Asymptotical Fluid Dynamics

- simplified equations ($Re \gg 1$, $\varepsilon \ll 1$)
- small disturbance theory
- Boundary Layer theory
- Interacting Boundary layer theory
- Triple Deck theory

Laminar Steady

Navier Stokes Equations

- non dimensional
- Reynolds number
- Boundary condition: no slip

$$(\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}})=0$$

$$(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}})=-\frac{\partial p}{\partial \bar{x}}+\frac{1}{Re}(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{u}}{\partial \bar{y}^2})$$

$$(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}})=-\frac{\partial \bar{p}}{\partial \bar{y}}+\frac{1}{Re}(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}),$$

Euler Equations

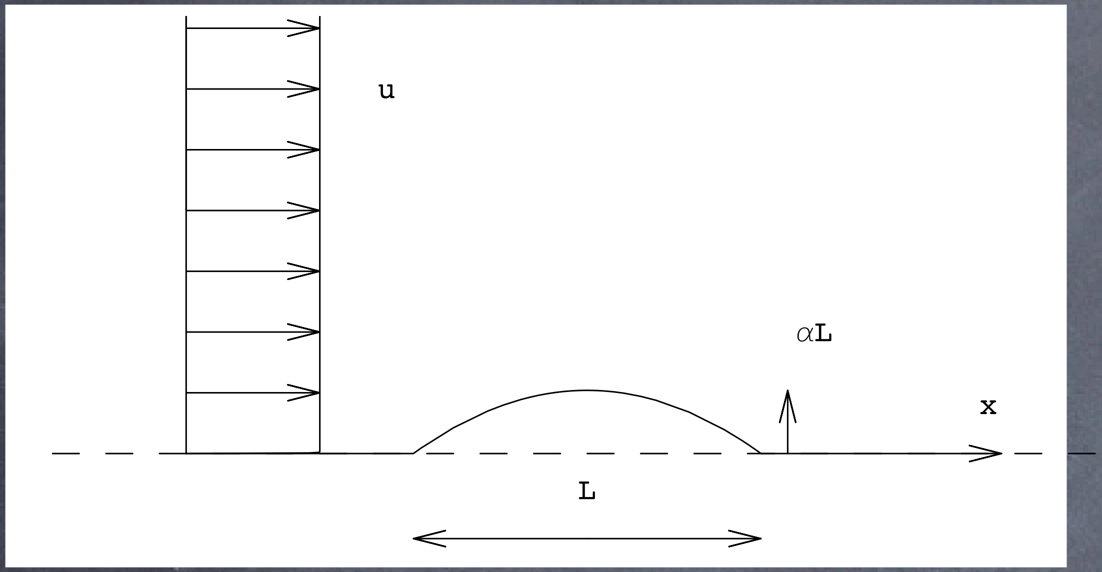
- 1/Re=0
- Boundary condition: slip

$$\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}}\right) = 0$$

$$\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = -\frac{\partial p}{\partial \bar{x}}$$

$$\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) = -\frac{\partial \bar{p}}{\partial \bar{y}}$$

simple ideal fluid flows



ideal fluid flow

small perturbation theory

$$\bar{u} = 1 + \alpha \bar{u}_1 + \alpha^2 \bar{u}_2 + \ldots$$

$$\bar{v} = 0 + \alpha \bar{v}_1 + \alpha^2 \bar{v}_2 + \ldots$$

$$\bar{p} = 0 + \alpha \bar{p}_1 + \alpha^2 \bar{p}_2 + \ldots$$

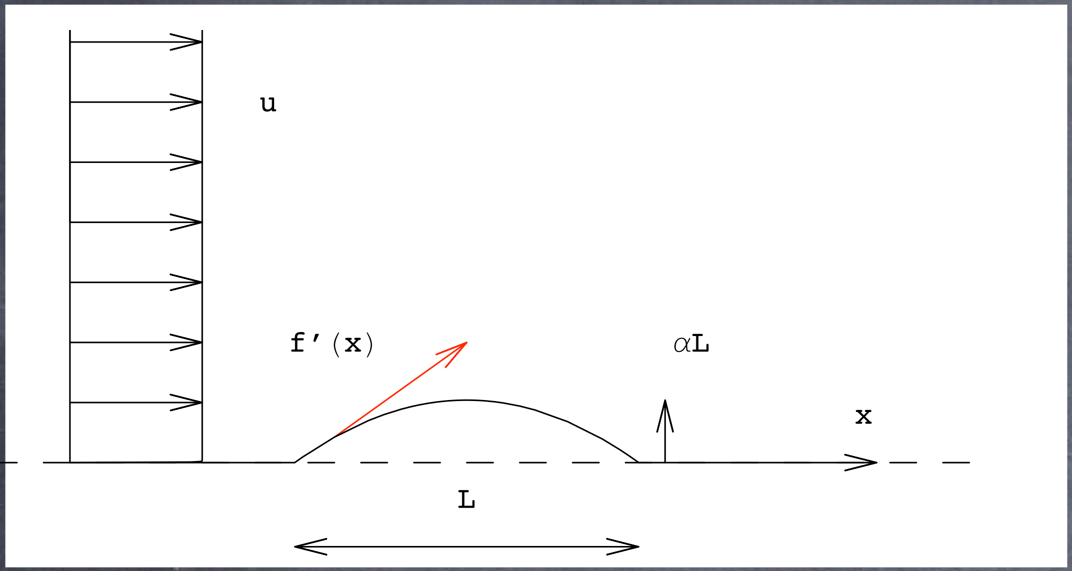
Linearized Euler

$$\frac{\partial}{\partial \bar{x}} \bar{u}_1 = -\frac{\partial}{\partial \bar{x}} \bar{p}_1$$

$$\frac{\partial}{\partial \bar{x}} \bar{v}_1 = -\frac{\partial}{\partial \bar{y}} \bar{p}_1$$

$$\frac{\partial}{\partial \bar{x}} \bar{u}_1 + \frac{\partial}{\partial \bar{y}} \bar{v}_1 = 0$$

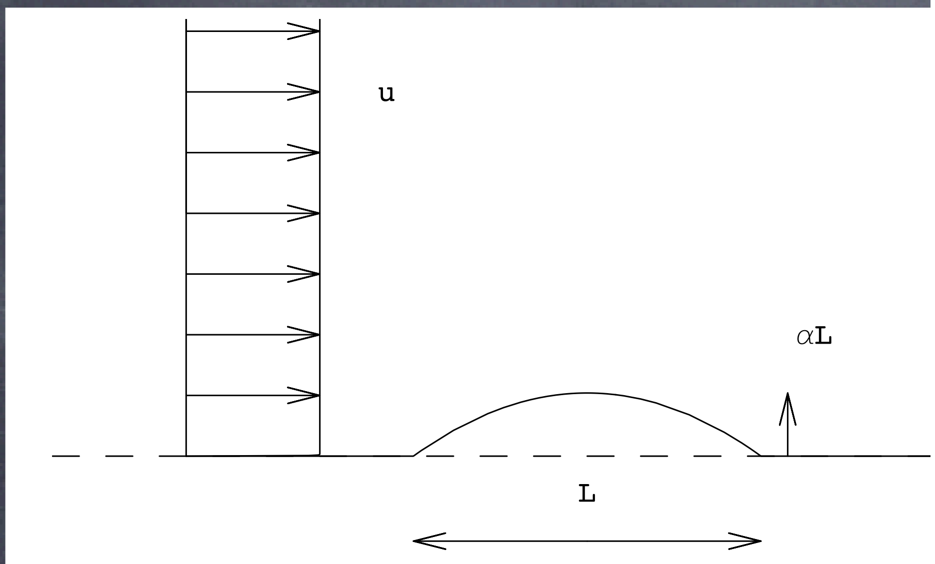
slip condition

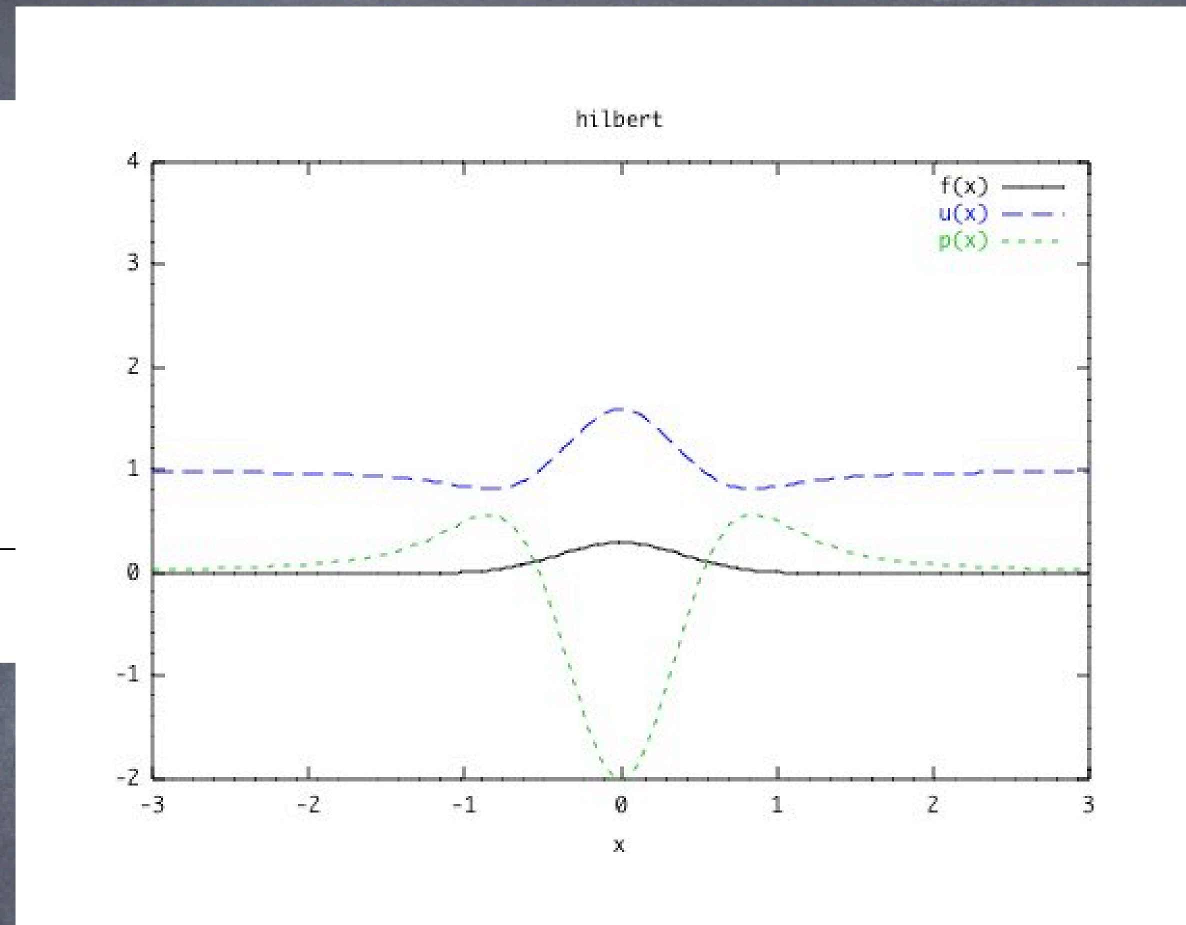


$$\frac{\bar{v}}{\bar{u}} = \alpha \bar{f}'(\bar{x})$$

$$\bar{v}_1 = \bar{f}'(\bar{x})$$

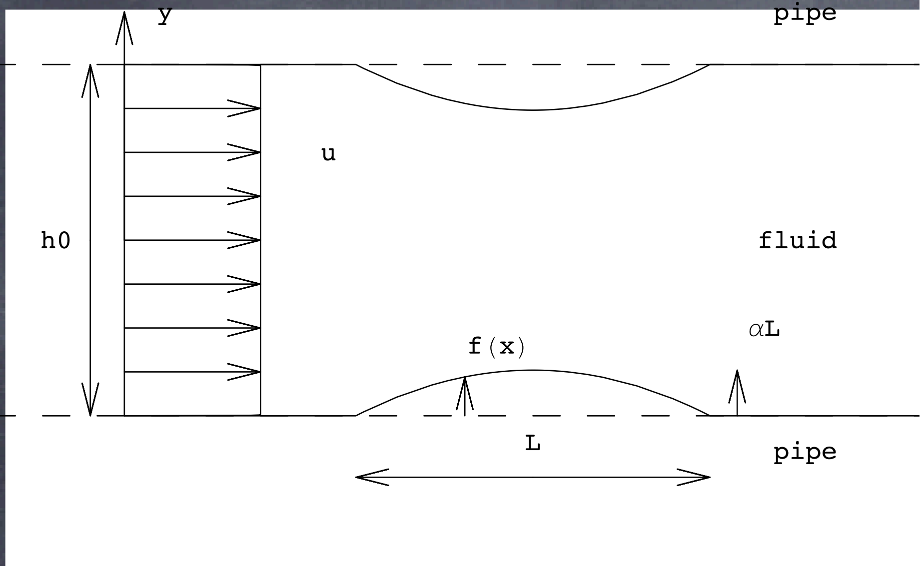
subsonic flow...

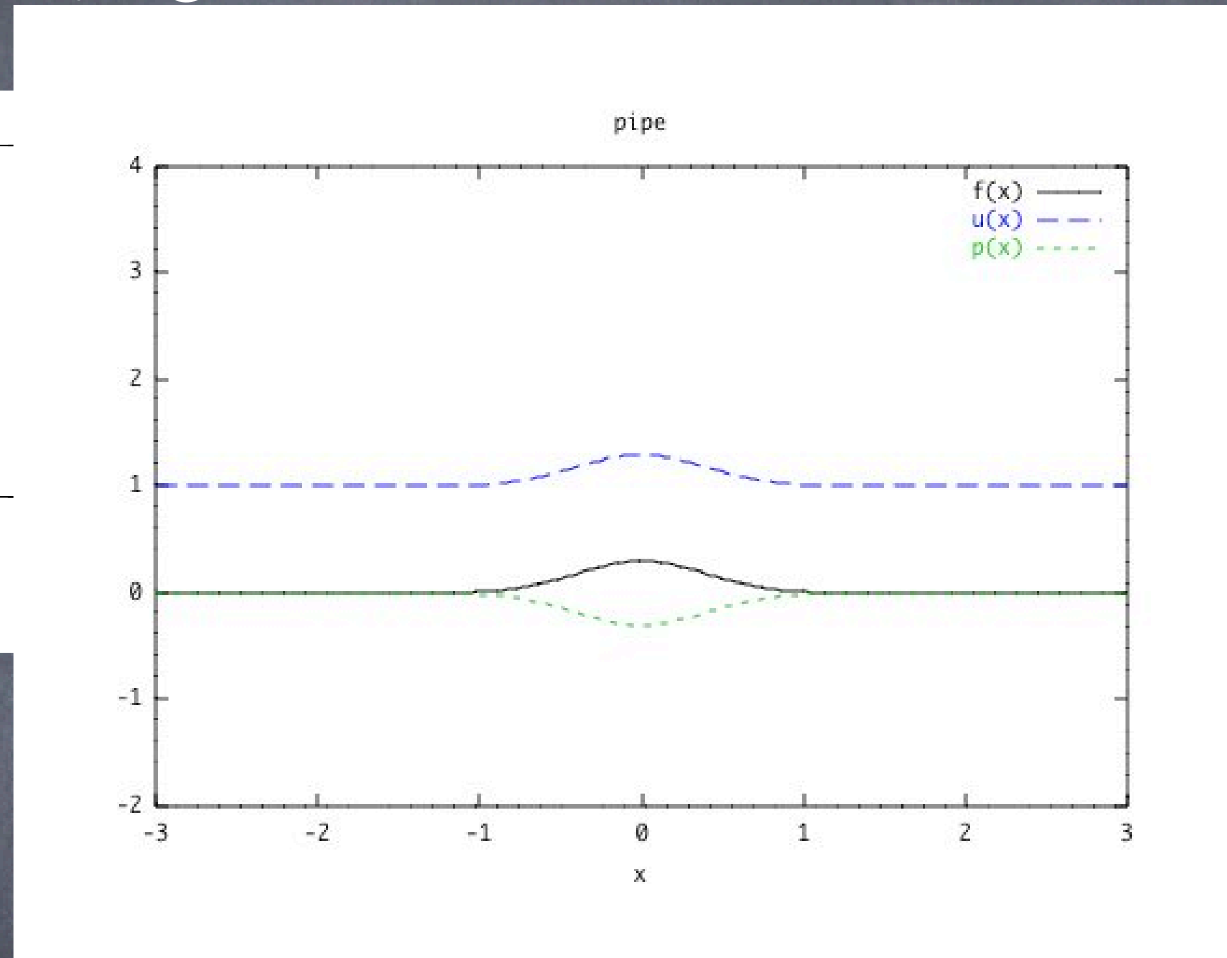




$$\bar{u} = 1 + \alpha \frac{1}{\pi} fp \int_{-\infty}^{\infty} \frac{\bar{f}'}{\bar{x} - \xi} d\xi$$

