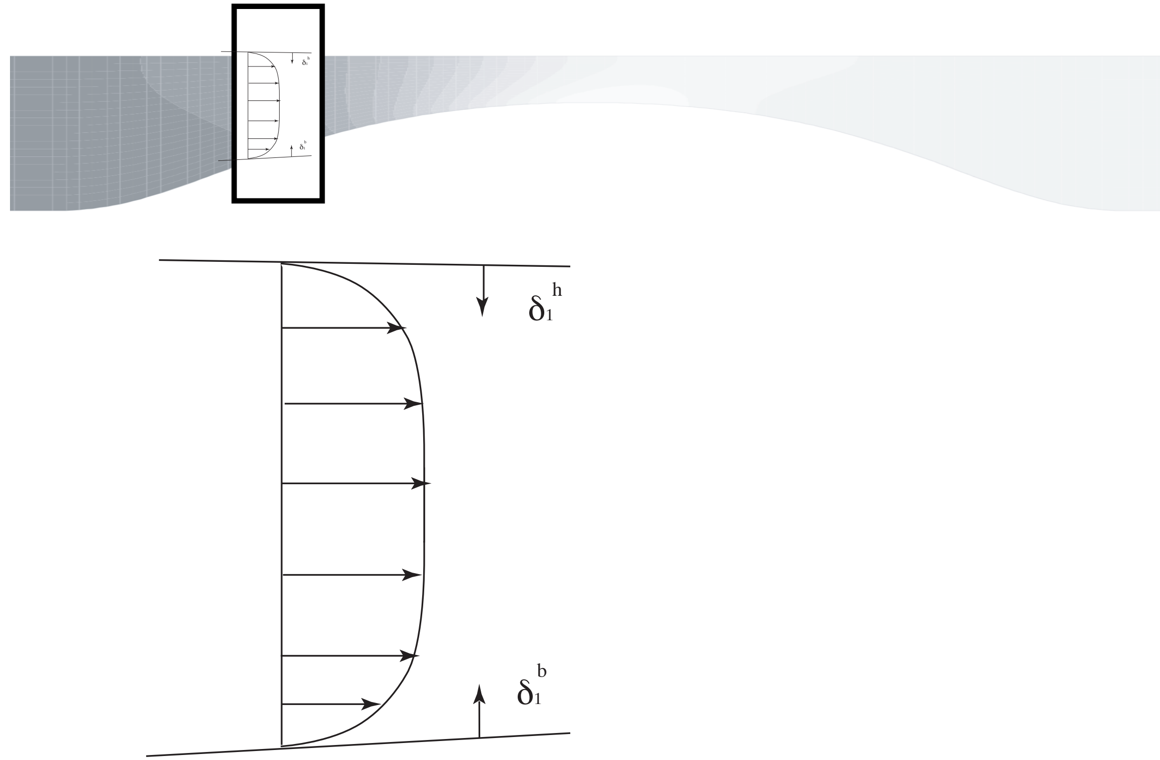
- Légère dissymétrie
- décomposition Fluide Parfait/Couche limite
- Utilisation d’une méthode intégrale

Epaisseur de déplacement de la couche limite

$$\delta_1^h = \int_0^\infty (1 - \frac{u}{u^h_{y=f^h}} dy) \qquad \delta_1^b = \int_0^\infty (1 - \frac{u}{u^b_{y=f^b}} dy)$$

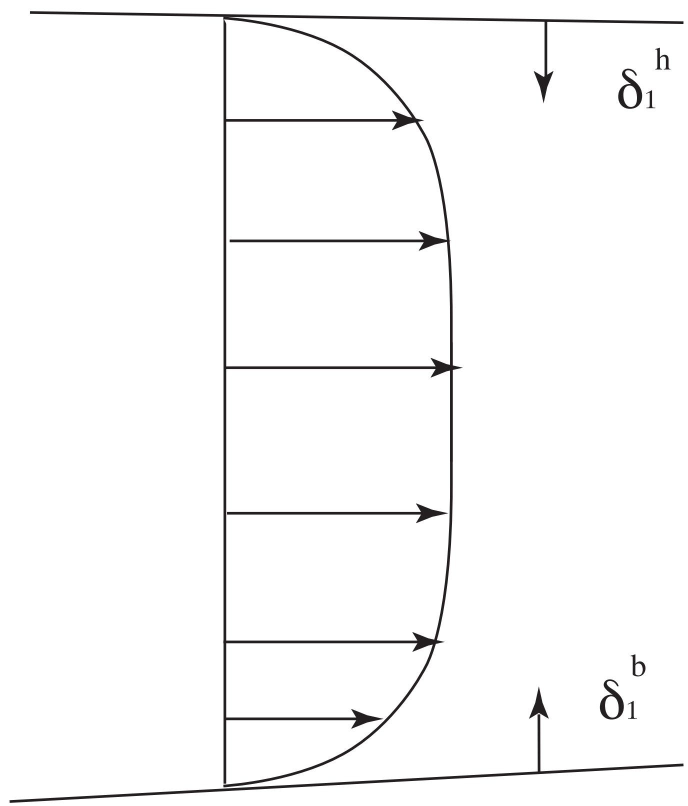
δ_1^h
δ_1^b


δ_1^b
δ_1^b

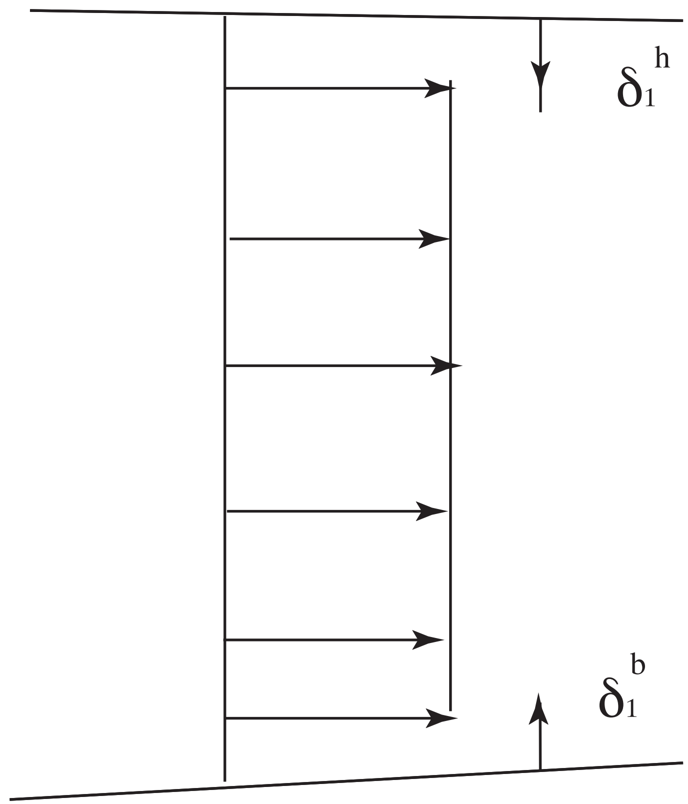


δ_1^h
δ_1^b

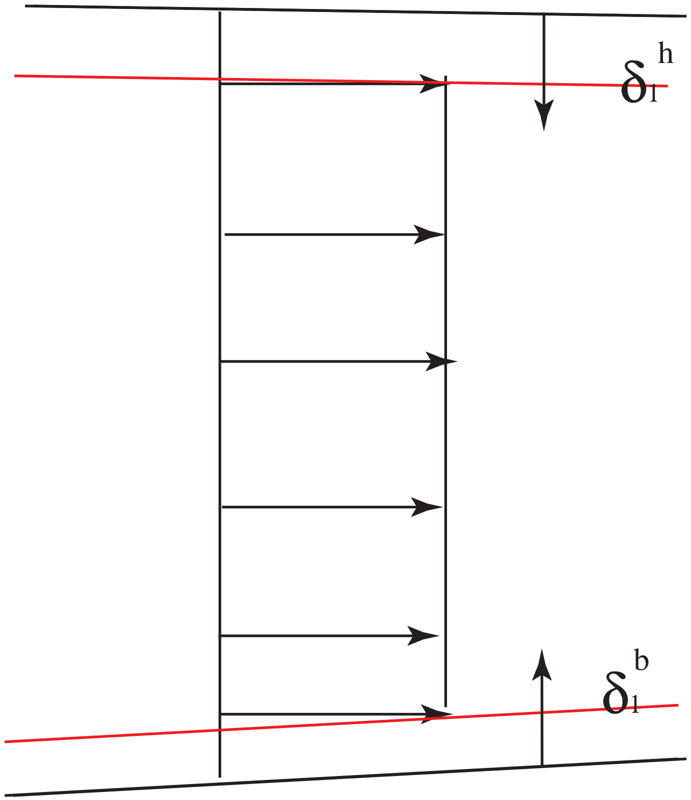
Utilisation d’une méthode intégrale

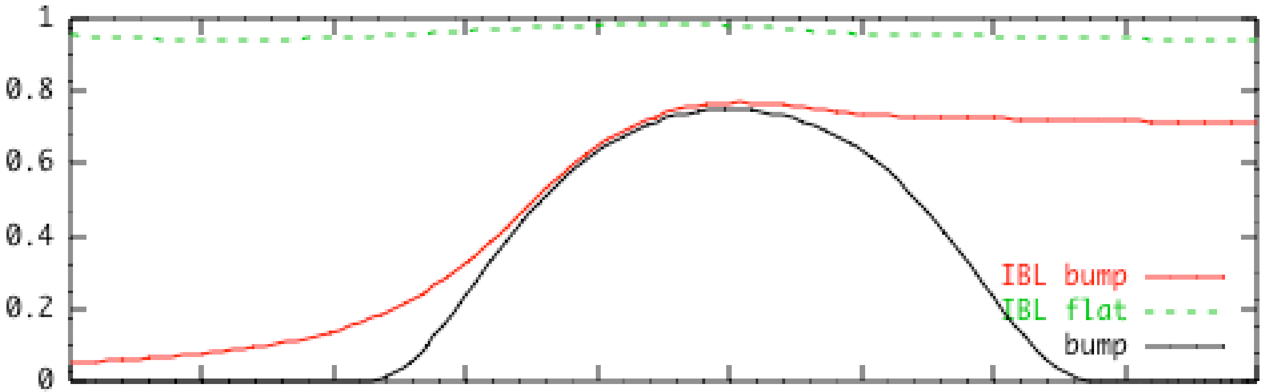


Utilisation d’une méthode intégrale



Utilisation d’une méthode intégrale



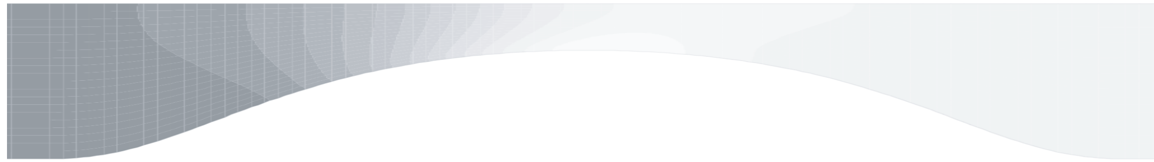


les deux couches limites finales
“conduit” réel

Relation de couplage

- Fluide parfait flux corrigé:

$$U_0(1-(f_h+\delta_1^h)-(f_b+\delta_1^b))=1$$

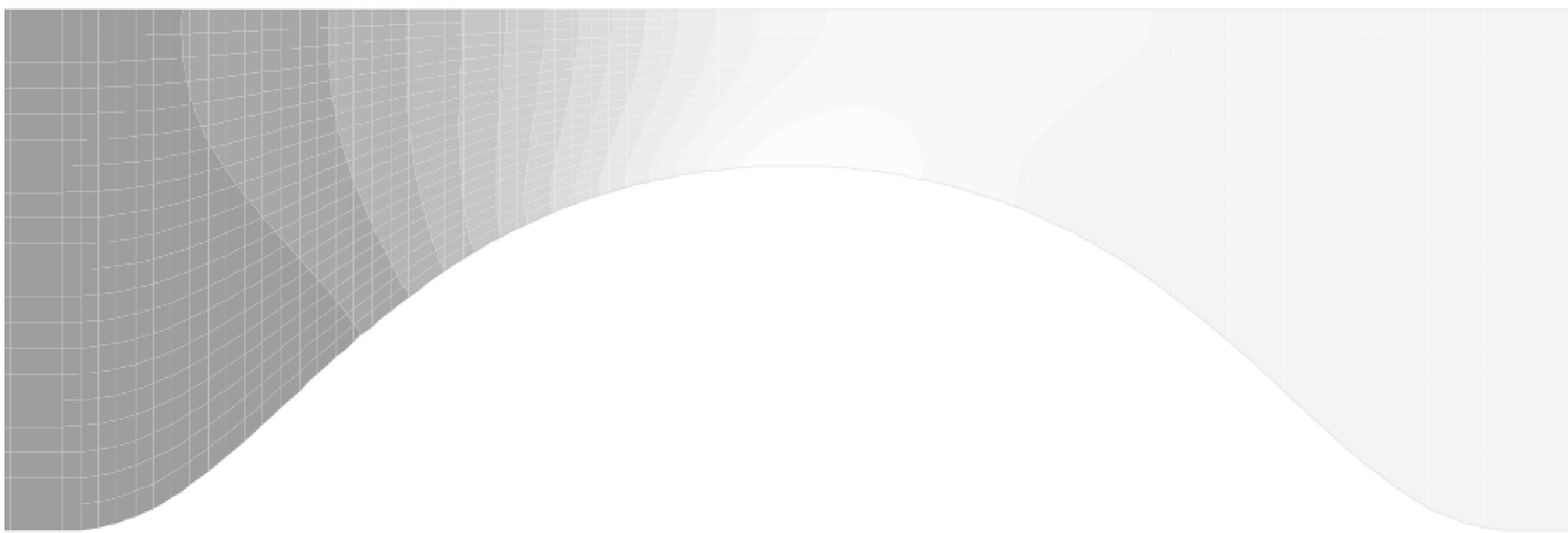


variation de pression au travers de la section

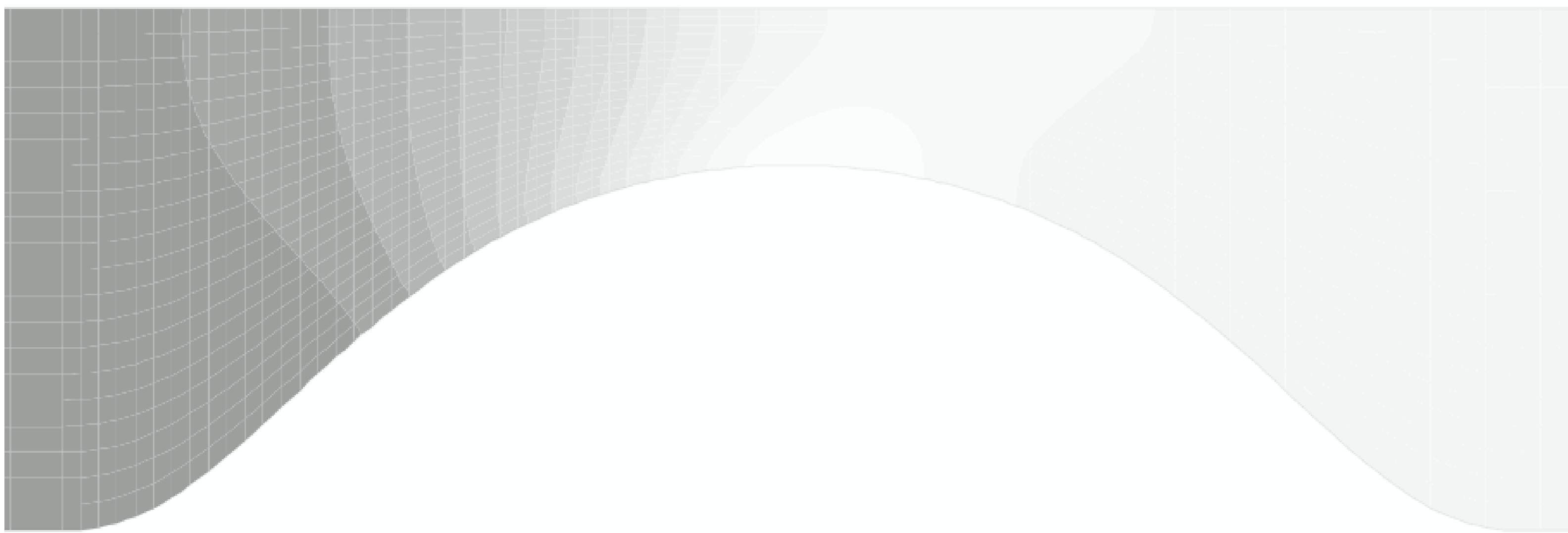
$$\Delta P_0 = \varepsilon^2 \Big(\frac{((f'_h + \delta_1'^h)^2 - (f'_b + \delta_1'^b)^2)}{1 - (f_b + \delta_1^b) - (f_h + \delta_1^h)}$$

$$+ \frac{(f''_h + \delta_1''^h - f''_b - \delta_1''^b)}{2} \Big).$$

$$\varepsilon = Re^{-1}$$

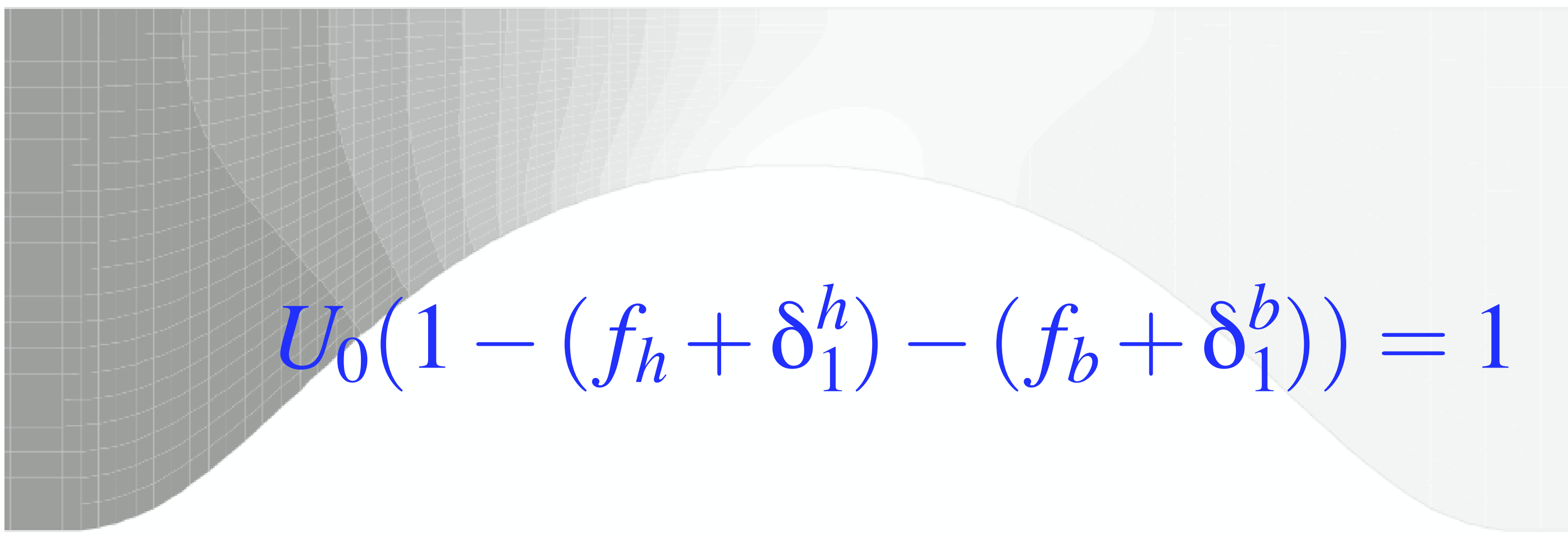


$$\frac{d}{dx}\left(\frac{\delta_1^h}{H}\right) + \frac{\delta_1^h}{u_e^h}\left(1 + \frac{2}{H}\right)\frac{du_e^h}{dx} = \frac{f_2 H}{\delta_1^h u_e^h}, \qquad \delta_1^h = F(p_e^h)$$



$$\frac{d}{dx}\left(\frac{\delta_1^b}{H}\right) + \frac{\delta_1^b}{u_e^b}\left(1 + \frac{2}{H}\right)\frac{du_e^b}{dx} = \frac{f_2 H}{\delta_1^b u_e^b}, \qquad \delta_1^b = F(p_e^b)$$

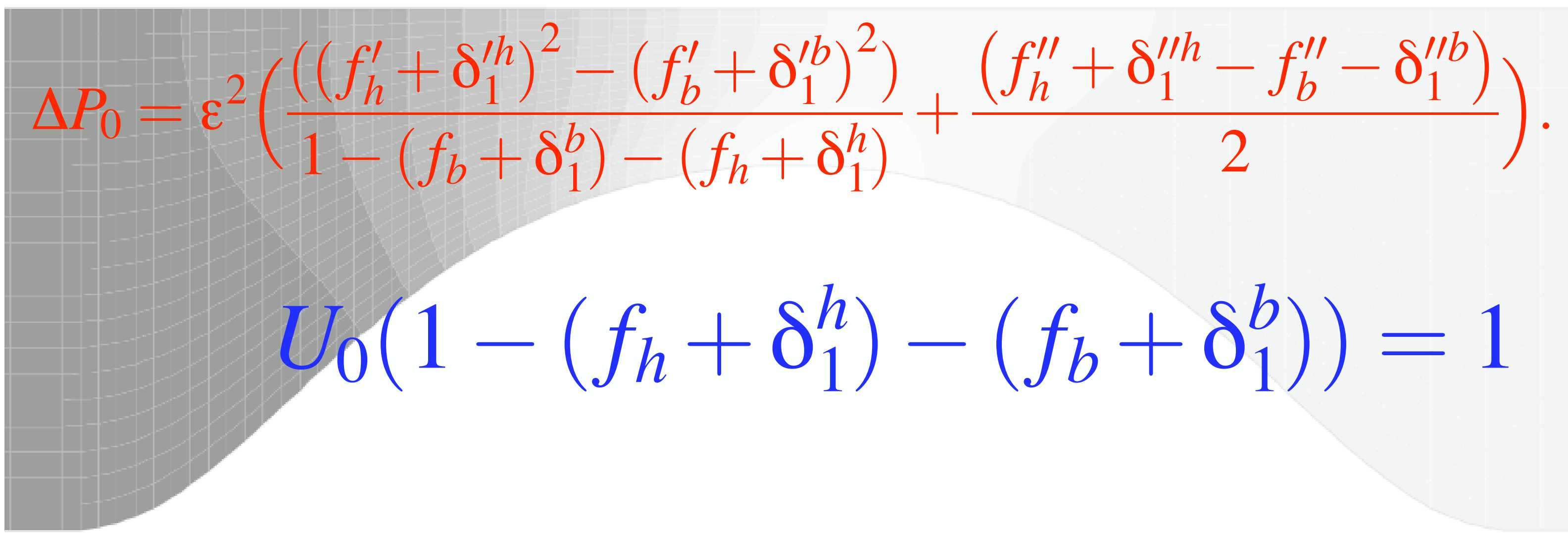
$$\frac{d}{dx}\left(\frac{\delta_1^h}{H}\right) + \frac{\delta_1^h}{u_e^h}\left(1 + \frac{2}{H}\right)\frac{du_e^h}{dx} = \frac{f_2 H}{\delta_1^h u_e^h}, \qquad \delta_1^h = F(p_e^h)$$



$$U_0(1 - (f_h + \delta_1^h) - (f_b + \delta_1^b)) = 1$$

$$\frac{d}{dx}\left(\frac{\delta_1^b}{H}\right) + \frac{\delta_1^b}{u_e^b}\left(1 + \frac{2}{H}\right)\frac{du_e^b}{dx} = \frac{f_2 H}{\delta_1^b u_e^b}, \qquad \delta_1^b = F(p_e^b)$$

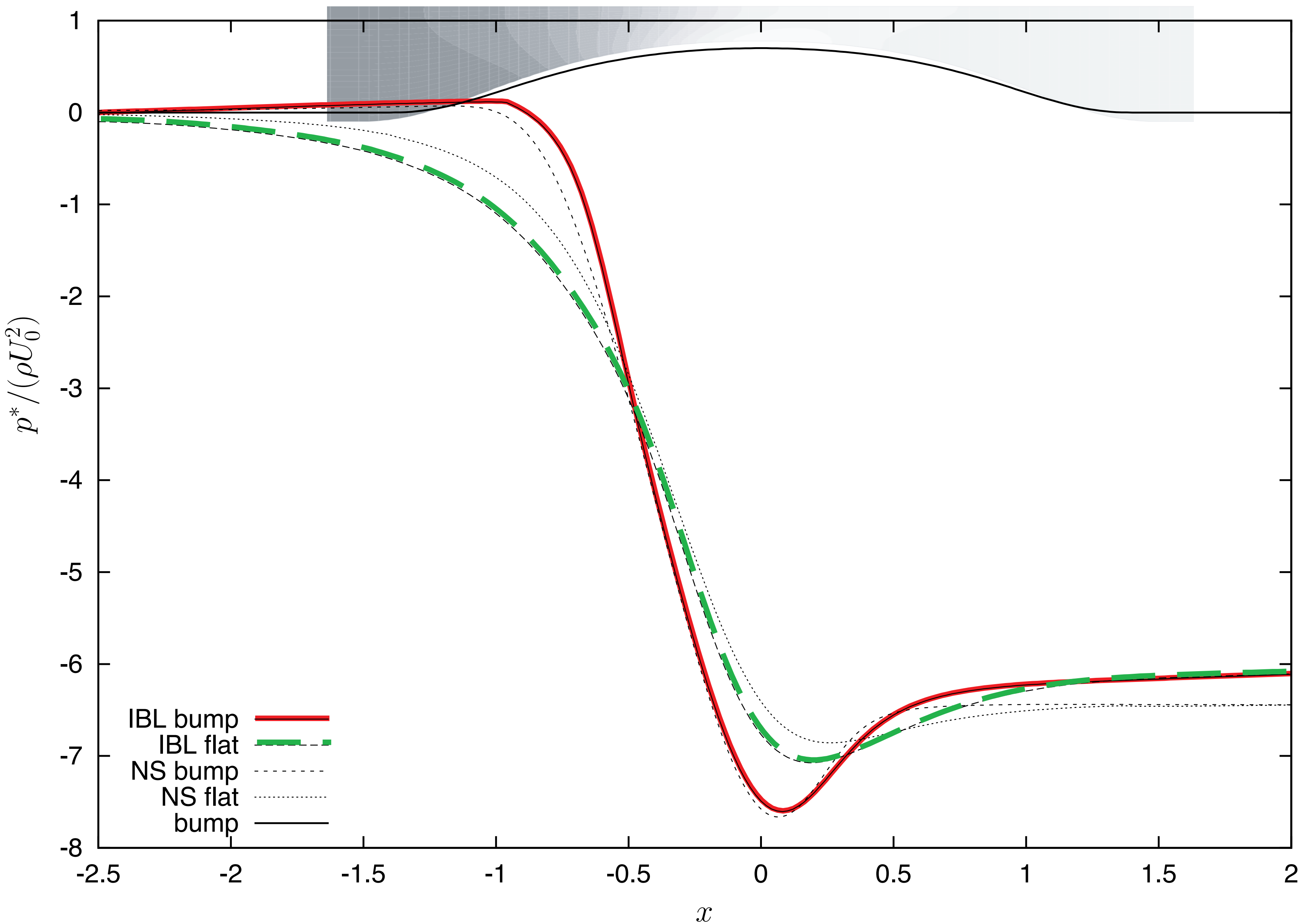
$$\frac{d}{dx}\left(\frac{\delta_1^h}{H}\right) + \frac{\delta_1^h}{u_e^h}\left(1 + \frac{2}{H}\right)\frac{du_e^h}{dx} = \frac{f_2 H}{\delta_1^h u_e^h}, \qquad \delta_1^h = F(p_e^h)$$