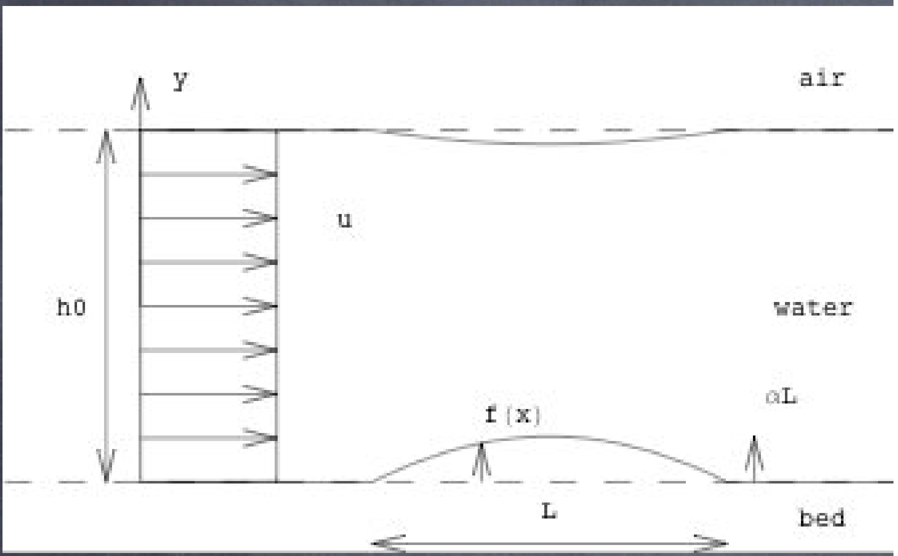
pipe flow

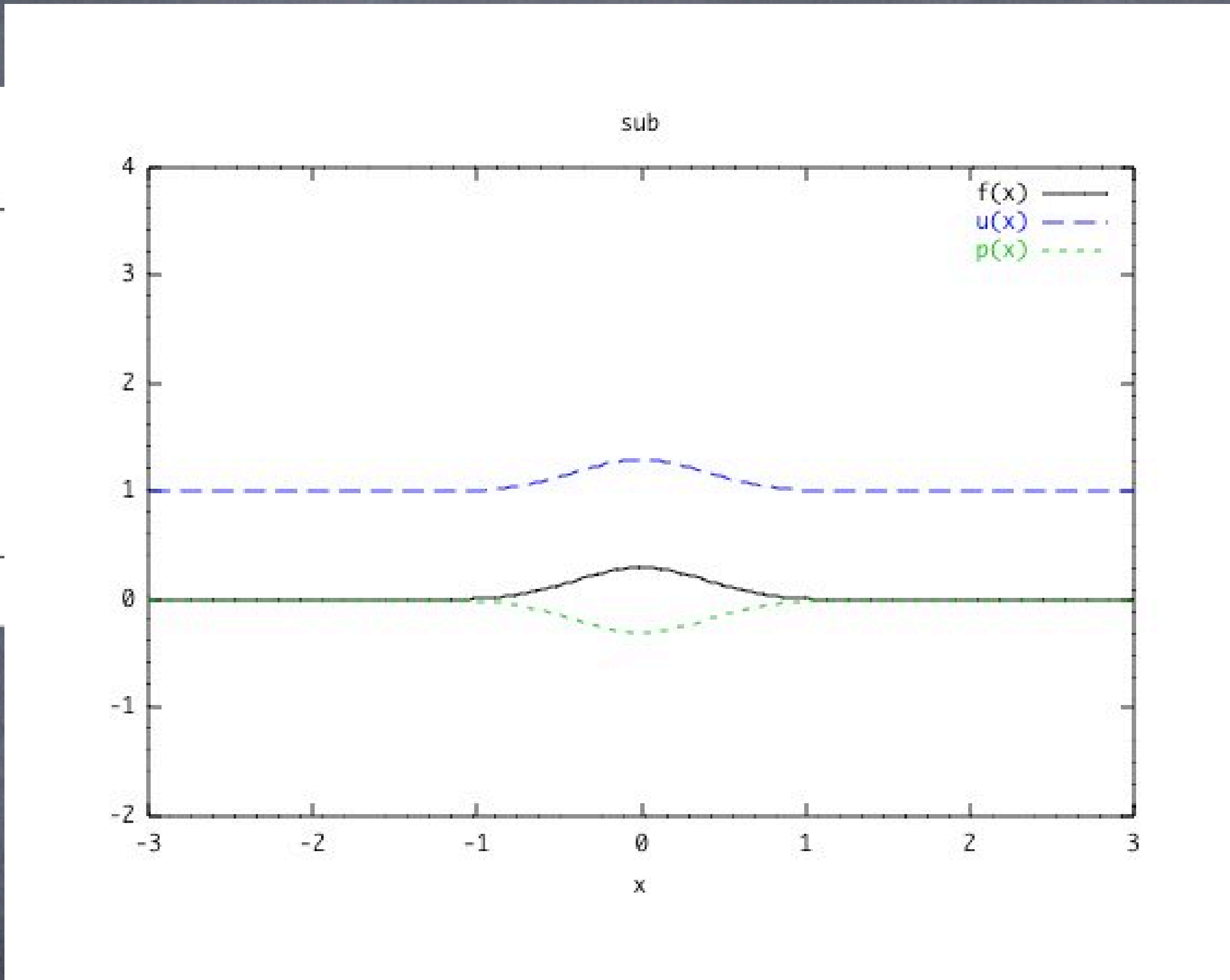




$$\bar{u} = 1 + \alpha \bar{f} + \ldots$$

sub critical flow   F<1

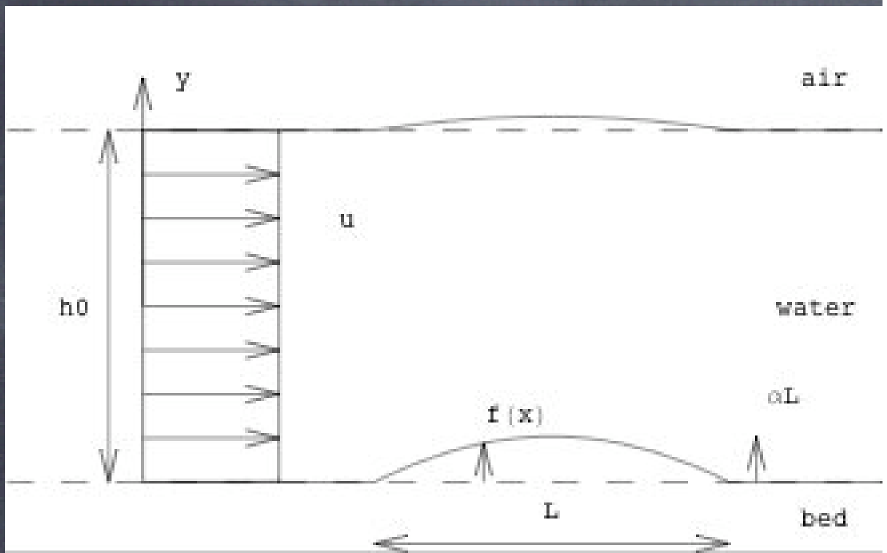


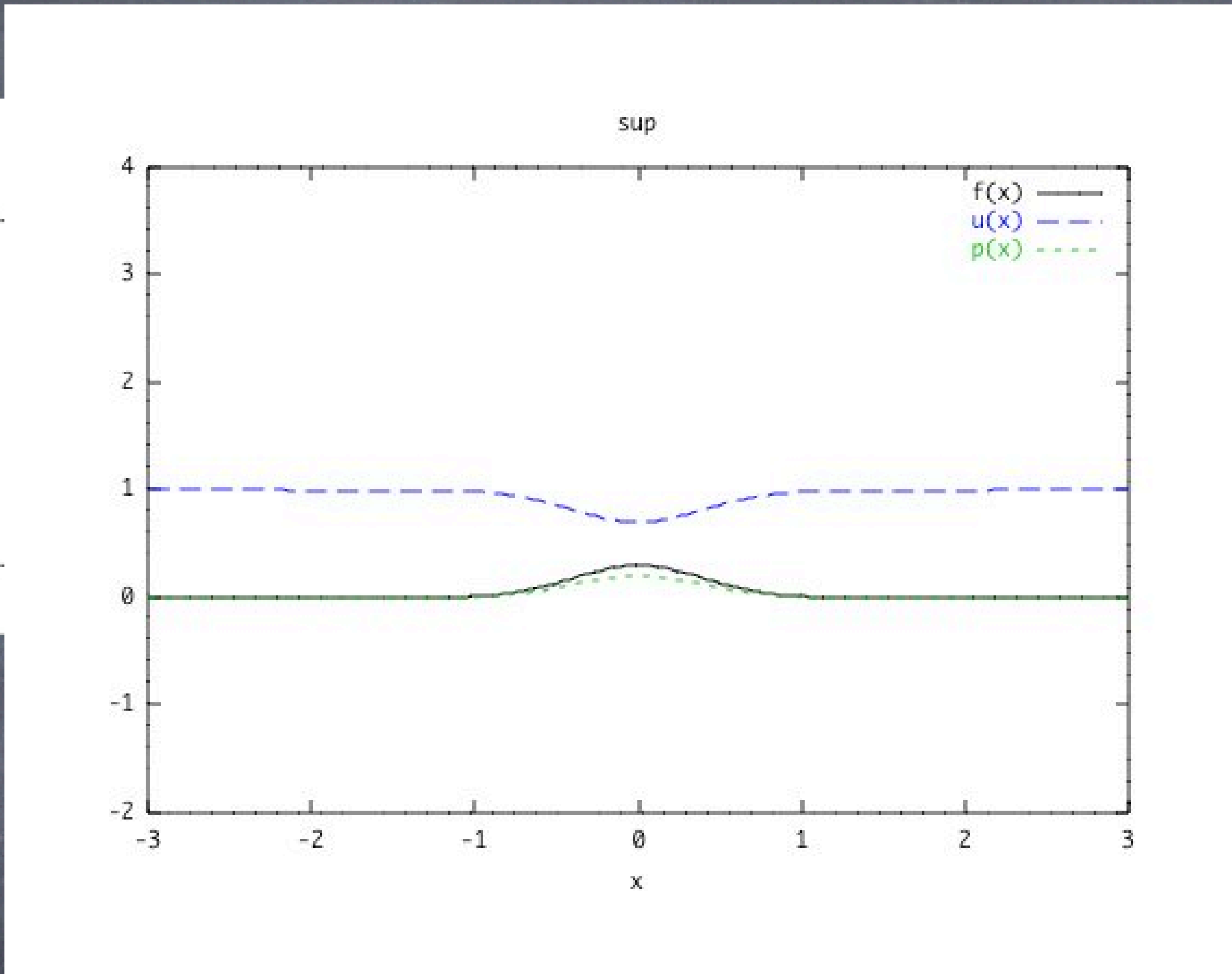


$$\bar{\eta} = F\frac{\bar{f}}{1-F}$$

$$\bar{u} = 1 + \frac{\alpha\bar{f}}{1-F} + ...$$

super critical flow F>1

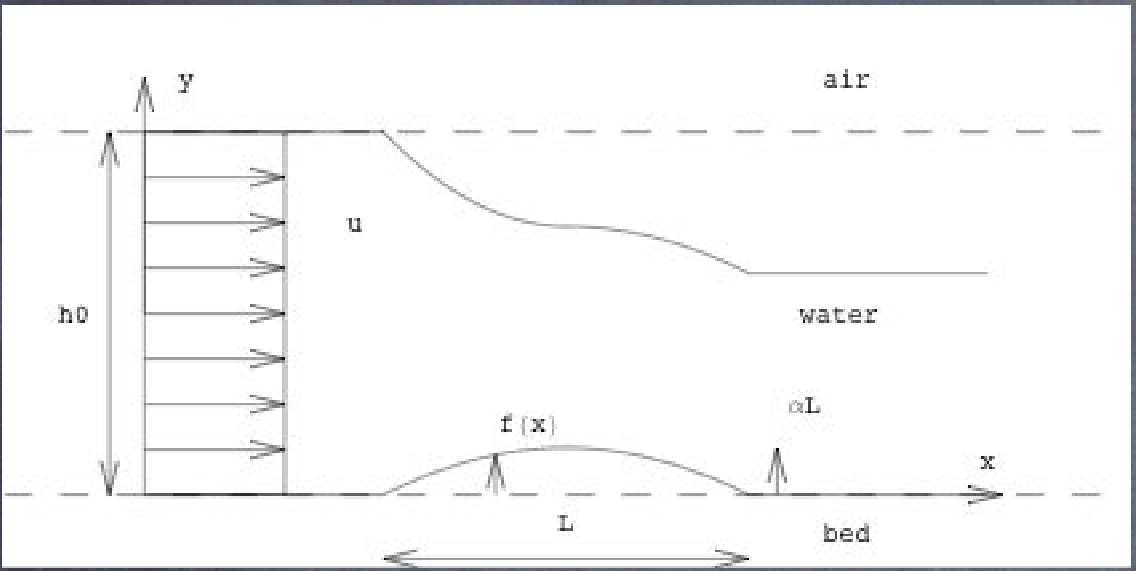




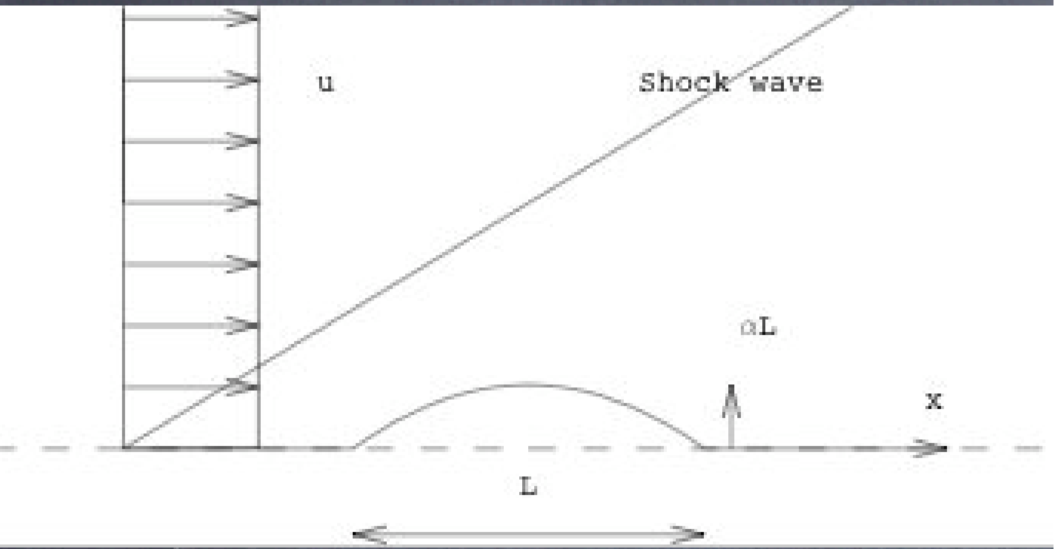
$$\bar{\eta} = F\frac{\bar{f}}{1-F}$$

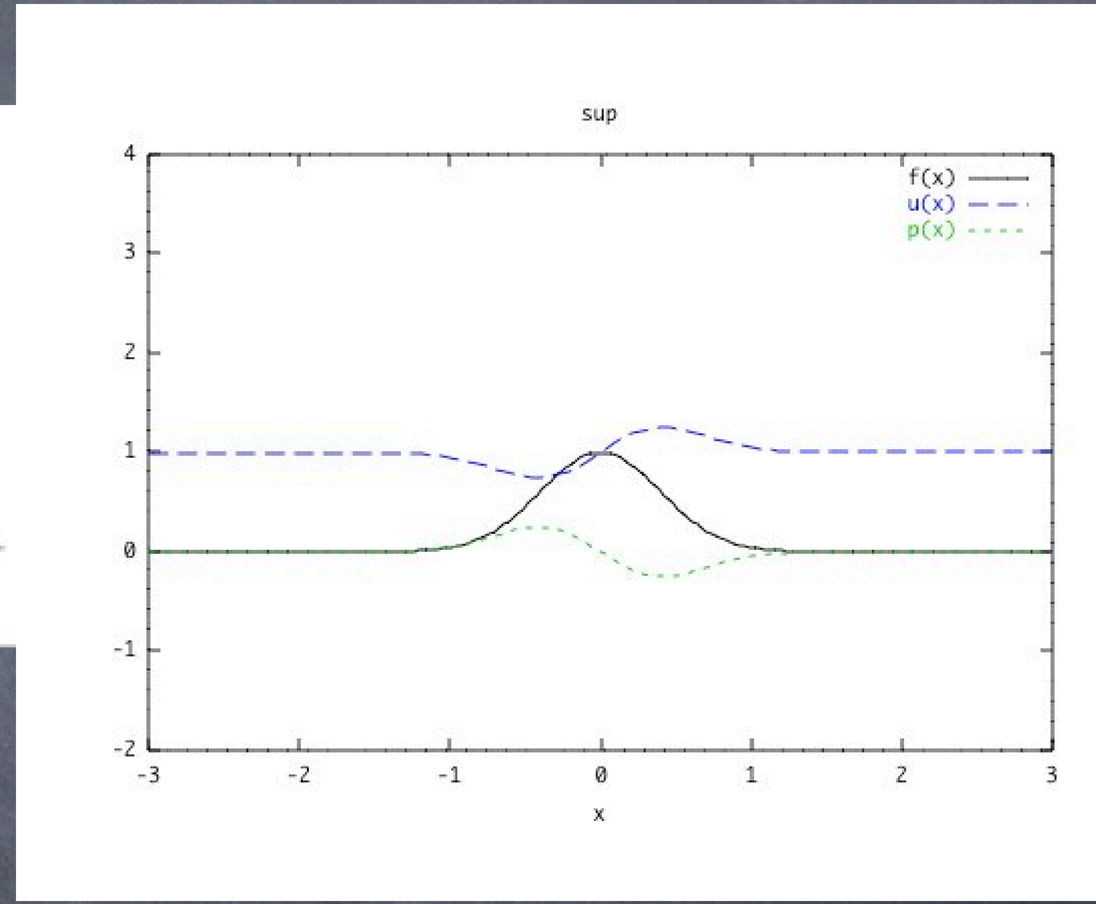
$$\bar{u} = 1 + \frac{\alpha \bar{f}}{1-F} + \ldots$$

trans critical flow F<>1



supersonic flow...





$$\bar{u} = 1 - \frac{M^2}{\sqrt{M^2 - 1}} \frac{\alpha d\bar{f}}{d\bar{x}} + \dots$$

Slip velocity

must have no slip condition on the wall

have to introduce a Boundary Layer

Boundary Layer

$$\left(\frac{\partial \tilde{u}}{\partial \bar{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}}\right) = 0,$$

$$\left(\tilde{u}\frac{\partial \tilde{u}}{\partial \bar{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right) = -\frac{\partial p}{\partial \bar{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

No slip boundary condition $\tilde{u}(\bar{x},0) = \tilde{v}(\bar{x},0) = 0$

Matching $\tilde{u}(\bar{x},\tilde{y} \to \infty) = \bar{u}(\bar{x},\bar{y} \to 0)$

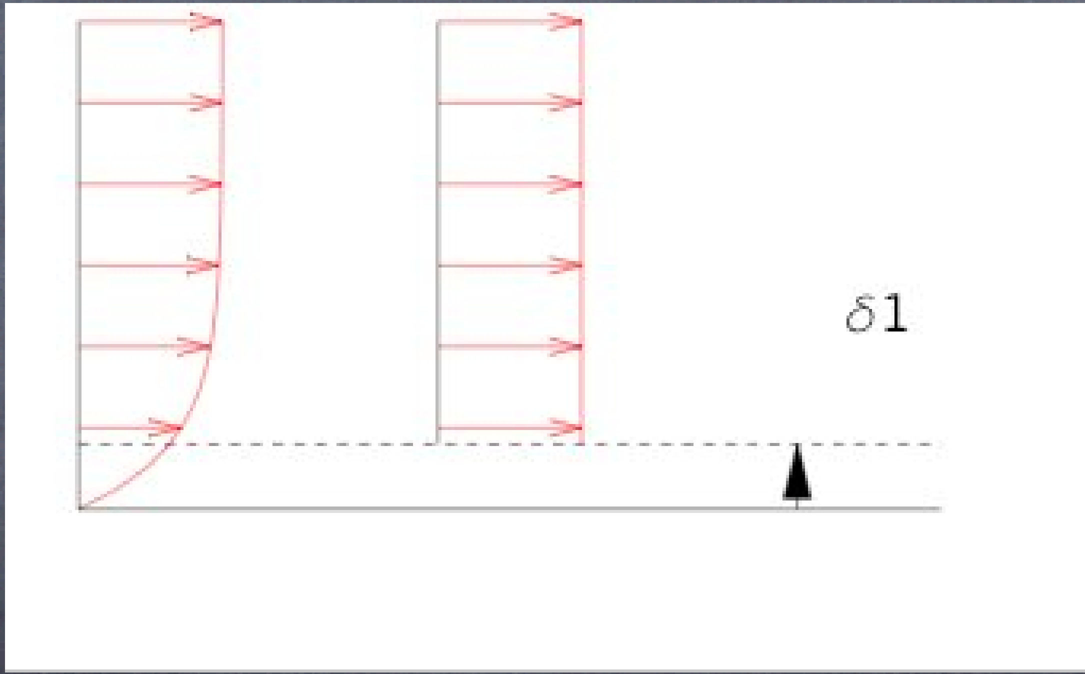
weak coupling

Ideal Fluid gives the outer edge velocity

the Boundary layer develops

weak coupling

the displacement thickness



$$\tilde{\delta}_1 = \int_0^{\infty} (1 - \frac{\tilde{u}}{\bar{U}_e}) d\tilde{y}$$

Blasius

Self similar solution

$$2f''' + ff'' = 0$$

$$f(0) = f'(0) = 0 \qquad \text{and} \qquad f'(\infty) = 1.$$

$$f''(0) = 0.332,$$

$$\int (1 - f) = 1.732$$

Blasius

Self similar solution

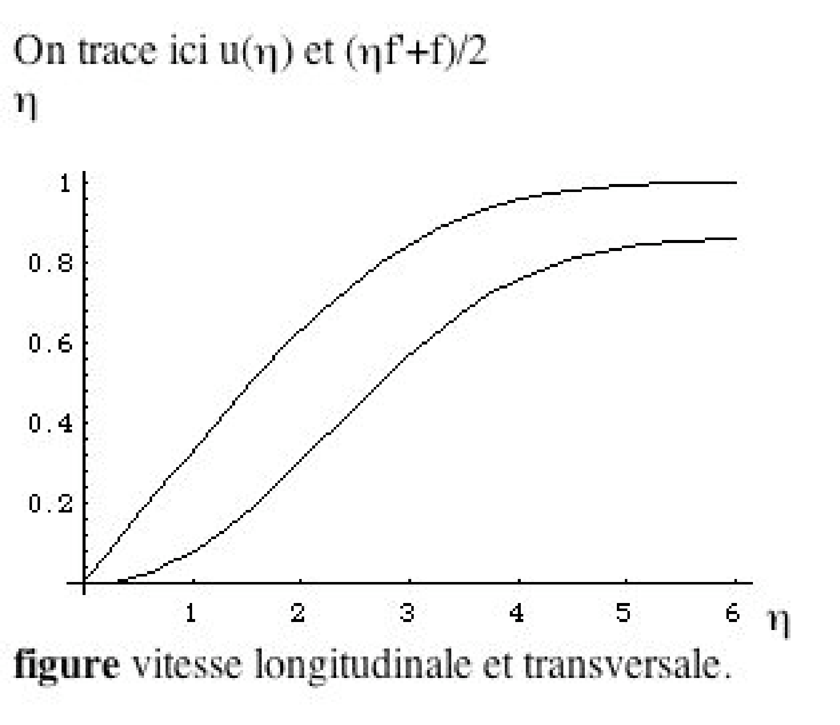
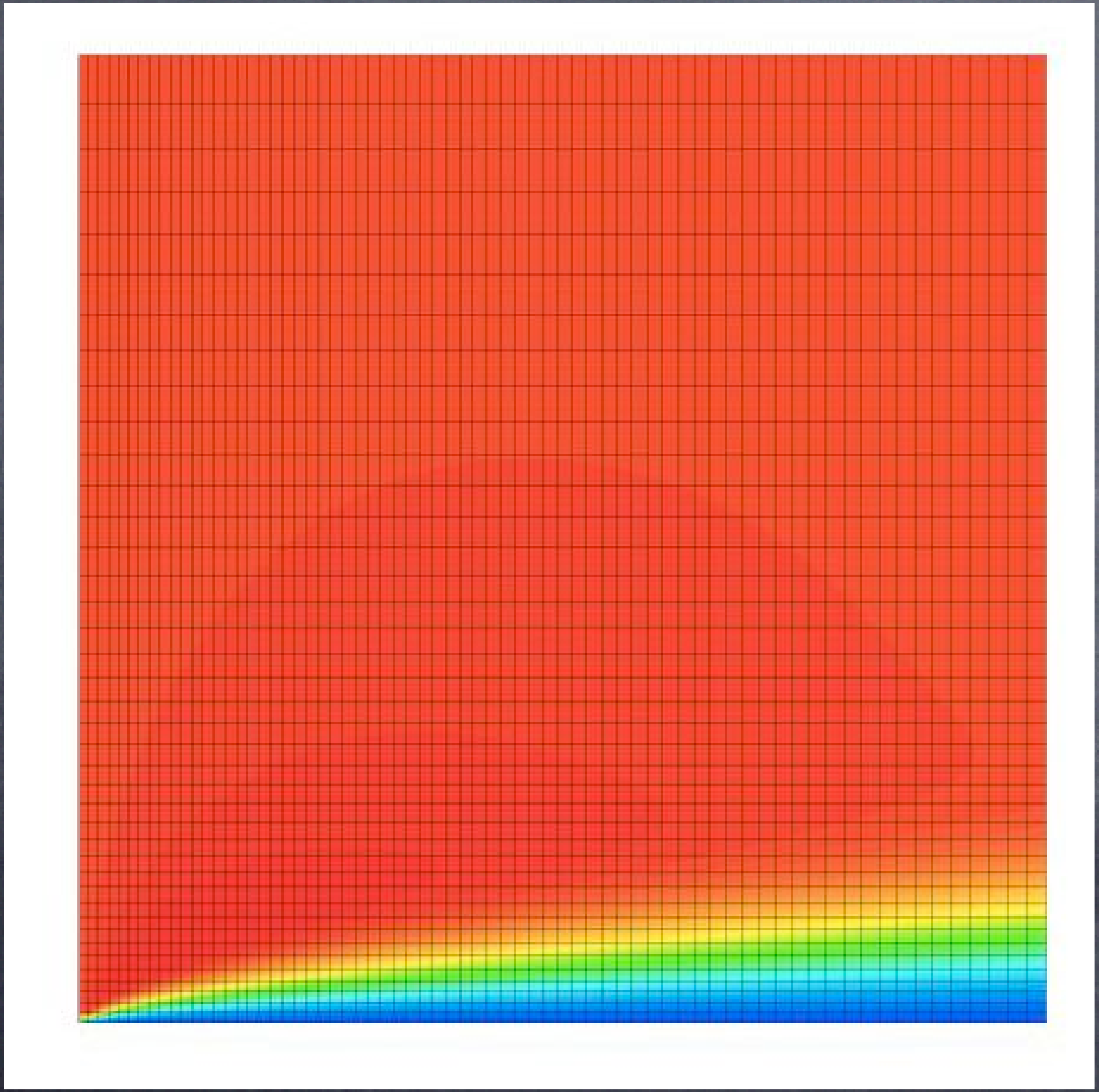


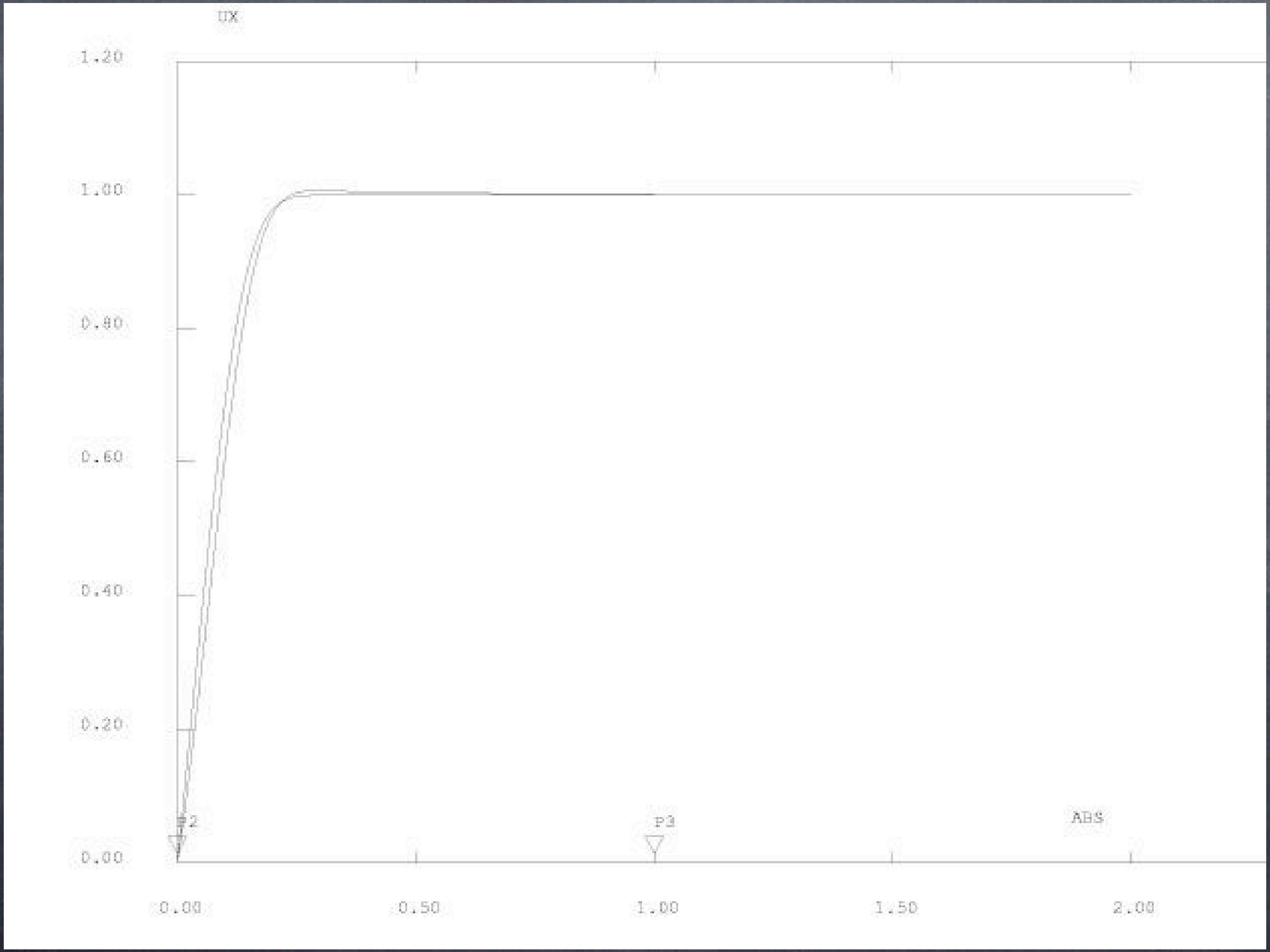
figure vitesse longitudinale et transversale.

Blasius



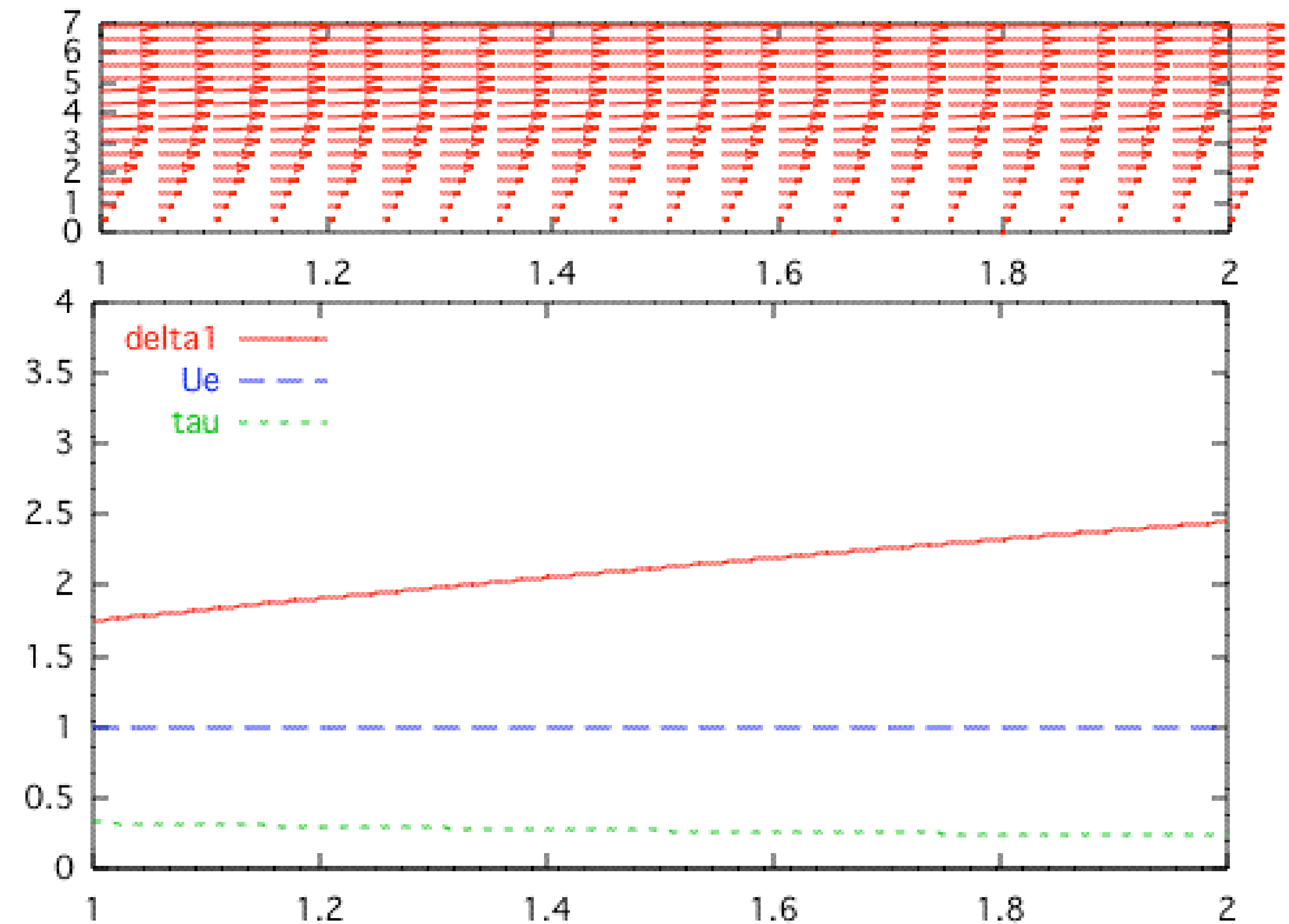
castem 2000

Blasius

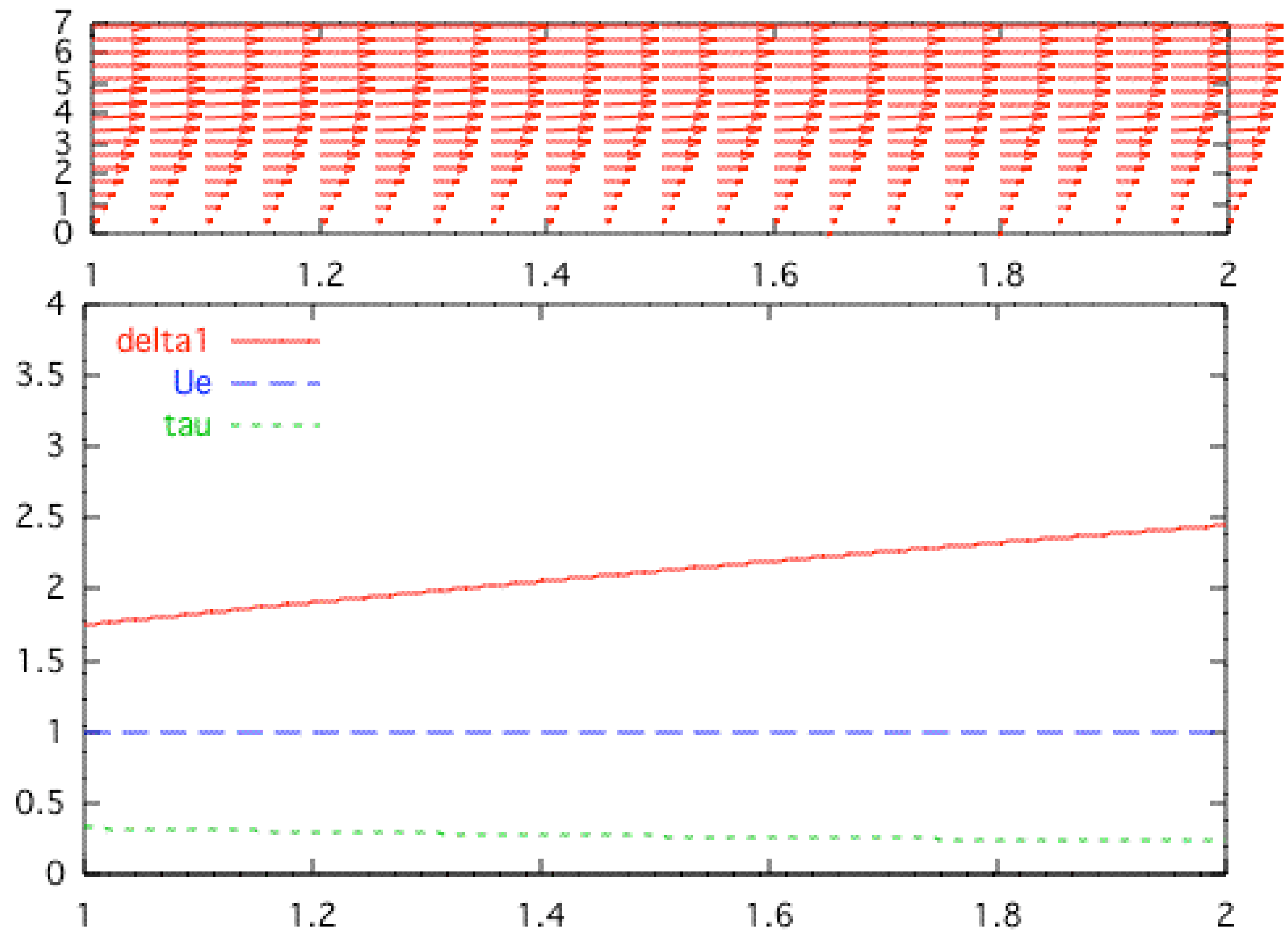


castem 2000

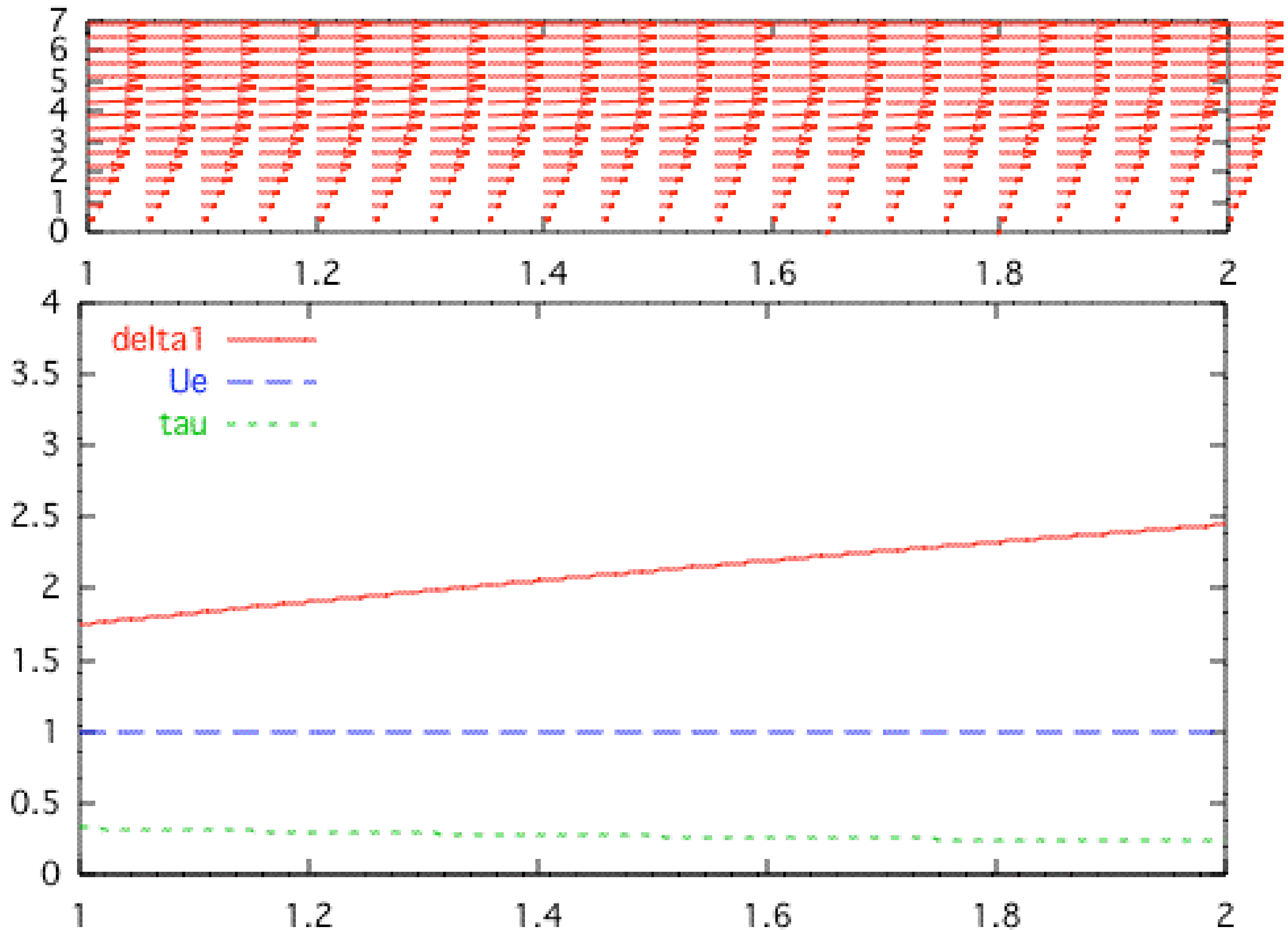
Ex. Boundary layer computations



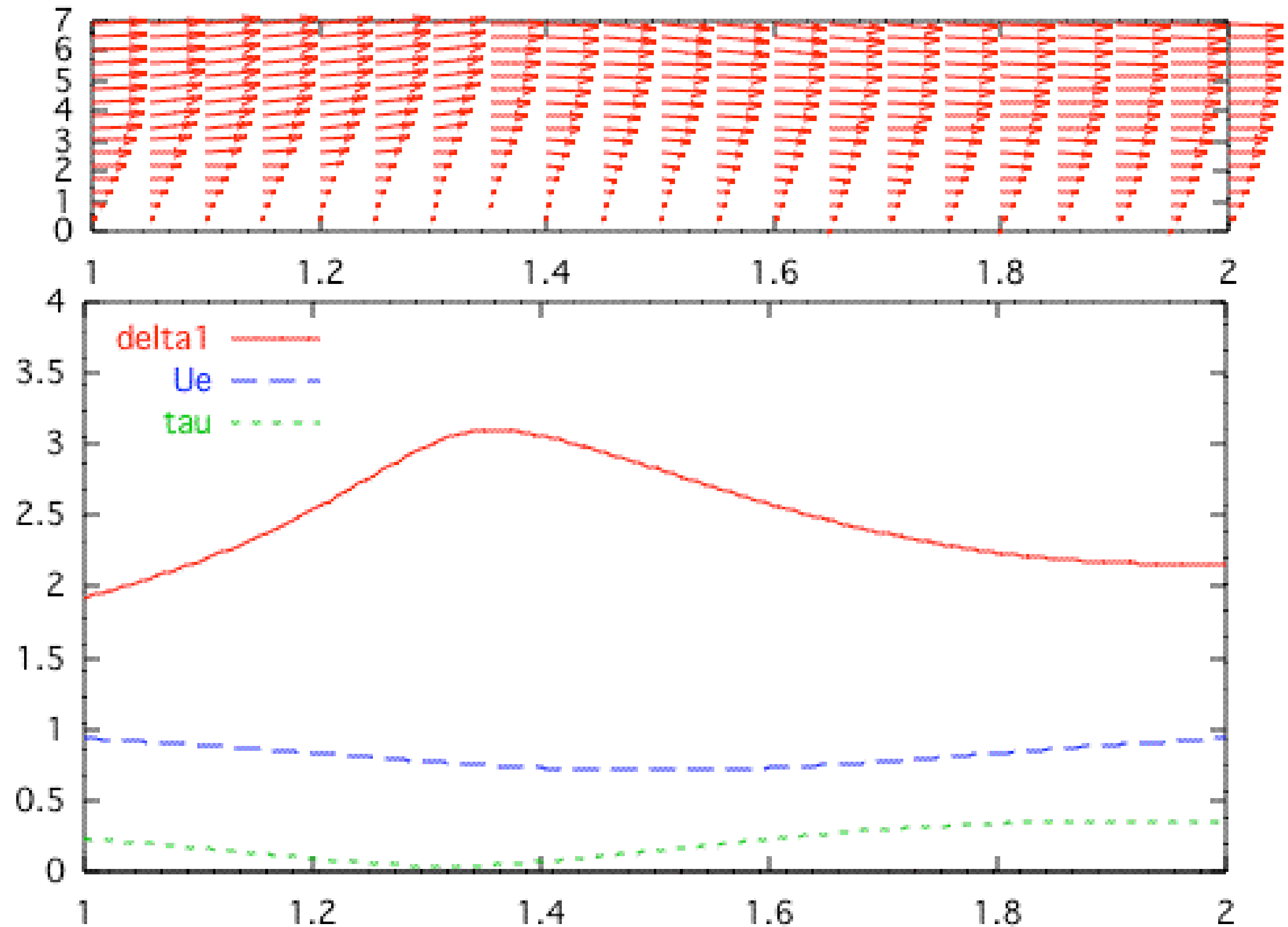
Ex. Boundary layer computations



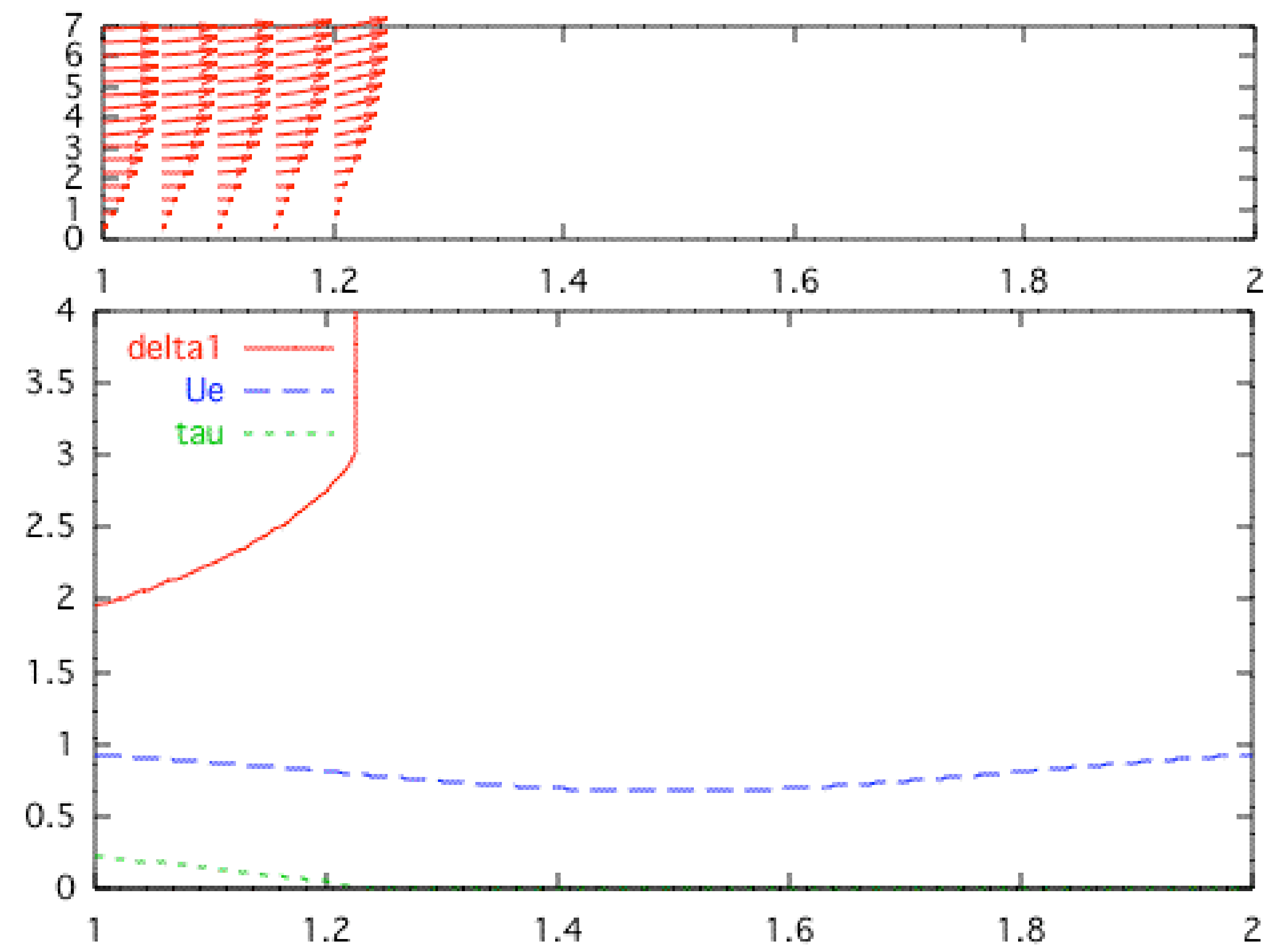
Ex. Boundary layer computations



Ex. Boundary layer computations



Ex. Boundary layer computations

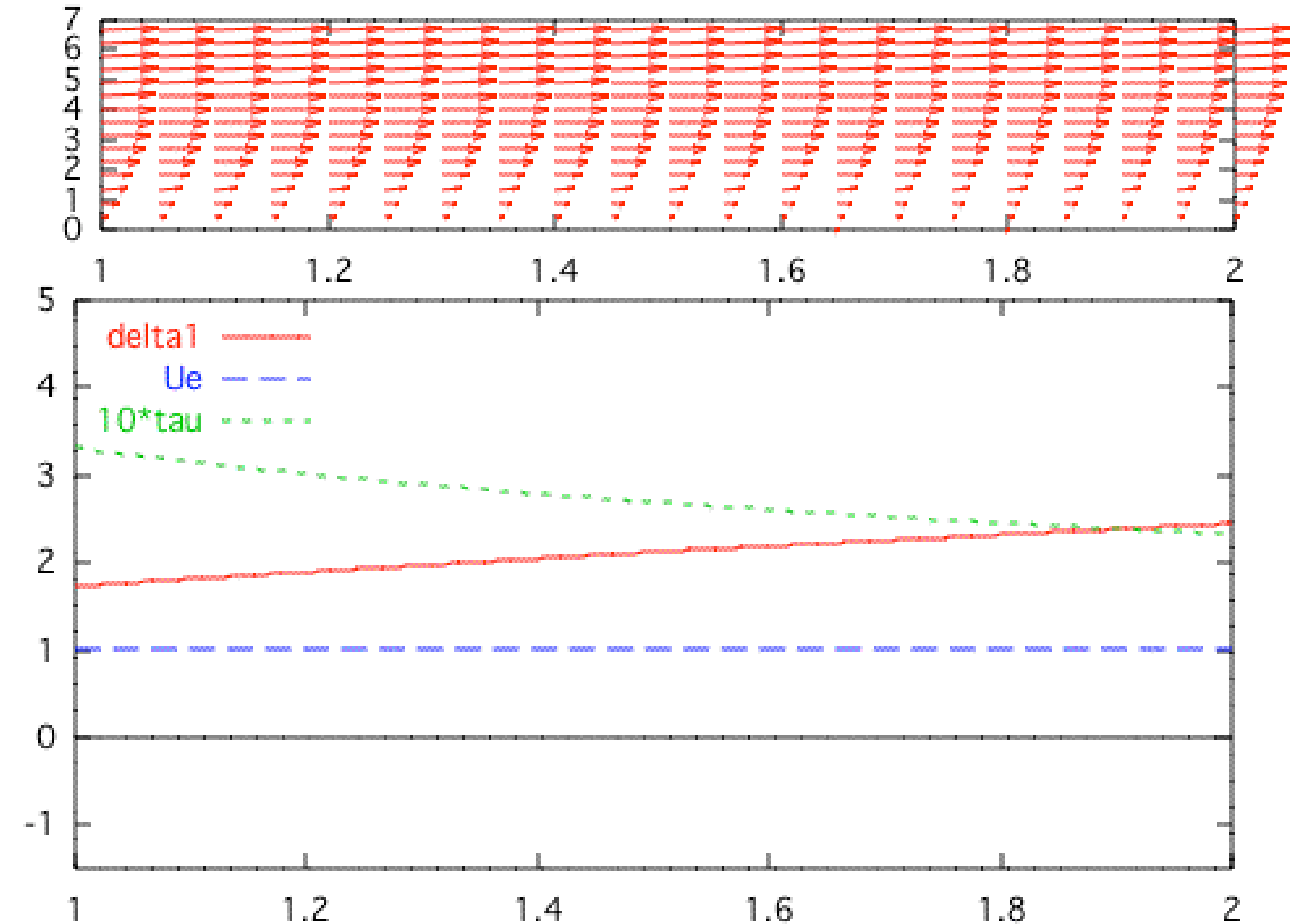


Goldstein Singularity

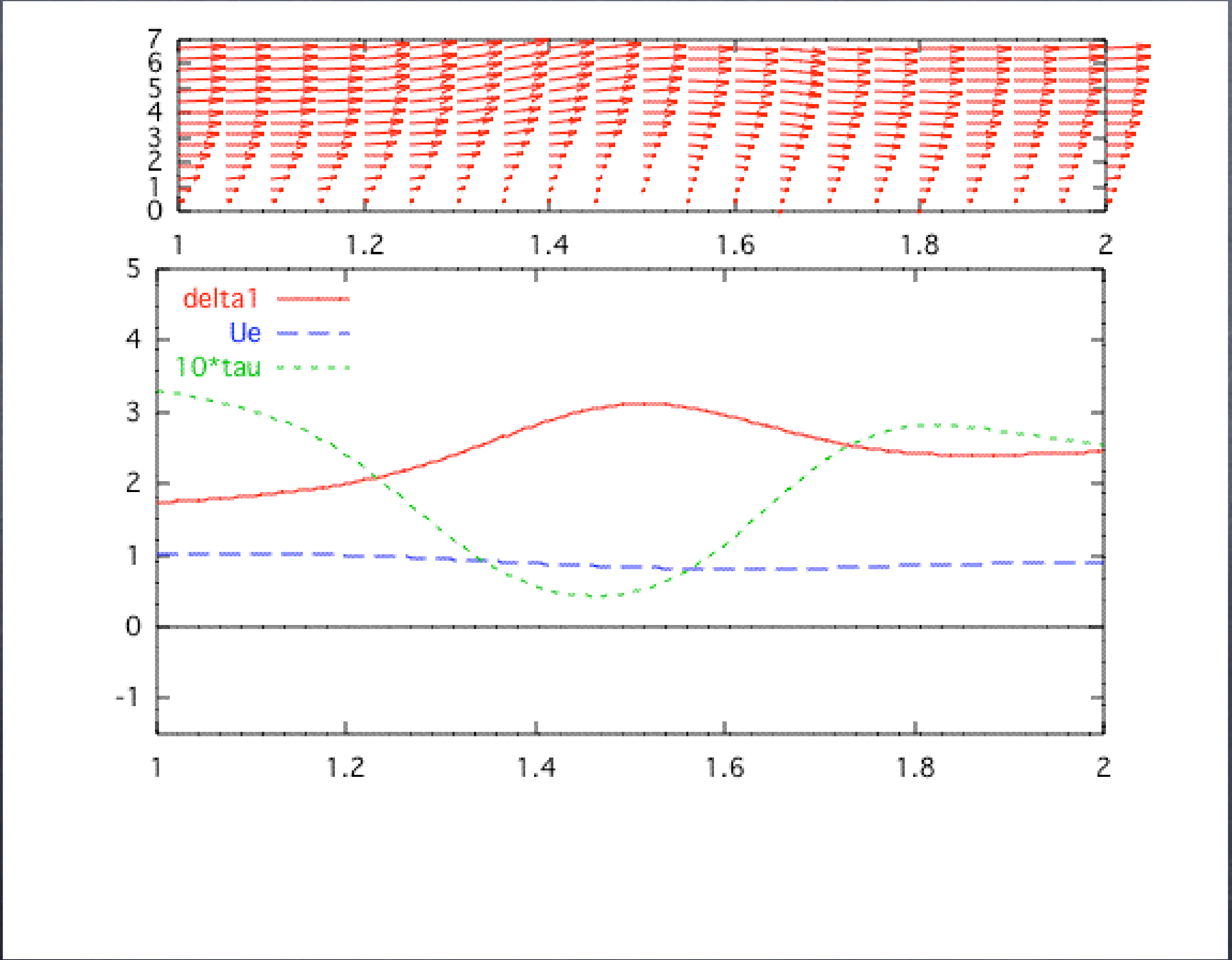
Impossible to compute Boundary layer separation

Inverse Boundary Layer!

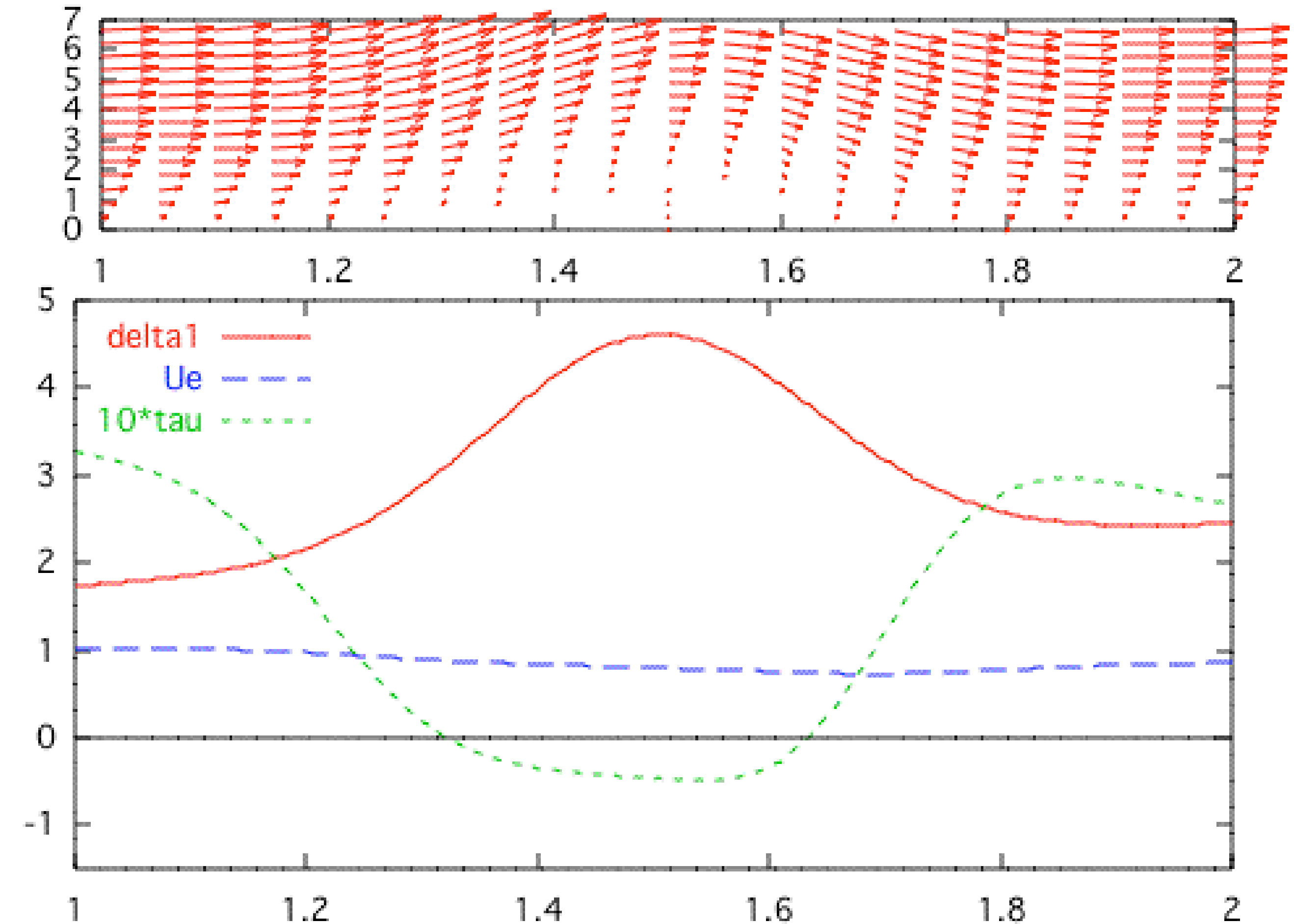
allows boundary layer separation,



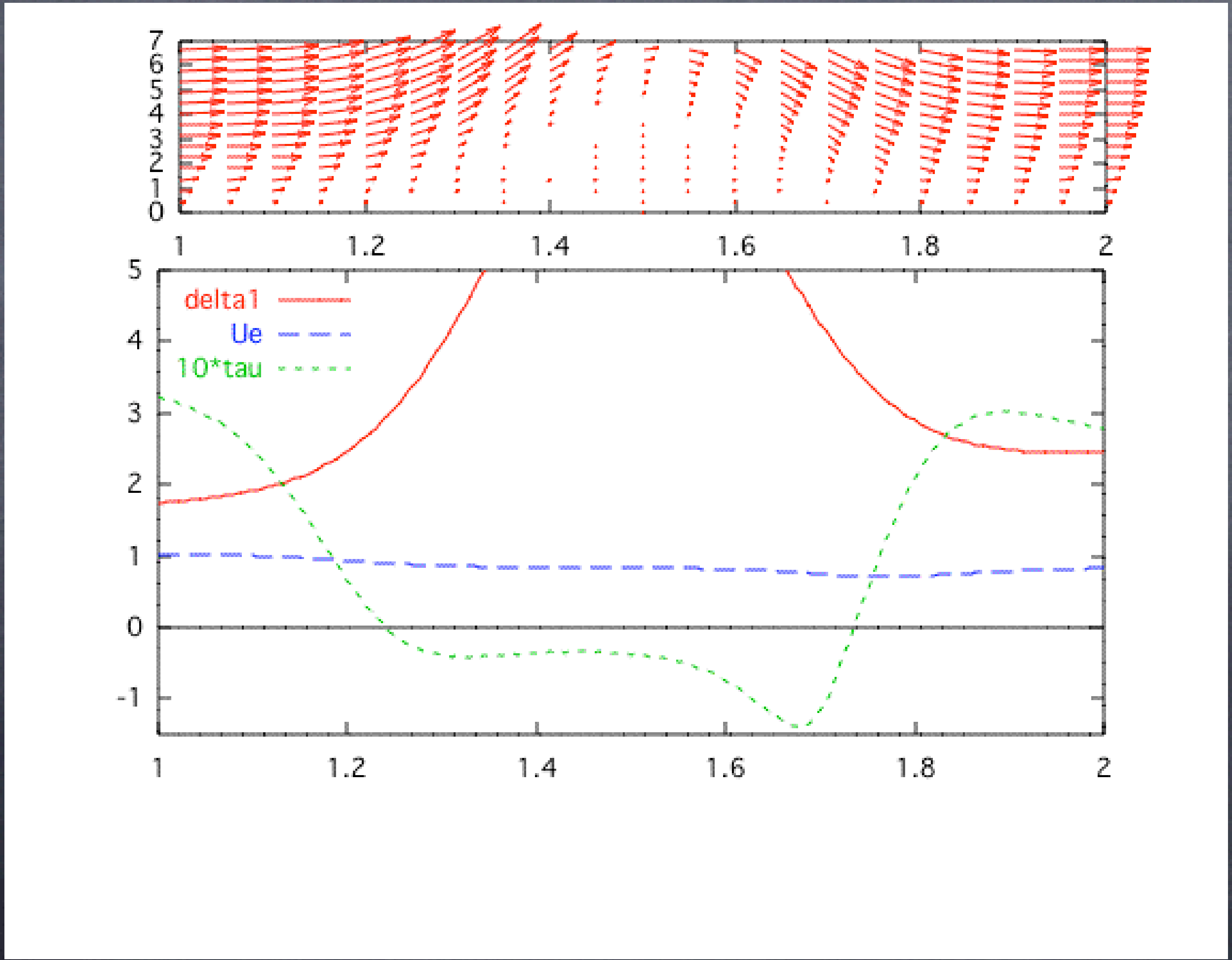
Inverse Boundary Layer!