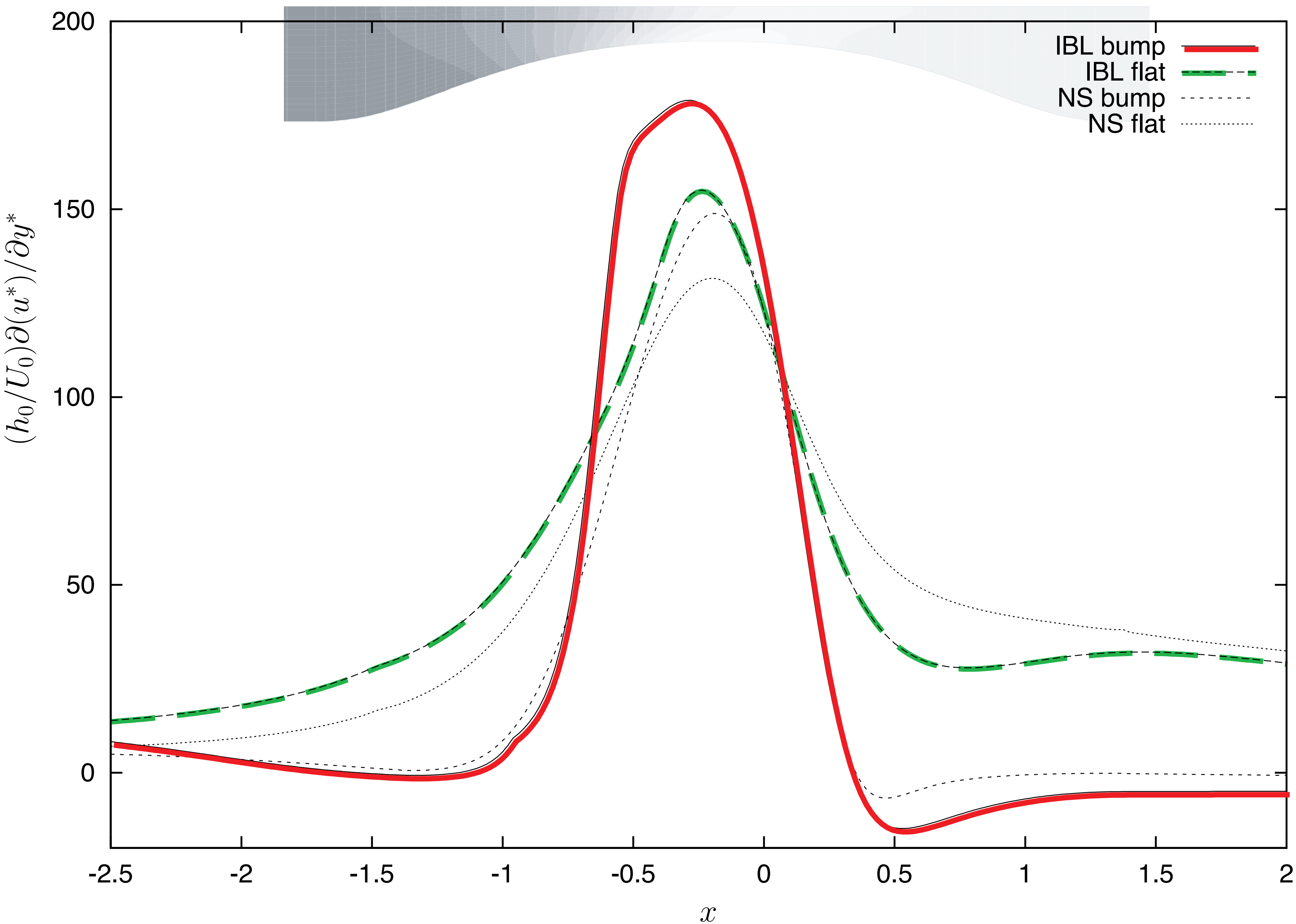
$$\Delta P_0 = \varepsilon^2\left(\frac{\left((f_h' + \delta_1'^h)^2 - (f_b' + \delta_1'^b)^2\right)}{1 - (f_b + \delta_1^b) - (f_h + \delta_1^h)} + \frac{\left(f_h'' + \delta_1''^h - f_b'' - \delta_1''^b\right)}{2}\right).$$

$$U_0\left(1 - (f_h + \delta_1^h) - (f_b + \delta_1^b)\right) = 1$$

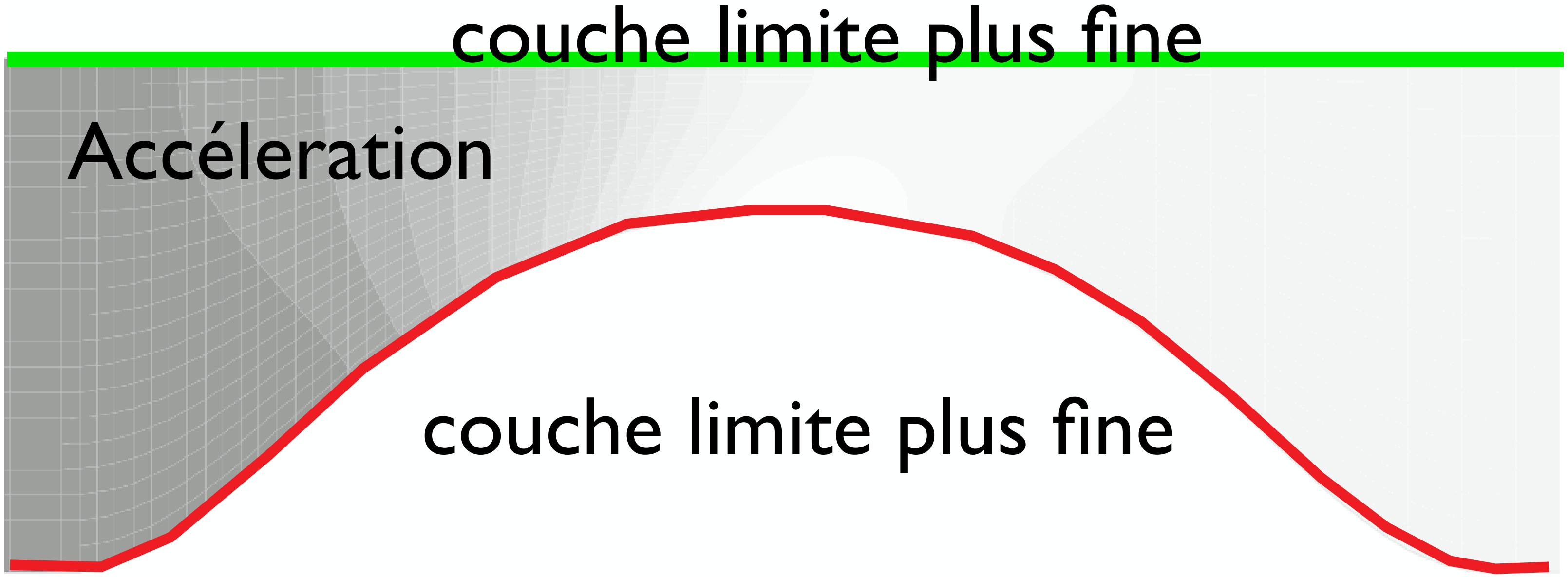
$$\frac{d}{dx}\left(\frac{\delta_1^b}{H}\right) + \frac{\delta_1^b}{u_e^b}\left(1 + \frac{2}{H}\right)\frac{du_e^b}{dx} = \frac{f_2 H}{\delta_1^b u_e^b}, \qquad \delta_1^b = F(p_e^b)$$

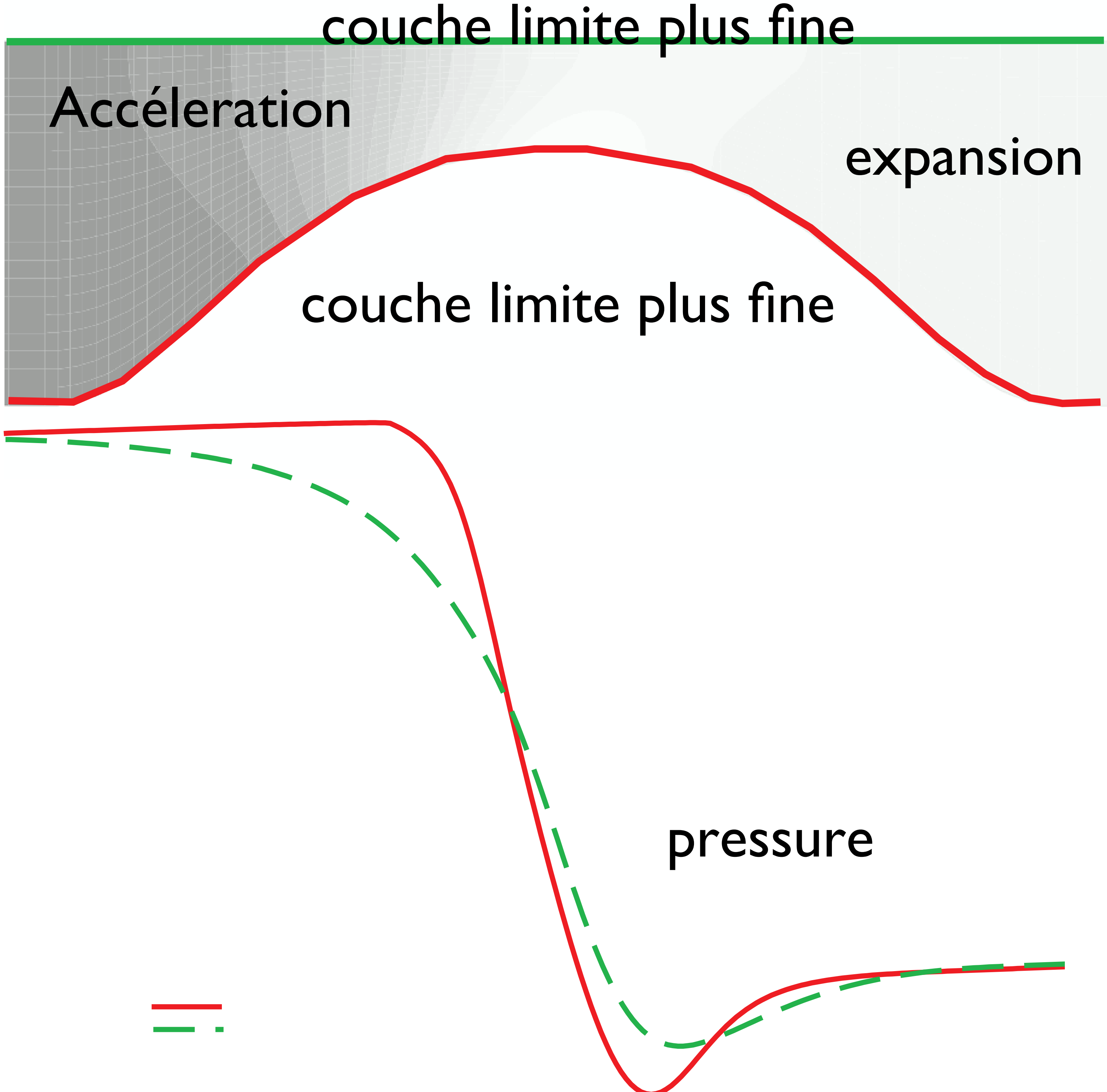


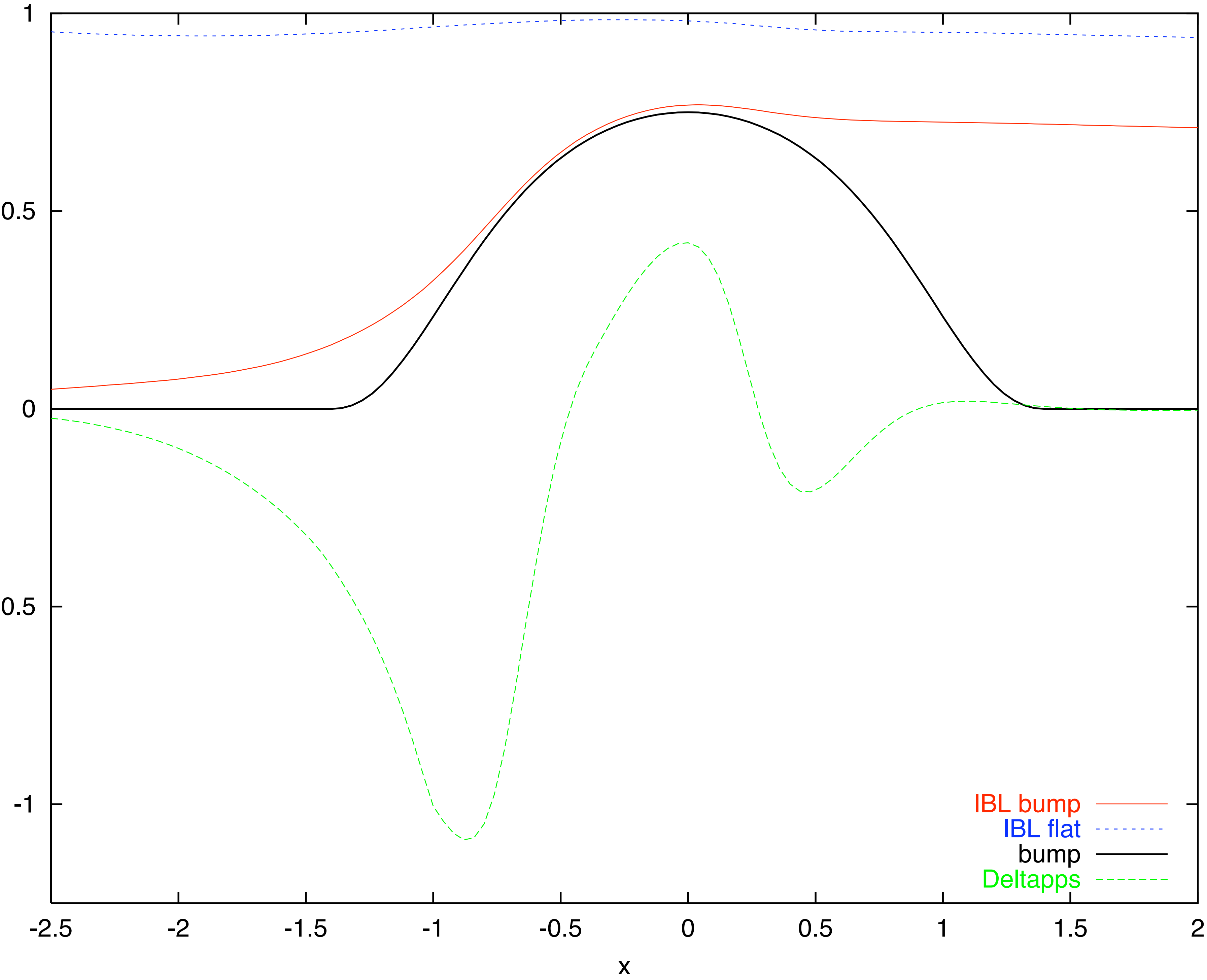
$p^*/(\rho U_0^2)$
x
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
IBL bump
IBL flat
NS bump
NS flat
bump



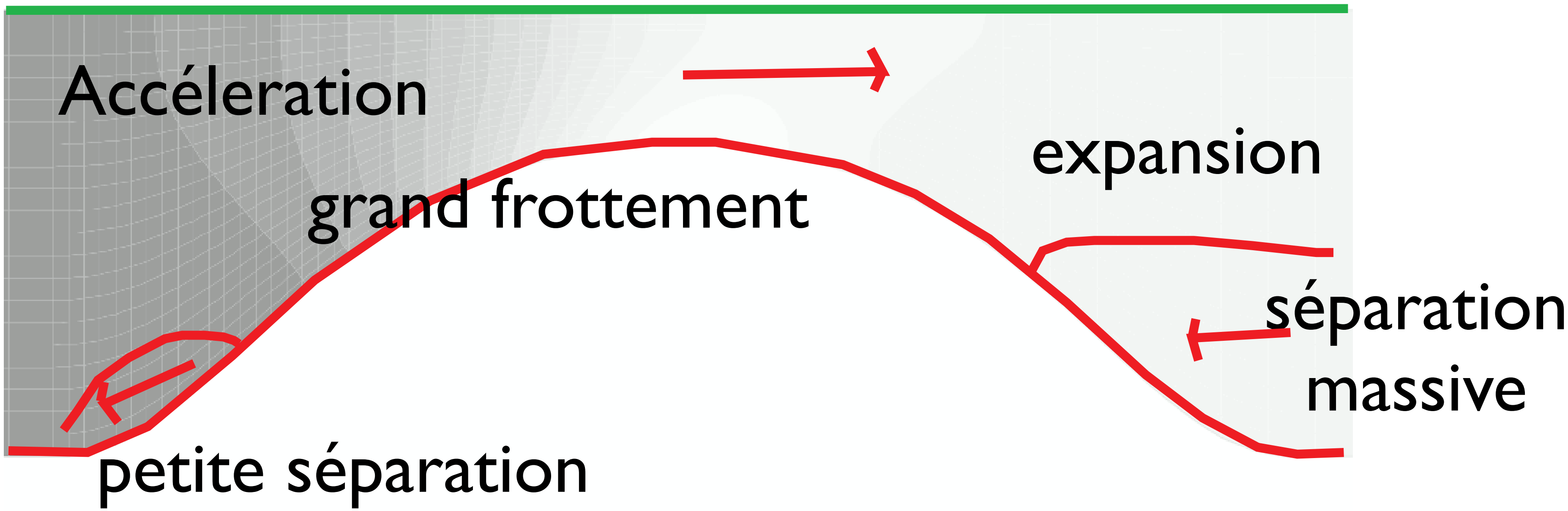
IBL bump
IBL flat
NS bump
NS flat
200
150
100
50
0
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
$(h_0/U_0)\partial(u^*)/\partial y^*$
x


couche limite plus fine
Accéleration
couche limite plus fine


couche limite plus fine
Accéleration
expansion
couche limite plus fine
pressure



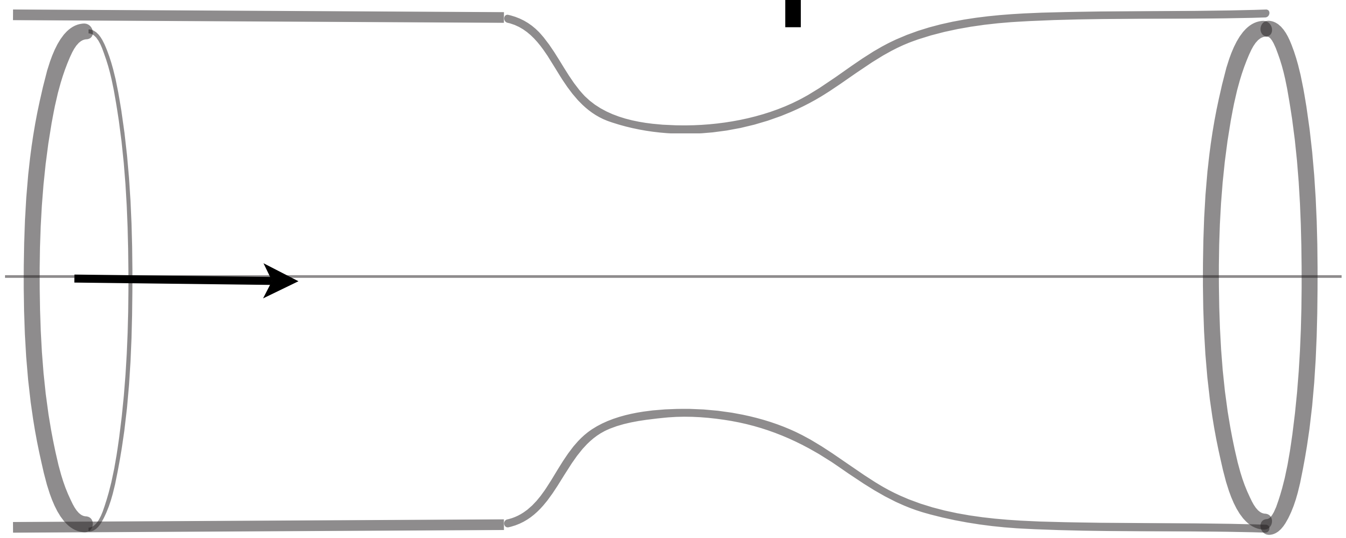
1
0.5
0
0.5
-1
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
IBL bump
IBL flat
bump
Deltapps


Accéleration
grand frottement
expansion
séparation
massive
petite séparation

Exemple 4

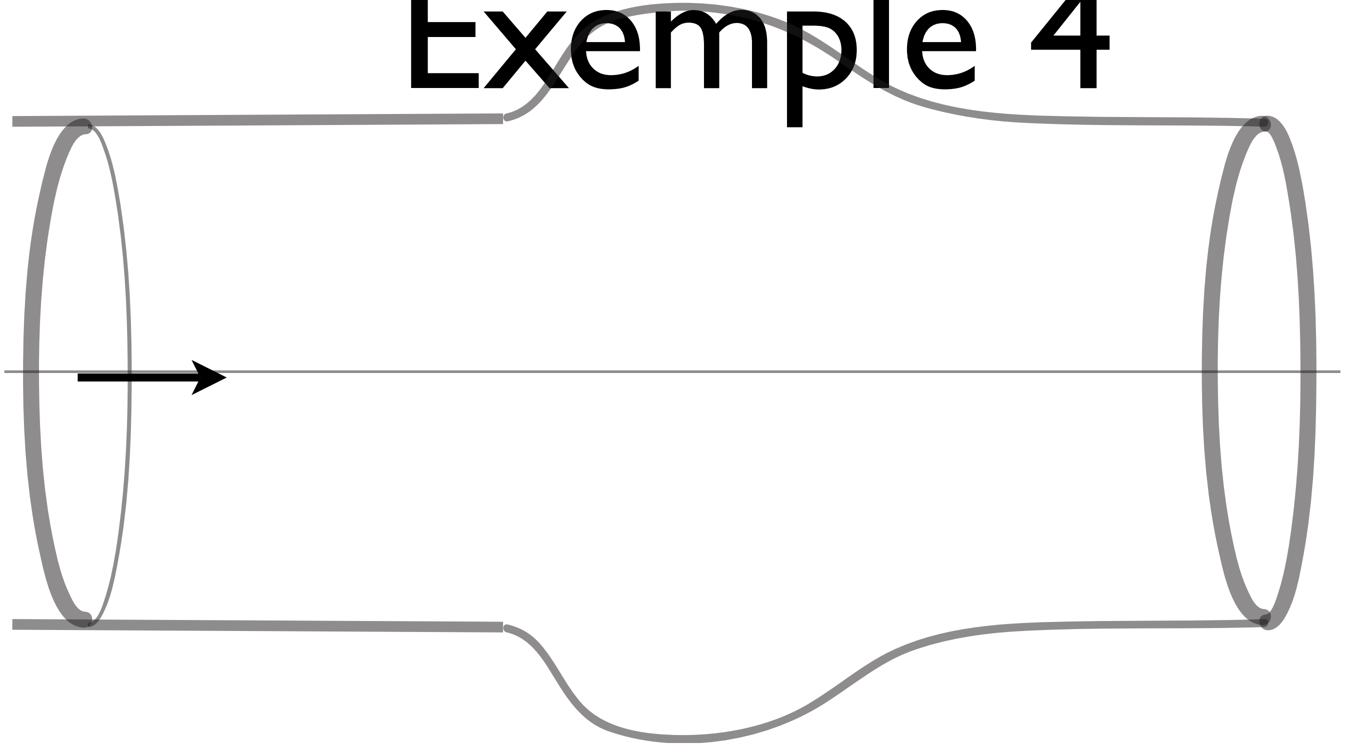
- écoulement dans un tuyau sténosé/anévrisme
- instationnaire, parois rigides

Exemple 4



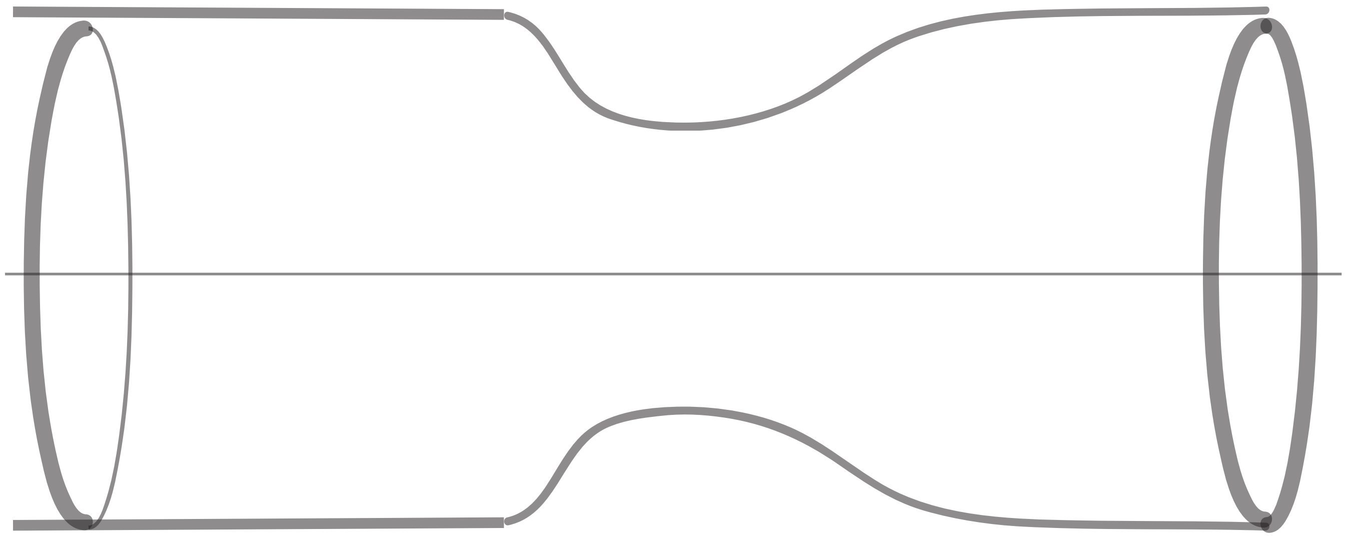
- écoulement dans un tuyau sténosé
- instationnaire, parois rigides

Exemple 4

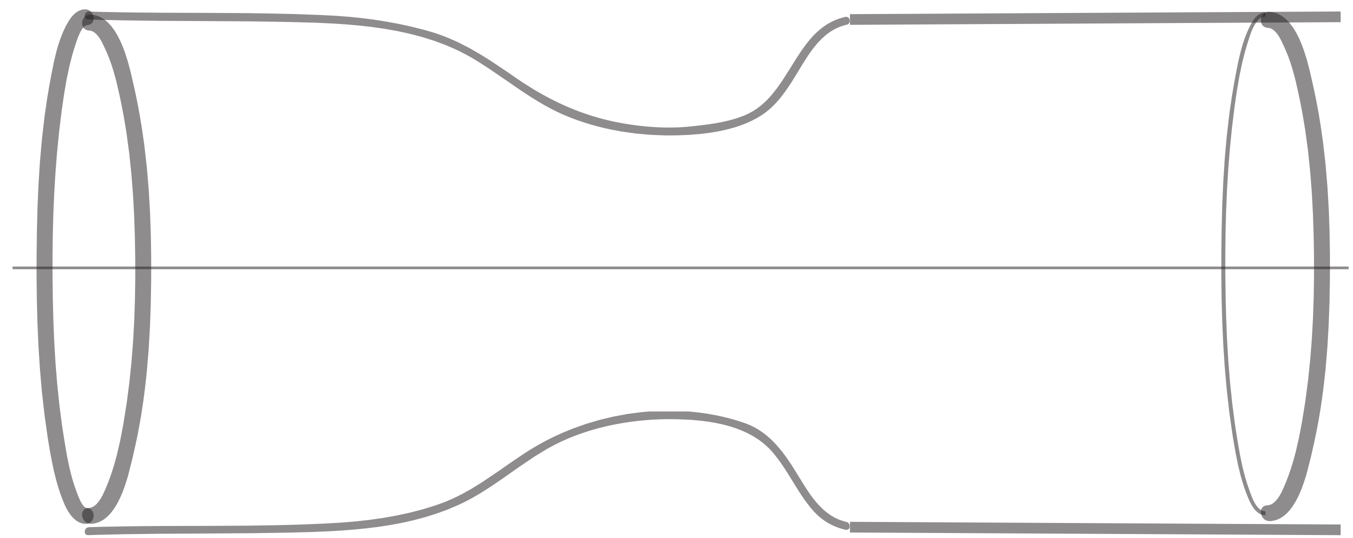


- écoulement dans un tuyau dilaté
- instationnaire, parois rigides

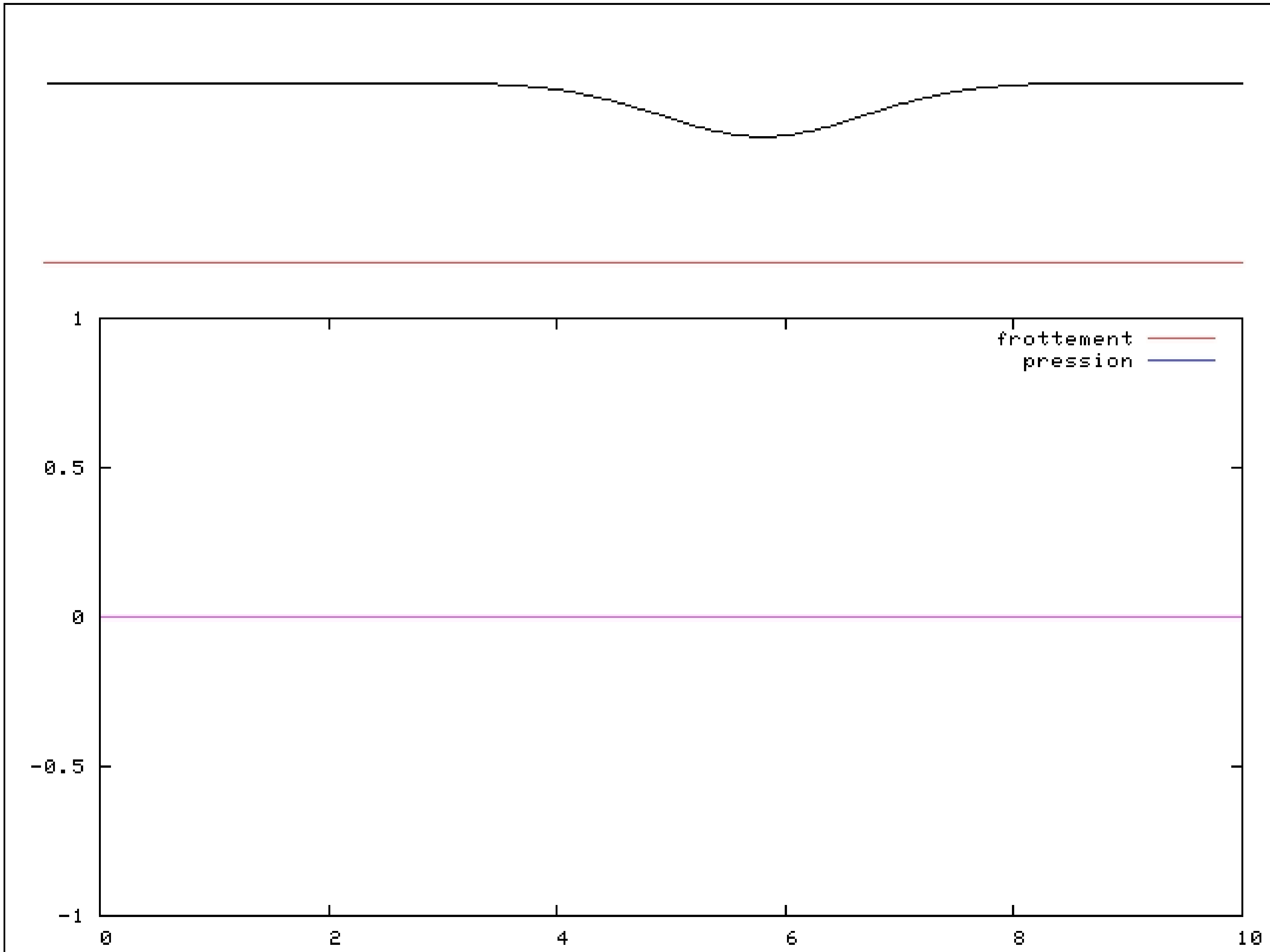
- Sténose



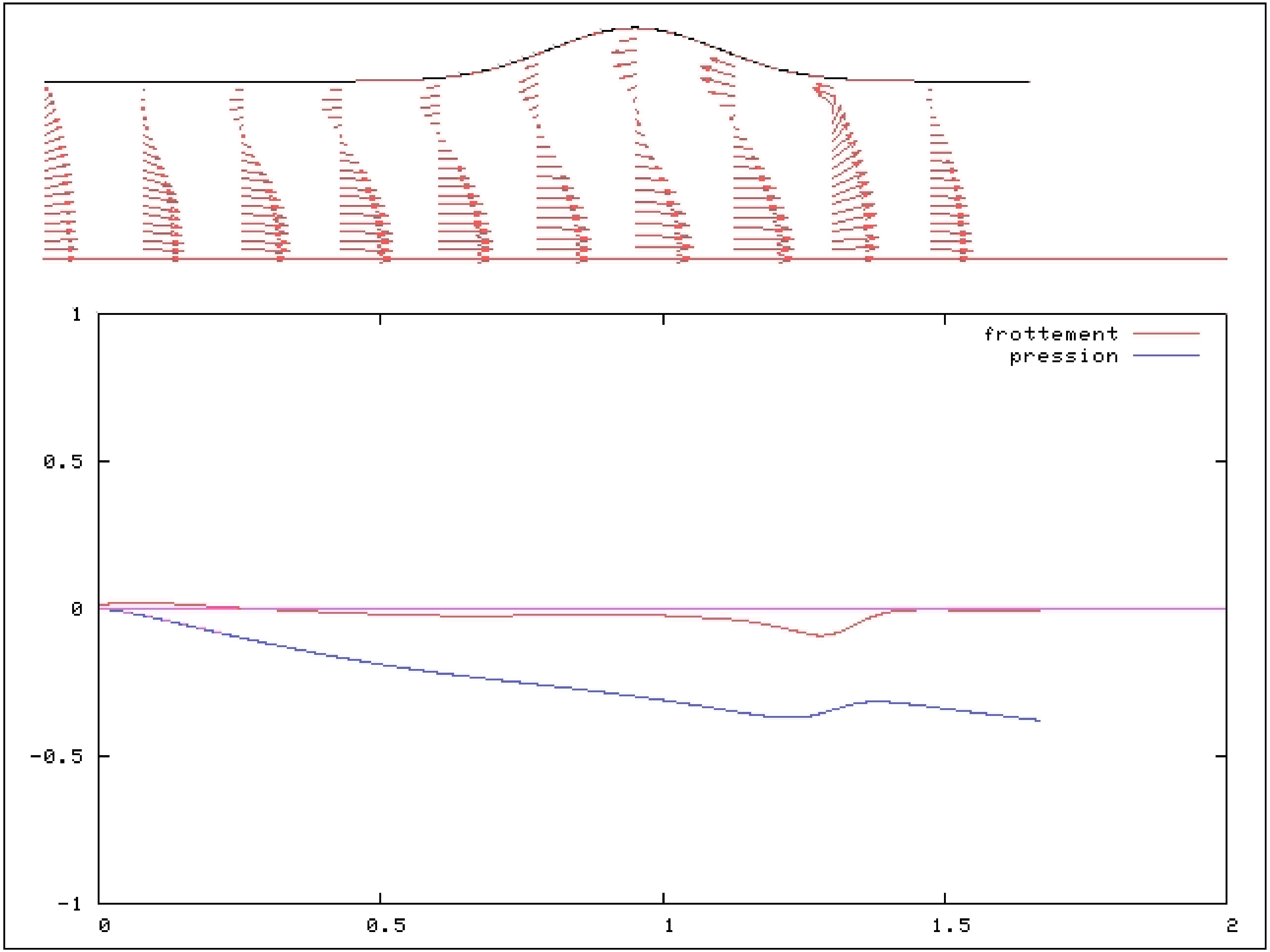
- Sténose



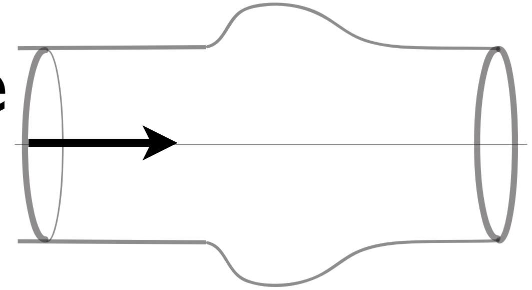
- Sténose

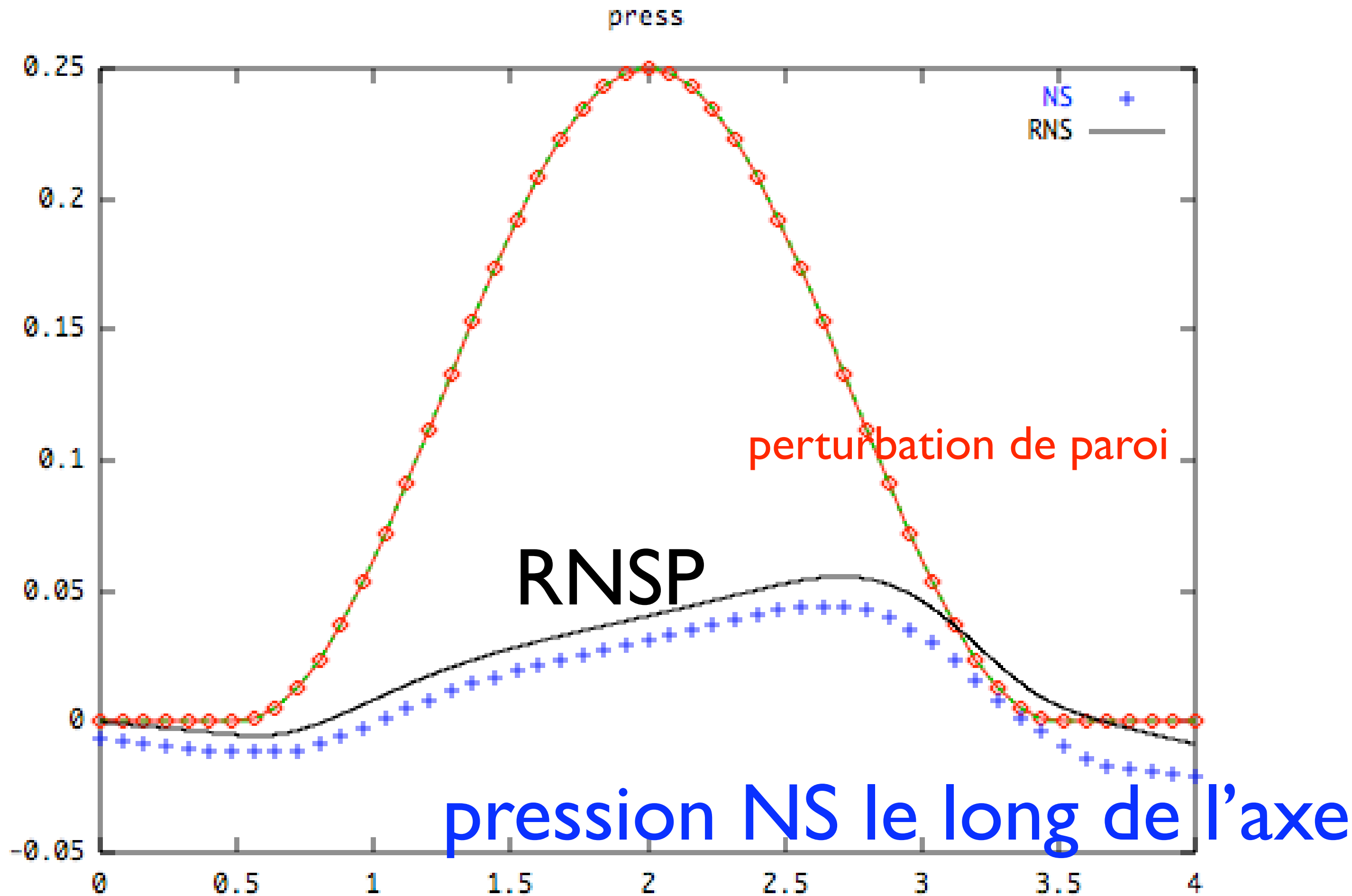


- Anévrisme

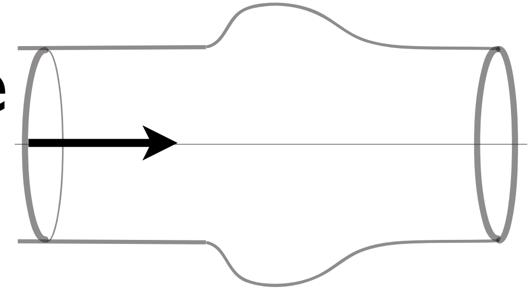


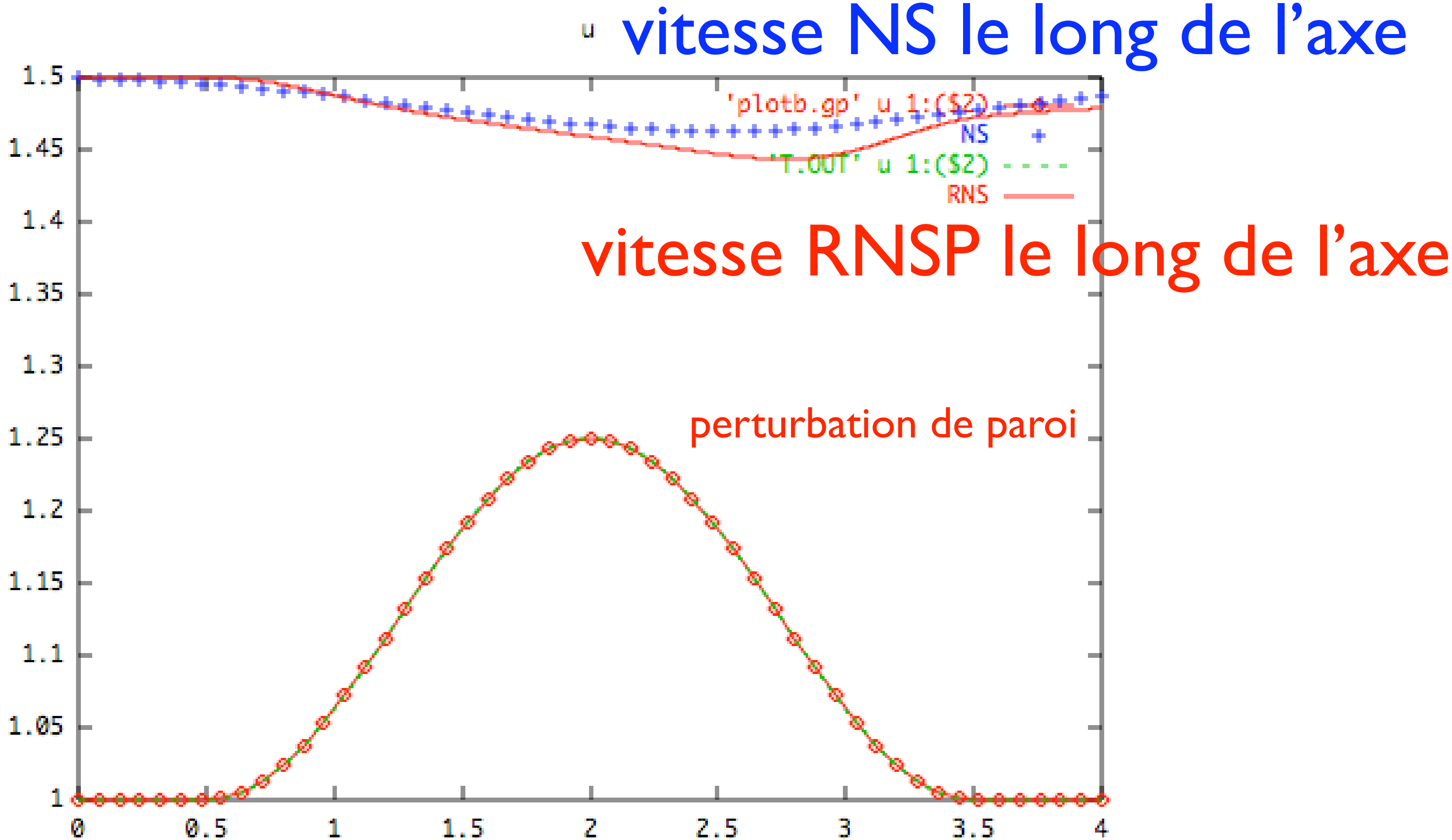
cas stationnaire



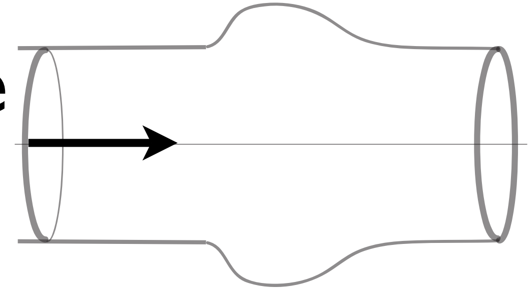


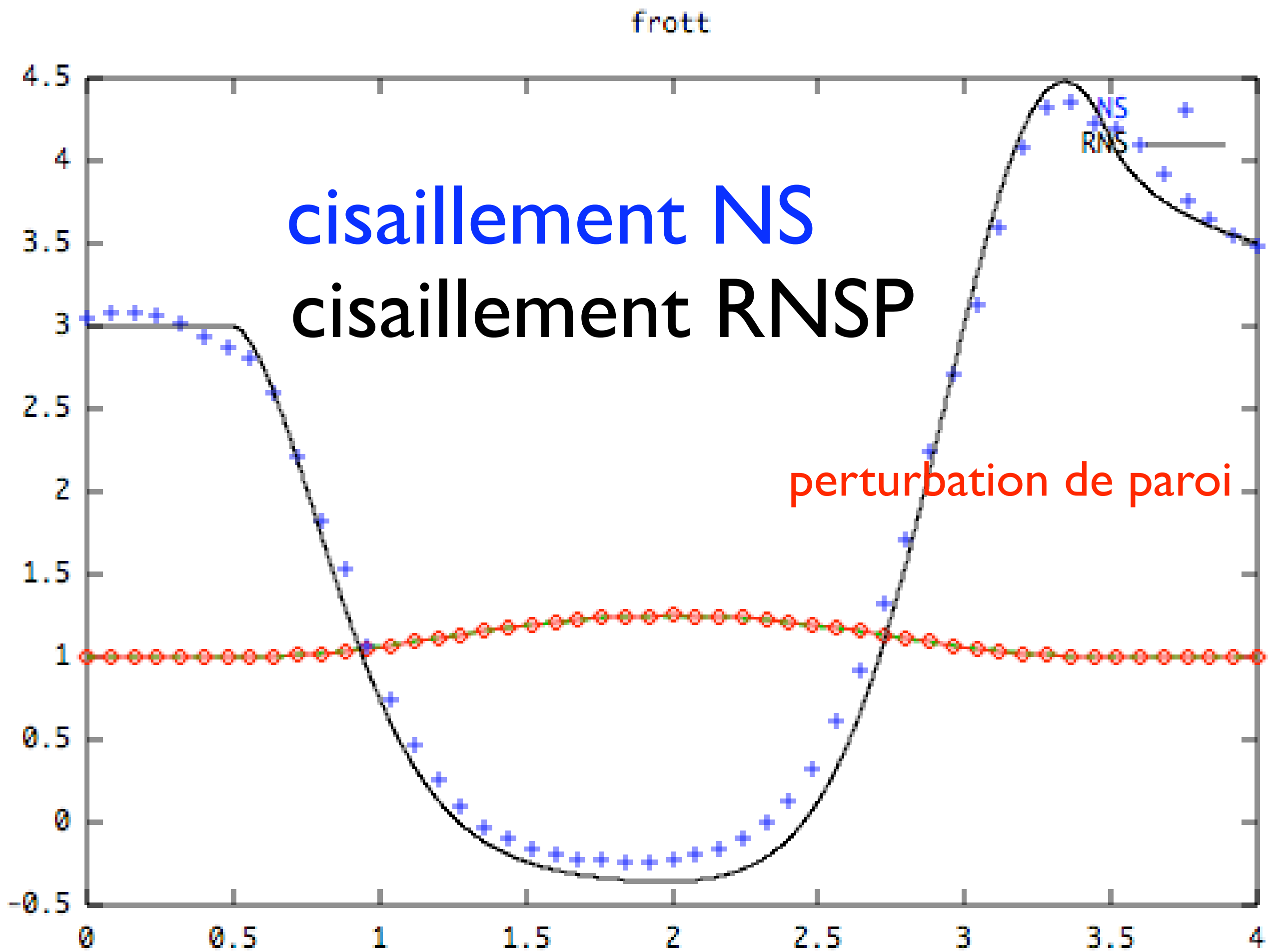
cas stationnaire

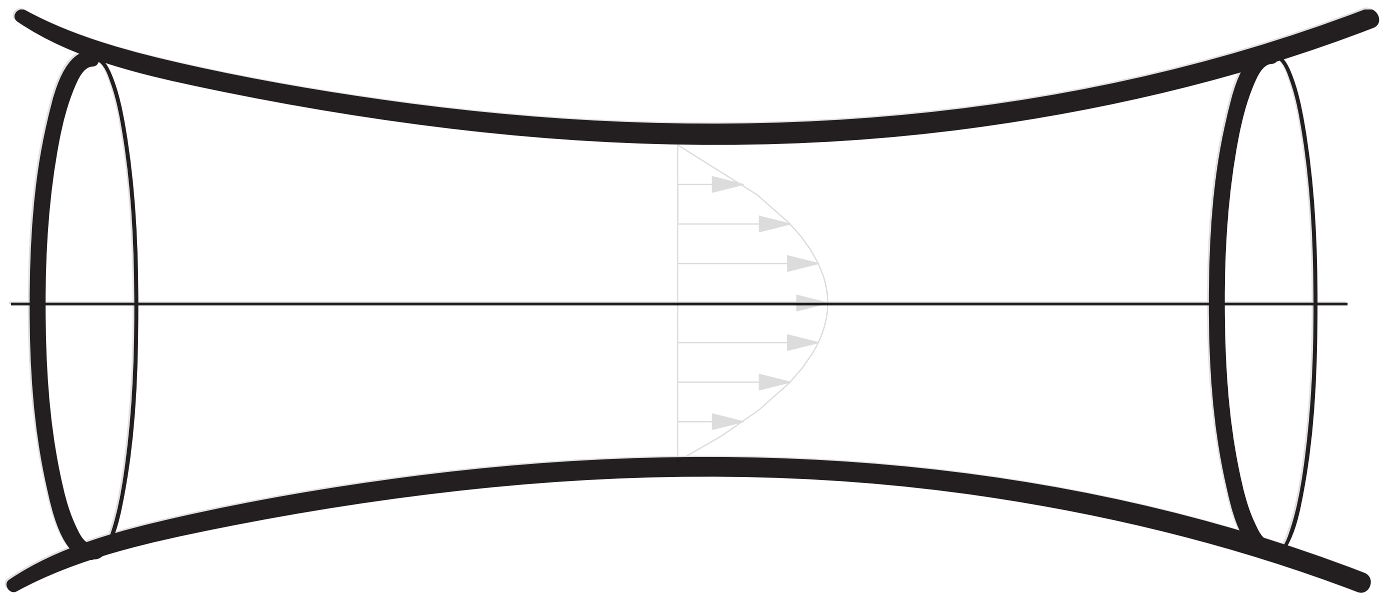




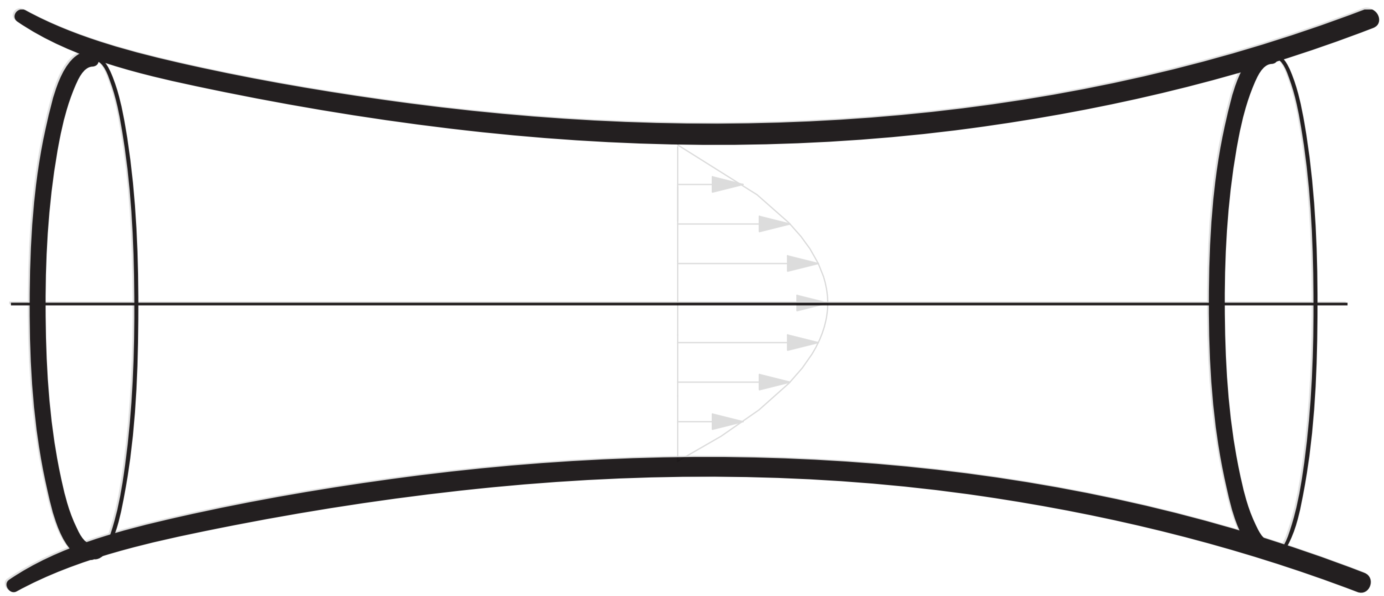
cas stationnaire



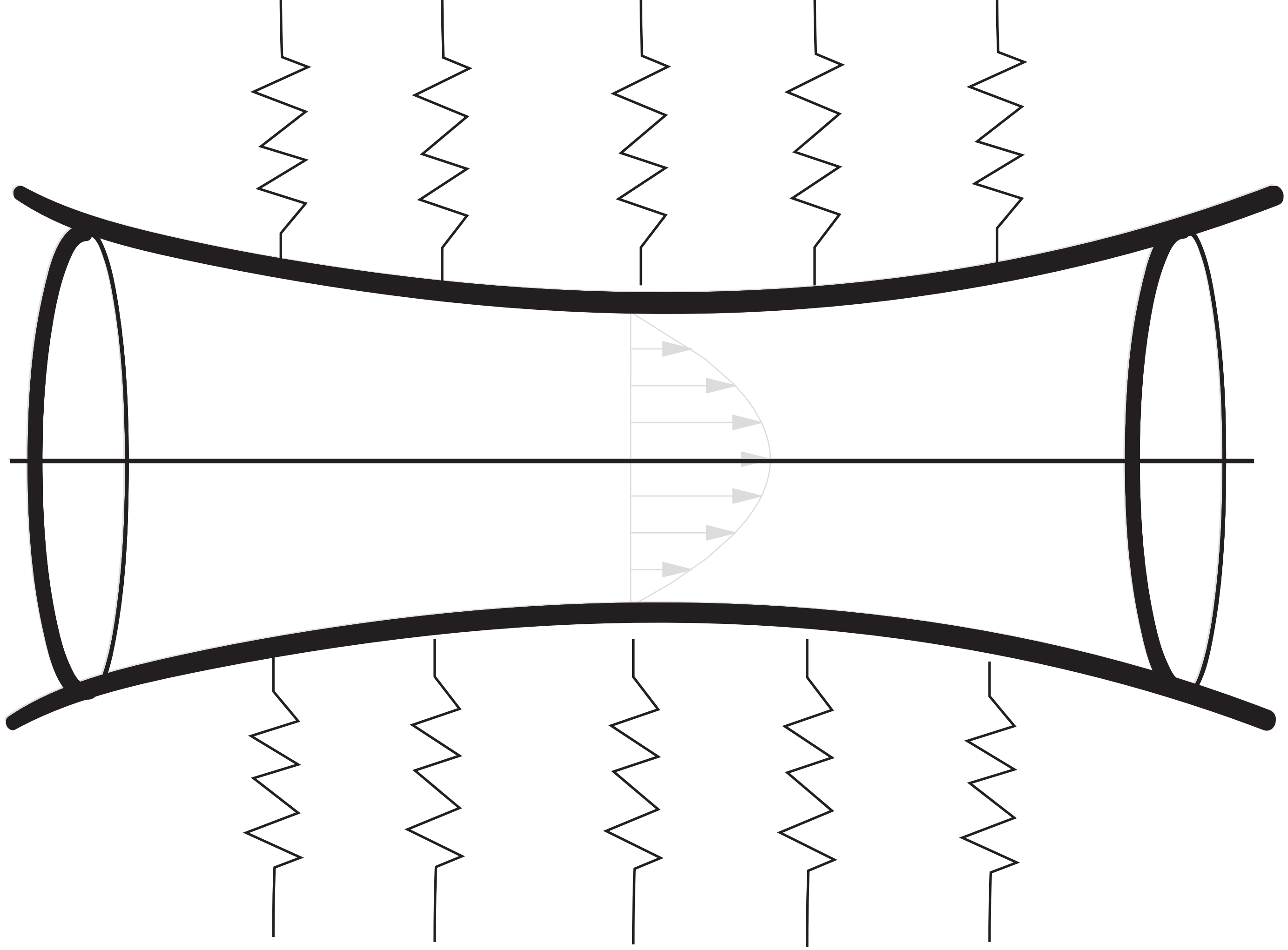




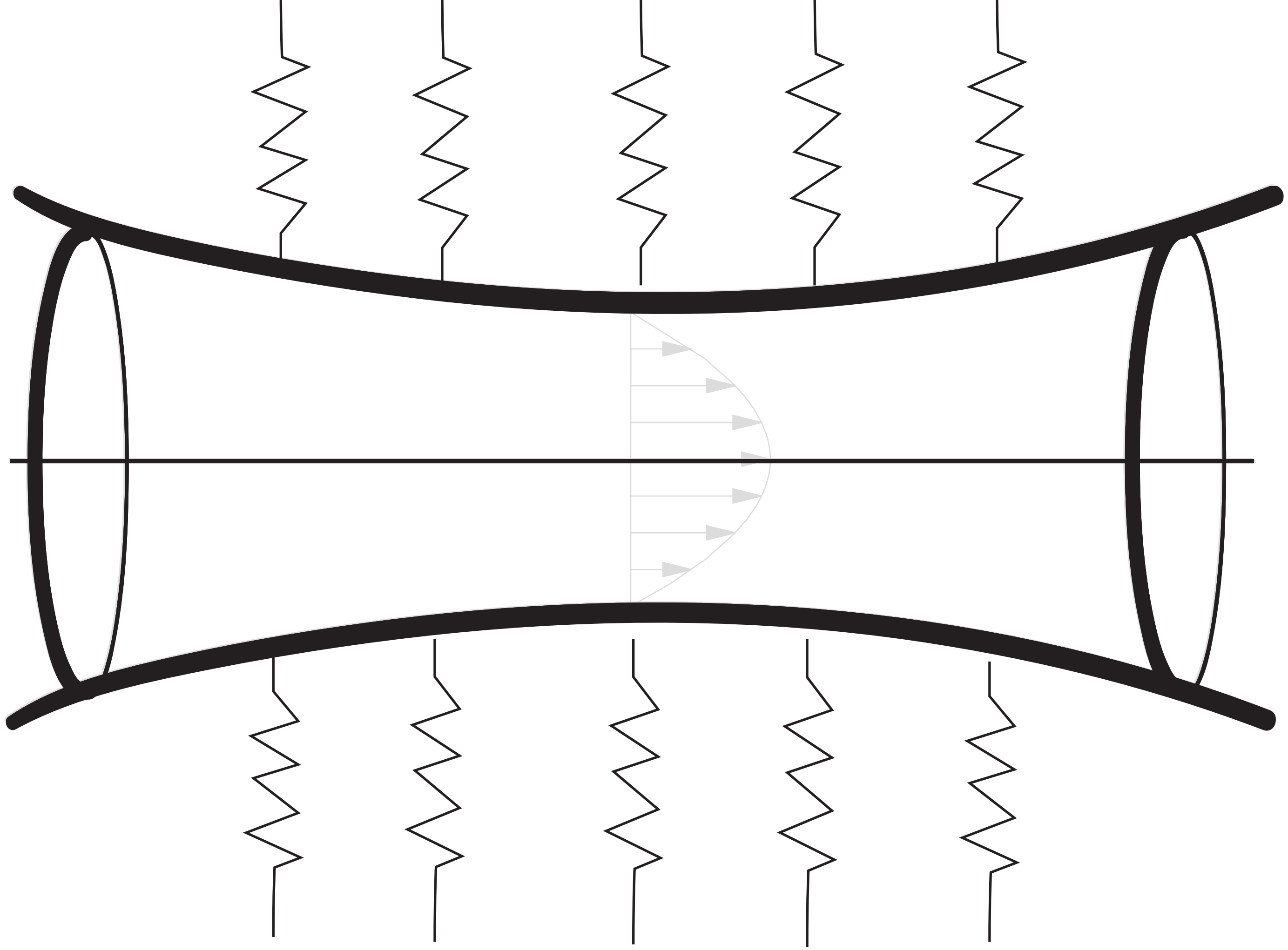
Jusqu'à maintenant la paroi était rigide