Inverse Boundary Layer!

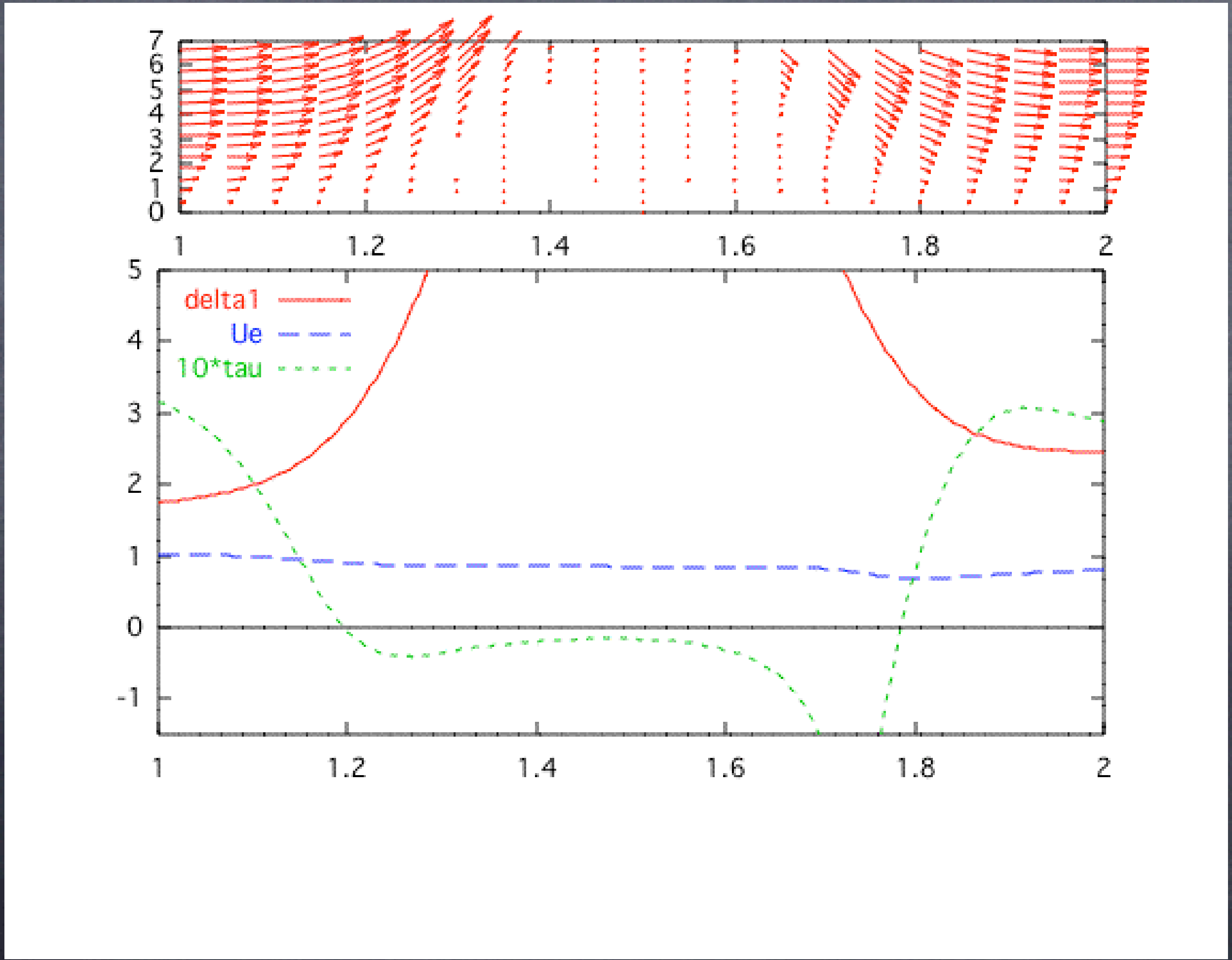


Inverse Boundary Layer!

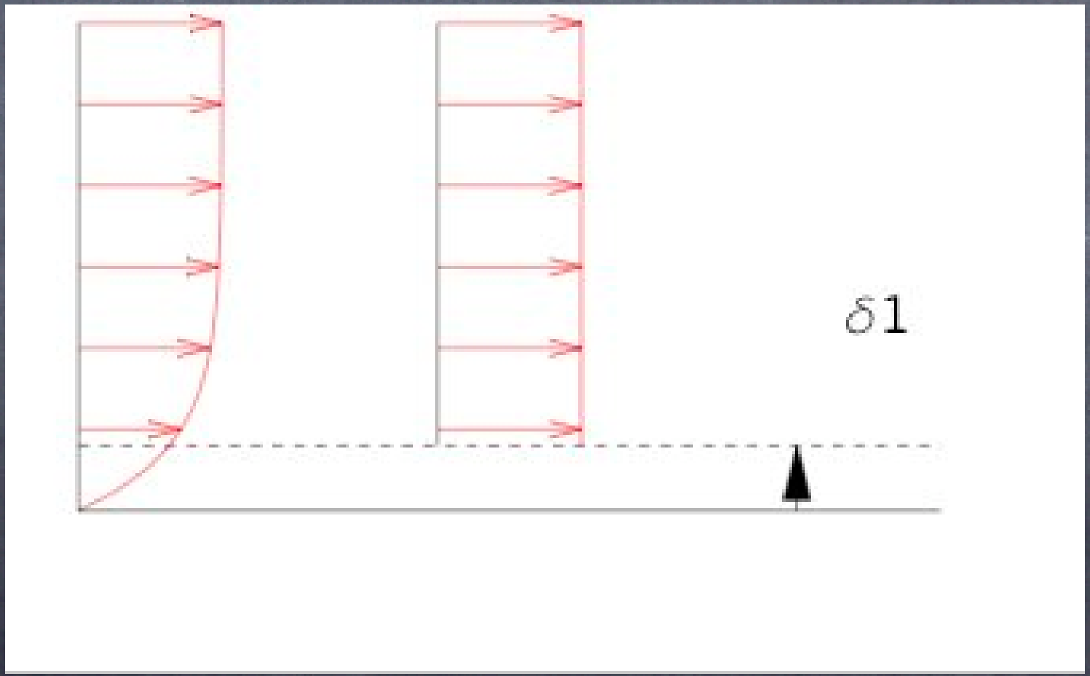


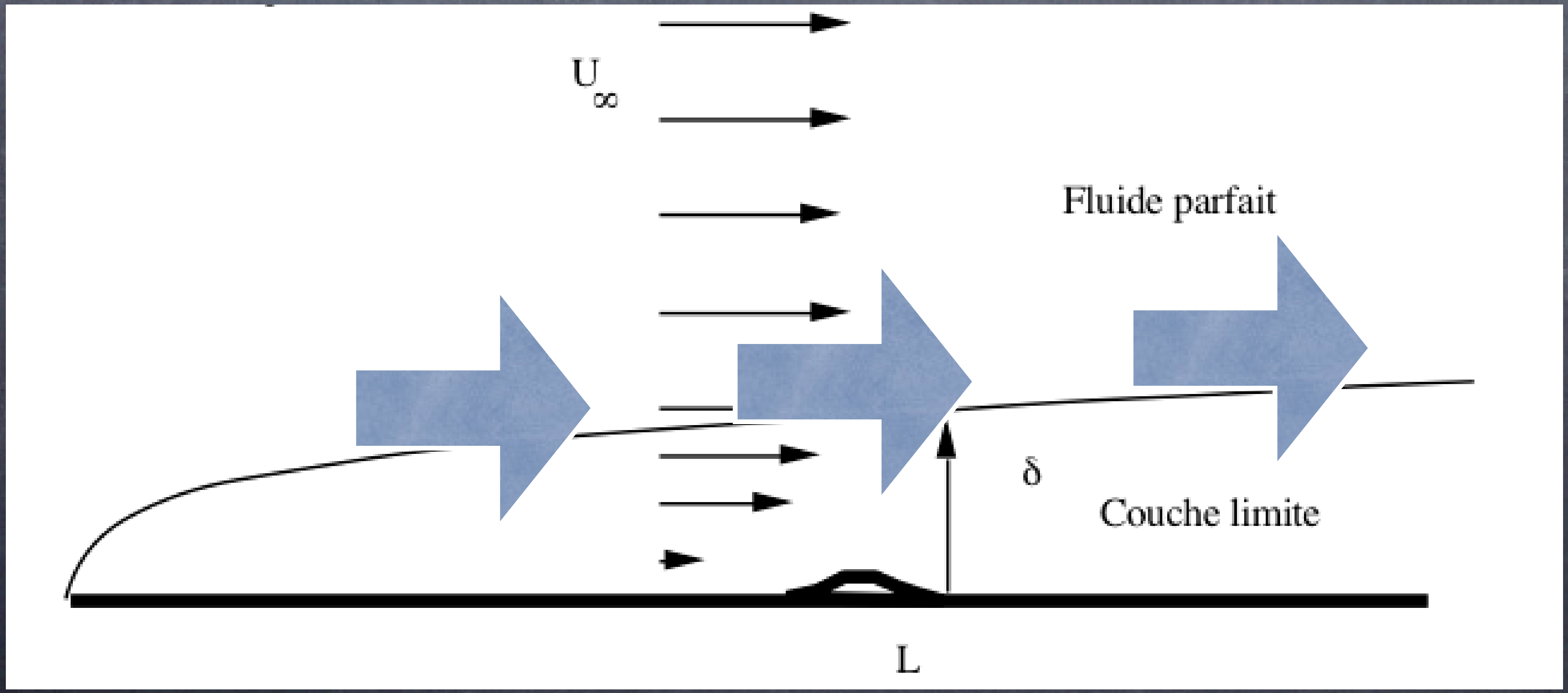
Inverse Boundary Layer!

allows boundary layer separation!!!



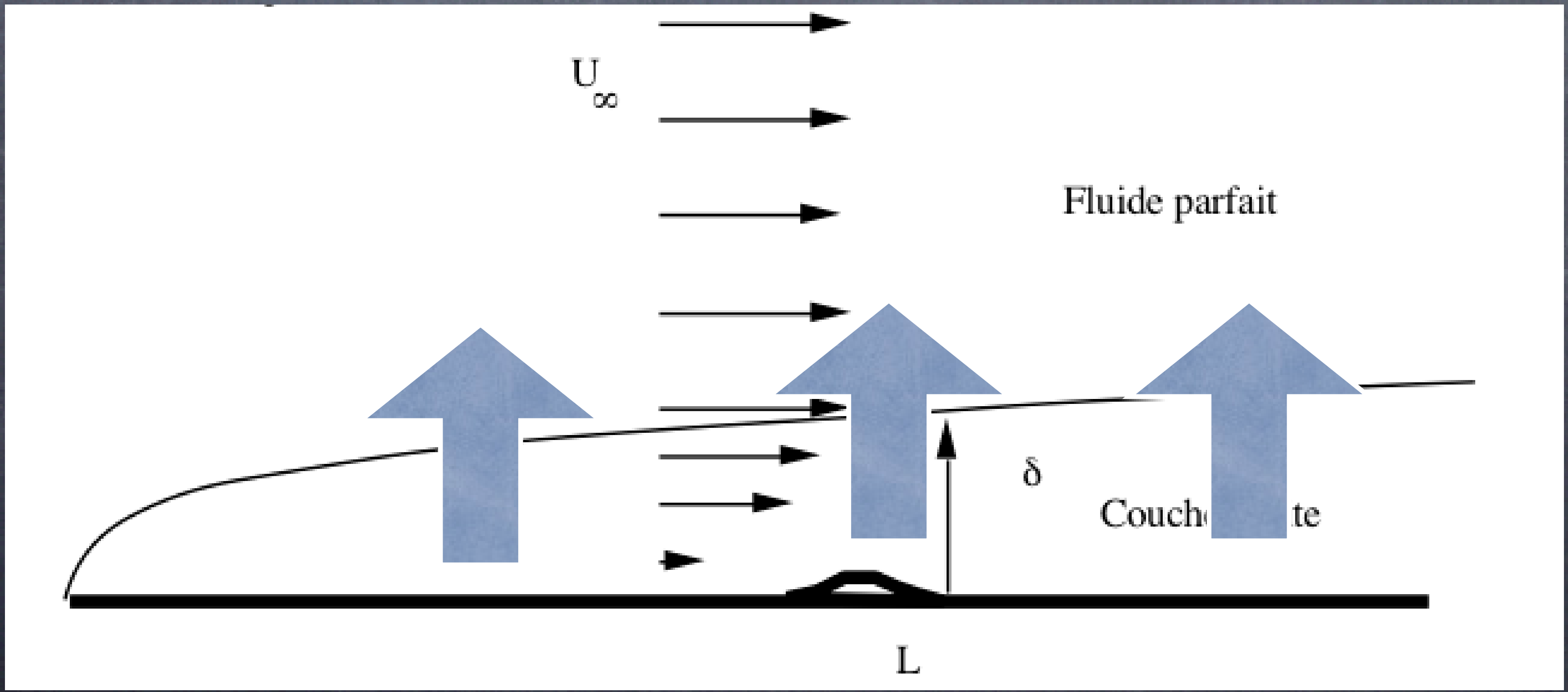
Perturbation of the Ideal fluid at the next order





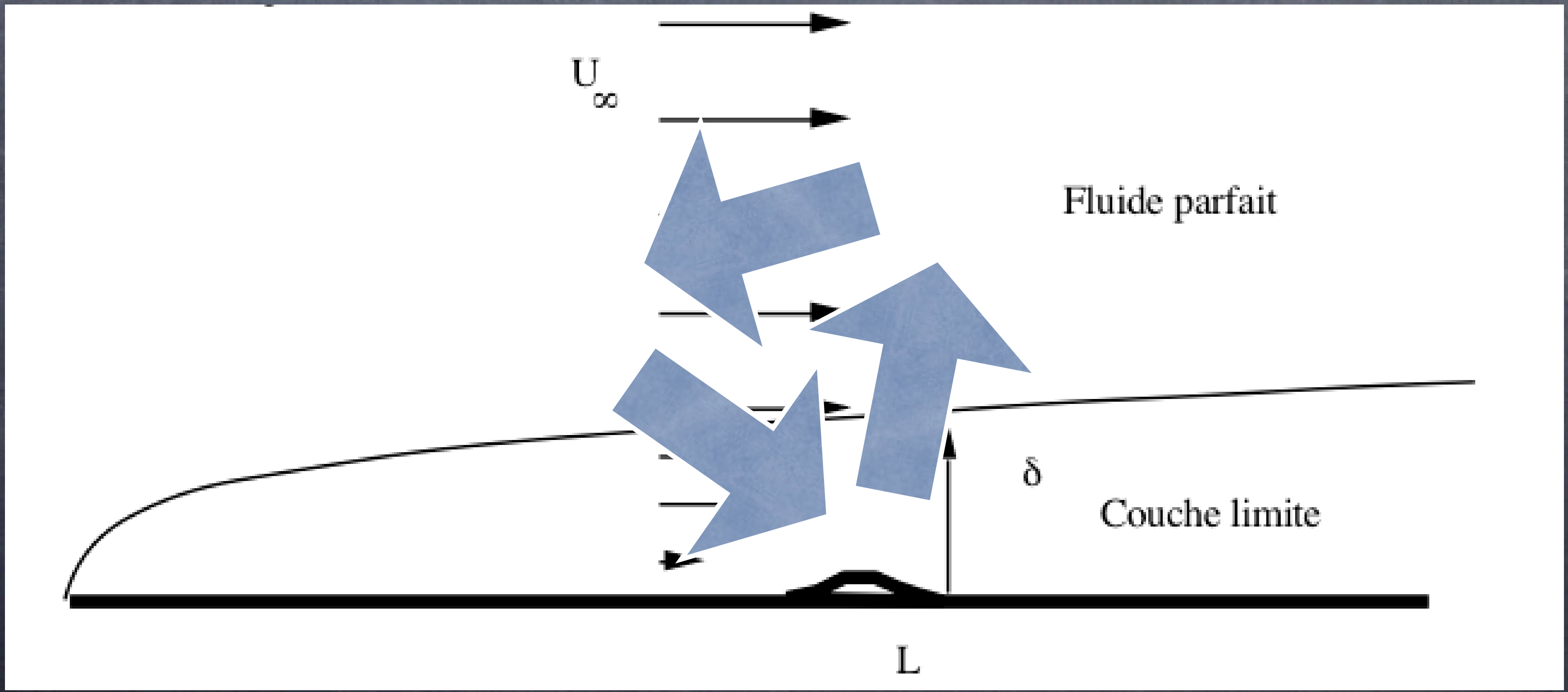
U_∞
Fluide parfait
δ
Couche limite
L

Perturbation of the Ideal fluid at the next order



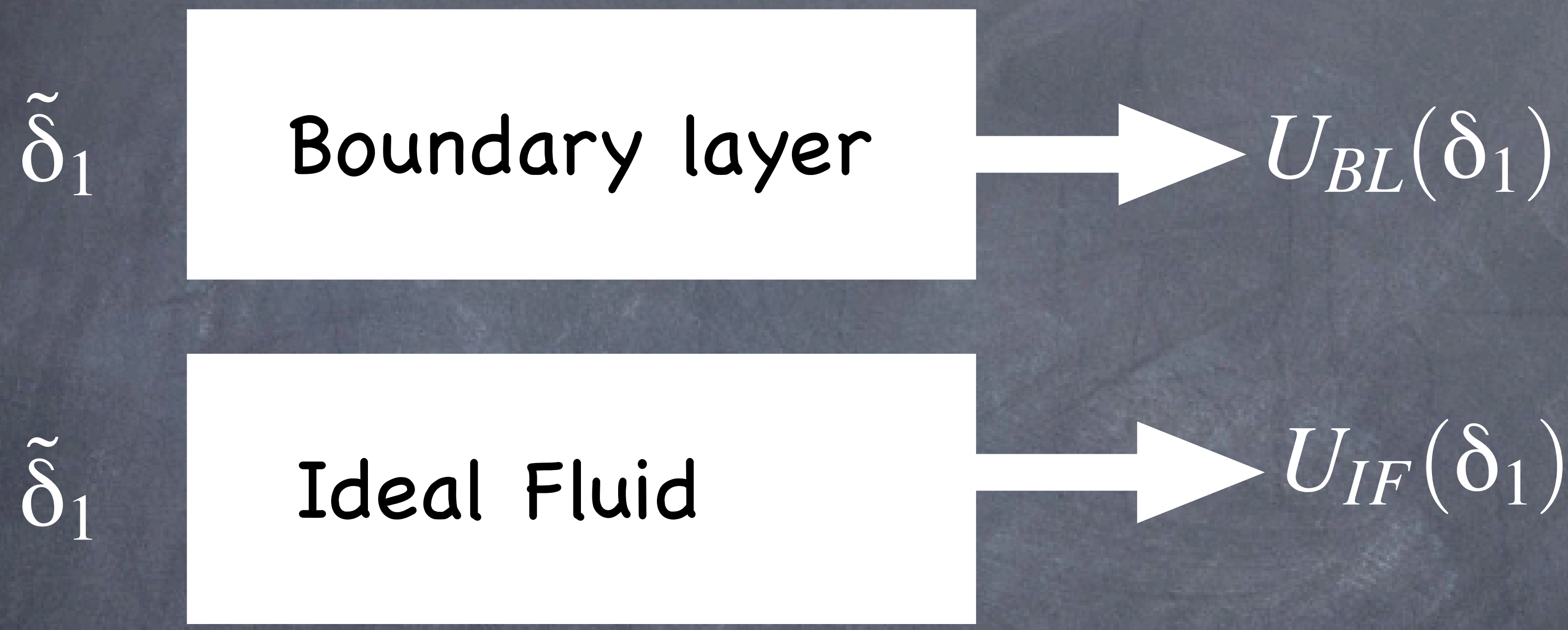
Interacting Boundary Layer

Ue <-> $\tilde{\delta}_1$



Interacting Boundary Layer

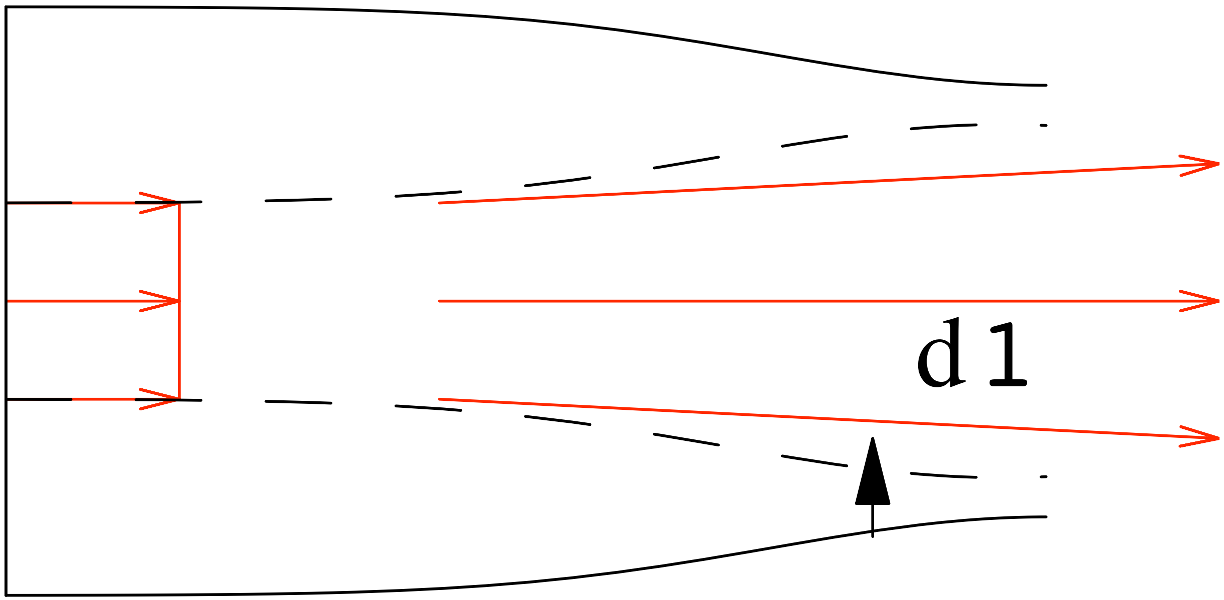
Semi inverse coupling



$$\tilde{\delta}_1^{n+1} = \tilde{\delta}_1^{n} + \mu(U_{BL}(\delta_1^n) - U_{IF}(\delta_1^n))$$

in stenoses

Boundary Layer/ Perfect Fluid



The displacement thickness acts as a "new" wall!
$\rightarrow$Interacting Boundary Layer (IBL)

After rescalling:
$r = R(\bar{x}) - (\lambda/Re)^{-1/2}\bar{y}$, $u = \bar{u}$, $v = (\lambda/Re)^{1/2}\bar{v}$ and $x - x_b = (\lambda/Re)\bar{x}$, $p = \bar{p}$, where x_b is the position of the bump, the RNSP(x) set gives the final IBL (interacting Boundary Layer) problem as follows:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{n}} = 0$$

$$(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{n}}) = \bar{u}_e \frac{d\bar{u}_e}{d\bar{s}} + \frac{\partial}{\partial \bar{n}}\frac{\partial \bar{u}}{\partial \bar{n}}$$

with: $\bar{u}(\bar{x}, 0) = 0$, $\bar{v}(\bar{x}, 0) = 0$ $\bar{u}(\bar{x}, \infty) = u_e$, where $\bar{\delta}_1 = \int_0^\infty (1 - \frac{\bar{u}}{\bar{u}_e}) d\bar{n}$, and

$$\bar{u}_e = \frac{1}{(R^2 - 2((\lambda/Re)^{-1/2})\bar{\delta}_1)}.$$

IBL integral: 1D equation

$$\frac{d}{d\bar{x}}(\frac{\bar{\delta}_1}{H}) + \bar{\delta}_1(1 + \frac{2}{H})\frac{d\bar{u}_e}{d\bar{x}} = \frac{f_2 H}{\bar{\delta}_1 \bar{u}_e},$$
$$\bar{u}_e = \frac{1}{(R^2 - 2(\lambda/Re)^{-1/2}\bar{\delta}_1)}.$$

To solve this system, a closure relationship linking H and f_2 to the velocity and the displacement thickness is needed.