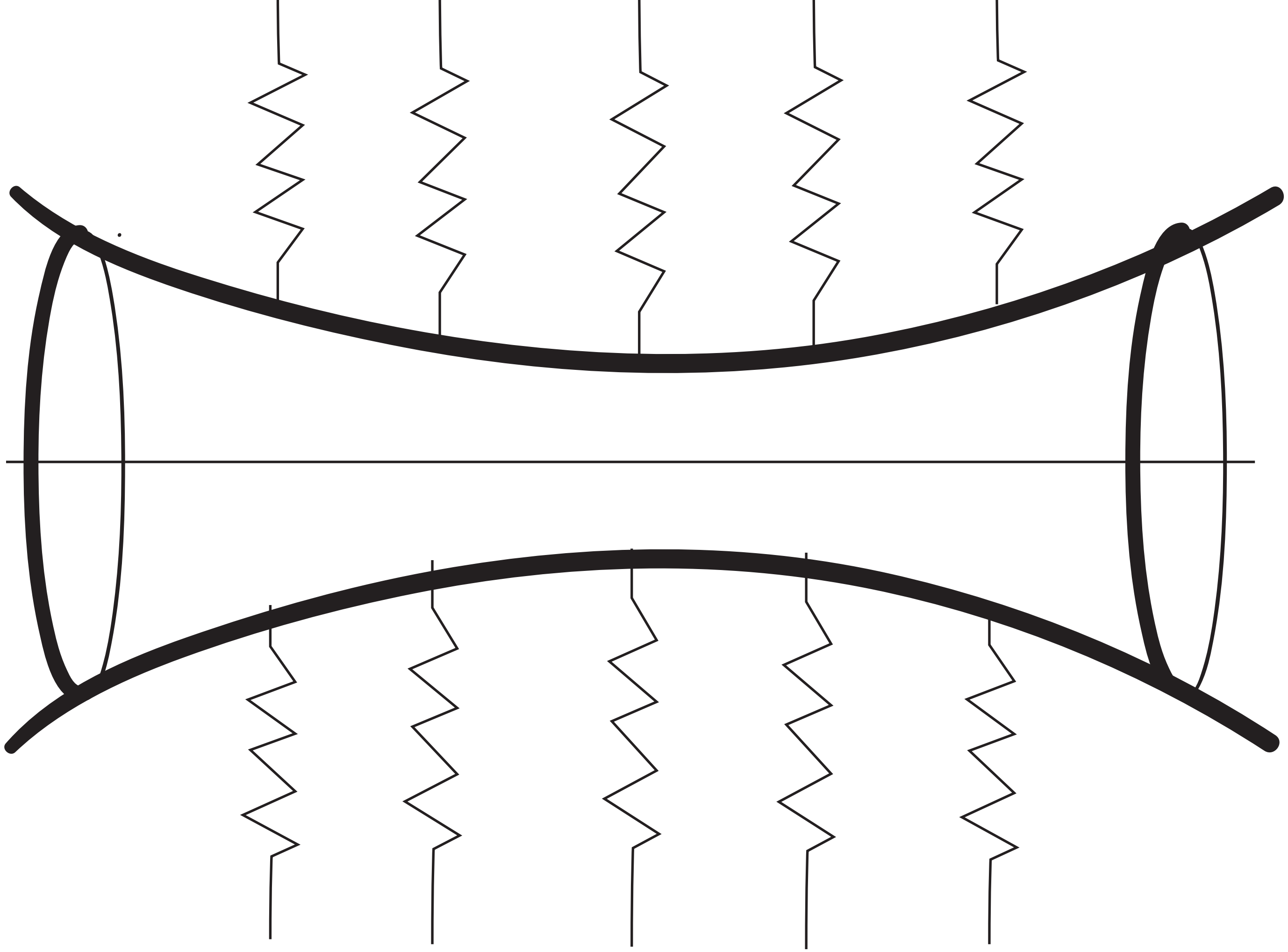


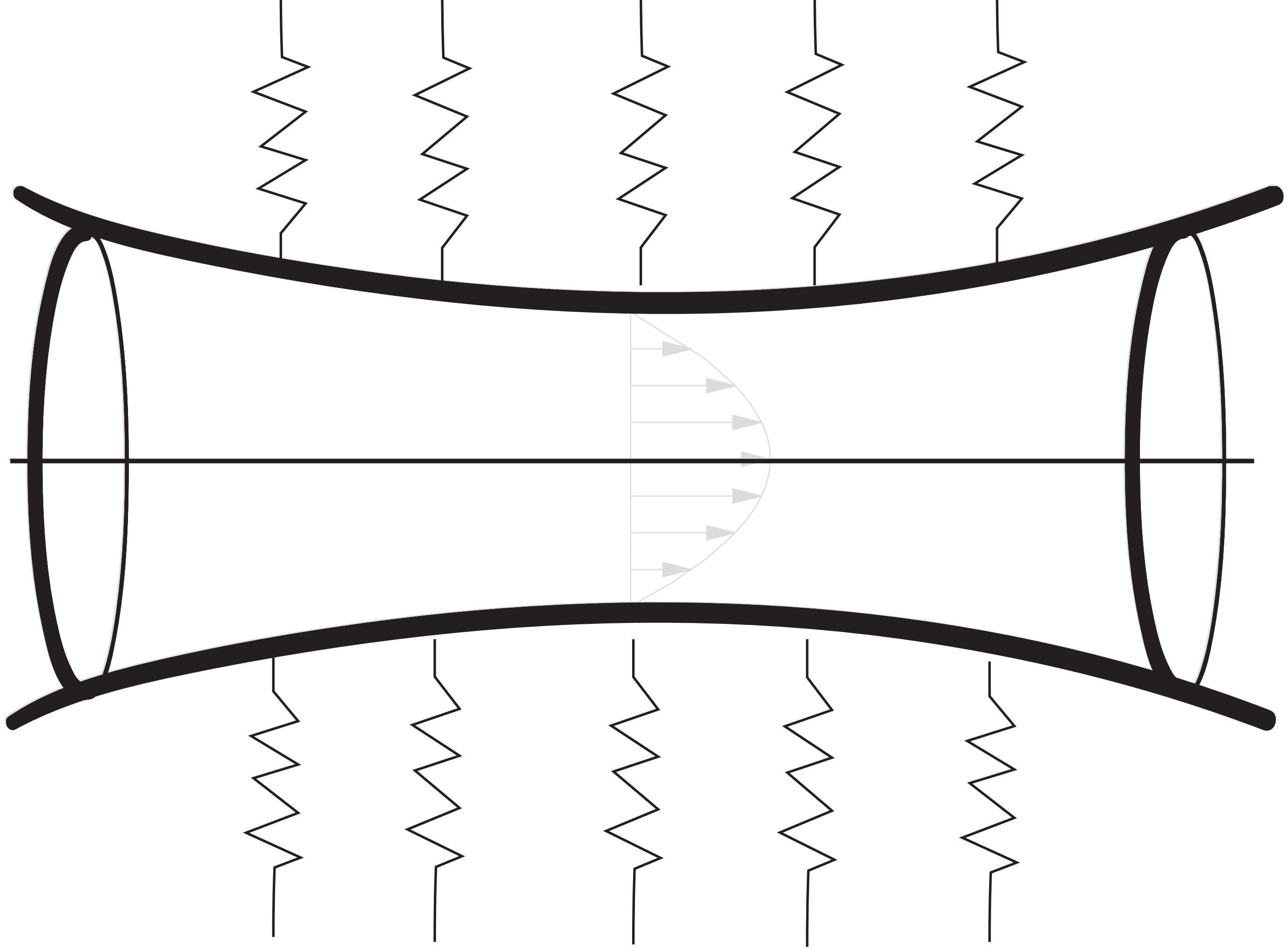
utilisons un modèle élastique simple

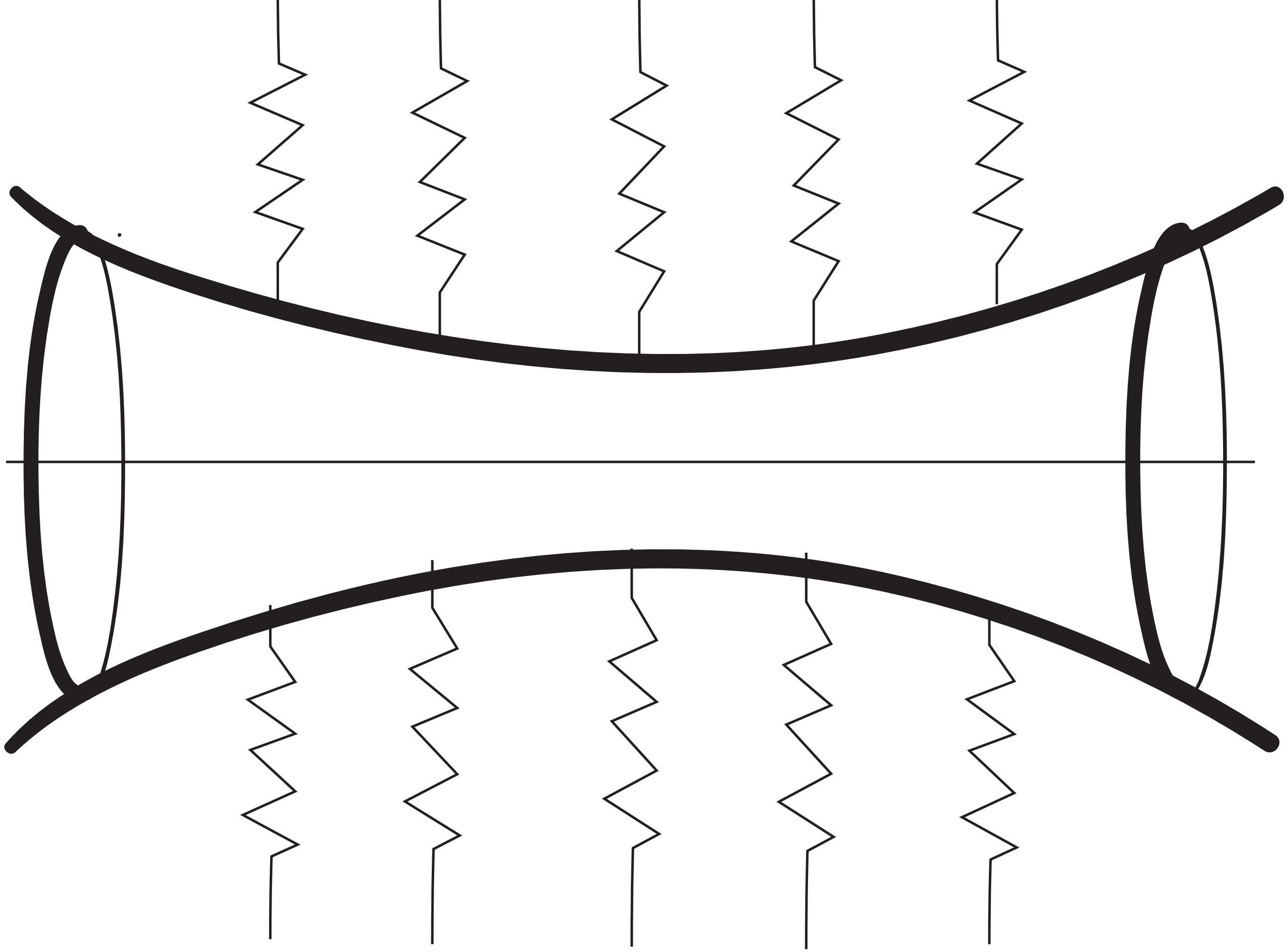


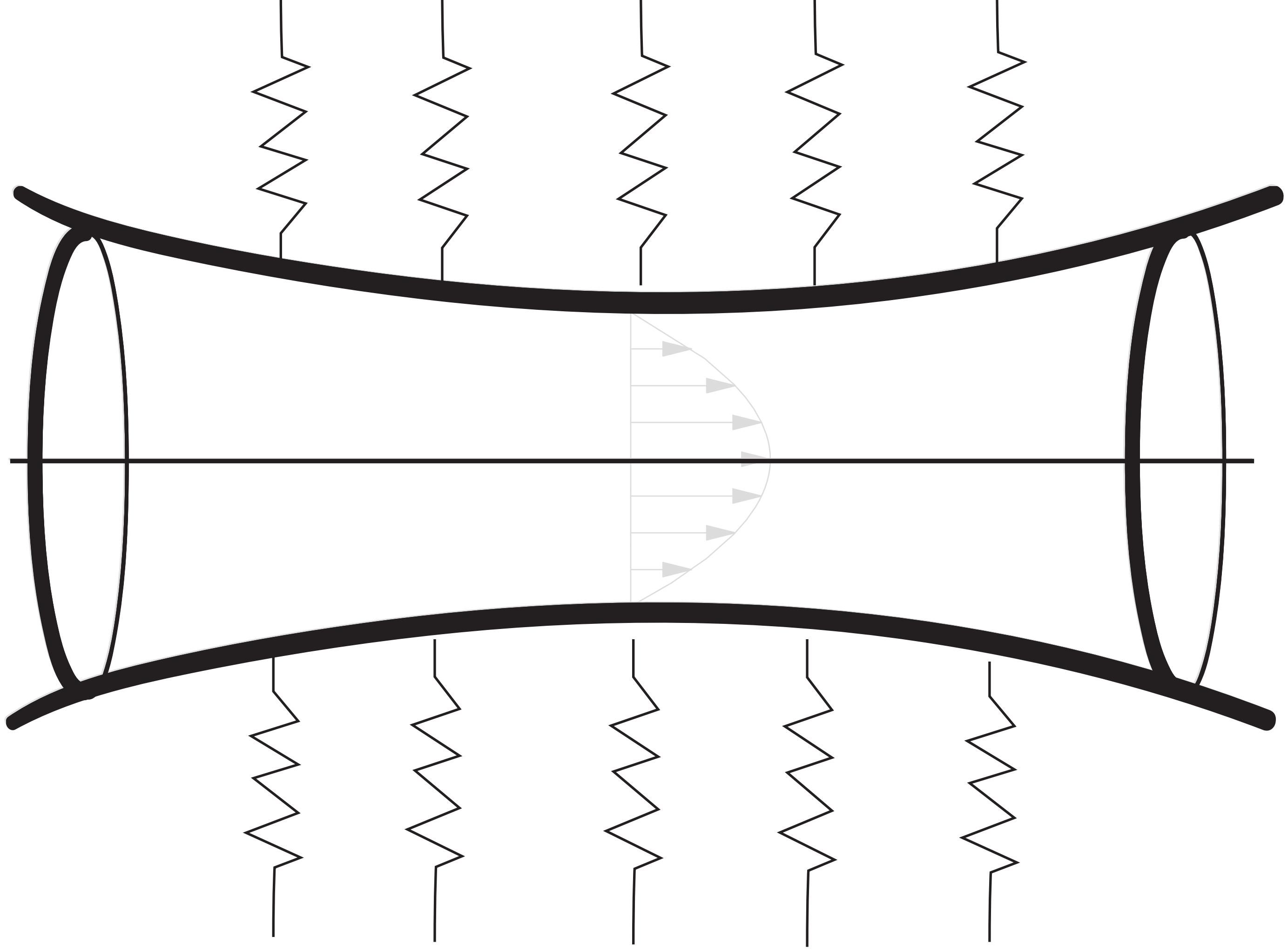
utilisons un modèle élastique simple



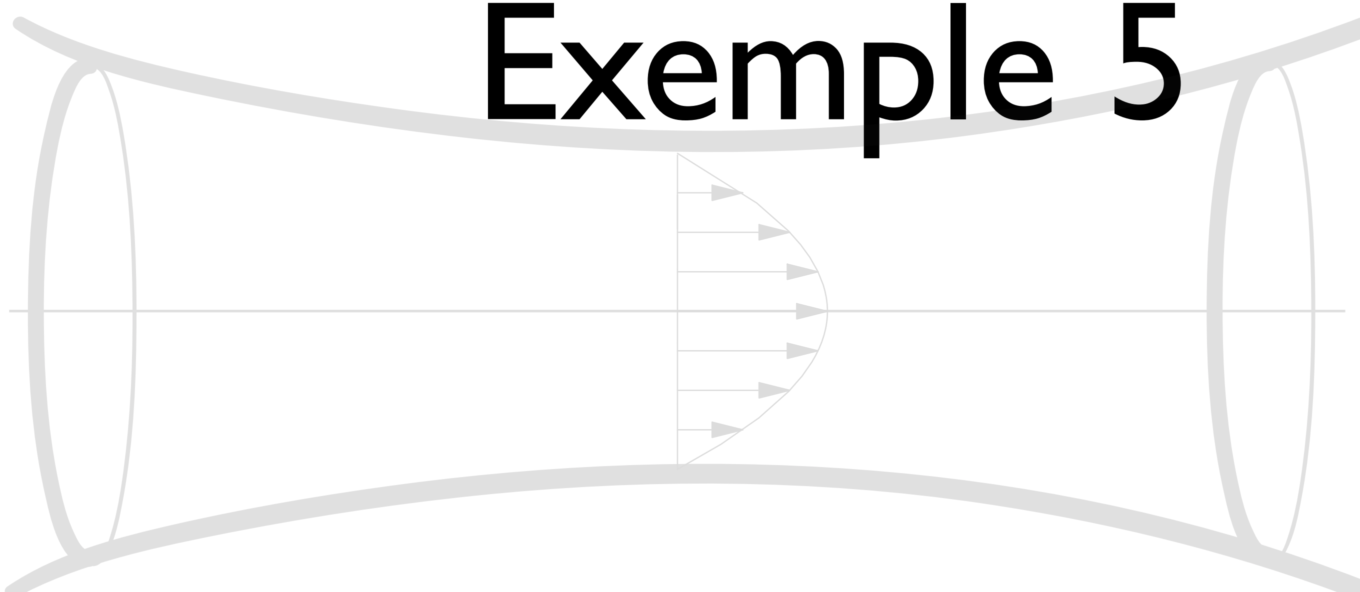




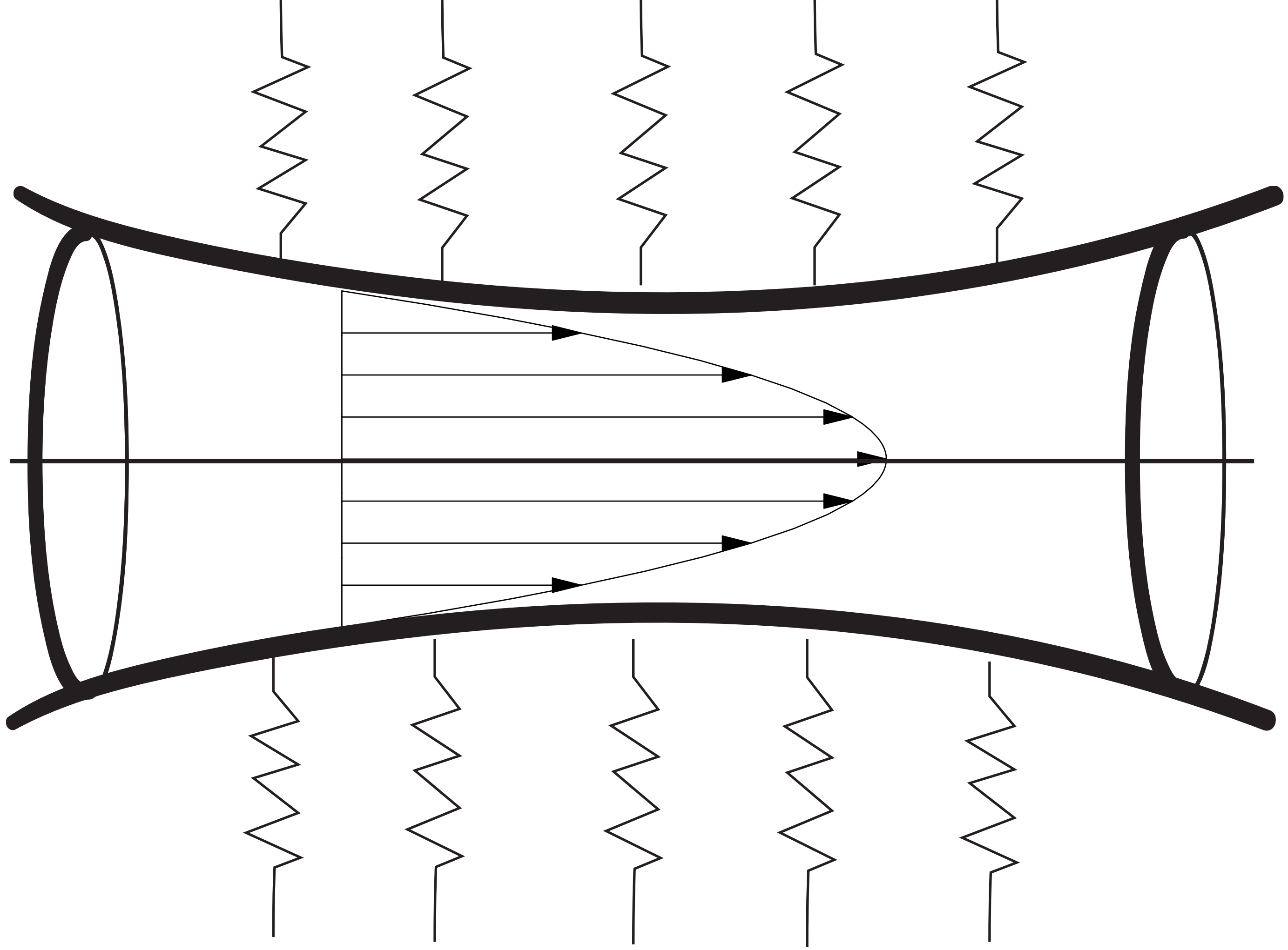




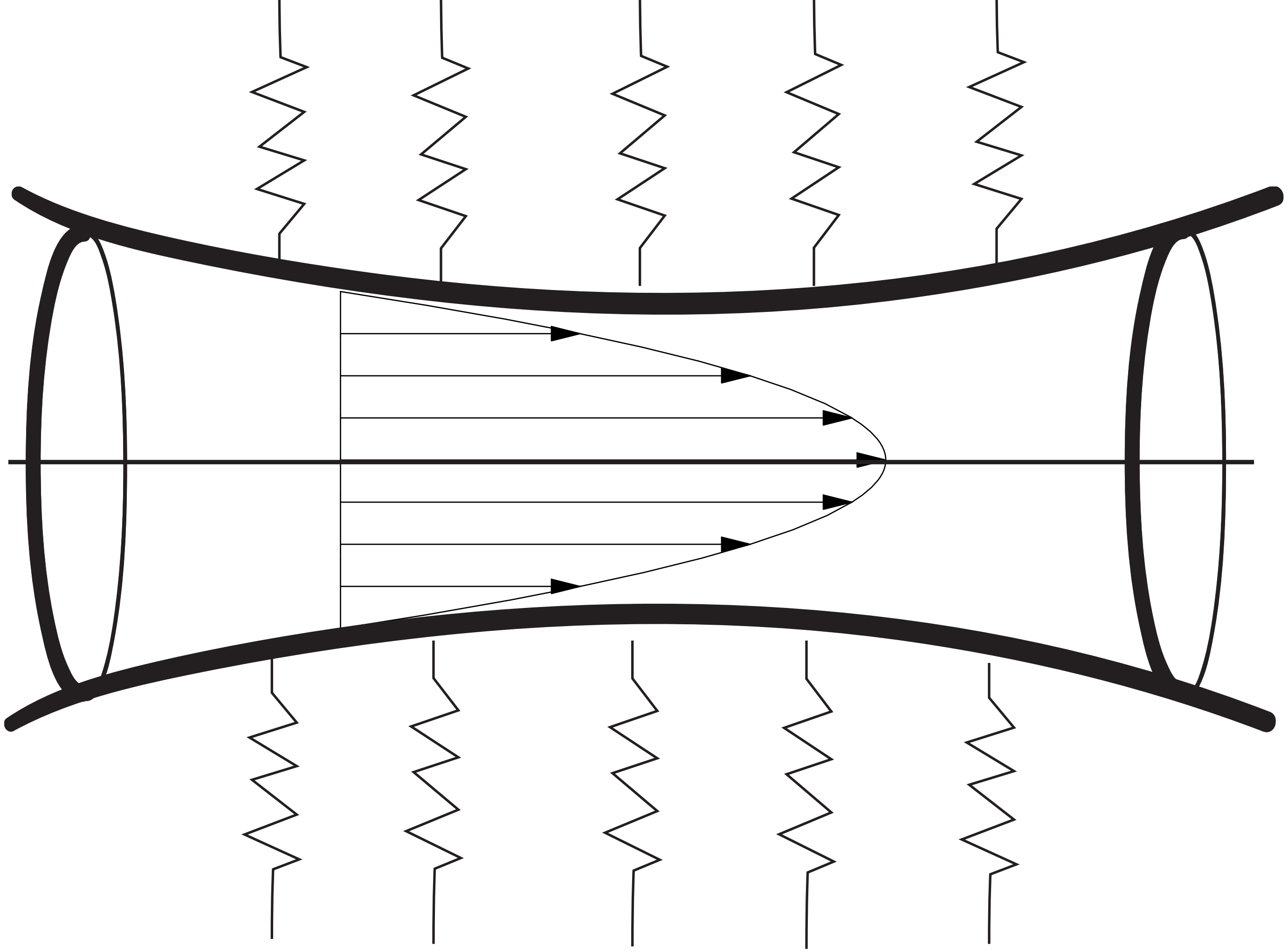
Exemple 5



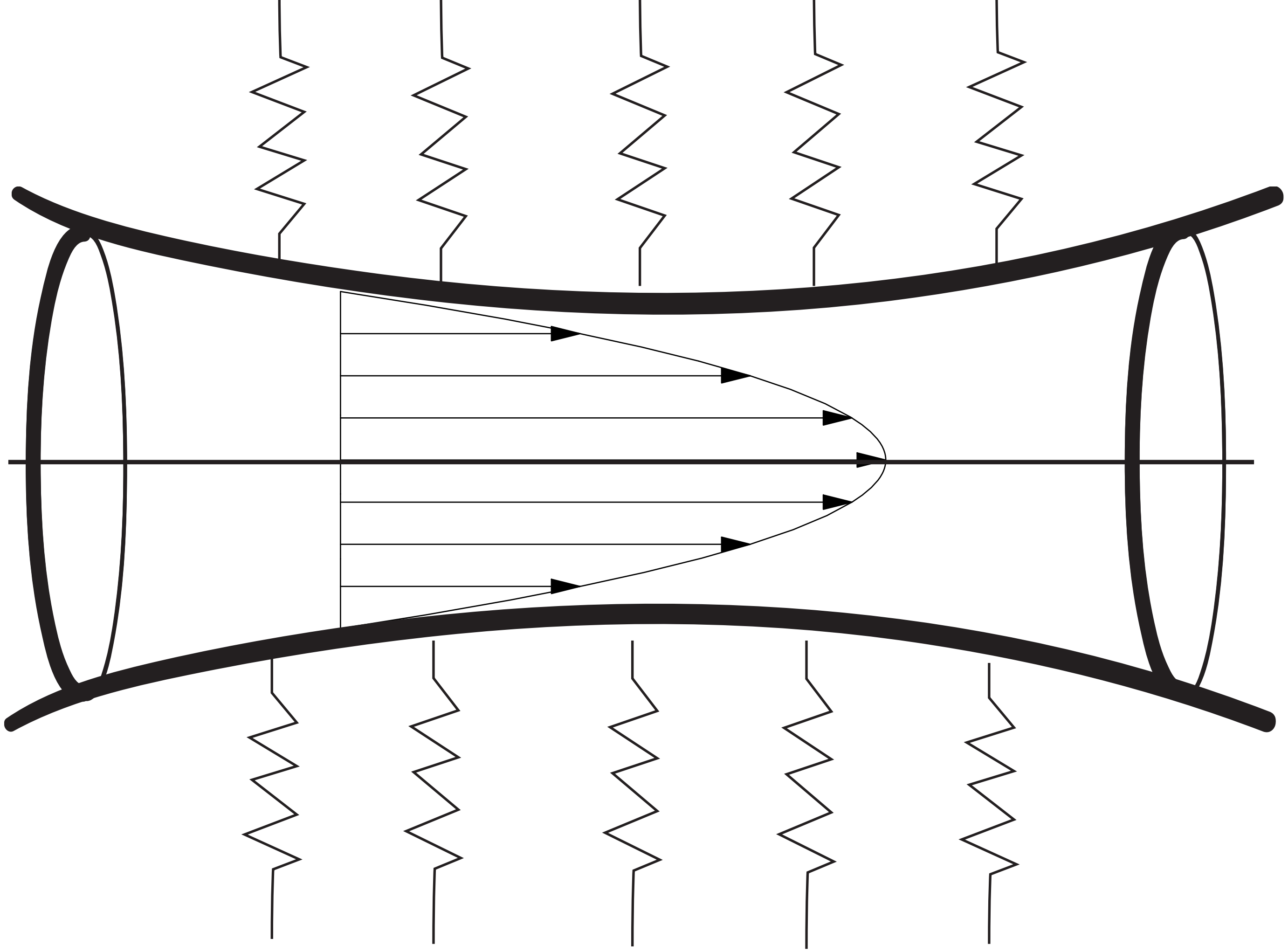
- Ecoulement dans un tuyau collable
- Instationnaire, paroi élastique, pas d’inertie



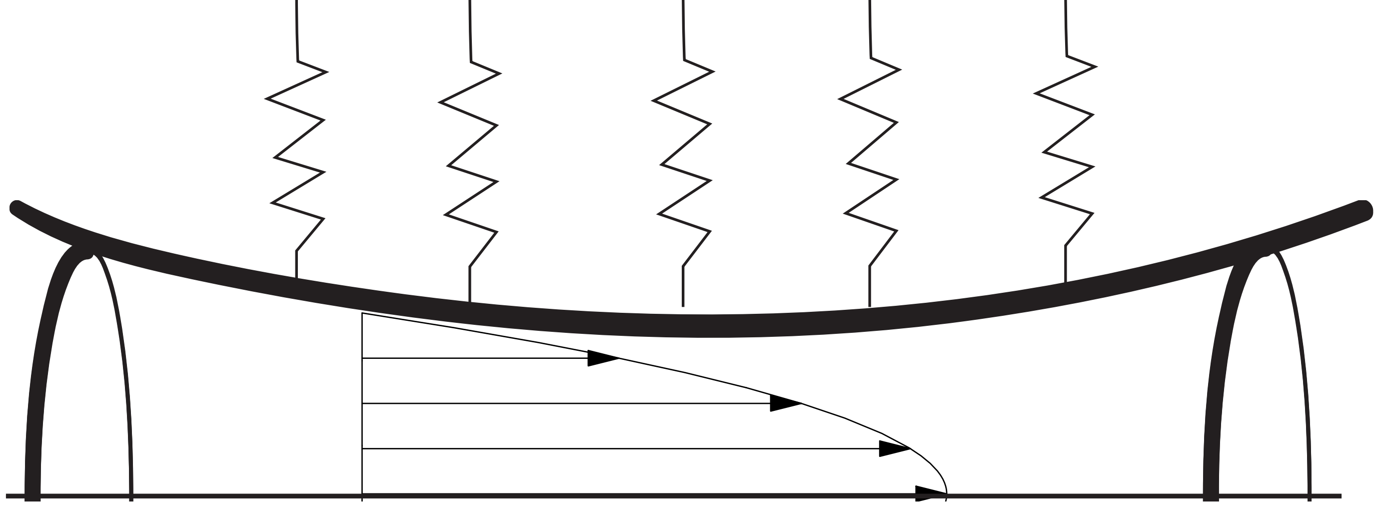
tuyau collable