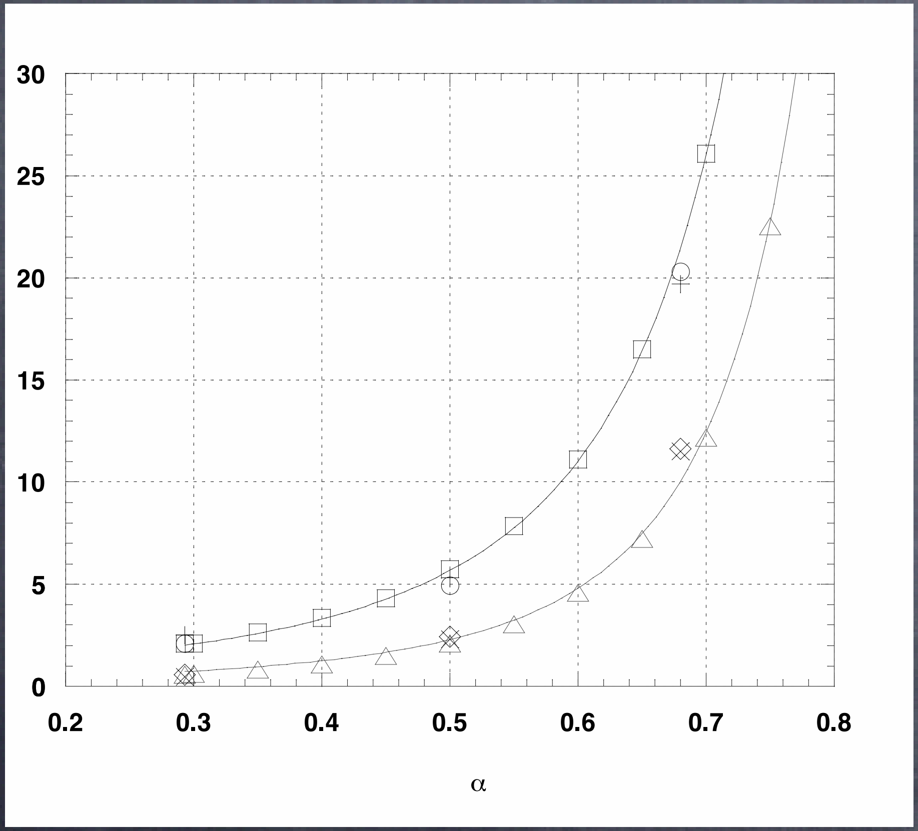
Defining $\Lambda_1 = \bar{\delta}_1^2 \frac{d\bar{u}_e}{d\bar{x}}$,

the system is closed from the resolution of the Falkner Skan system as follows:

if $\Lambda_1 < 0.6$ then $H = 2.5905exp(-0.37098\Lambda_1)$, else $H = 2.074$.

From H,f_2 is computed as $f_2 = 1.05(-H^{-1} + 4H^{-2})$.

IBL integral: Comparison with Navier Stokes (Siegel et al. 1994)

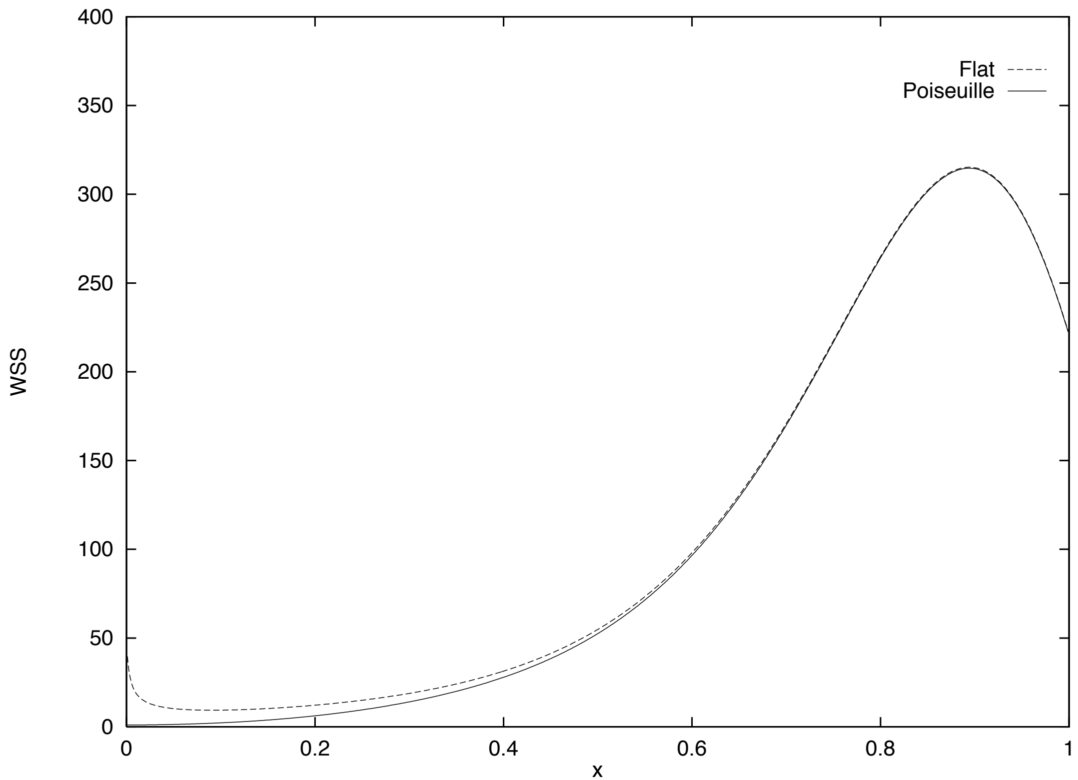


$$WSS = aRe^{1/2} + b$$

Coefficient a and b for the maximum WSS.
solid lines with $\triangle$ and "square" : coefficient a and b obtained using the IBL integral method ;

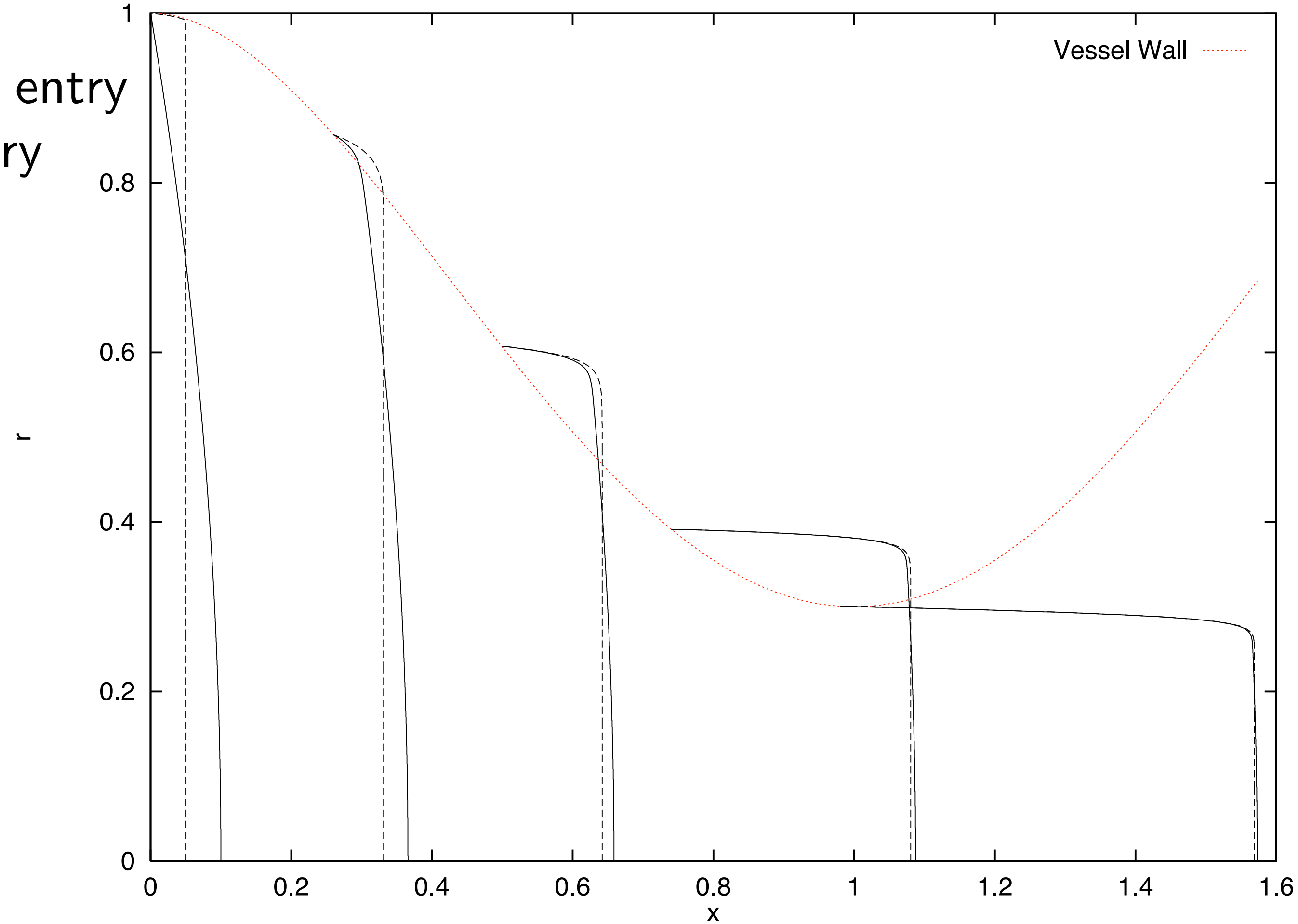
$\diamond$: coefficient a derived from Siegel for $\lambda = 3$;
$\times$: coefficient a derived from Siegel for $\lambda = 6$;
$\bigcirc$: coefficient b derived from Siegel for $\lambda = 3$;
$+$: coefficient b derived from Siegel for $\lambda = 6$.

Wall Shear Stress

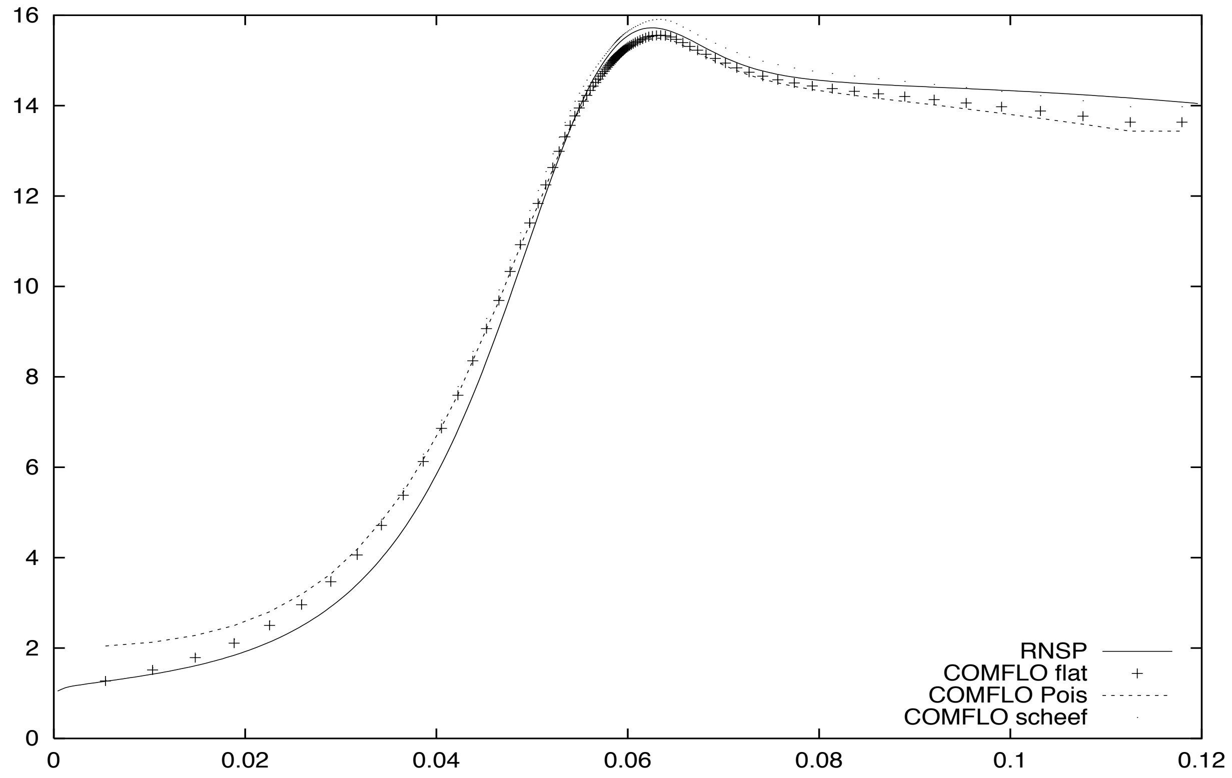


Evolution of the WSS distribution along the convergent part of a 70% stenosis ($Re = 500$) ; solid line: Poiseuille entry profile ; broken line: flat entry profile.

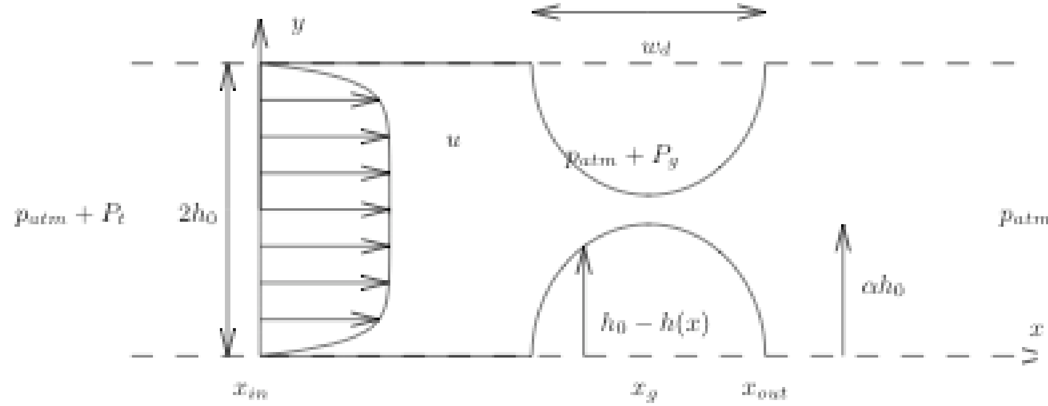
Evolution of the velocity profile along the convergent part of a 70% stenosis ($Re = 500$) ;
solid line: Poiseuille entry
broken line: flat entry



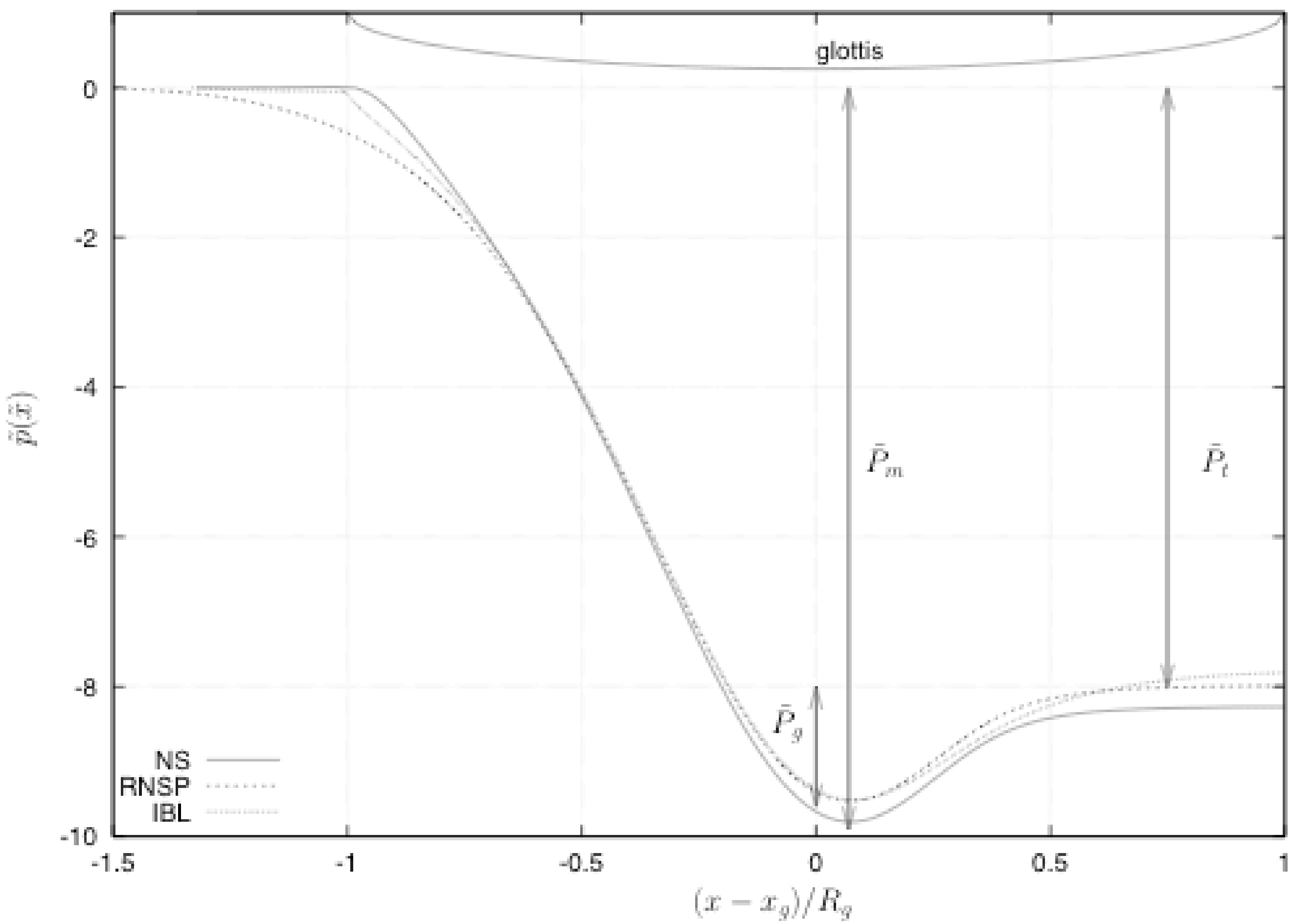
Testing asymmetry in the entry profile



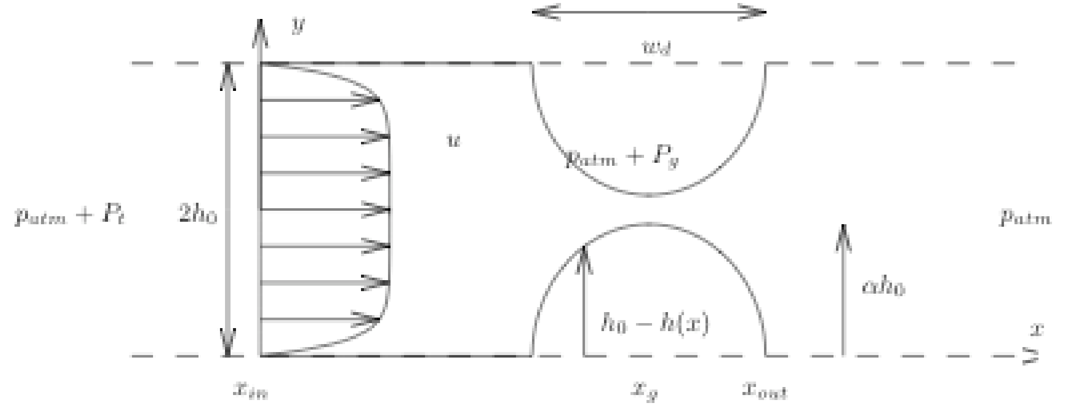
The velocities in the middle for Comflo and RNS.
Comflo uses here 50X50X100 points. Dimensionless scales!



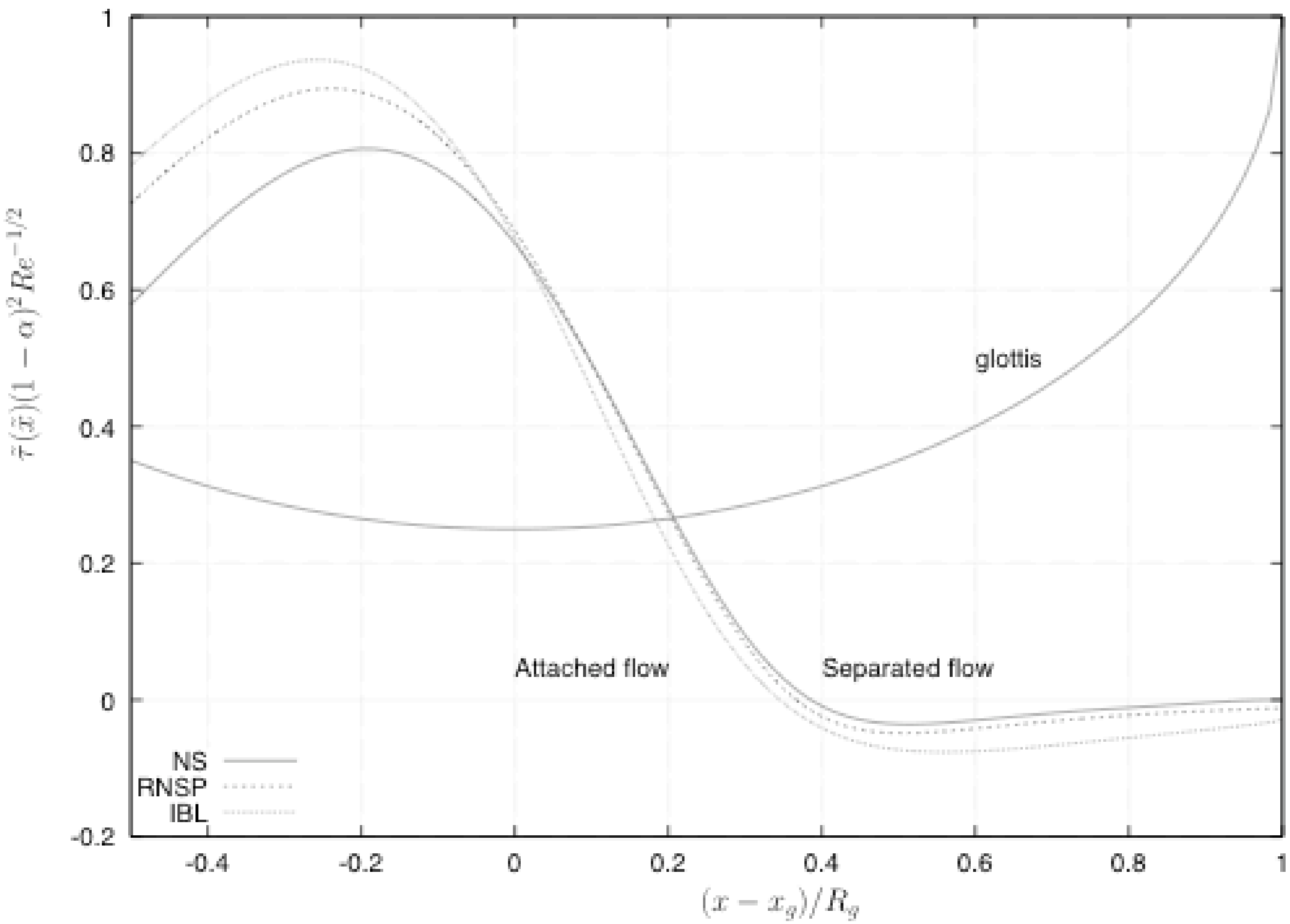
y
w_d
u
$p_{atm} + P_g$
$p_{atm} + P_t$
$2h_0$
p_{atm}
αh_0
$h_0 - h(x)$
x
x_{in}
x_g
x_{out}



glottis
0
-2
-4
-6
-8
-10
$\tilde{p}(x)$
$\tilde{P}_m$
$\tilde{P}_t$
$\tilde{P}_g$
NS
RNSP
IBL
-1.5
-1
-0.5
0
0.5
1
$(x - x_g)/R_g$

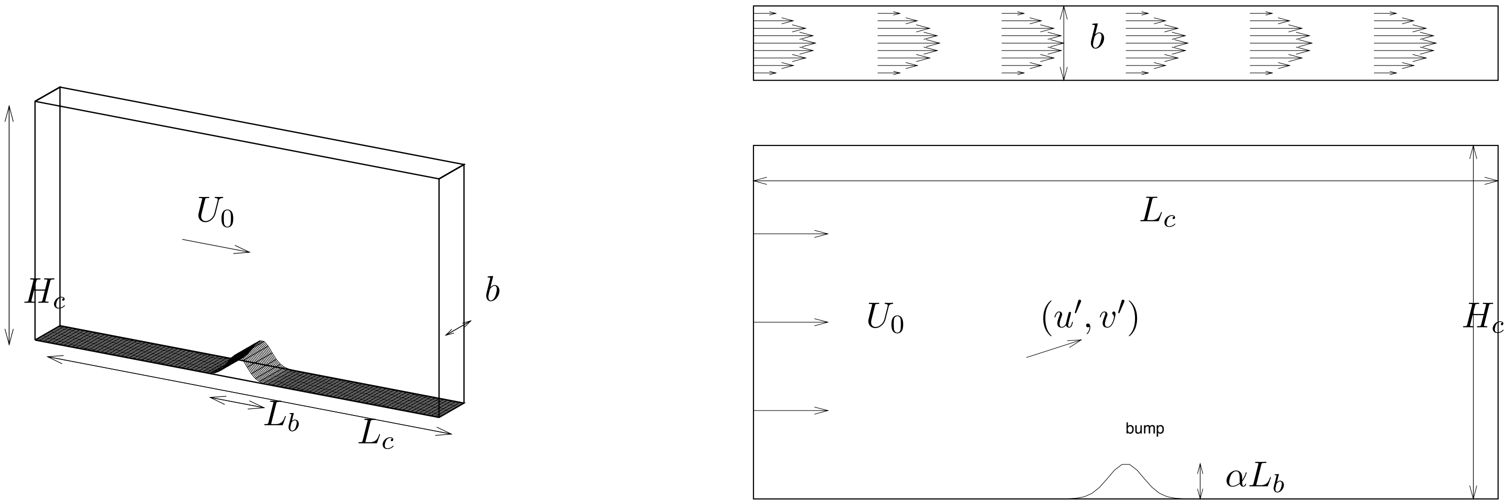


y
w_d
u
$p_{atm} + P_g$
$p_{atm} + P_t$
$2h_0$
p_{atm}
αh_0
$h_0 - h(x)$
x
x_{in}
x_g
x_{out}



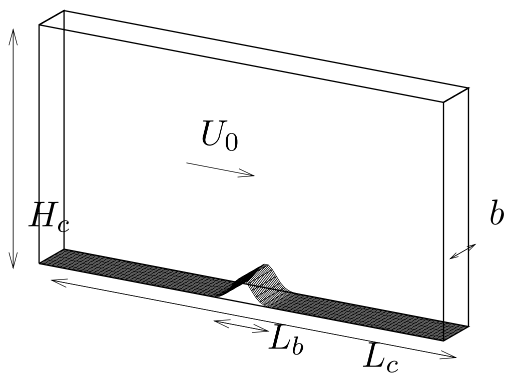
$\tilde{\tau}(\tilde{x})(1-\alpha)^2 Re^{-1/2}$
$(x - x_g)/R_g$
glottis
Attached flow
Separated flow
NS
RNSP
IBL

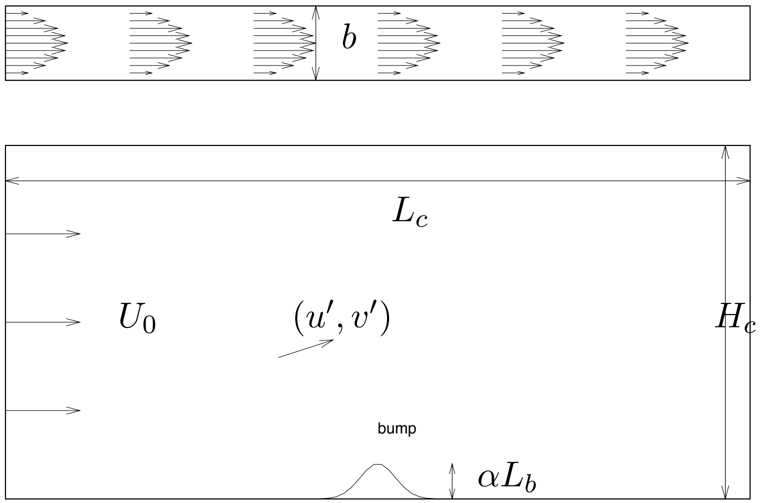
exemple Hele Shaw



$$
\begin{aligned}
(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}}) &= 0 \\
(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}) &= -\frac{\partial p}{\partial \bar{x}} - \bar{u} + \frac{1}{Re}(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}) \\
(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}) &= -\frac{\partial \bar{p}}{\partial \bar{y}} - \bar{v} + \frac{1}{Re}(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}),
\end{aligned}
$$

exemple Hele Shaw

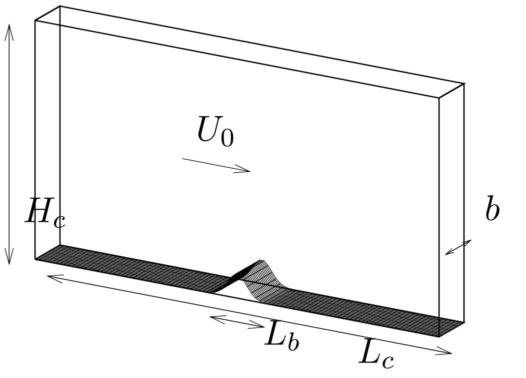


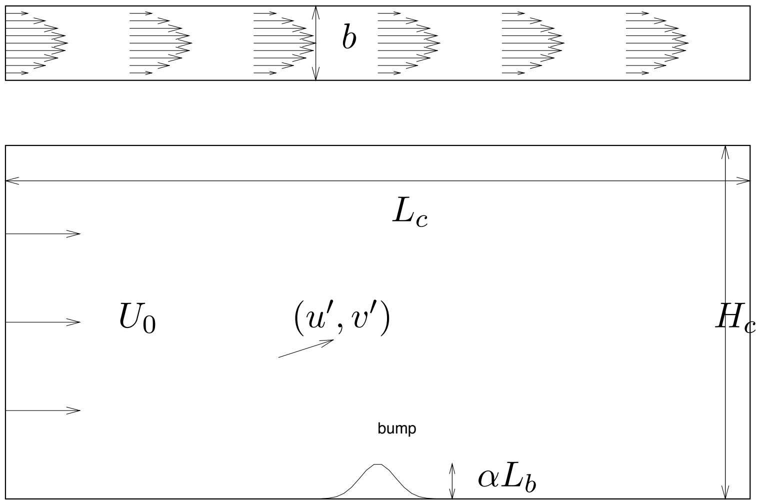


$$
\begin{aligned}
\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}}\right) &= 0 \\
\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) &= -\frac{\partial p}{\partial \bar{x}} - \bar{u} \\
\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) &= -\frac{\partial \bar{p}}{\partial \bar{y}} - \bar{v}
\end{aligned}
$$

$$
\bar{U}_e = 1 + \frac{1}{\pi} fp \int_{-\infty}^{\infty} \frac{\frac{d}{d\bar{x}}(\alpha \bar{f})}{\bar{x} - \xi} d\xi .
$$

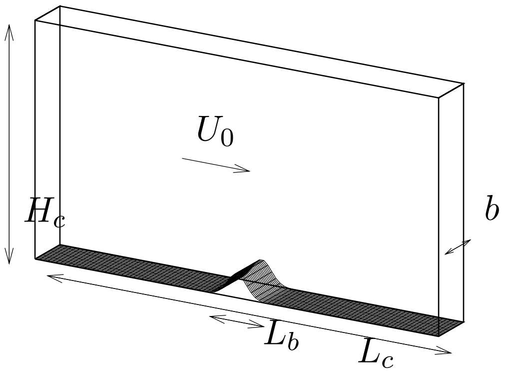
exemple Hele Shaw

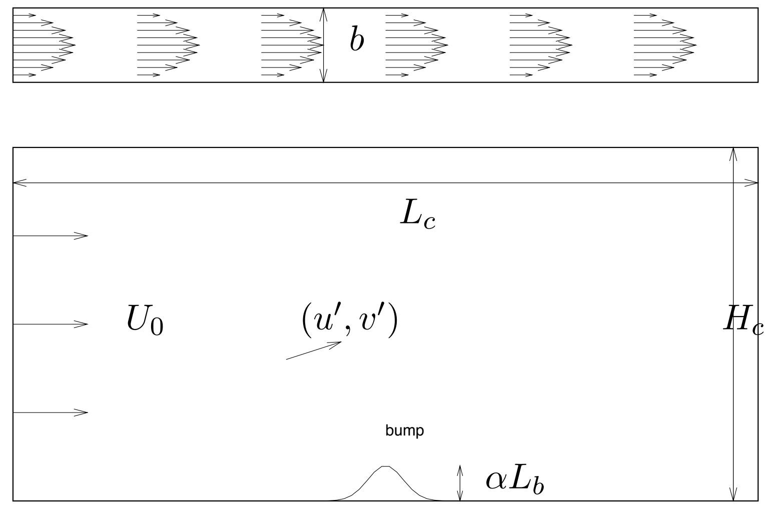




$$
\begin{aligned}
\left(\frac{\partial \tilde{u}}{\partial \bar{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}}\right) &= 0, \\
\left(\tilde{u}\frac{\partial \tilde{u}}{\partial \bar{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right) &= -\frac{\partial p}{\partial \bar{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} - \tilde{u}, \\
0 &= -\frac{\partial \tilde{p}}{\partial \tilde{y}}.
\end{aligned}
$$

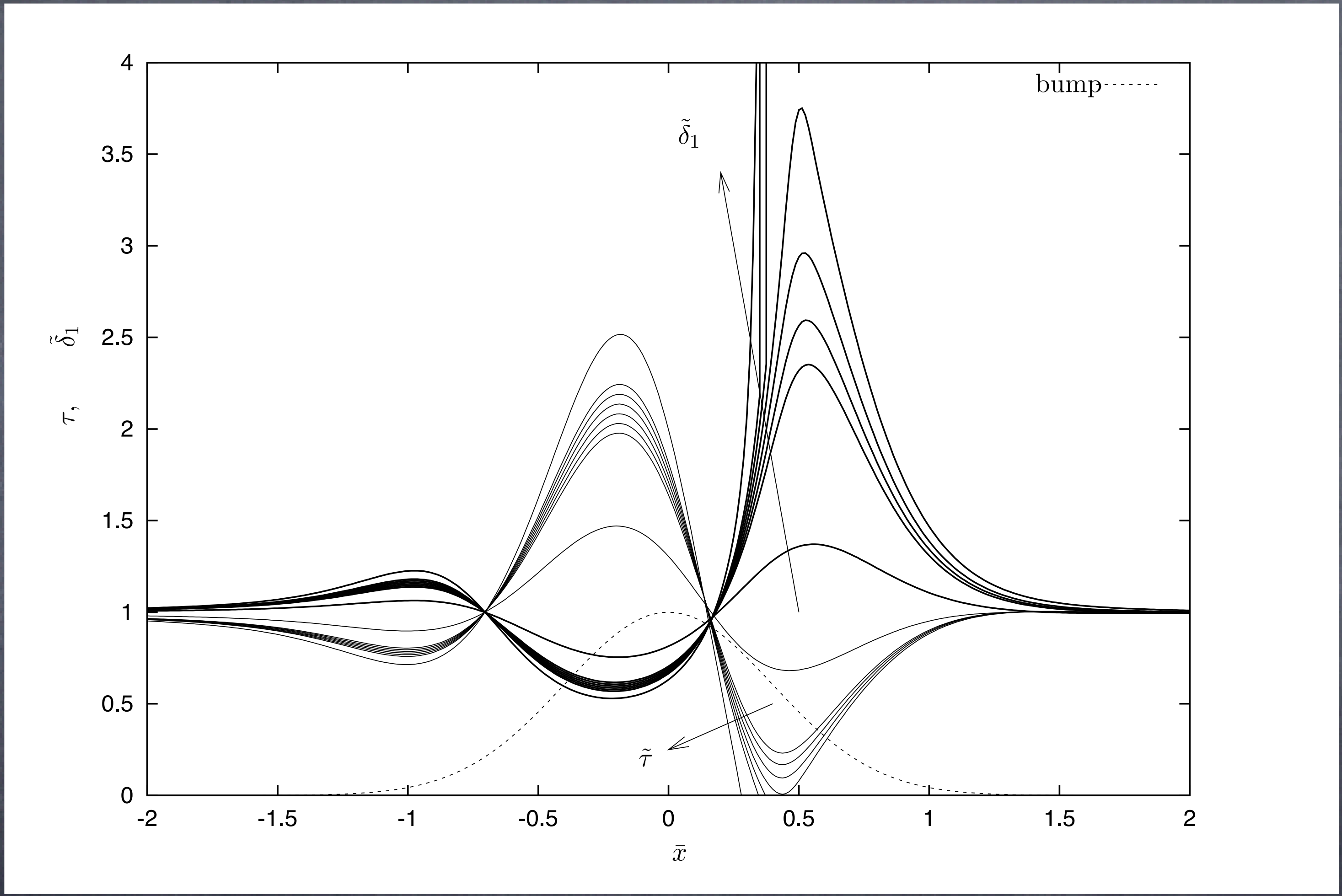
exemple Hele Shaw



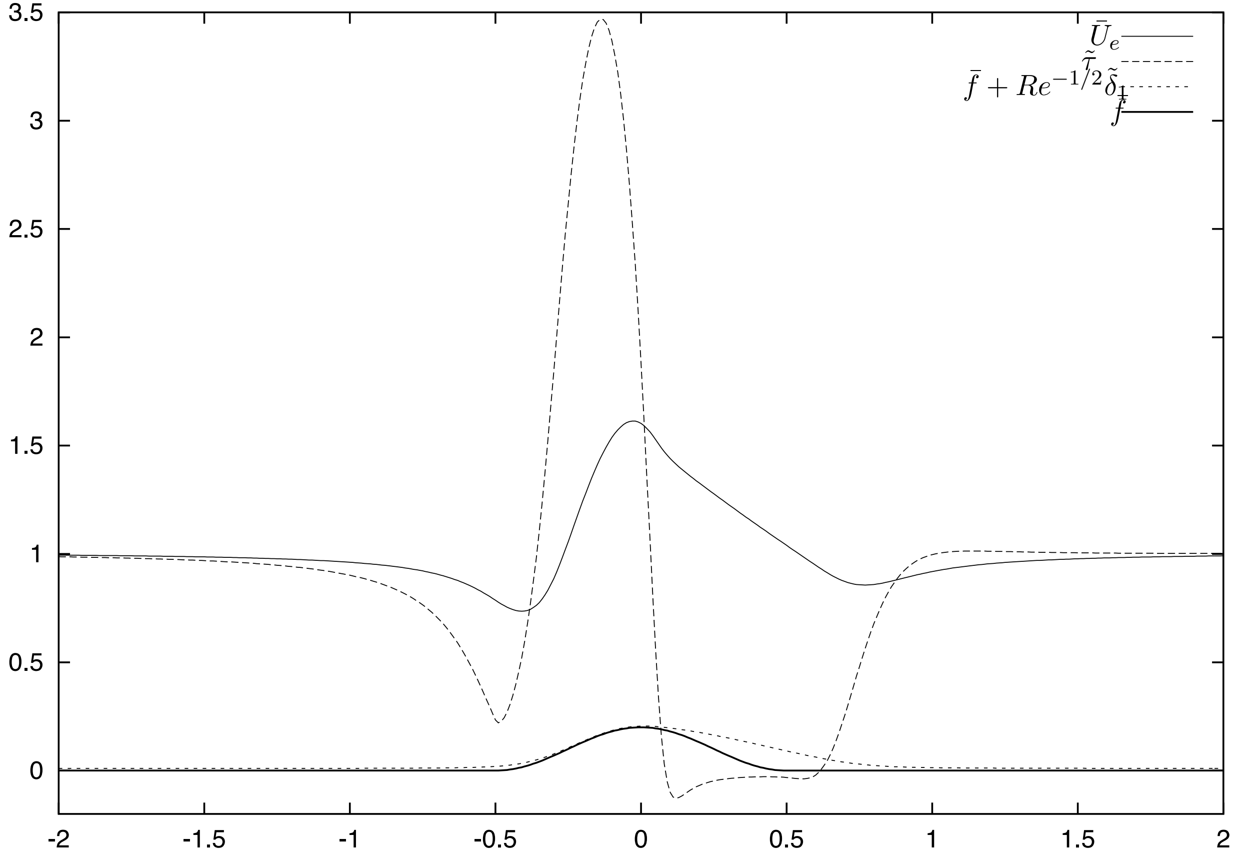


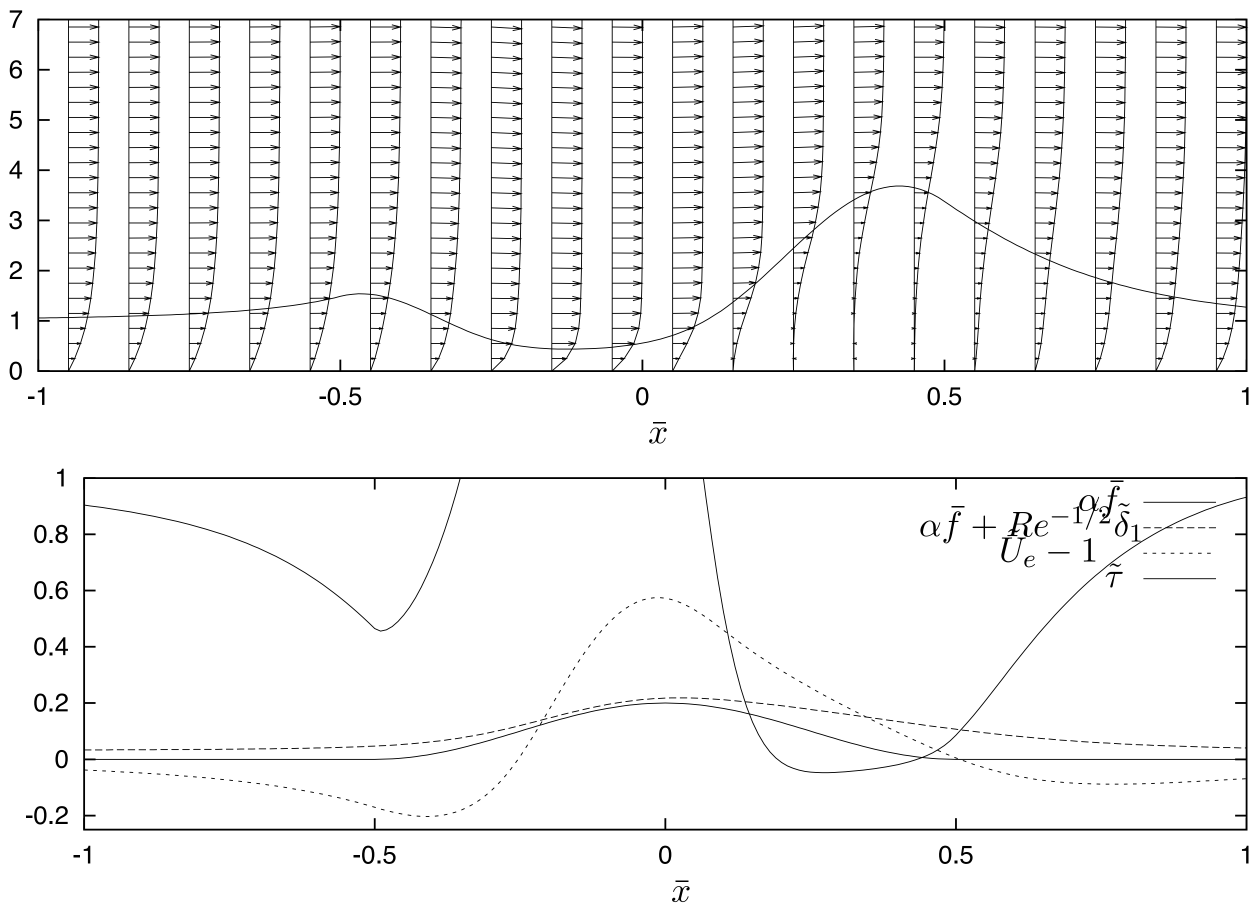
$$\bar{U}_e = 1 + \frac{1}{\pi} f p \int_{-\infty}^{\infty} \frac{\frac{d}{d\bar{x}}(\alpha \bar{f} + \tilde{\delta}_1 Re^{-1/2})}{\bar{x} - \xi} d\xi.$$

$$\begin{aligned}
\left(\frac{\partial \tilde{u}}{\partial \bar{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}}\right) &= 0, \\
\left(\tilde{u}\frac{\partial \tilde{u}}{\partial \bar{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right) &= -\frac{\partial p}{\partial \bar{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} - \tilde{u}, \\
0 &= -\frac{\partial \tilde{p}}{\partial \tilde{y}}.
\end{aligned}$$

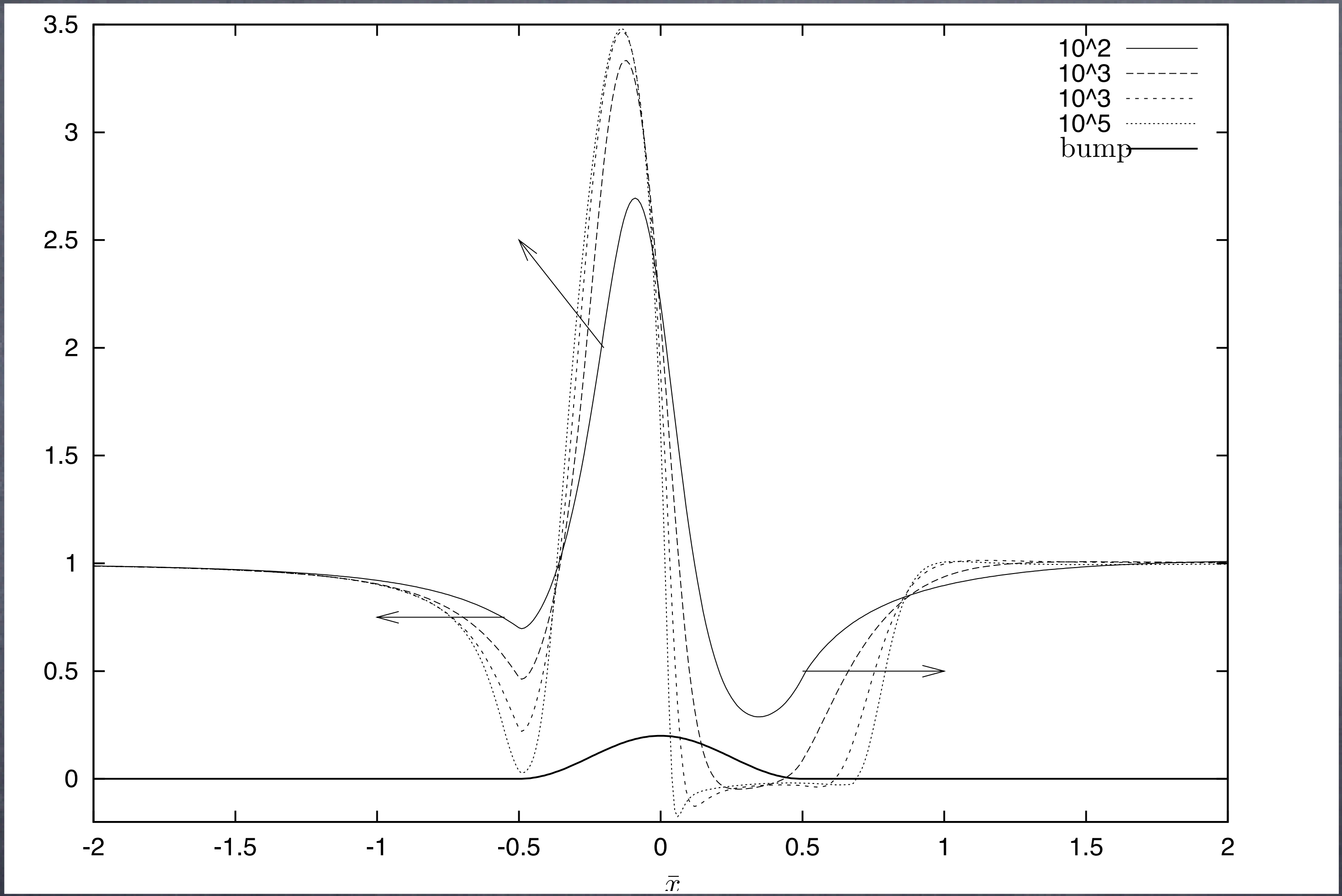


bump
$\tilde{\delta}_1$
$\tilde{\tau}$
$\tau, \quad \tilde{\delta}_1$
$\bar{x}$
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

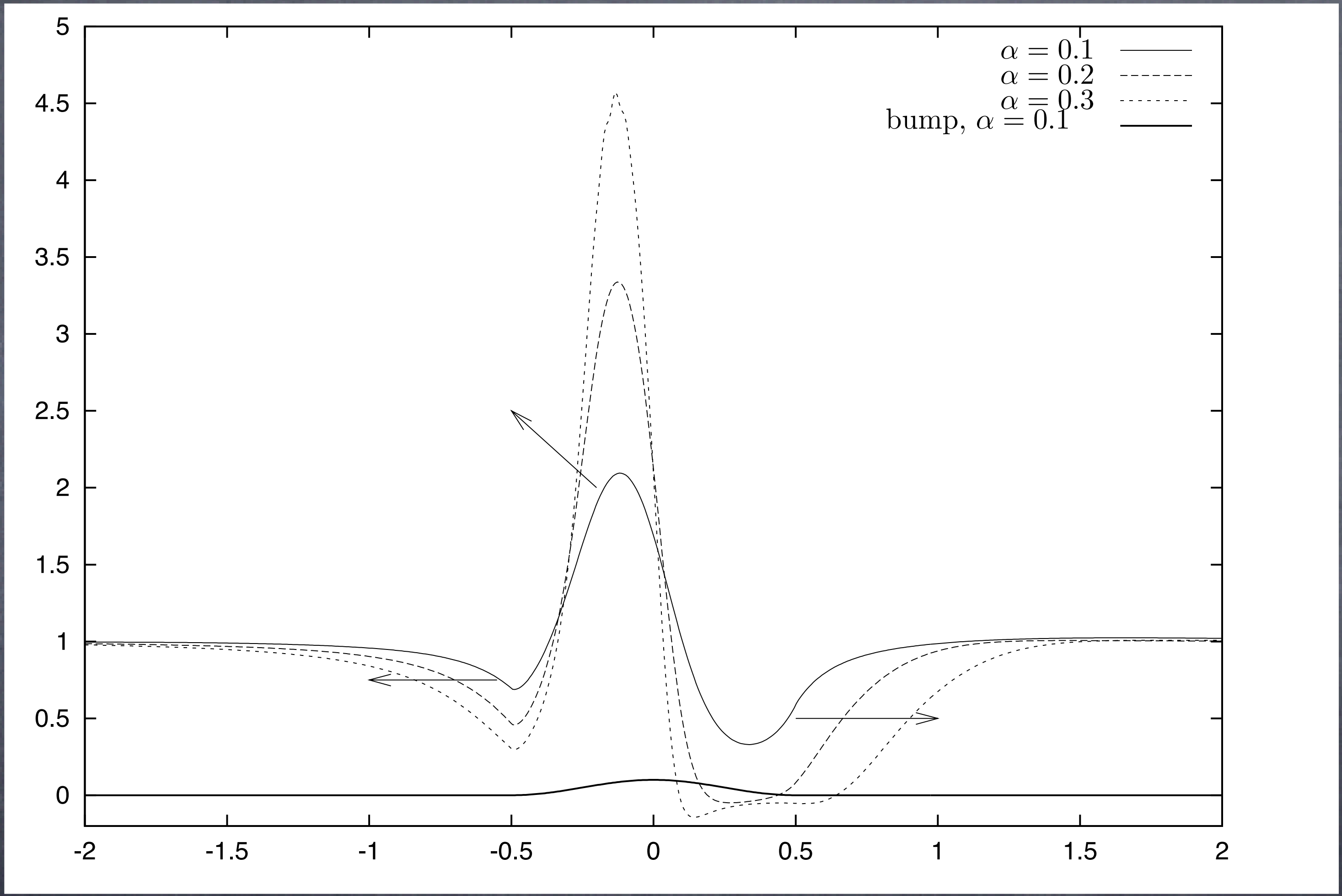

$\bar{U}_e$
$\tilde{\tau}$
$\bar{f} + Re^{-1/2}\tilde{\delta}_1$
$\bar{f}$
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2



7
6
5
4
3
2
1
0
-1
-0.5
0
0.5
1
$\bar{x}$
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.5
0
0.5
1
$\bar{x}$
$\alpha\bar{f}$
$\alpha\bar{f} + Re^{-1/2}\tilde{\delta}_1$
$\bar{U}_e - 1$
$\tilde{\tau}$



10^2
10^3
10^3
10^5
bump
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
$\bar{x}$



$\alpha = 0.1$
$\alpha = 0.2$
$\alpha = 0.3$
bump, $\alpha = 0.1$
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Looking at AB functions

$$(\bar{U}_e^2 \frac{d}{d\bar{x}}(\frac{\tilde{\delta}_1}{H}) + (\tilde{\delta}_1 + \frac{2\tilde{\delta}_1}{H})\bar{U}_e \frac{d\bar{U}_e}{d\bar{x}}) = f_1 \frac{\bar{U}_e}{\tilde{\delta}_1}.$$

$$\bar{U}_e = 1 + \frac{1}{\pi} f p \int_{-\infty}^{\infty} \frac{\frac{d}{d\bar{x}}(\alpha \bar{f} + \tilde{\delta}_1 Re^{-1/2})}{\bar{x} - \xi} d\xi$$