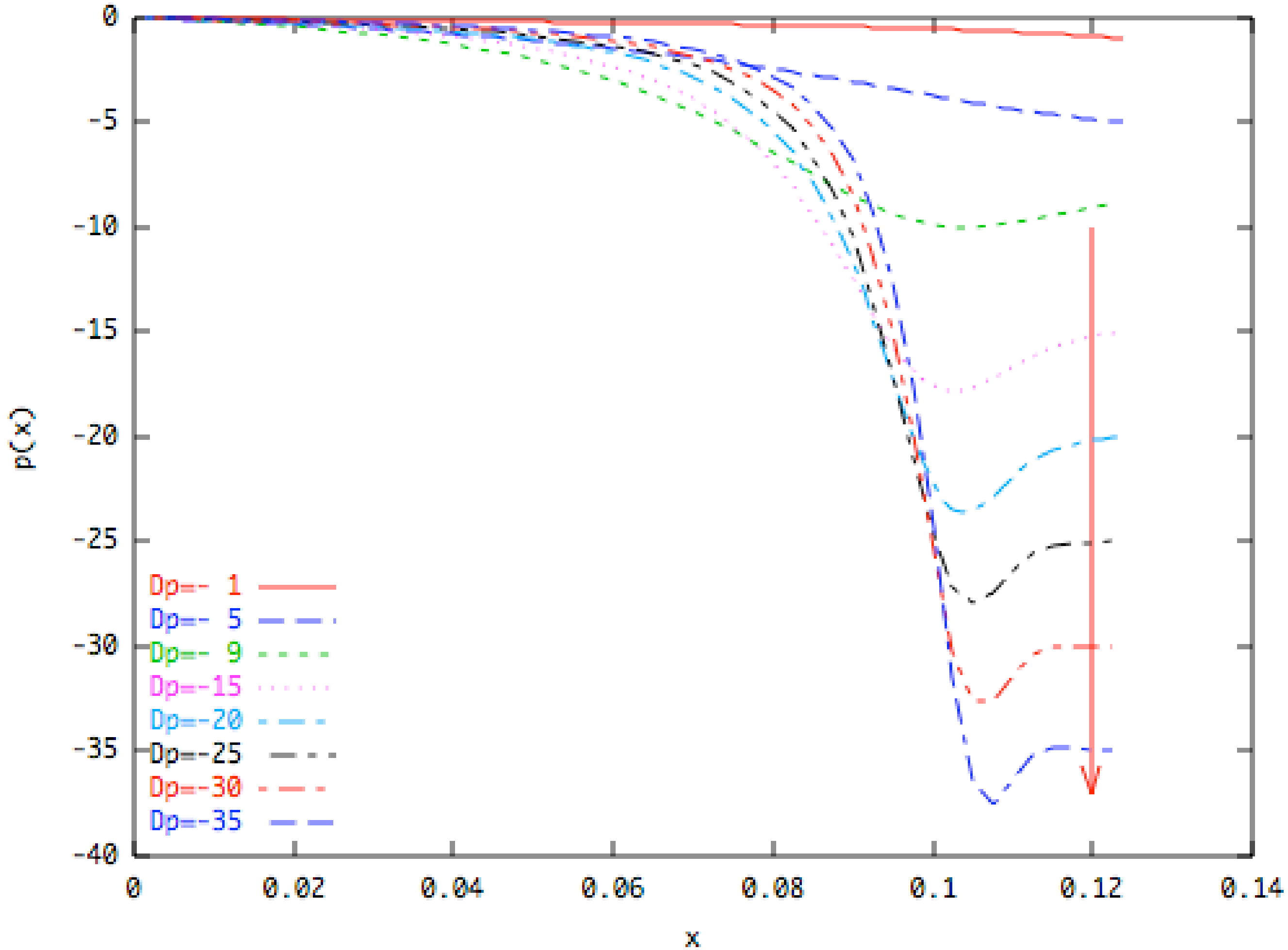


R^n donne p^{n+1}

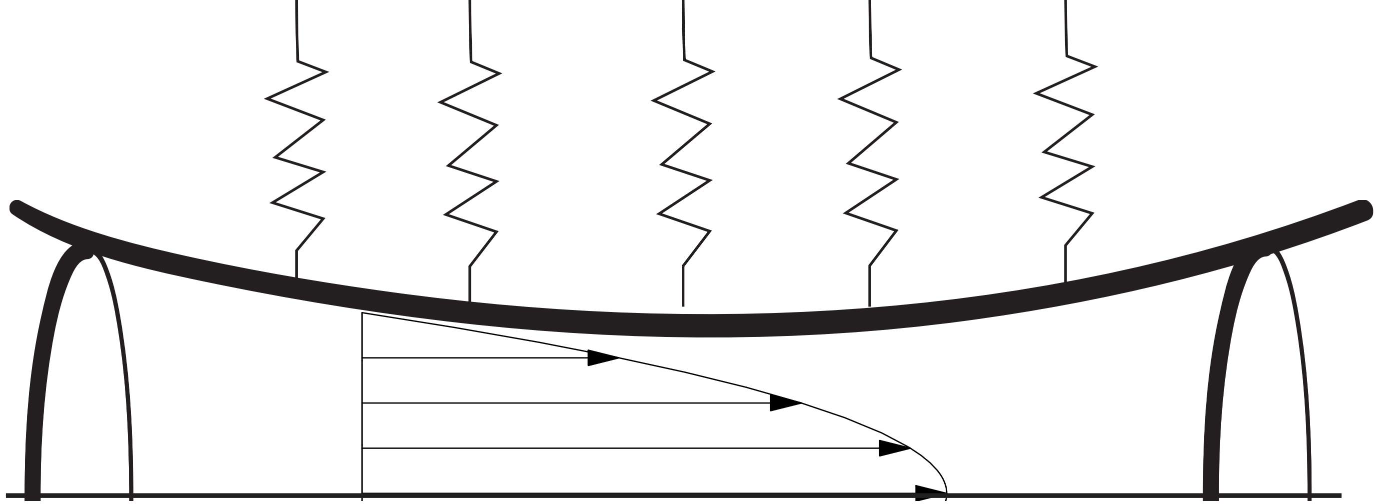


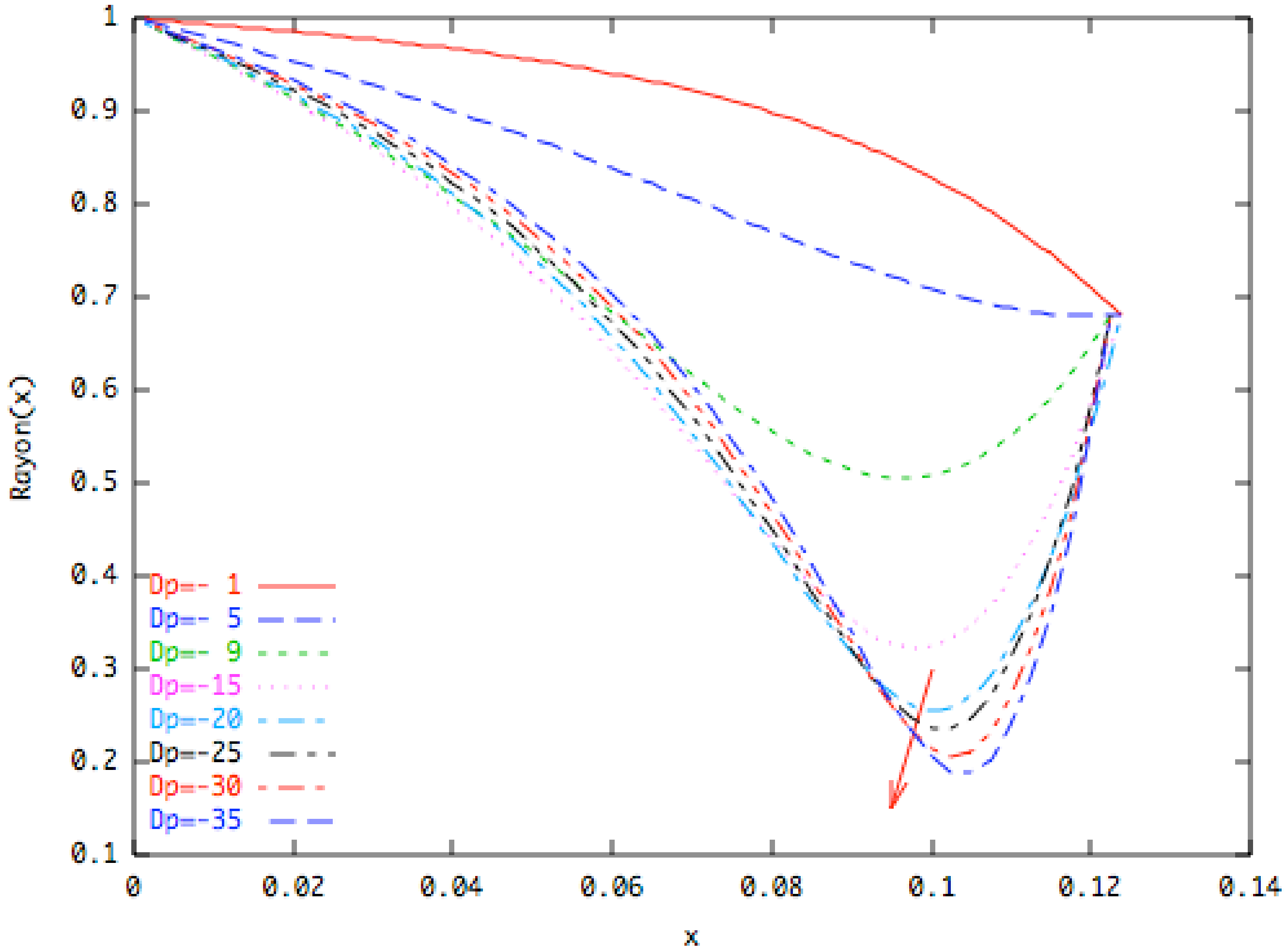
R^n donne p^{n+1} $\longrightarrow$ $p^{n+1} = k(R^{n+1} - 1)$



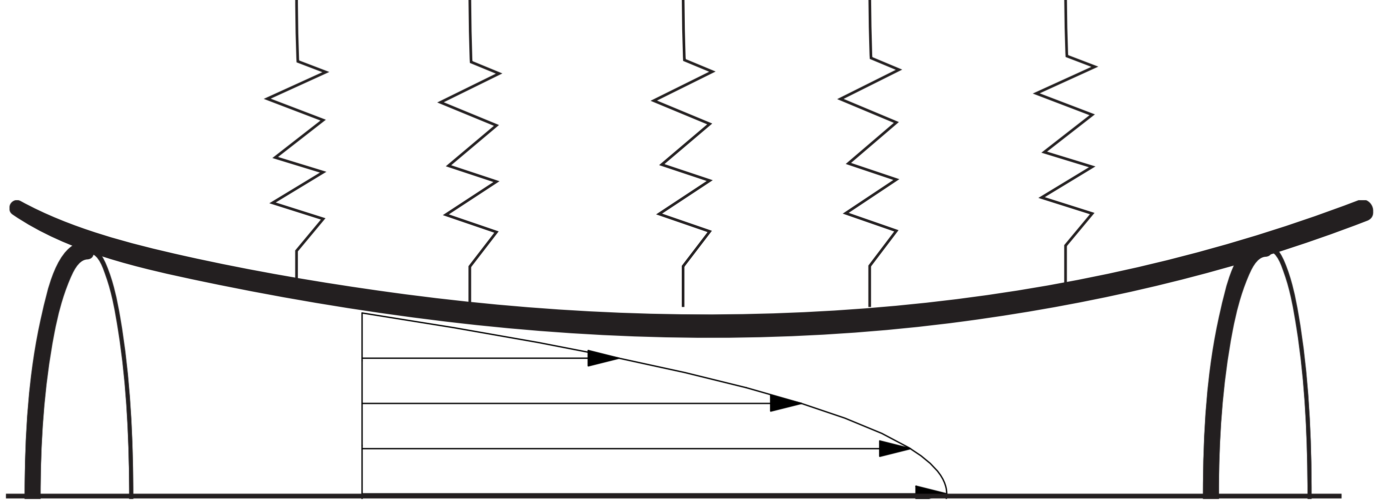


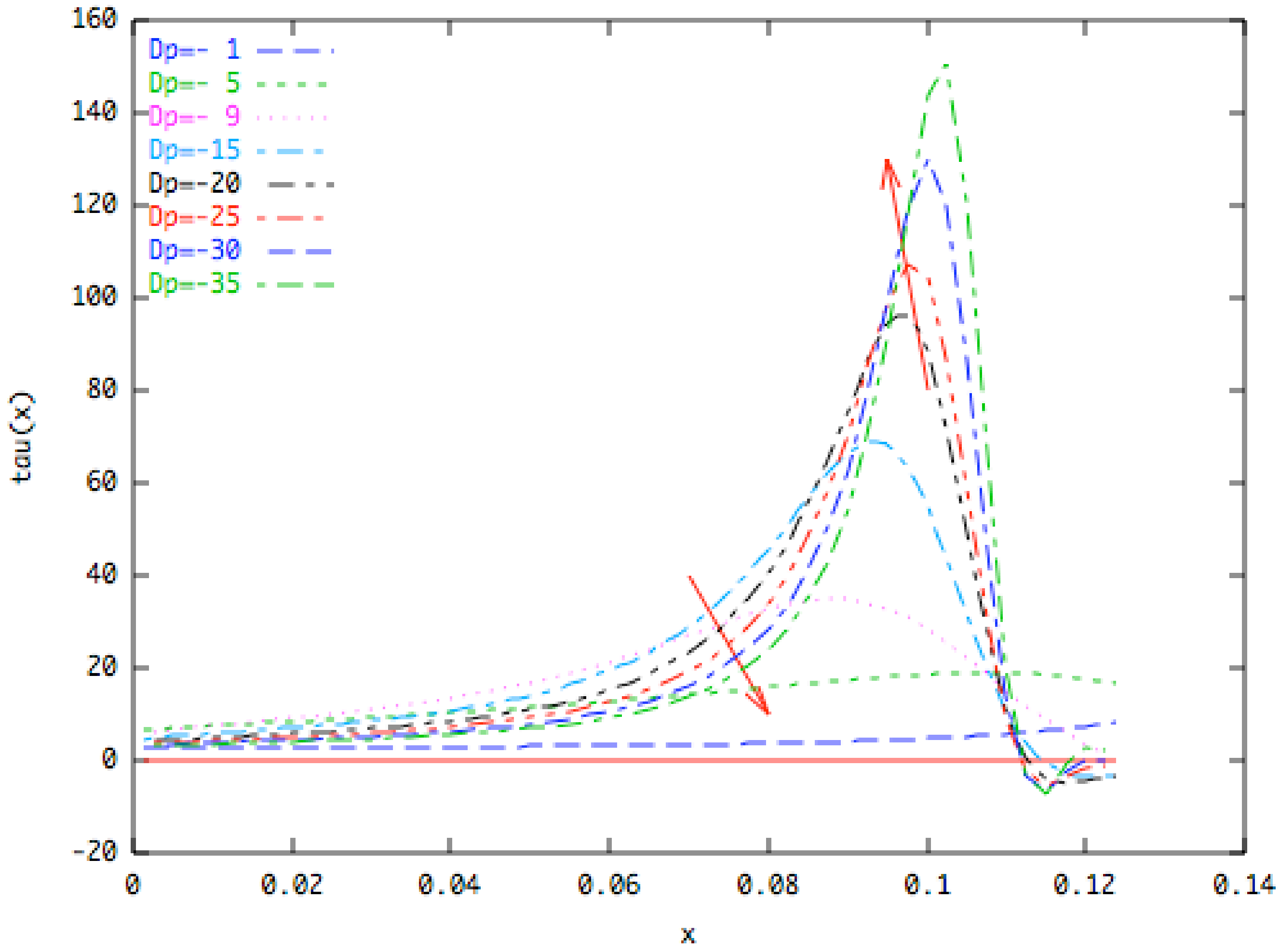
p(x)
x
0
-5
-10
-15
-20
-25
-30
-35
-40
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
Dp=- 1
Dp=- 5
Dp=- 9
Dp=-15
Dp=-20
Dp=-25
Dp=-30
Dp=-35



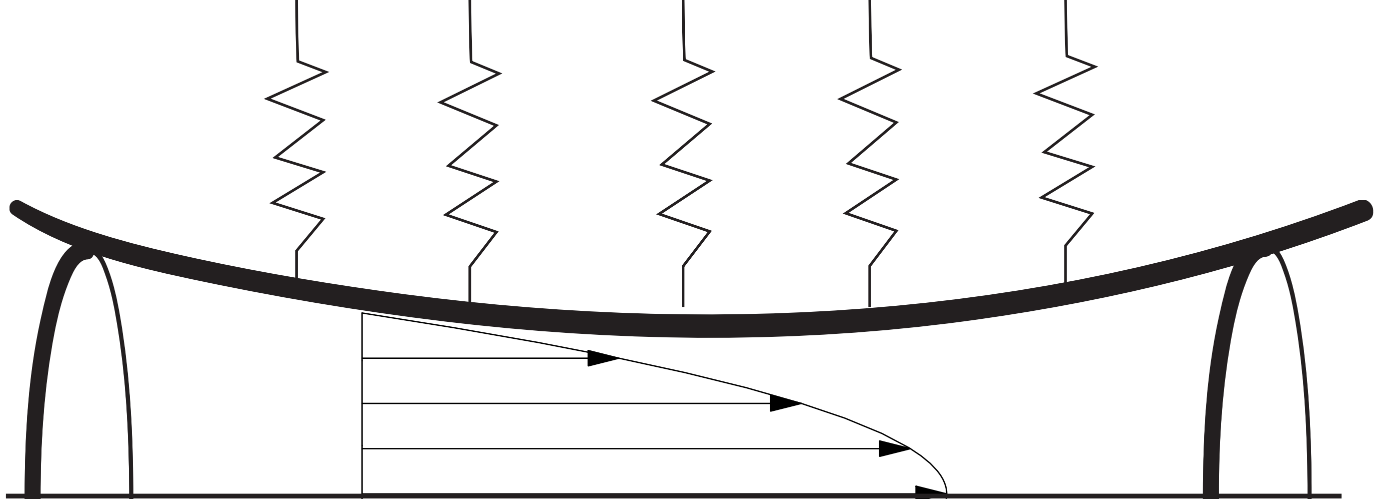


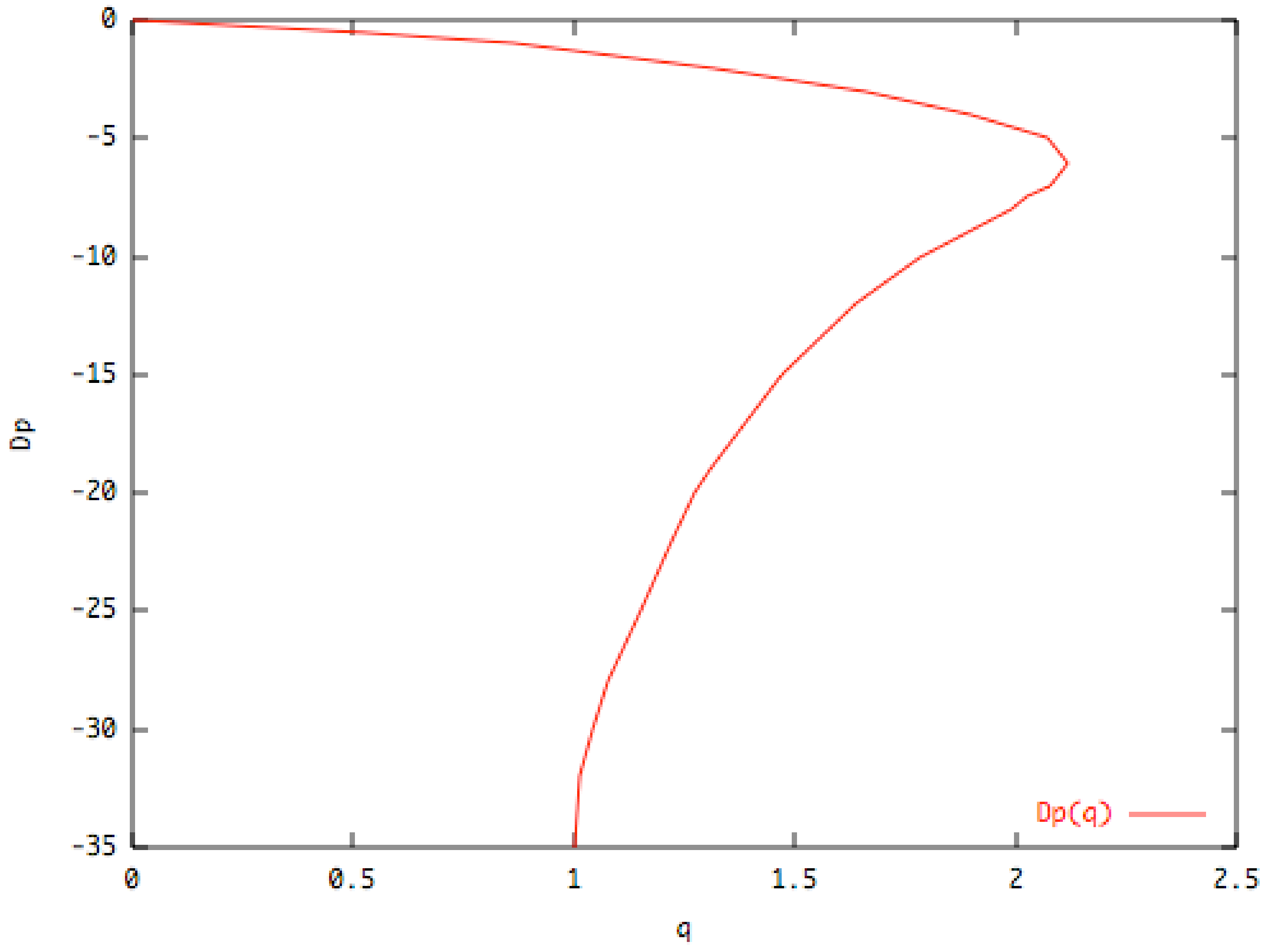
Rayon(x)
x
Dp=- 1
Dp=- 5
Dp=- 9
Dp=-15
Dp=-20
Dp=-25
Dp=-30
Dp=-35