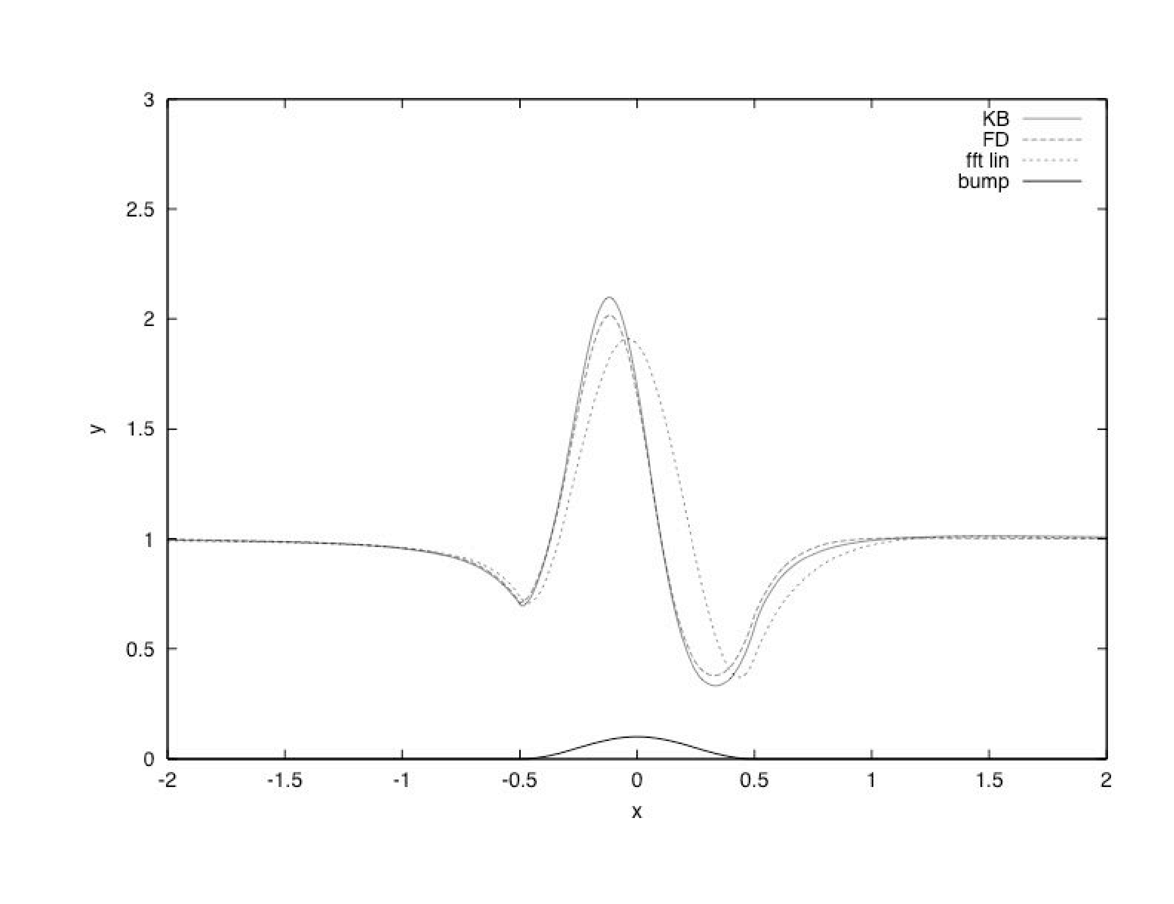
Suppose the profile remains the same exponential profile

Looking at AB functions

Linearized AB

$$\tilde{\tau} = 1 + FT^{-1}[(1 - \frac{C(k)|k|(1-|k|Re^{-1/2})}{1-C(k)|k|Re^{-1/2}})|k|FT[\alpha\bar{f}]].$$

$$C(k) = -\frac{(-ik)}{\frac{(-ik)}{2}+2}$$



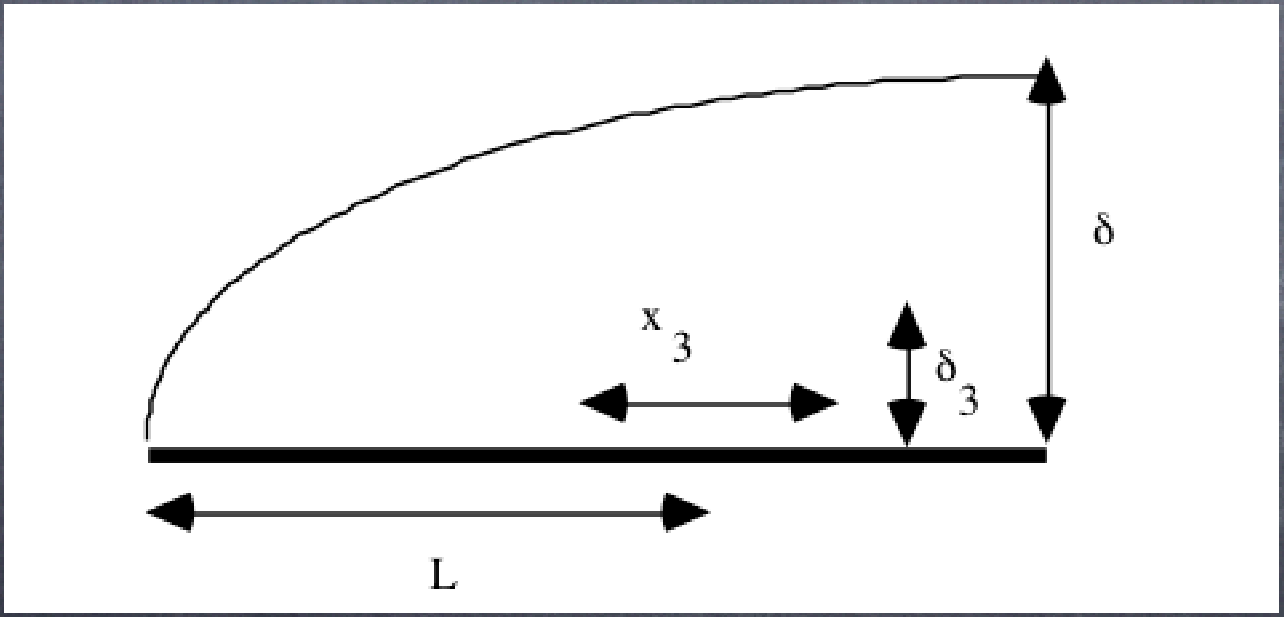
The TRIPLE DECK

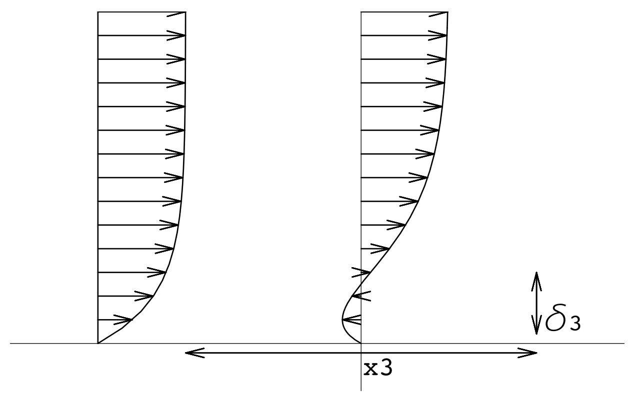
justification of the Interacting Boundary Layer

Rational way to look at Looking at AB functions in laminar flow at high Re

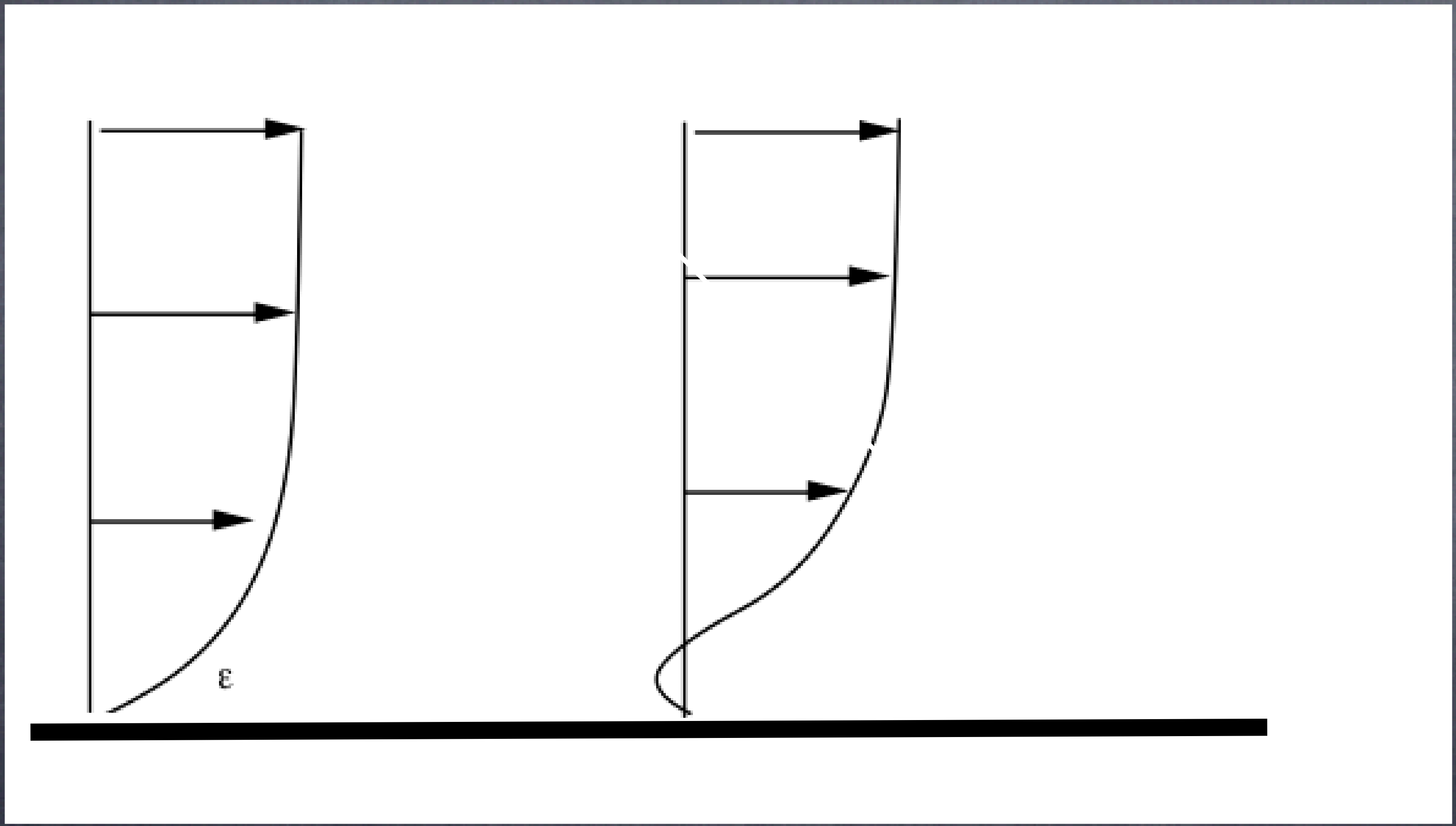
Triple Deck

new scales

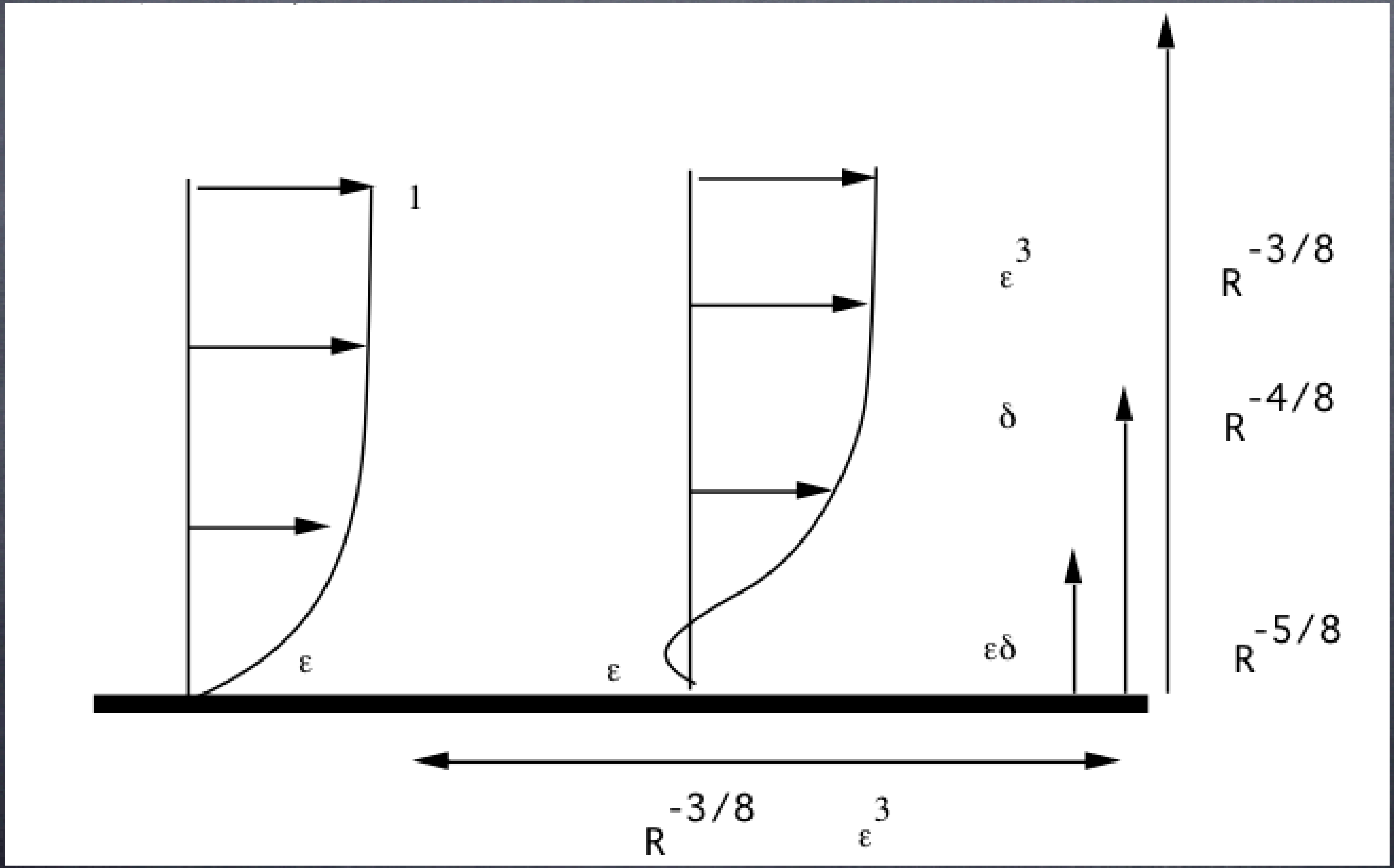




δ_3
x3


ε

triple deck



equations

lower Deck

$$\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v=0,$$

$$u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial y}u=-\frac{d}{dx}p+\frac{\partial^2}{\partial y^2}u.$$

$$u(x,y=f(x))=0, \qquad v(x,y=f(x))=0$$

$$\& \qquad \lim_{y\to\infty}u(x,y)=y+A.$$

coupling relation

Upper Deck

$$p = \frac{1}{\pi}\int \frac{\frac{dA(\xi)}{dx}}{x-\xi} d\xi$$

$$p = \pm A$$

$$A = 0$$

$$p = -\frac{dA}{dx}$$

Linearised Fourier Solution

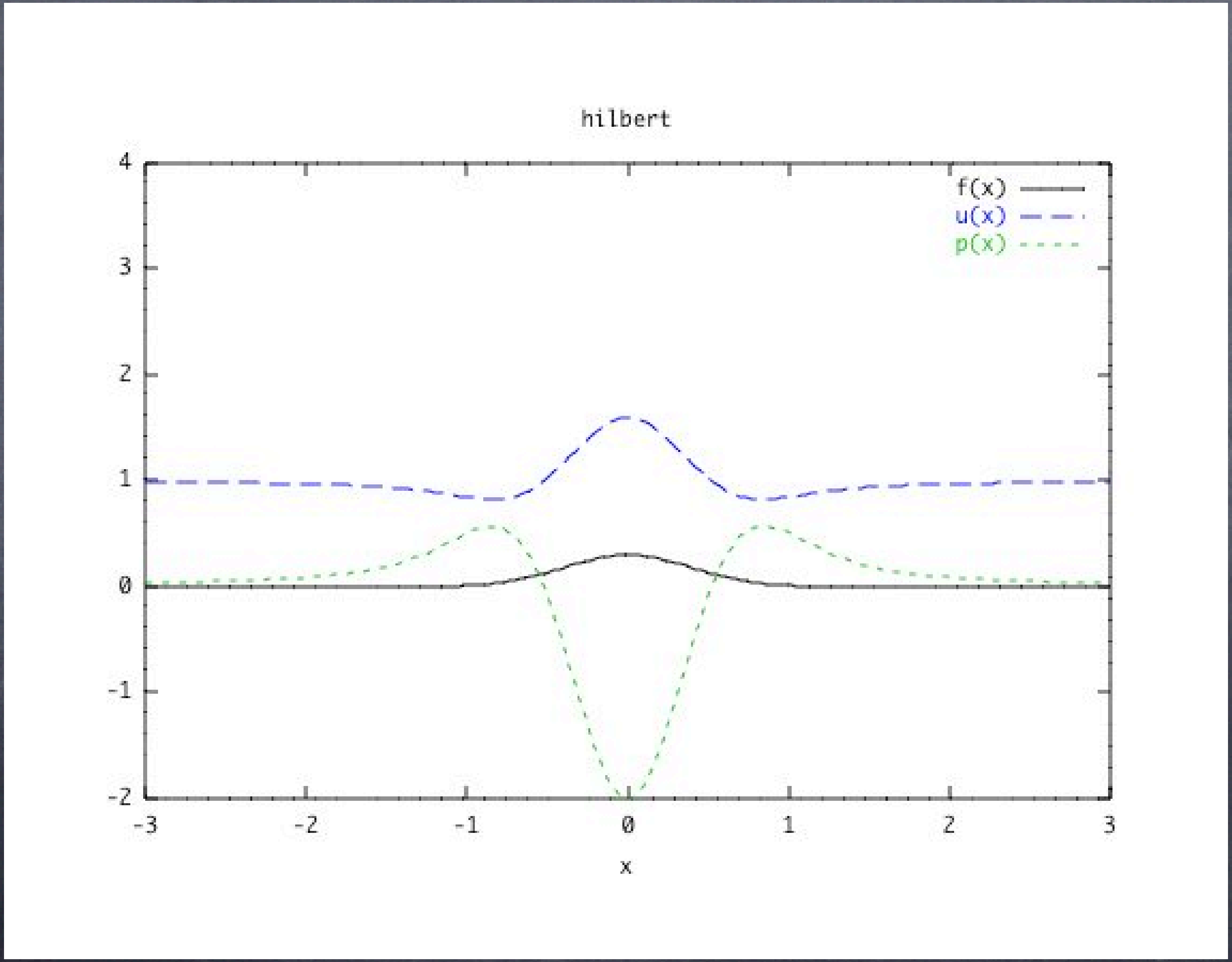
"Andreotti Bruno Functions"