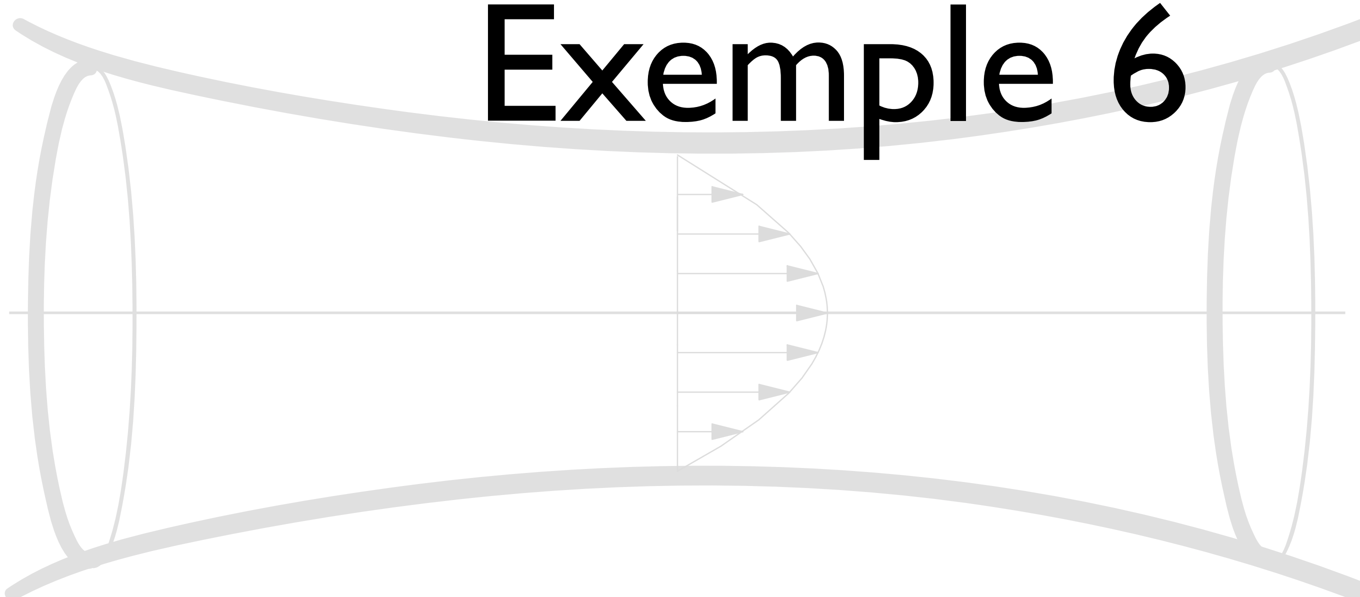
Dp=- 1
Dp=- 5
Dp=- 9
Dp=-15
Dp=-20
Dp=-25
Dp=-30
Dp=-35
tau(x)
x
160
140
120
100
80
60
40
20
0
-20
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14



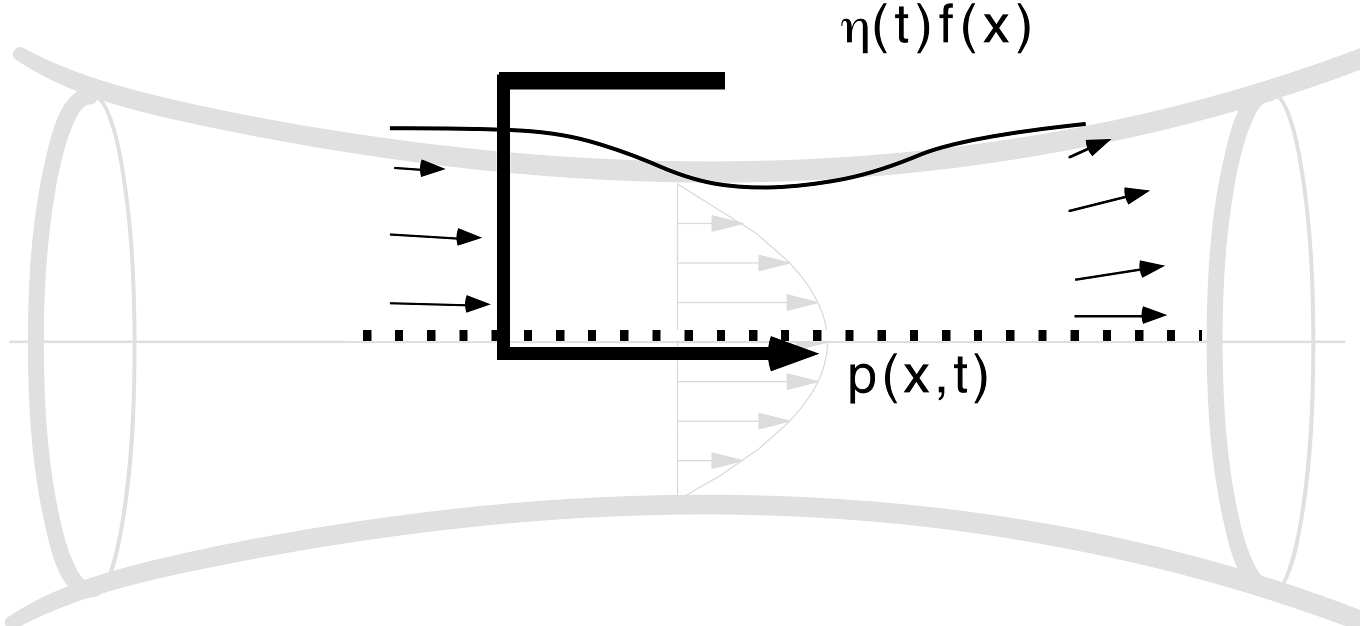


0
-5
-10
-15
-20
-25
-30
-35
Dp
0
0.5
1
1.5
2
2.5
q
Dp(q)

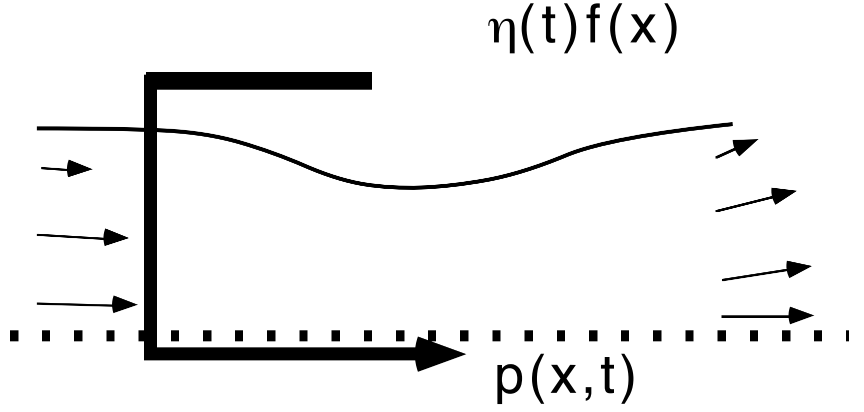
Exemple 6

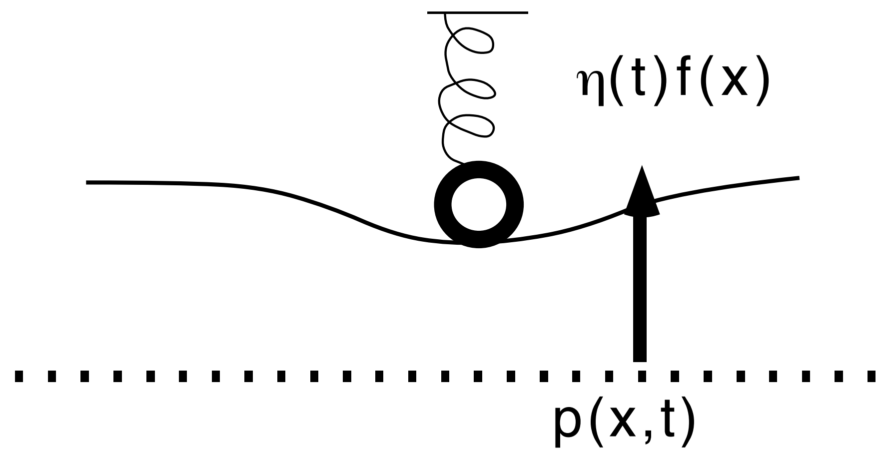


- Ecoulement avec paroi élastique avec masse (glotte?)

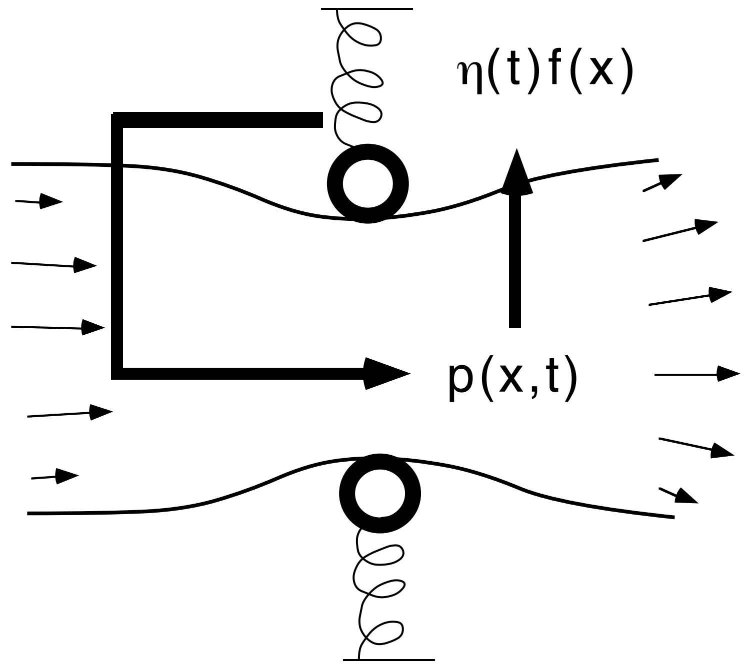


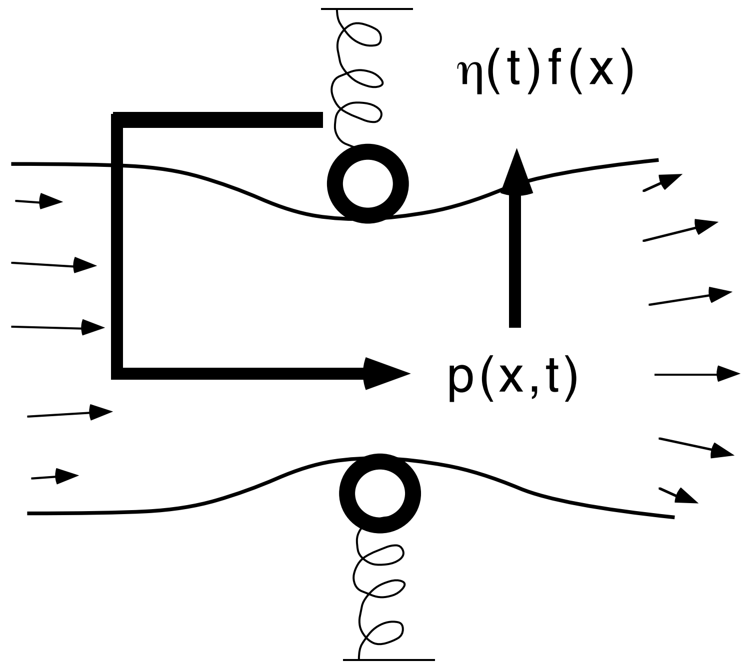
η(t)f(x)
p(x,t)


$\eta(t)f(x)$
$p(x,t)$

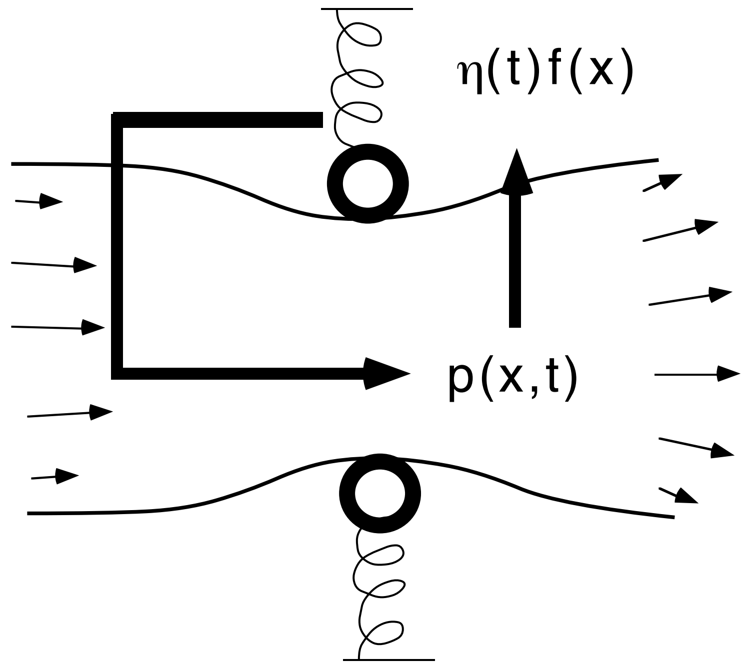


η(t)f(x)
p(x,t)

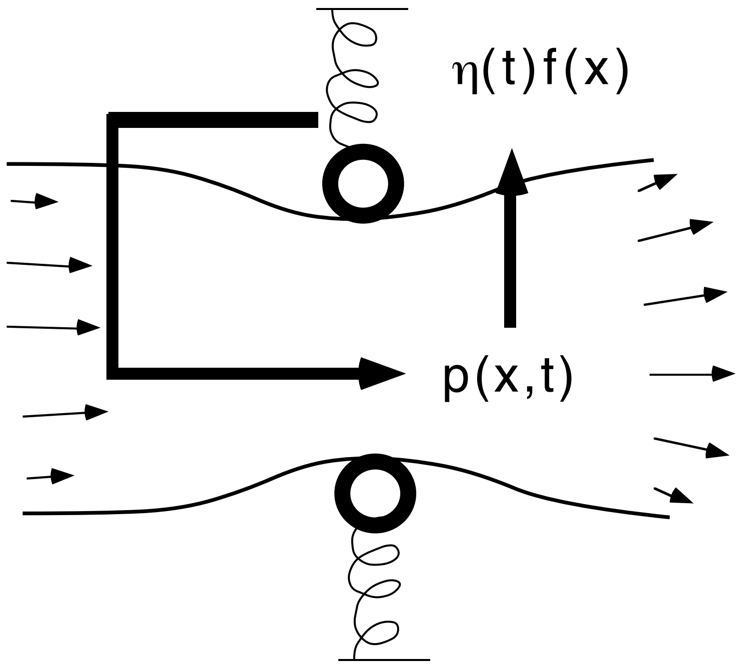

η(t) f(x)
p(x,t)



$$\mu \frac{\partial^2 \eta}{\partial t^2} + k\eta = -p$$

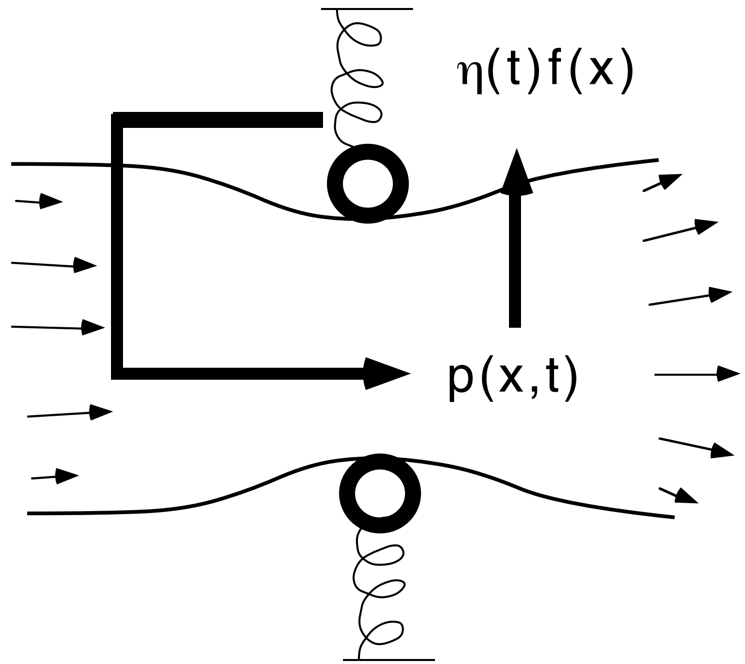


méthode de Newmark pour le ressort: prédiction/ correction



méthode de Newmark pour le ressort:
prédiction/ correction

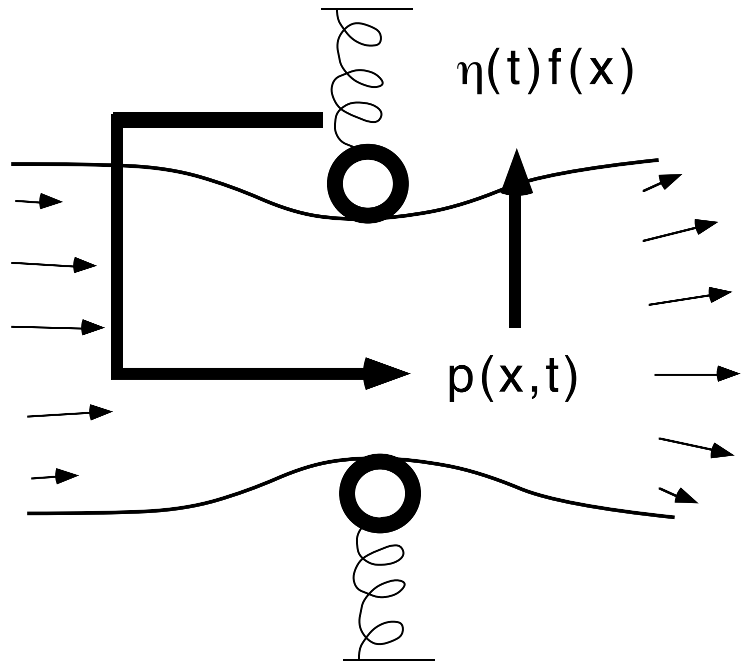
$$\eta^n, \frac{\partial \eta^n}{\partial t} \xrightarrow{\text{fluide}} p$$



méthode de Newmark pour le ressort:
prédiction/ correction

$$\eta^n, \frac{\partial \eta^n}{\partial t} \xrightarrow{\text{fluide}} p \qquad \eta^e, \frac{\partial \eta^e}{\partial t}$$

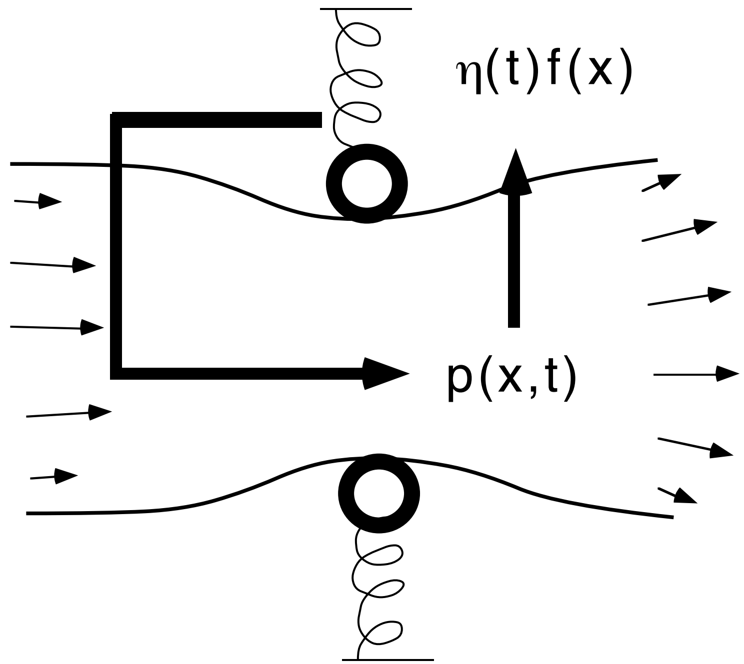
ressort-prédiction



méthode de Newmark pour le ressort:
prédiction/ correction

$$\eta^n, \frac{\partial \eta^n}{\partial t} \xrightarrow{\text{fluide}} p \qquad \eta^e, \frac{\partial \eta^e}{\partial t} \xrightarrow{\text{fluide}} p^e$$

ressort-prédiction

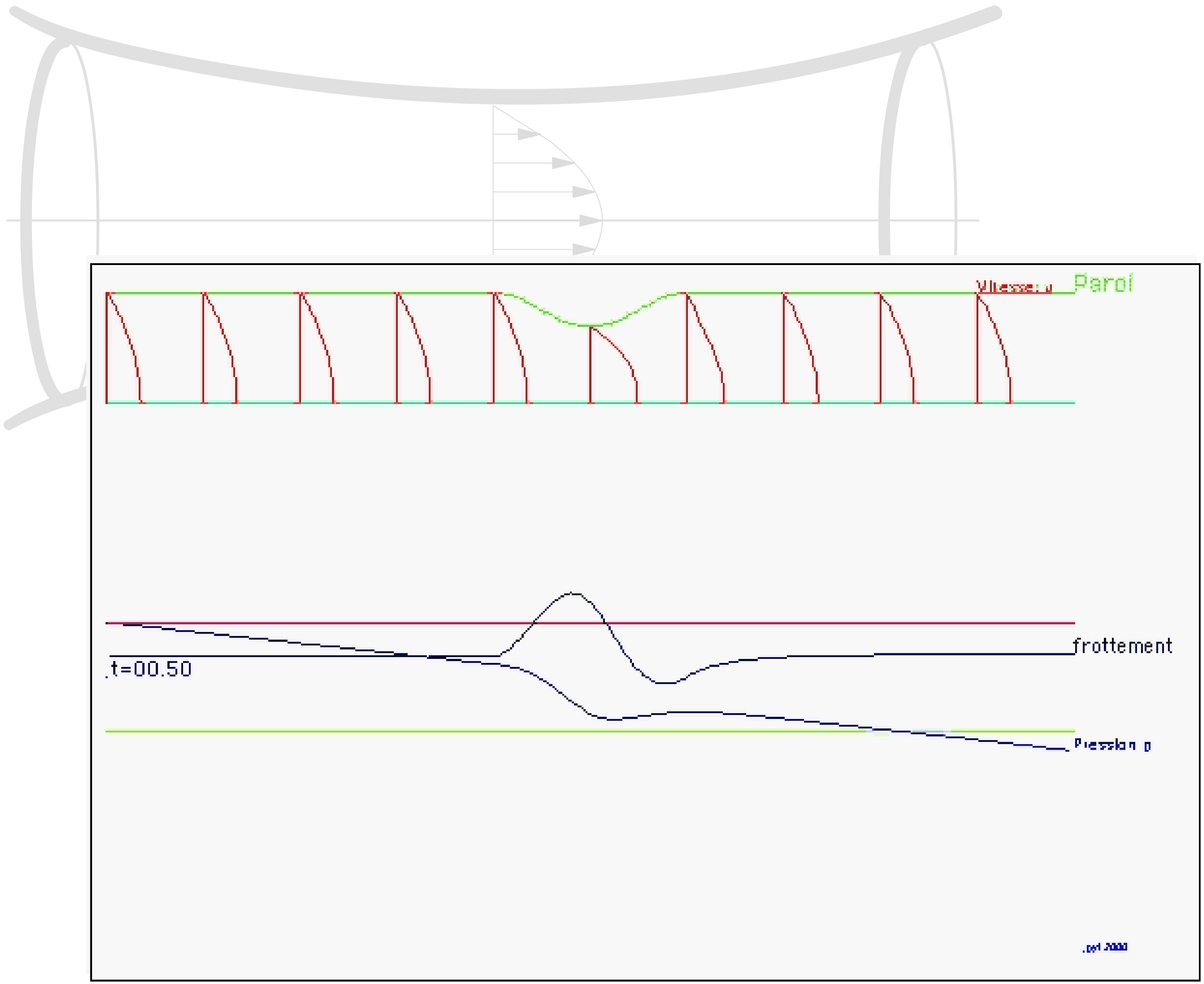


méthode de Newmark pour le ressort:
prédiction/ correction

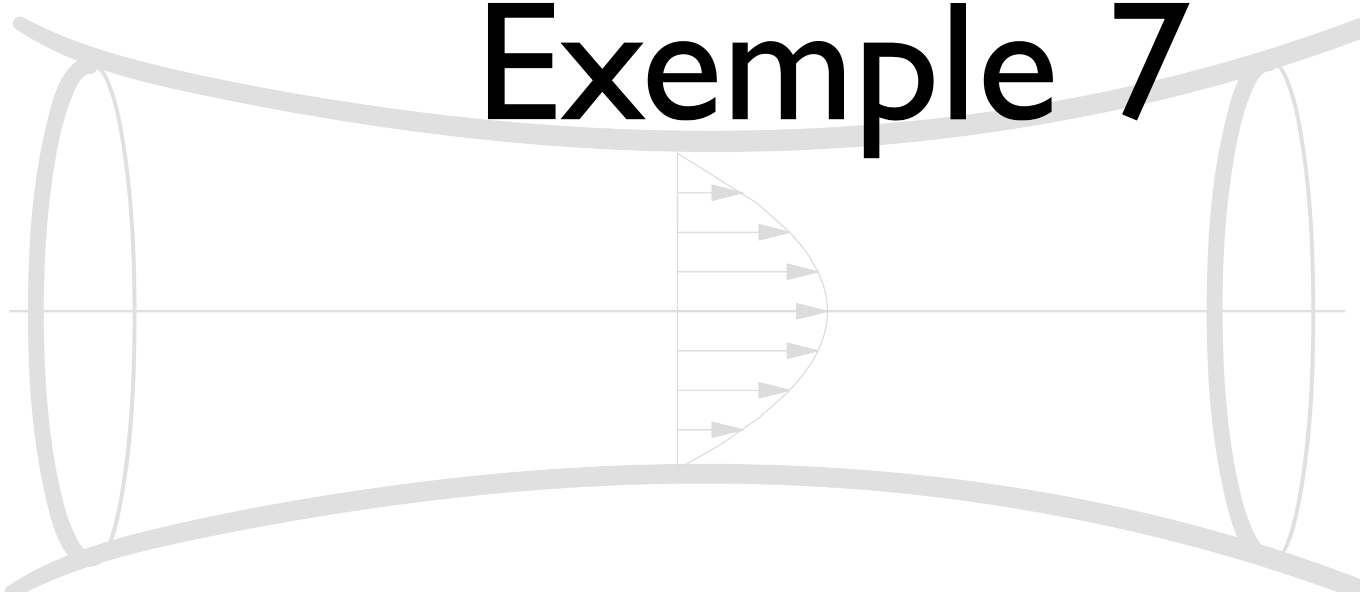
$$\eta^n, \frac{\partial \eta^n}{\partial t} \xrightarrow{\text{fluid}} p \qquad \eta^e, \frac{\partial \eta^e}{\partial t} \xrightarrow{\text{fluide}} p^e \qquad \eta^{n+1}, \frac{\partial \eta^{n+1}}{\partial t}$$

ressort-prédiction

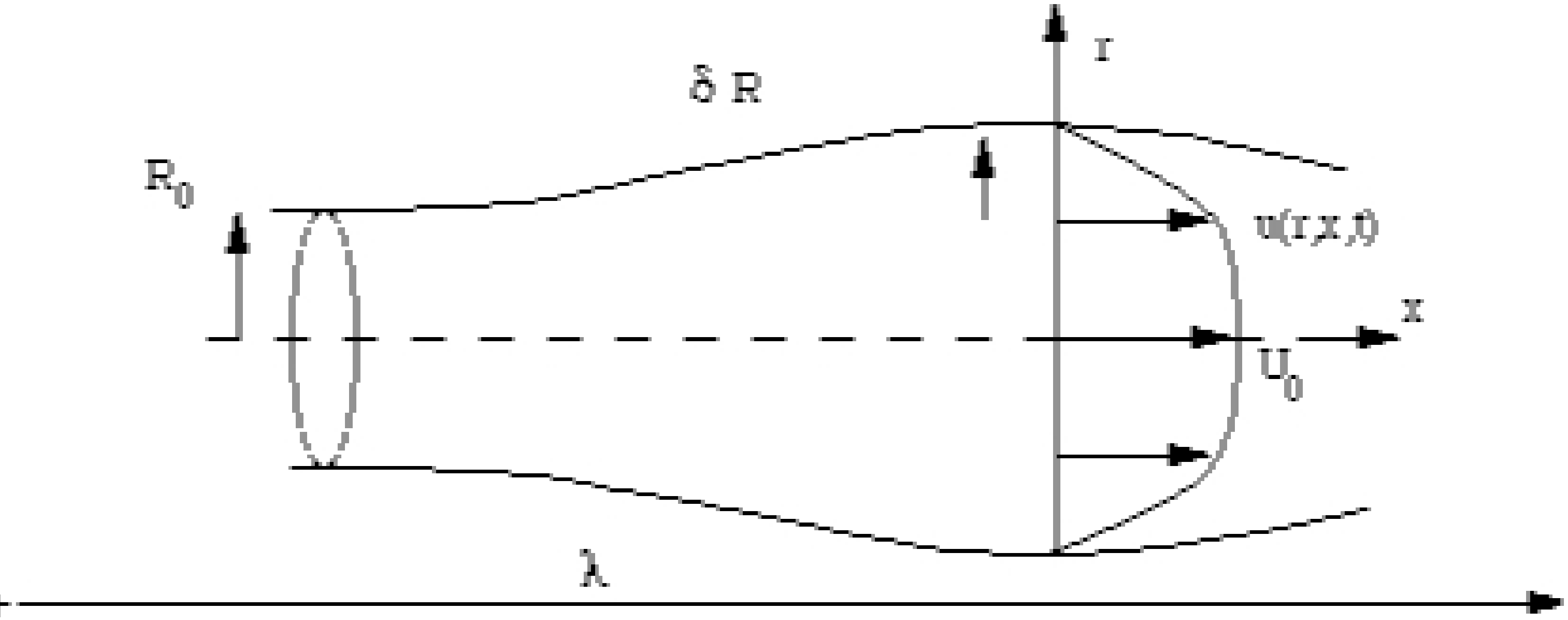
ressort- correction


Vitesse u
Paroi
frottement
t=00.50
Pression p

Exemple 7

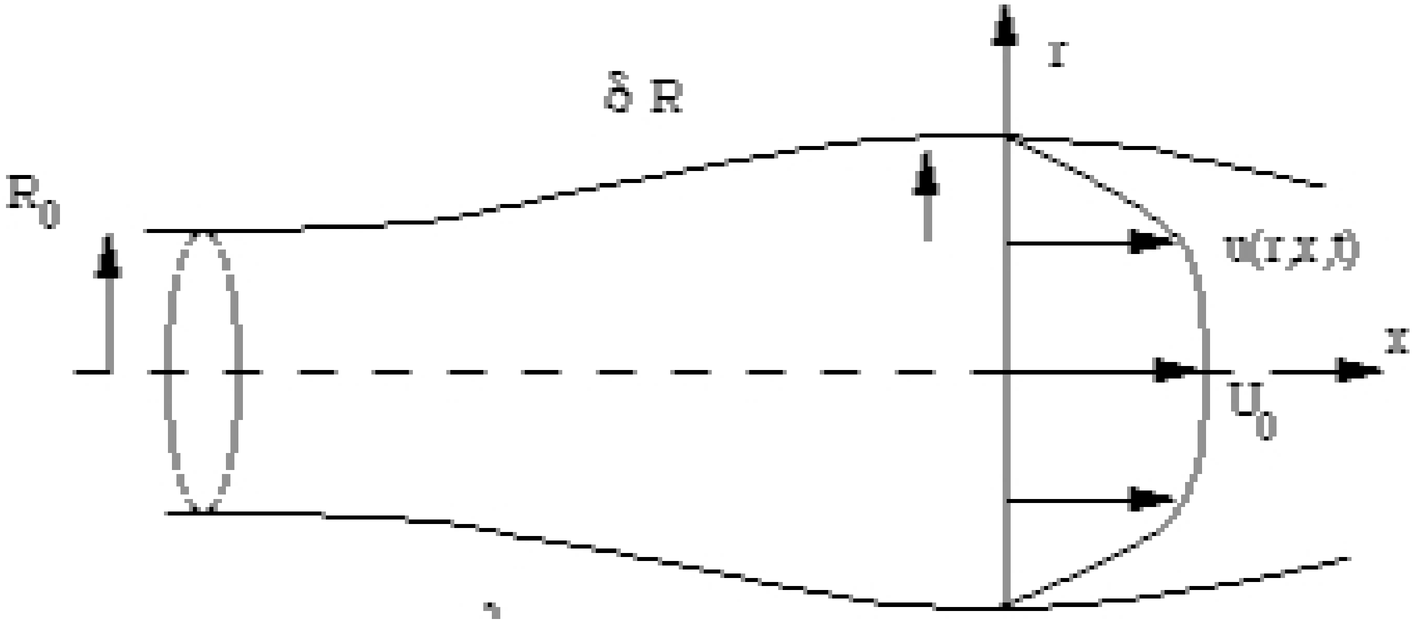


écoulement dans les artères



Divergence de la vitessse: $\frac{U_0}{\lambda} \sim \frac{(\delta R/T)}{R_0}$
Conservation de qt. mvt.: $\rho\frac{U_0}{T} \sim k\frac{\delta R}{\lambda}$ donc $\lambda = T\sqrt{(}(kR_0)/\rho)$

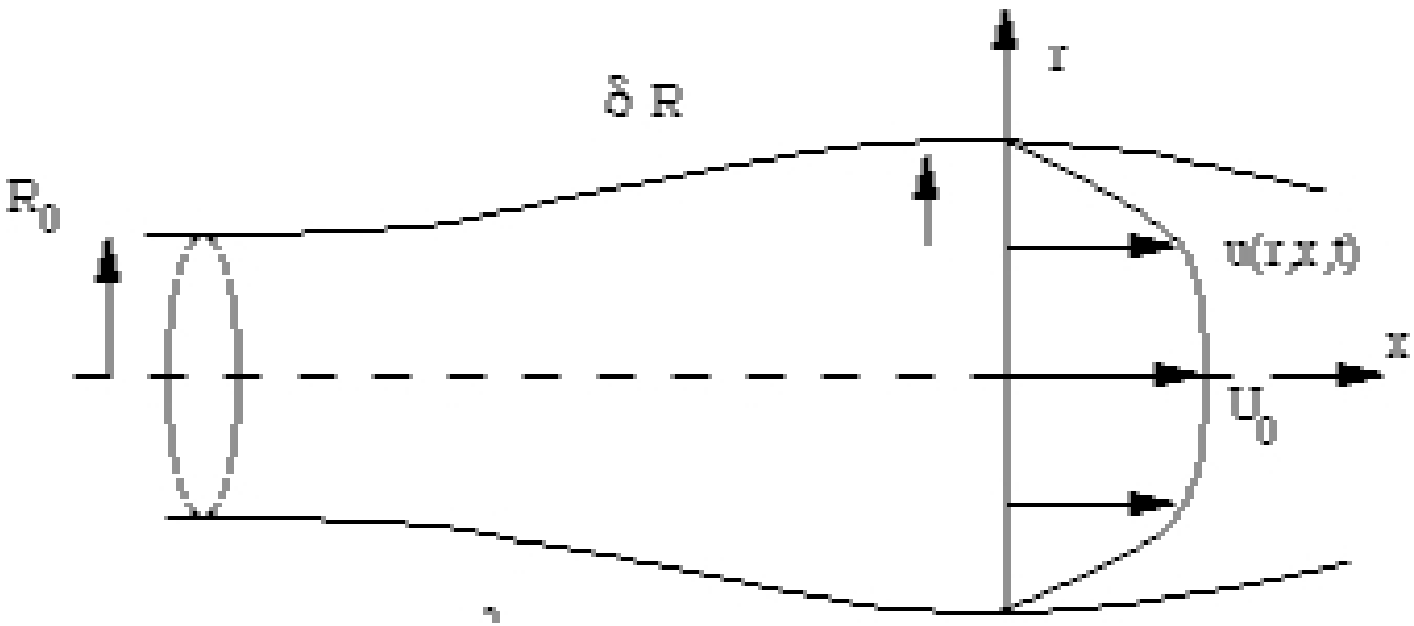
Non linéarité $(\frac{U_0^2}{\lambda})/(\frac{U_0}{T}) = \frac{\delta R}{R_0}$
Viscosité: $(\mu\frac{U_0}{R_0^2})/(\rho\frac{U_0}{T}) = \nu\frac{T}{R_0^2}$



$$\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv = 0,$$

$$\frac{\partial u}{\partial t} + \varepsilon_2(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) = -\frac{\partial p}{\partial x} + \frac{2\pi}{\alpha^2}\frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), 0 = -\frac{\partial p}{\partial r}.$$

$$\varepsilon_2 = \frac{\delta R}{R_0}, \qquad \alpha = R_0\sqrt{\frac{2\pi/T}{\nu}}$$



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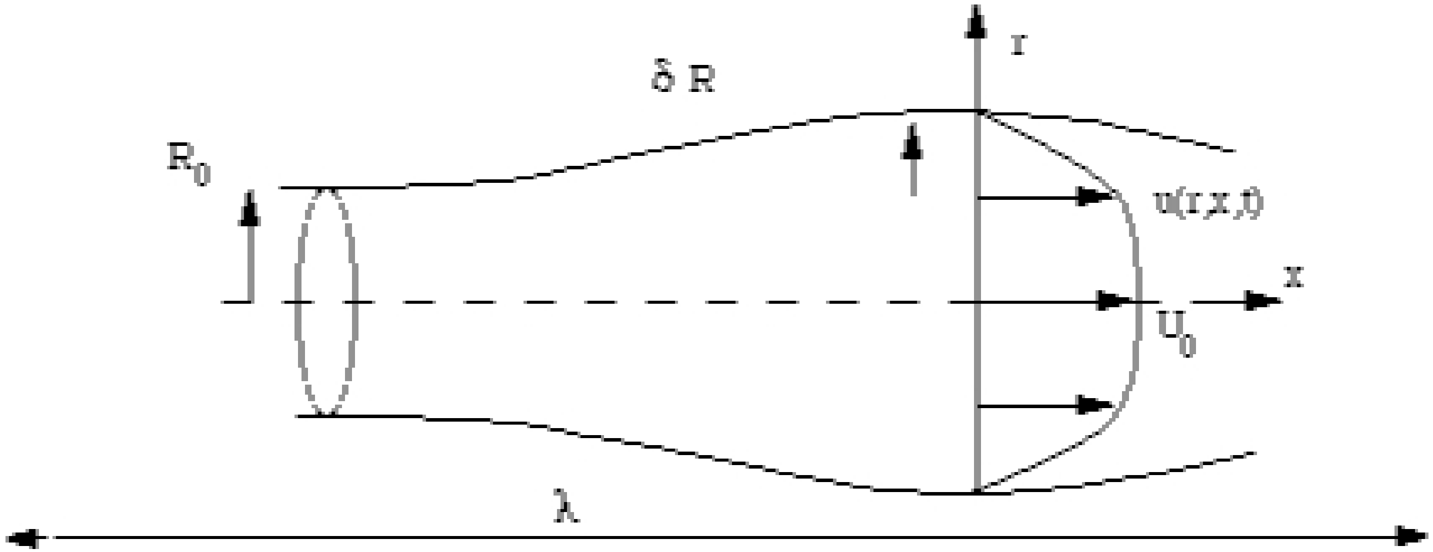
$$\varepsilon_2 = \frac{\delta R}{R_0}, \qquad \alpha = R_0\sqrt{\frac{2\pi/T}{\nu}}$$

élasticité de la paroi: $p(x,t) = k(R(x,t) - R_0)$

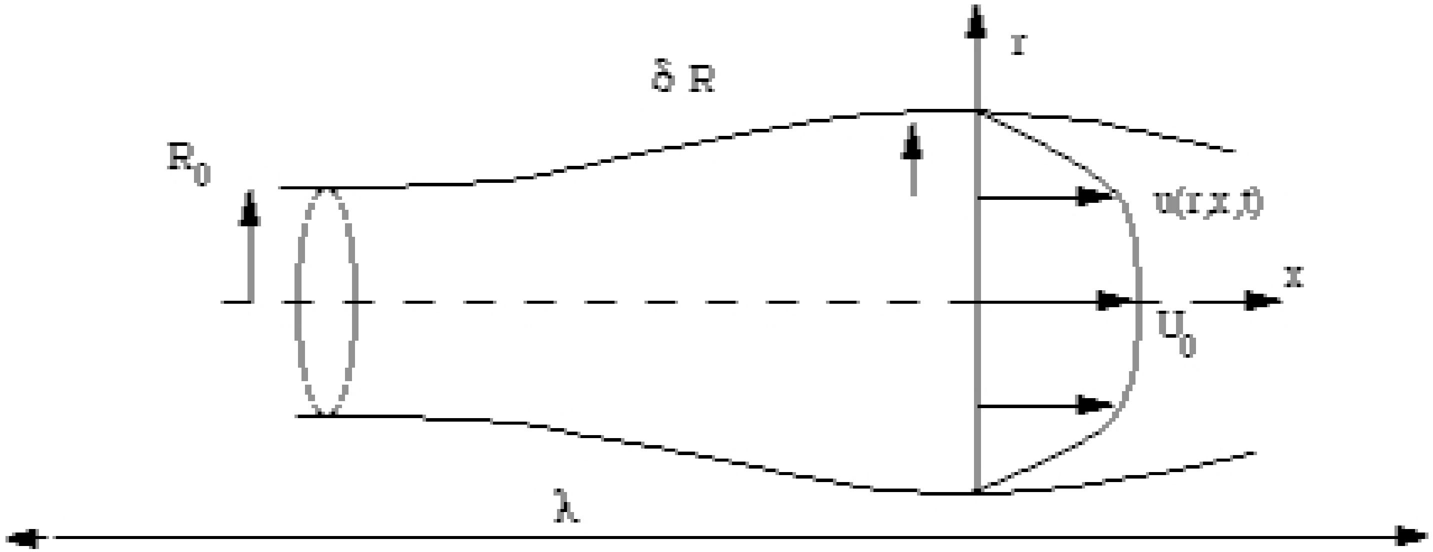
+ Conditions aux limites: ici hyperboliques($R(x_{in},t)$ and $R(x_{out},t)$) données.



couplage faible

$$\frac{u^{n+1}-u^n}{\Delta t}+NL(u^n)=-\frac{\partial p^n}{\rho\partial x}+\nu\frac{\partial}{r\partial r}r\frac{\partial u^{n+1}}{\partial r}$$

$$v^{n+1}(R^n)=-\int_0^{R^n}\frac{r}{R^n}\frac{\partial u^{n+1}}{\partial x}dr$$



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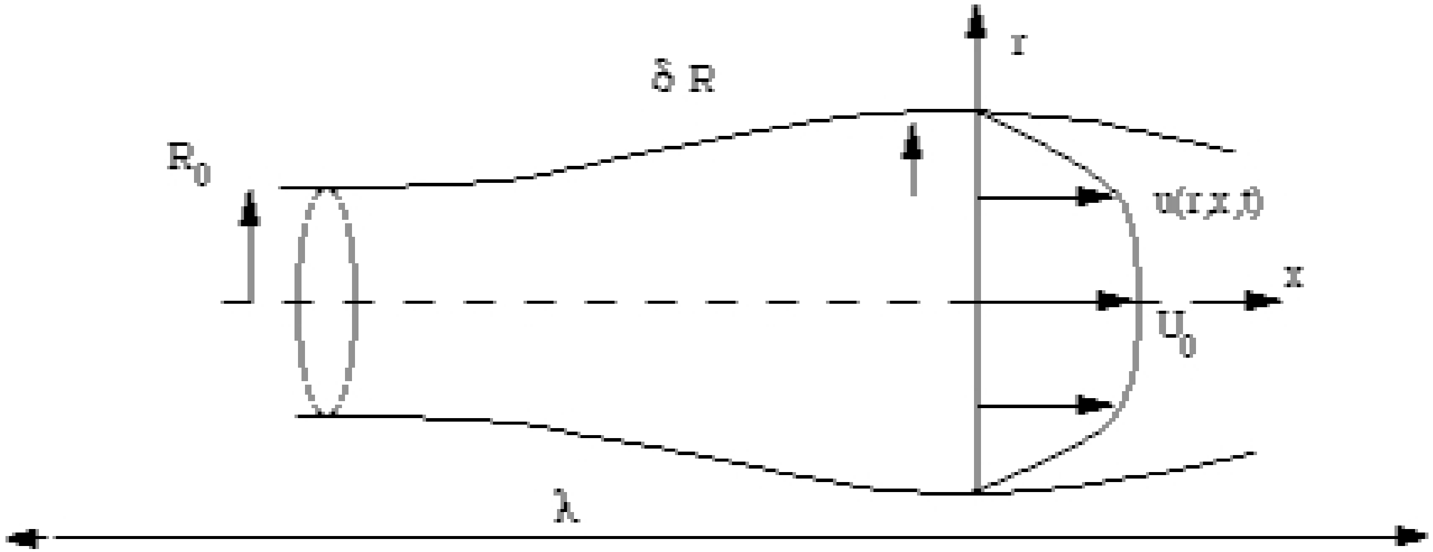
$$R^{n+1}=R^n+v^{n+1}(R^n)\Delta t$$



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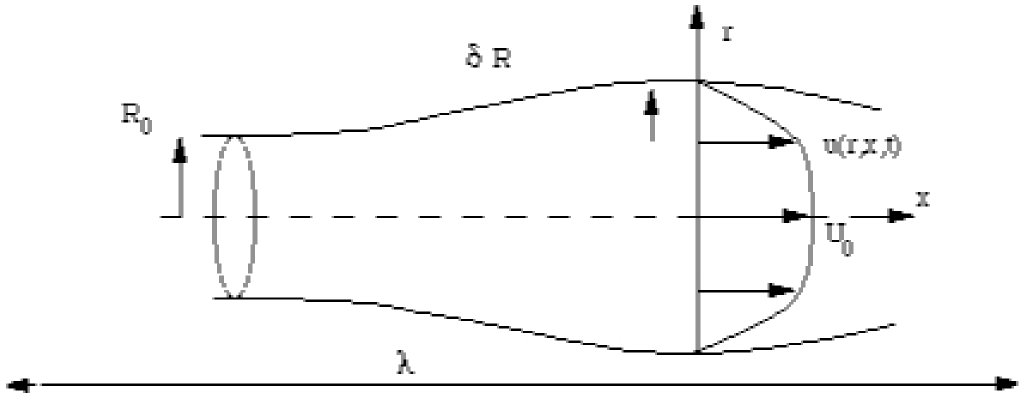
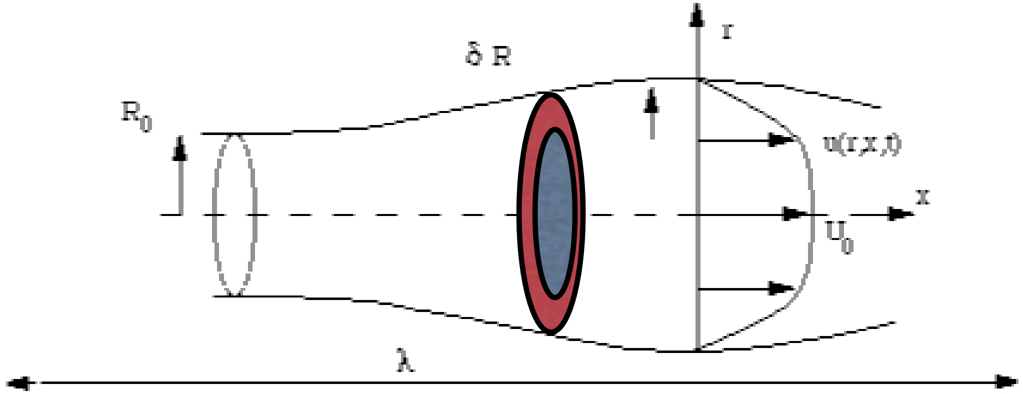
$$R^{n+1}=R^n+v^{n+1}(R^n)\Delta t \qquad p^{n+1}=k(R^{n+1}-R_0)$$



Ecoulement dans une artère élastique: relations intégrales

- relations intégrales: adaptées des relations de Von Kármán

L'idée consiste à intégrer transversalement les équation (par rapport à la variable réduite $\eta = r/R$) du centre du tuyau à la paroi ($0 \leq \eta \leq 1$).



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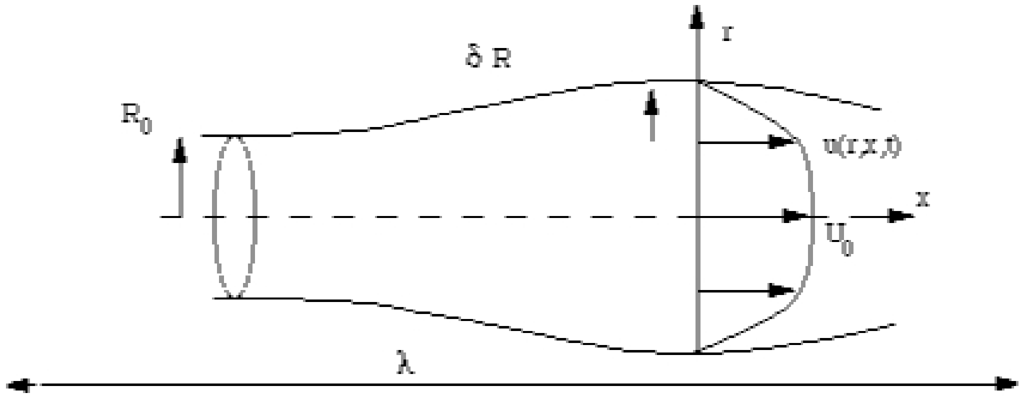
L'idée consiste à intégrer transversalement les équation (par rapport à la variable réduite $\eta = r/R$) du centre du tuyau à la paroi ($0 \leq \eta \leq 1$).

- U_0, la vitesse le long de l'axe,

- q une sorte de perte de flux de masse (δ_1),

- Γ une sorte de perte de flux de quantité de mouvement (δ_2):

$$U_0(x,t) = u(x, \eta = 0, t), \quad q = R^2(U_0 - 2\int_0^1 u\eta d\eta) \;\;\&\;\; \Gamma = R^2(U_0^2 - 2\int_0^1 u^2\eta d\eta).$$



Ecoulement dans une artère élastique: relations intégrales

En intégrant l'équation de la masse:

$$\frac{\partial R^2}{\partial t} + \varepsilon_2 \frac{\partial}{\partial x}(R^2 U_0 - q) = 0, \quad R = 1 + \varepsilon_2 h.$$

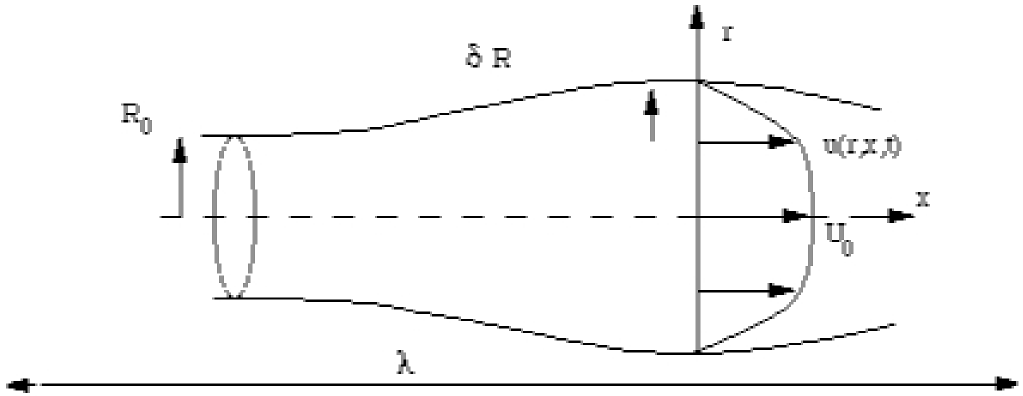
En intégrant l'équation de quantité de mouvement, grâce aux conditions aux limites:

$$\frac{\partial q}{\partial t} + \varepsilon_2 (\frac{\partial}{\partial x}\Gamma - U_0 \frac{\partial}{\partial x} q) = -2\frac{2\pi}{\alpha^2}\tau, \qquad \tau = (\frac{\partial u}{\partial \eta})|_{\eta=1} - (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$

De la même équation évaluée sur l'axe de symétrie (en $\eta = 0$), on obtient une équation pour la vitesse le long de l'axe $U_0(x,t)$:

$$\frac{\partial U_0}{\partial t} + \varepsilon_2 U_0 \frac{\partial U_0}{\partial x} = -\frac{\partial p}{\partial x} + 2\frac{2\pi}{\alpha^2}\frac{\tau_0}{R^2}, \qquad \tau_0 = (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$

Conditions aux limites: donnée de ($h(x_{in}, t)$ and $h(x_{out}, t)$).

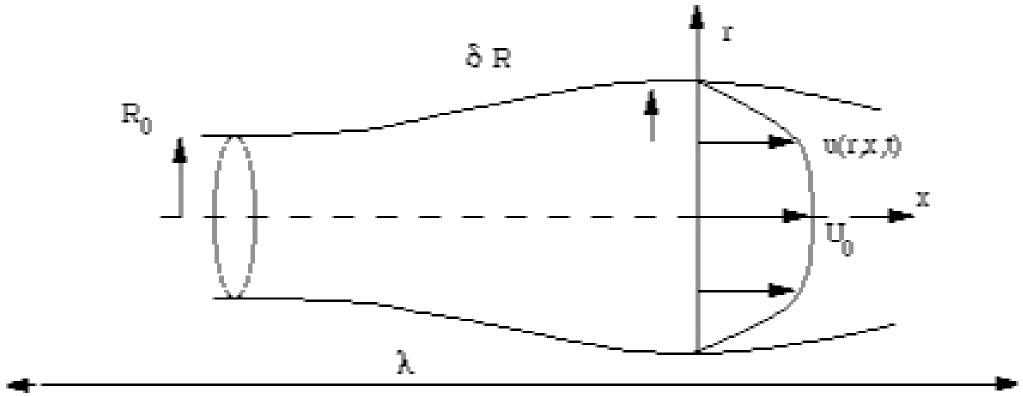


Fermeture

Les deux équations précédentes introduisent la valeur du frottement en $\eta = 0$, sur l'axe: $((\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0})$ le frottement en $\eta = 1$, sur la paroi: $((\frac{\partial u}{\partial \eta})|_{\eta=1})$.

- de l'information a été perdue, nous avons besoin d'une relation de fermeture entre (Γ, τ, τ_0) et (q, R, U_0).

- on doit imaginer des profils de vitesse et en déduire des relations liant Γ, τ etτ_0 et q, U_0 et R.

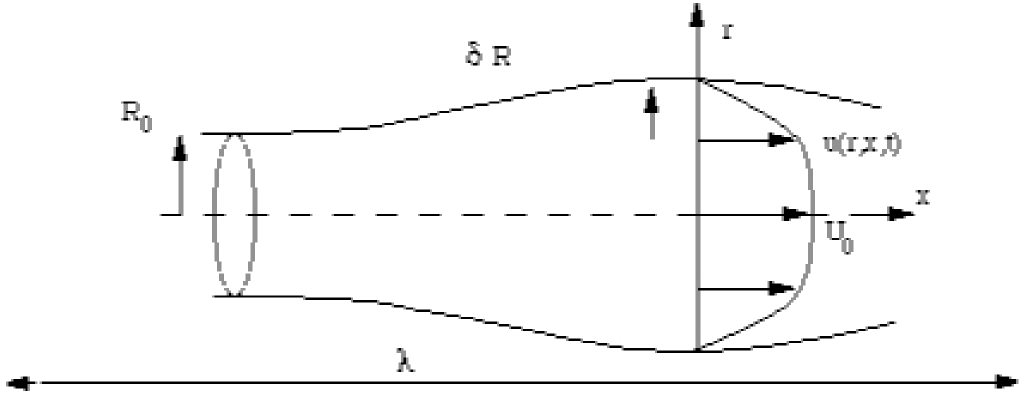


Fermeture: Womersley

• L'idée la plus simple consiste à utiliser les profils de la solution linéarisée donnée par Womersley (1955):

$$(j_r + i j_i) = \left(\frac{1 - \frac{J_0(i^{3/2}\alpha\eta)}{J_0(i^{3/2}\alpha)}}{1 - \frac{1}{J_0(i^{3/2}\alpha)}} \right).$$

• On suppose alors que la distribution de vitesse a la même dépendance en η. Cela veut dire que l'on suppose que le mode fondamental impose la structure radiale de l'écoulement.

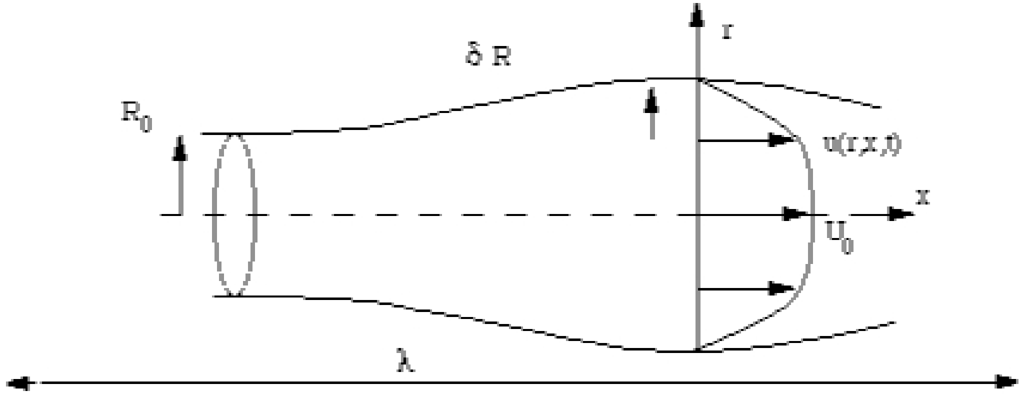


Les coefficients de fermeture

- par intégration/ dérivation, on trouve:

$$\Gamma = \gamma_{qq}\frac{q^2}{R^2} + \gamma_{qu} q U_0 + \gamma_{uu} R^2 U_0^2, \quad \tau = \tau_q \frac{q}{R^2} + \tau_u U_0 \quad \tau_0 = \tau_{0q}\frac{q}{R^2} + \tau_{0u} U_0.$$

Les coefficients $((\gamma_{qq}, \gamma_{qu}, \gamma_{uu}), (\tau_q, \tau_u), (\tau_{0q}, \tau_{0u}))$ ne dépendent que de α.



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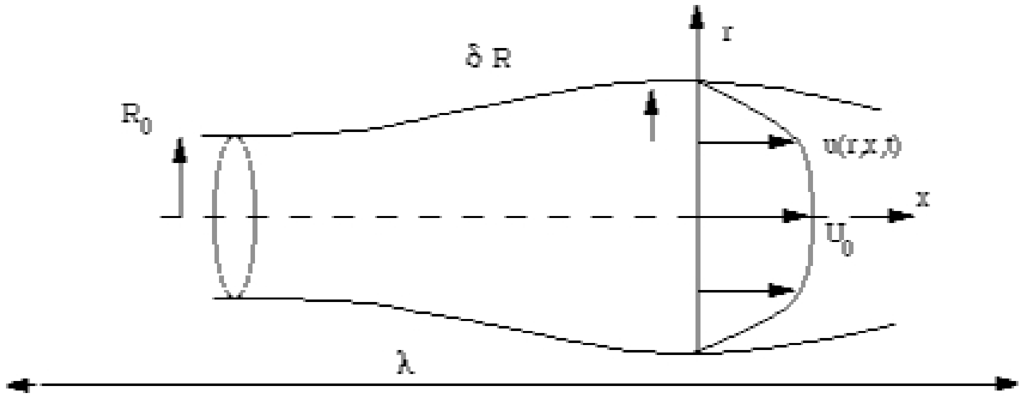
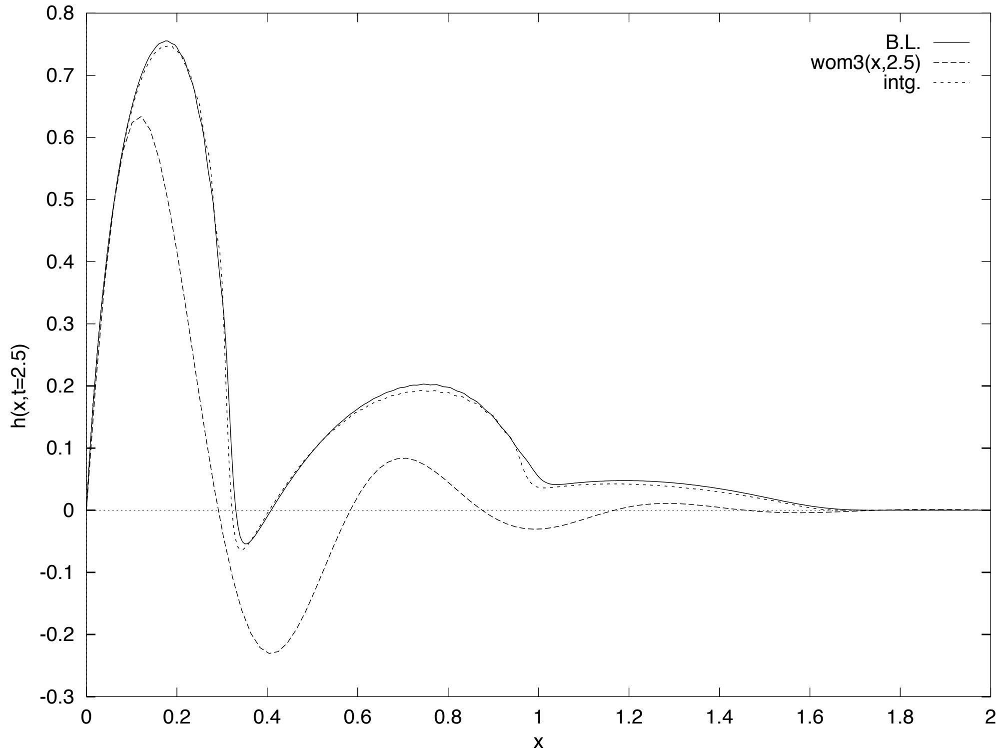
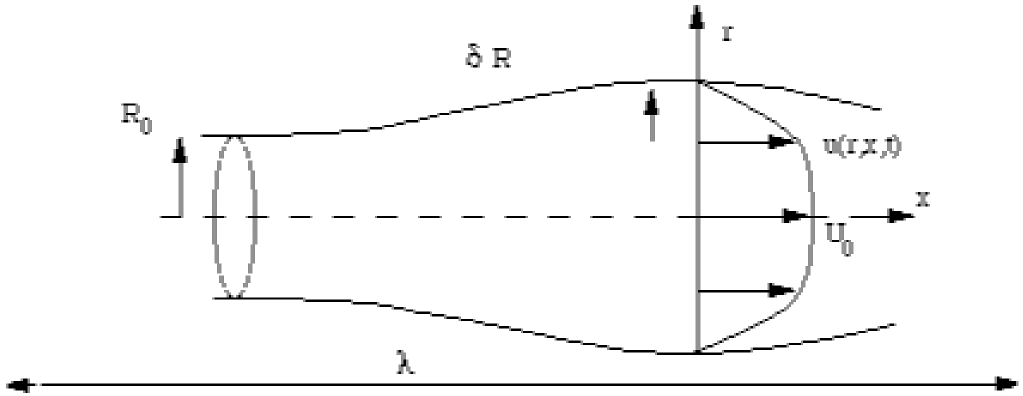
Les coefficients $((\gamma_{qq}, \gamma_{qu}, \gamma_{uu}), (\tau_q, \tau_u), (\tau_{0q}, \tau_{0u}))$ ne dépendent que de α.

$$\begin{aligned}
\gamma_{uu} &= 1 - \int j_i^2/(\int j_i)^2 - (2\int j_r j_i)/\int j_i - \int j_r^2 + \\
&+(2\int j_i^2 \int j_r)/(\int j_i)^2 + (2\int j_i j_r \int j_r)/\int j_i - \\
&-(\int j_i^2 (\int j_r)^2)/(\int j_i), \\
\tau_{0u} &= \partial_\eta^2 j_{r\eta=0} + \partial_\eta^2 j_{i\eta=0}/\int j_i - (\partial_\eta^2 j_{i\eta=0}\int j_r)/\int j_i.
\end{aligned}$$



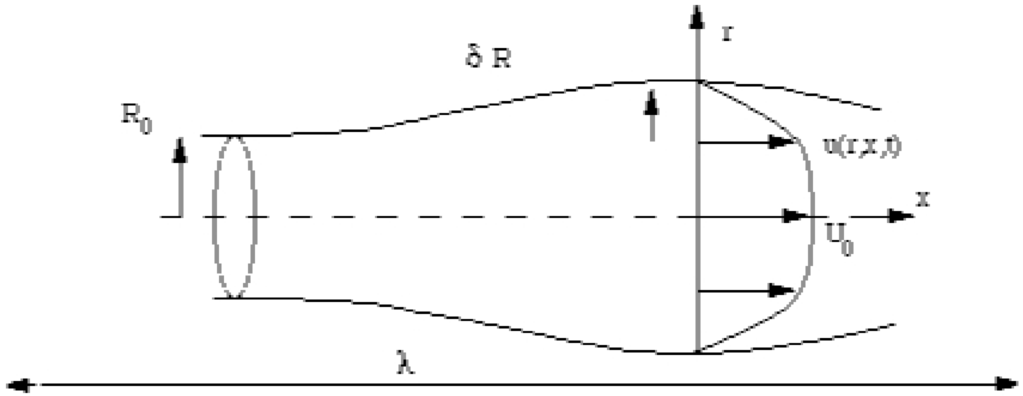


Le déplacement de la paroi au temps $t = 2.5$ ($h(x, t = 2.5)$) en fonction de x est tracé. Le tiré (wom3(x,2.5)) est la solution de Womersley de référence, le trait plein est le résultat de code de couche mince RNSP et les points (intg) sont le résultat de la solution intégrale ($\alpha = 3$, $k_1 = 1$, $k_2 = 0$ and $\varepsilon_2 = 0.2$).



Méthode inverse

En utilisant les équations RNSP comme données synthétiques on cherche à retrouver les paramètres par une méthode inverse (rétropropagation...)



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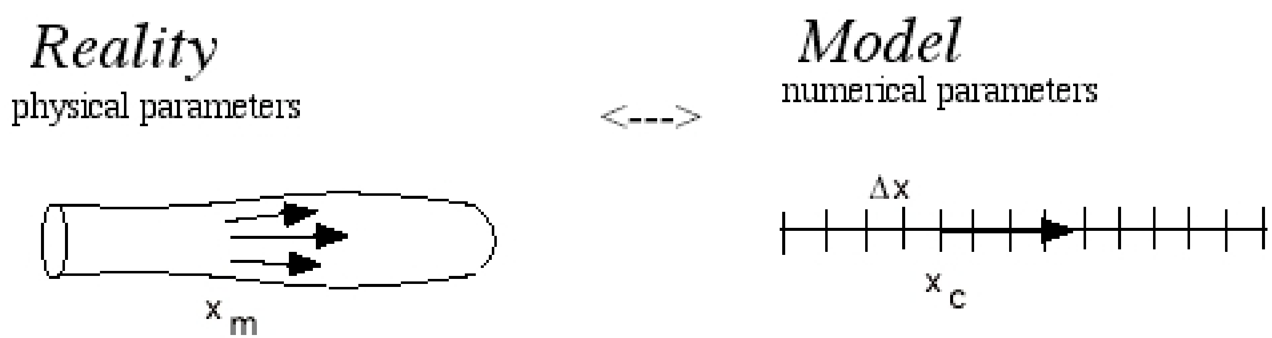
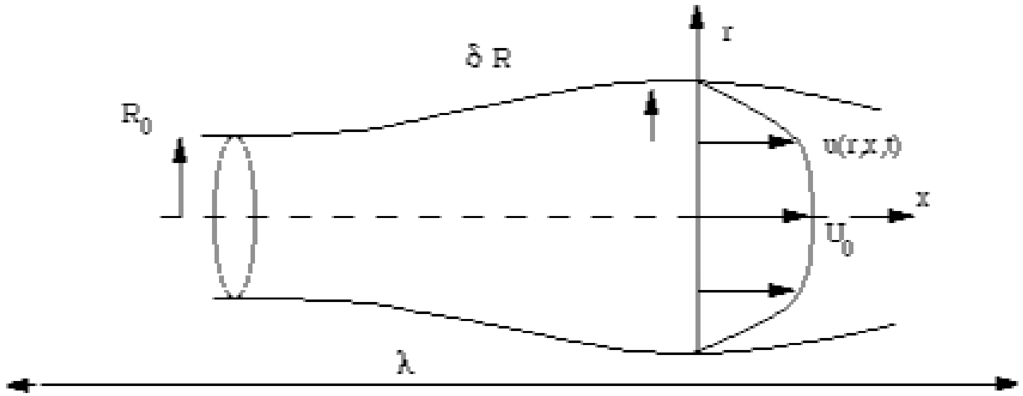
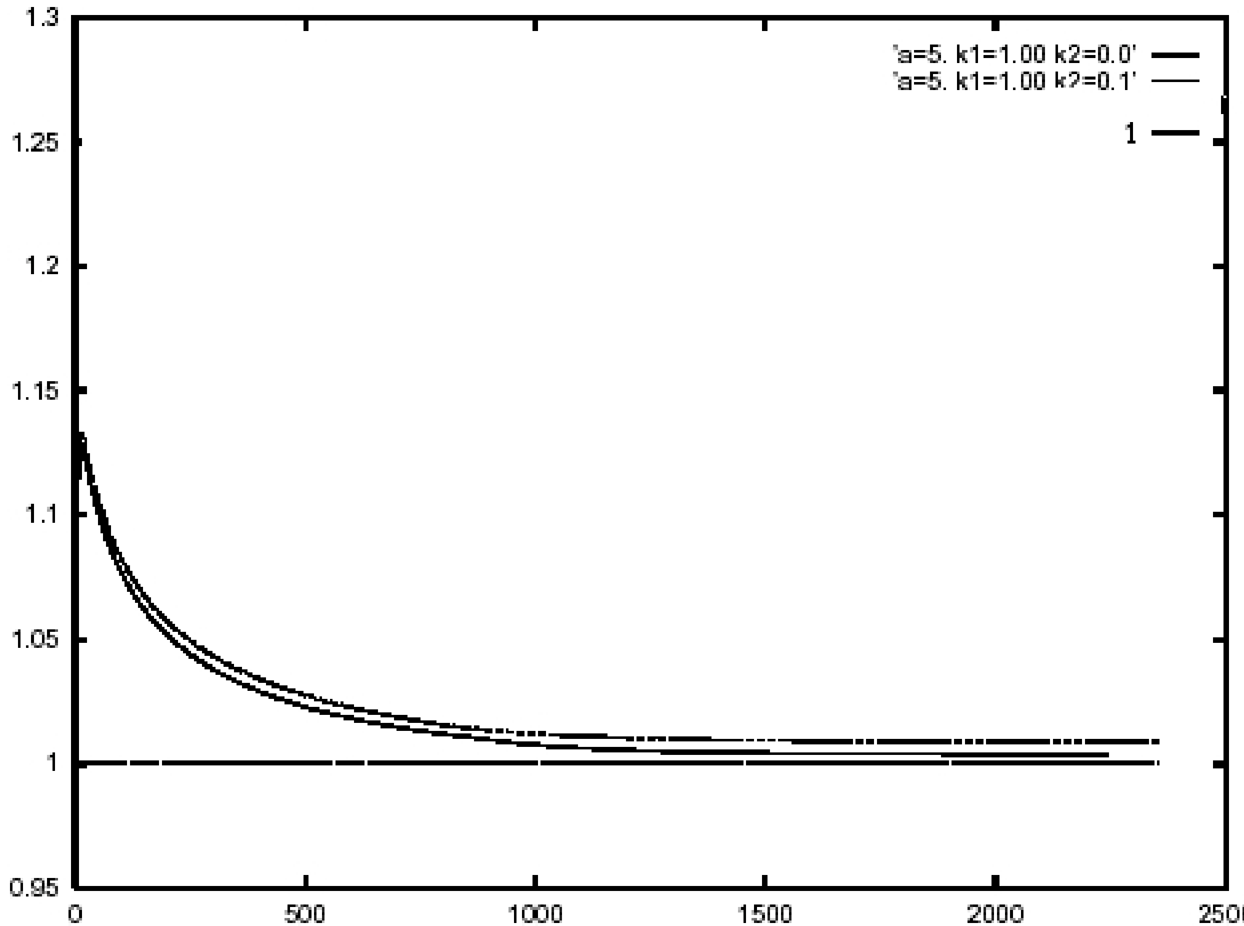
mise en oeuvre d'une méthode non invasive



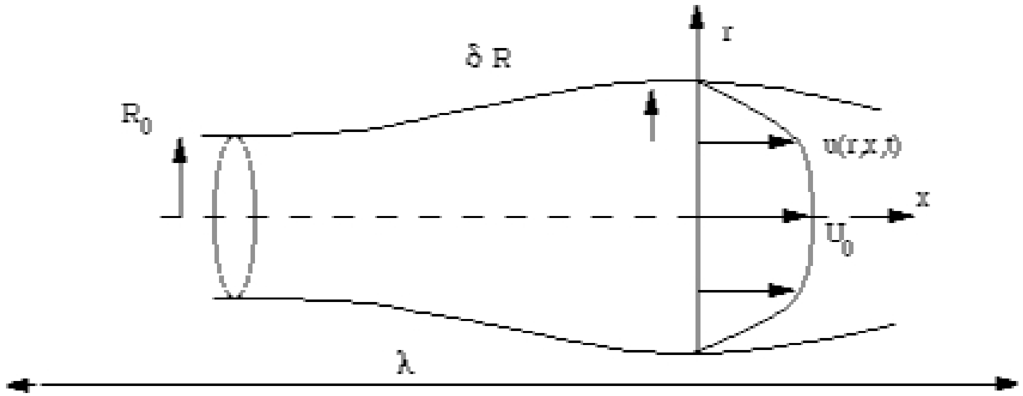
Minimisation entre la "mesure" et le calcul 1D en un point.

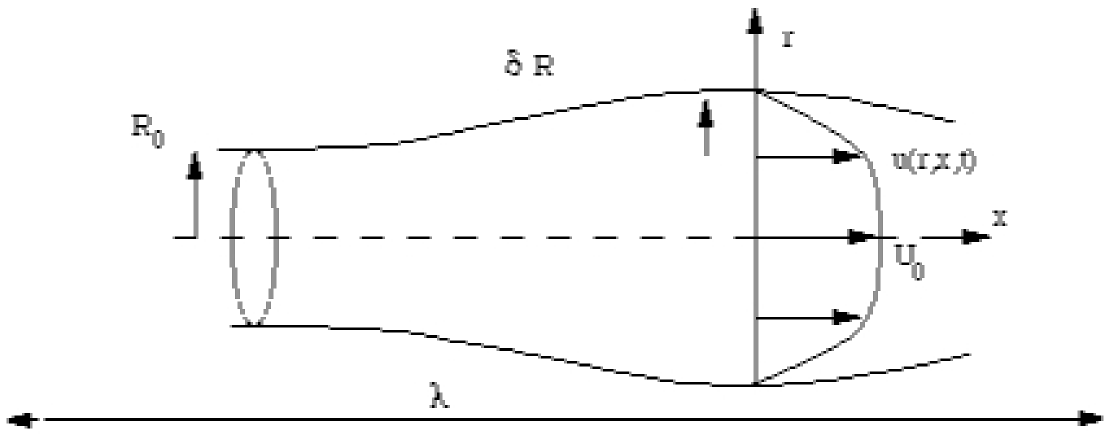


Méthode inverse



Exemple de résolution: historique des itérations pour retrouver la valeur visée $k = 1$.


r
δ R
R_0
u(r,x,t)
x
U_0
λ



δ R
r
R0
u(r,x,t)
x
U0
λ

Conclusion

- partant de Navier Stokes
- système plus simple d’équations: RNSP
- système plus simple d’équations Intégrales

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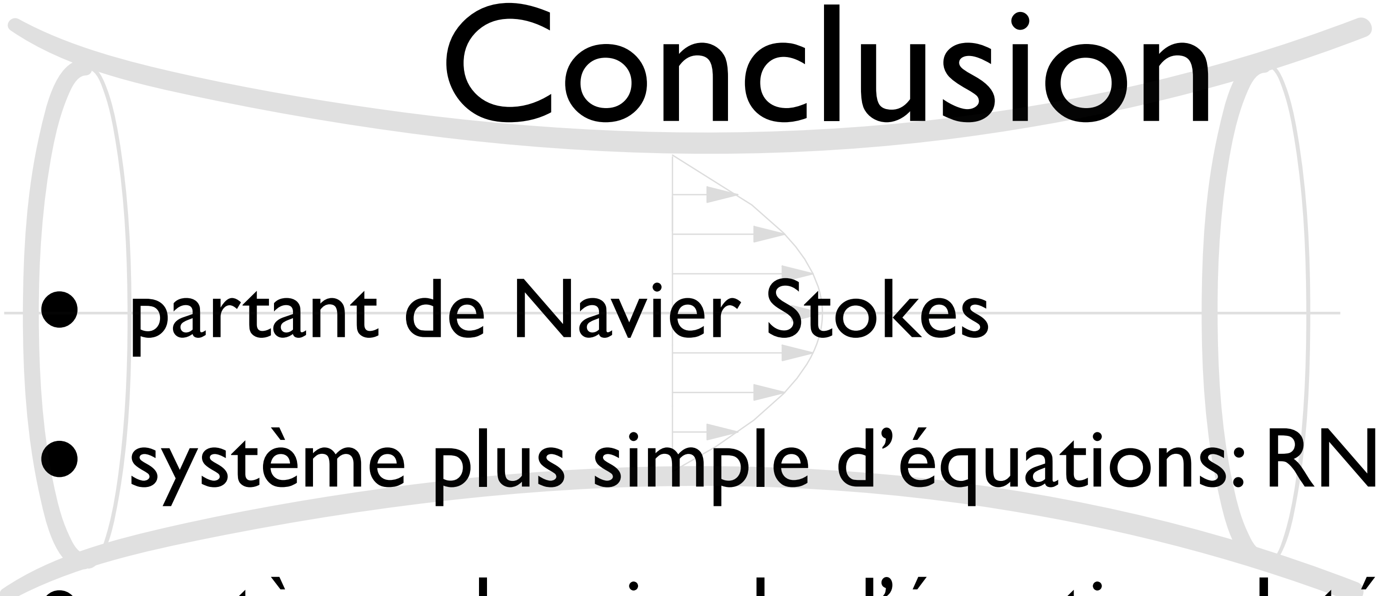

- partant de Navier Stokes
- système plus simple d'équations: RNSP
- système plus simple d'équations Intégrales
- bon agrément avec Navier Stokes complet
- "explique" les caractéristiques du flot
- conditions aux limites pour NS complet
- vers des simulations temps réel

- P.-Y. Lagrée & S. Lorthois (2005):
"The RNS/Prandtl equations and their link with other asymptotic descriptions. Application to the computation of the maximum value of the Wall Shear Stress in a pipe", Int. J. Eng Sci., Vol 43/3-4 pp 352-378.
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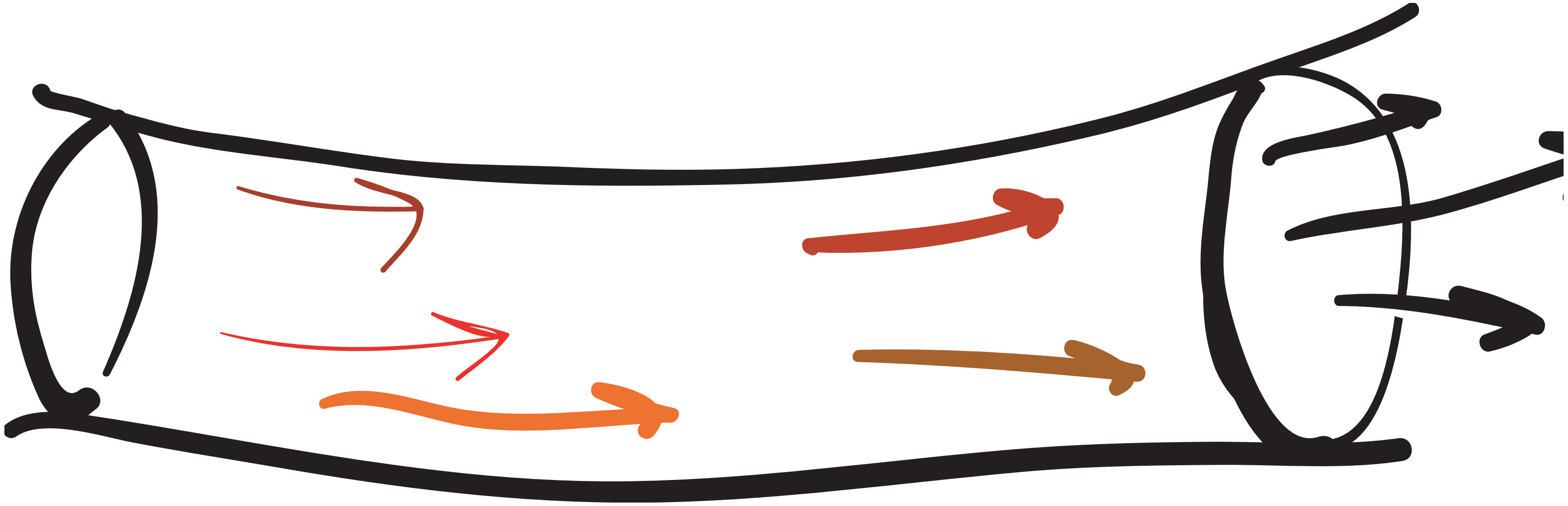
"In-vitro validation of some flow assumptions for the prediction of the pressure distribution during obstructive sleep apnea". Medical & biological engineering & computing no 43(1) pp. 162-171
- P.-Y. Lagrée, E. Berger, M. Deverge, C. Vilain & A. Hirschberg (2005):
"Characterization of the pressure drop in a 2D symmetrical pipe: some asymptotical, numerical and experimental comparisons". Z. Angew. Math. Mech. 85, No. 2, pp. 141-146
- M. Deverge, X. Pelorson, C. Vilain, P.-Y. Lagrée, F. Chentouf, J. Willems & A. Hirschberg (2003):
"Influence of the collision on the flow through in-vitro rigid models of the vocal folds".
J. Acoust. Soc. Am. 114 , pp. 3354 - 3362.
- S. Lorthois, P.-Y. Lagrée, J.-P. Marc-Vergnes & F. Cassot. (2000):
"Maximal wall shear stress in arterial stenoses: Application to the internal carotid arteries", Journal of Biomechanical Egineering, Volume 122, Issue 6, pp. 661-666.
- P.-Y. Lagrée (2000):
"An inverse technique to deduce the elasticity of a large artery ", European Physical Journal, Applied Physics 9, pp. 153-163
- Lorthois S. & Lagrée P.-Y. (2000):
"Écoulement dans un convergent axisymétrique: calcul de la contrainte de cisaillement pariétal maximale/ Flow in a axisymmetric convergent: evaluation of maximum wall shear stress", C. R. Acad. Sci. Paris, t328, Série II b, p33-40, 2000
- Lagrée P-Y and Rossi M. (1996):
"Etude de l'écoulement du sang dans les artères: effets nonlinéaires et dissipatifs", C. R. Acad. Sci. Paris, t322, Série II b, p401- 408, 1996.



- Utiliser Acrobat Reader 7.05 pour voir les animations
- version à jour ici.