$$\beta^{*}=(3Ai'(0))^{-1}(-ik)^{1/3}$$

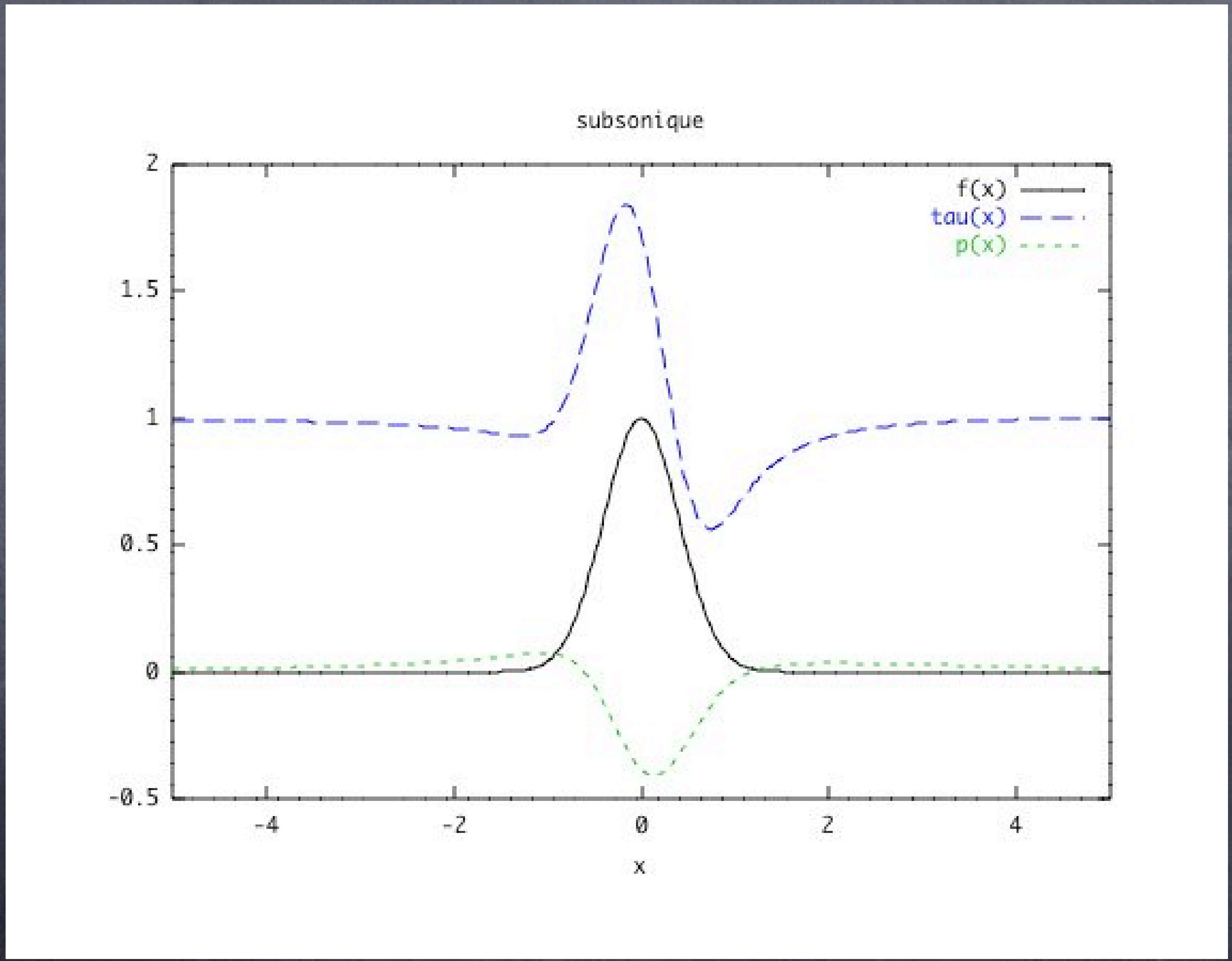
$$\beta_{pf}=1/|k|,0,1,-1,1/(ik)$$

$$FT[\tau]=\frac{(-ik)^{2/3}}{Ai'(0)}Ai(0)\frac{FT[f]}{\beta^{*}-\beta_{pf}}$$

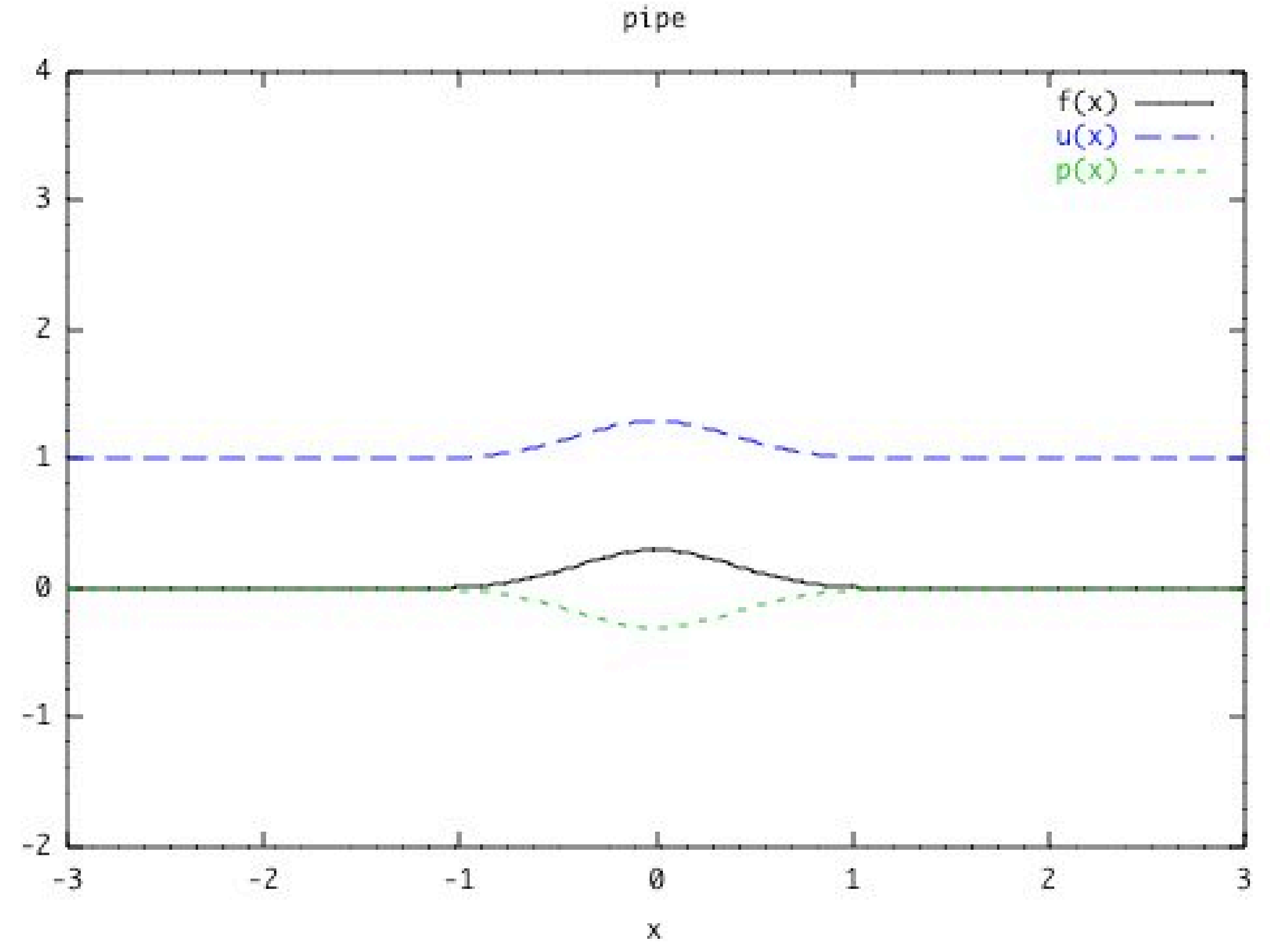
incompressible



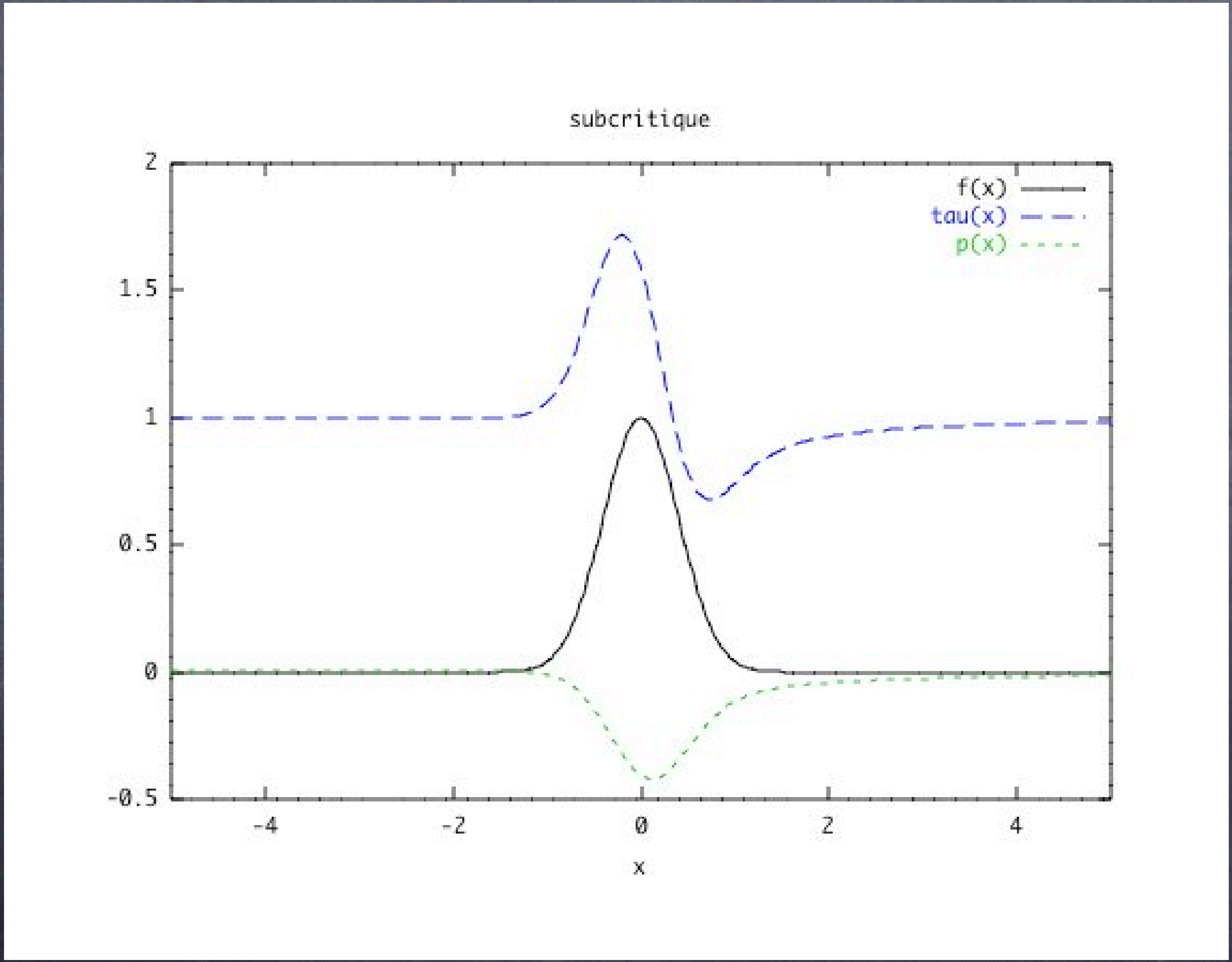
incompressible $p = \frac{1}{\pi}\int \frac{\frac{dA(\xi)}{dx}}{x-\xi}d\xi$



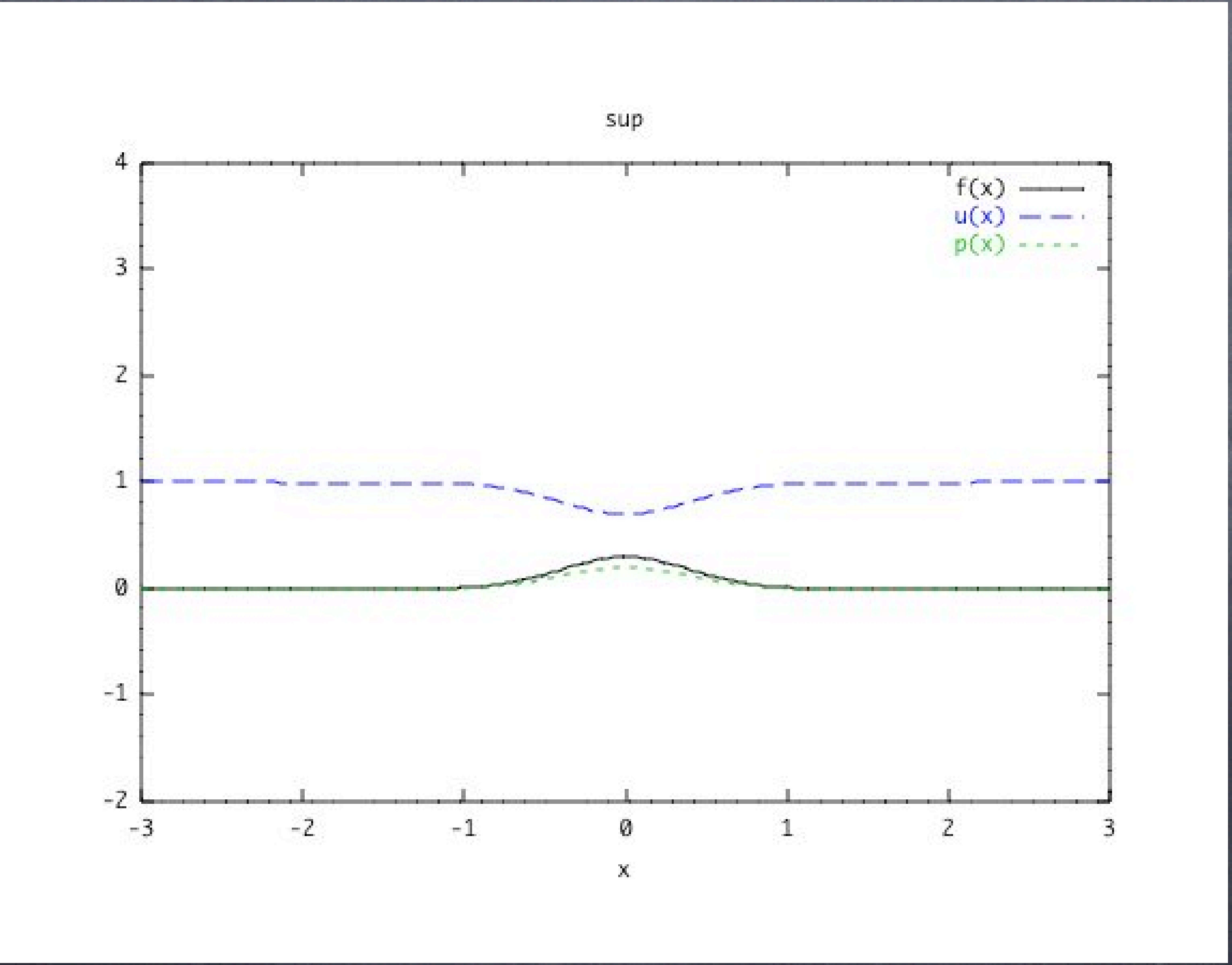
pipe/ subcritical



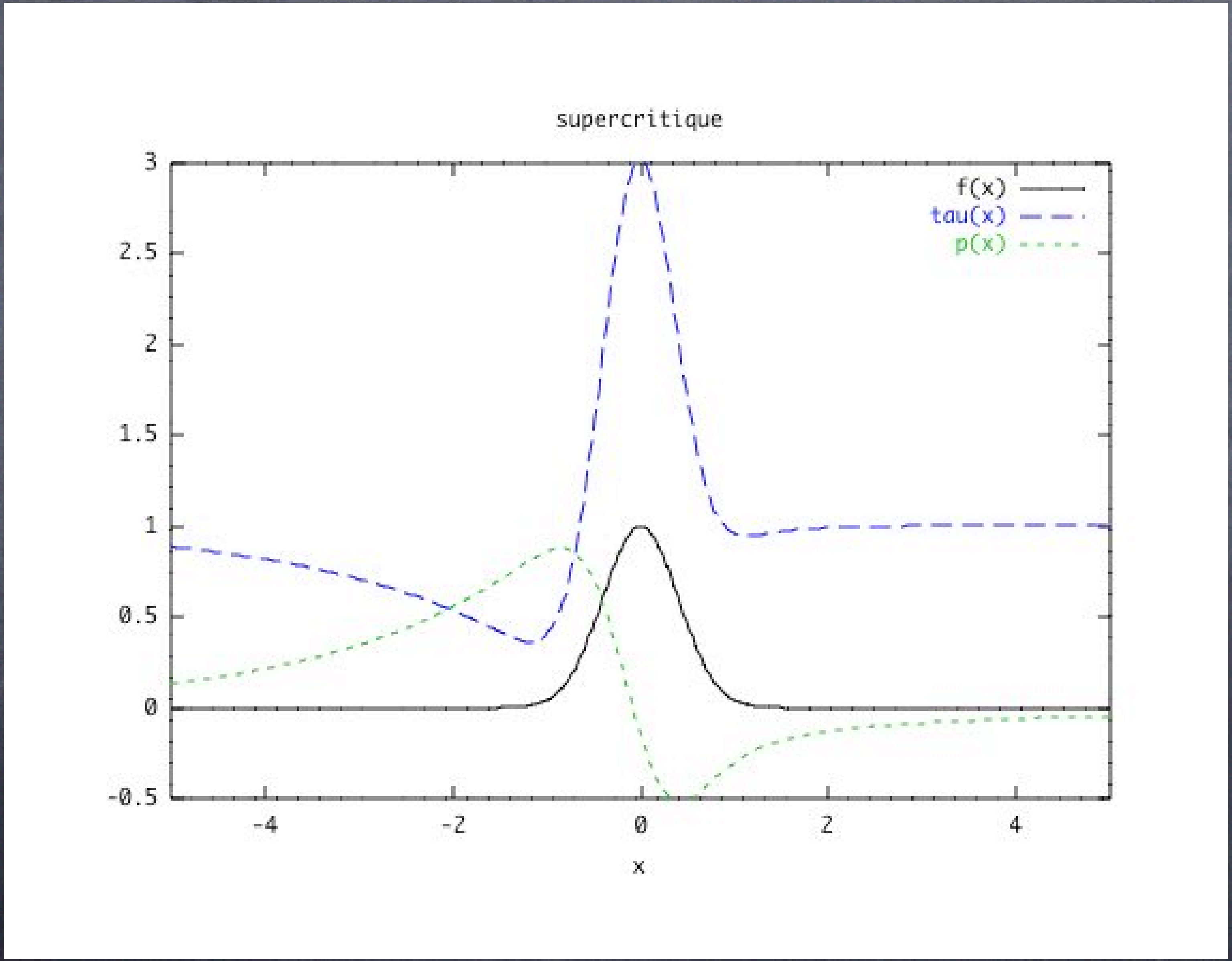
pipe/ subcritical $p = A$



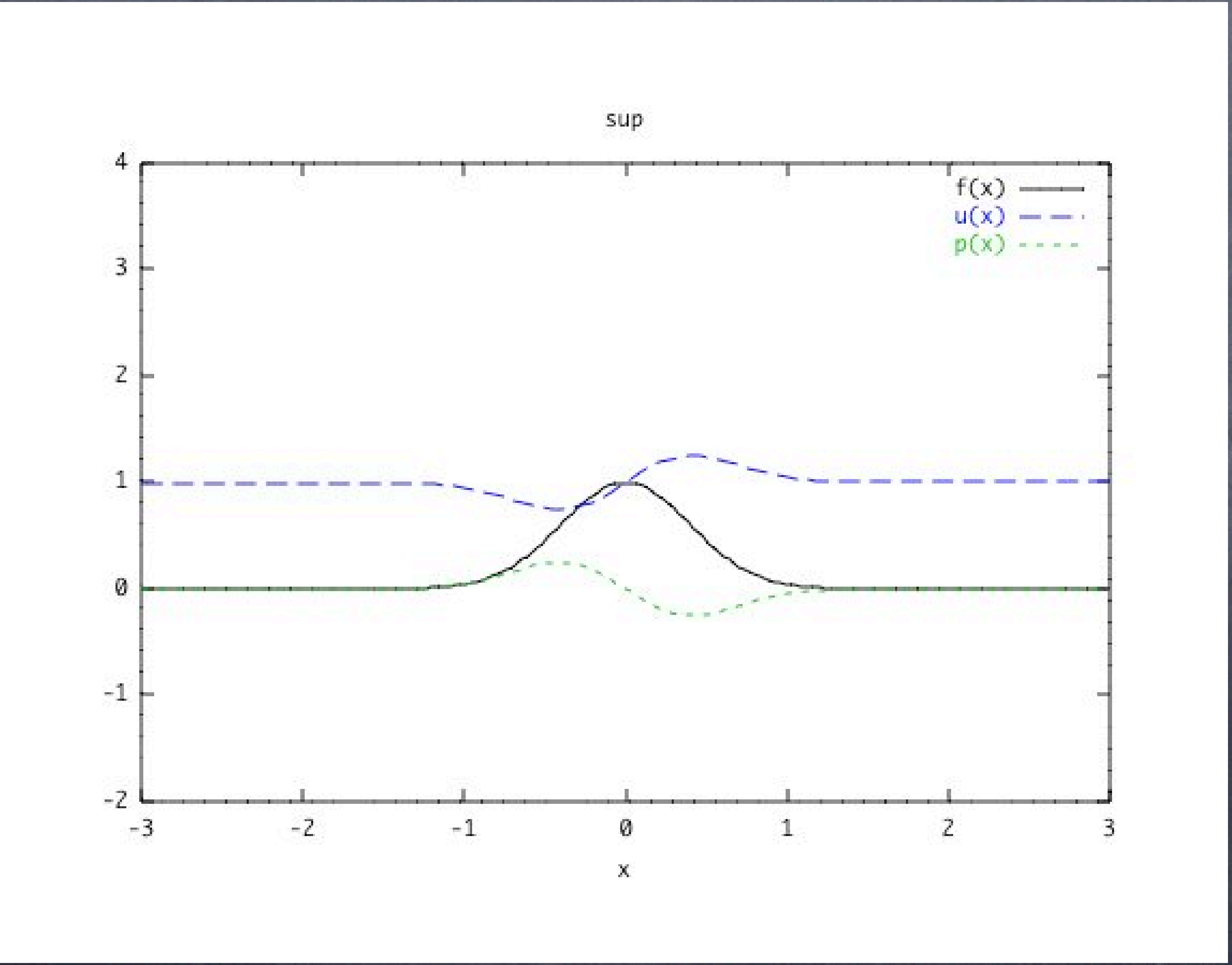
supercritical



supercritical $p = -A$

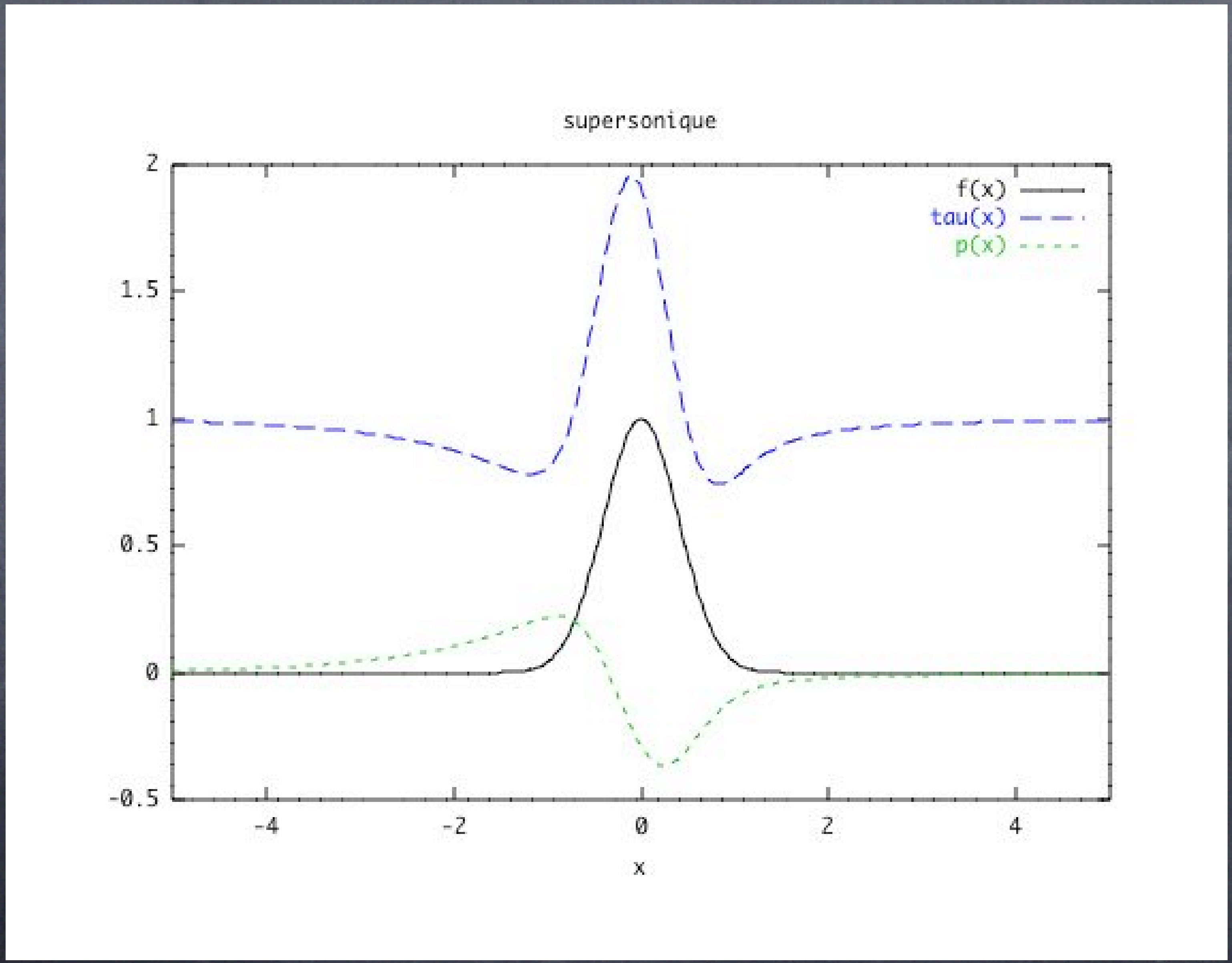


supersonic



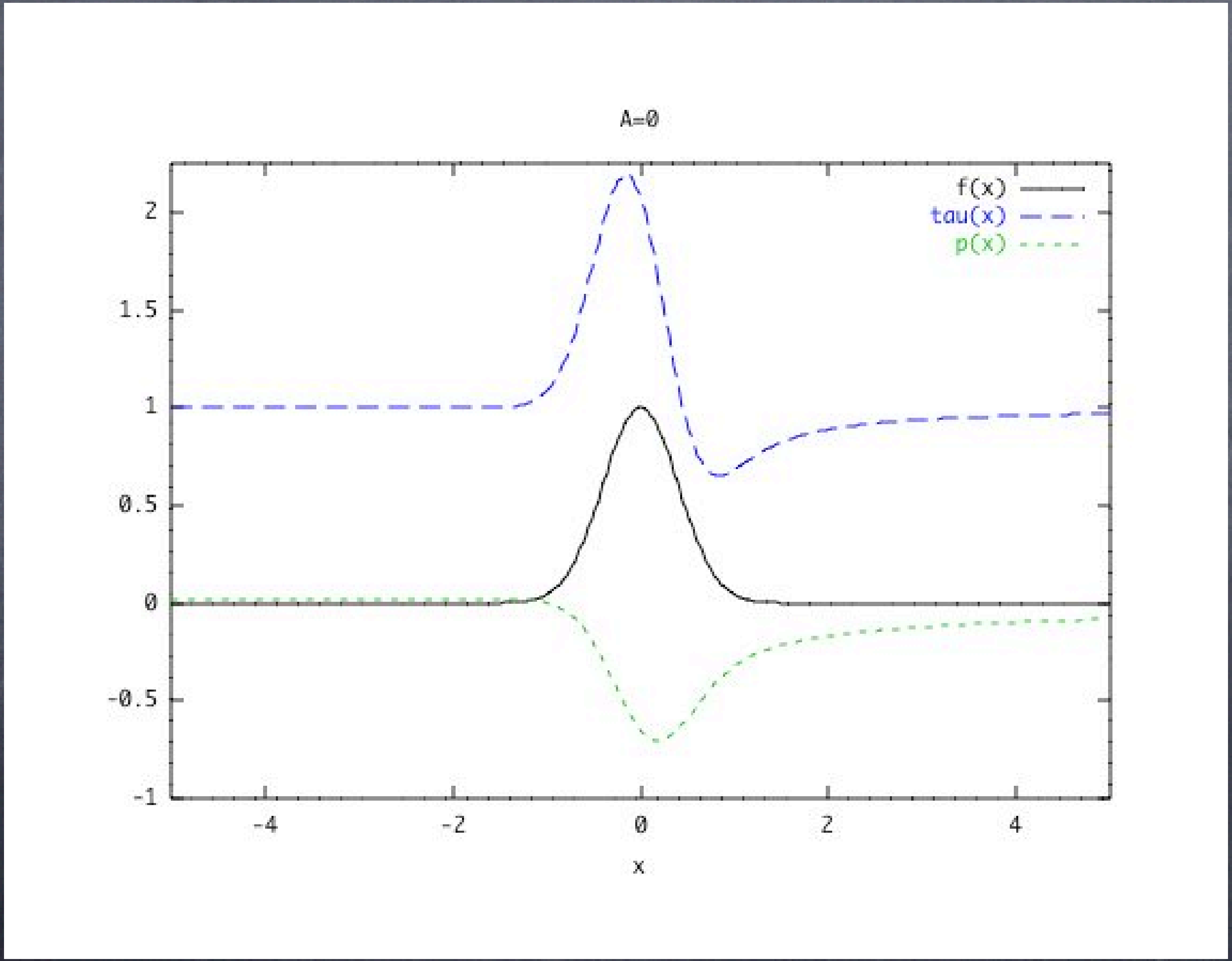
supersonic

$$p = \frac{-dA}{dx}$$



shear flow

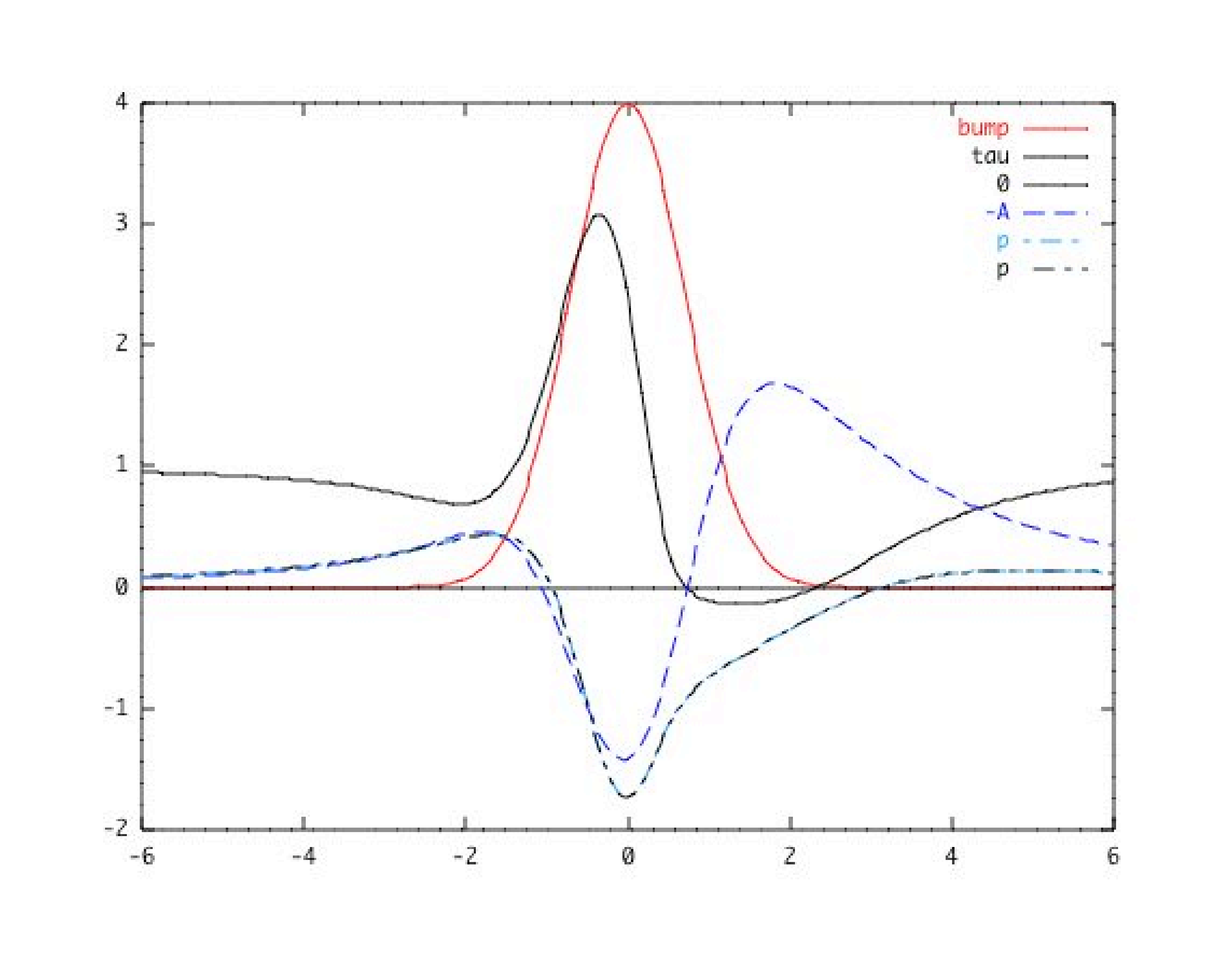
$A = 0$



Exemples with Boundary layer separation

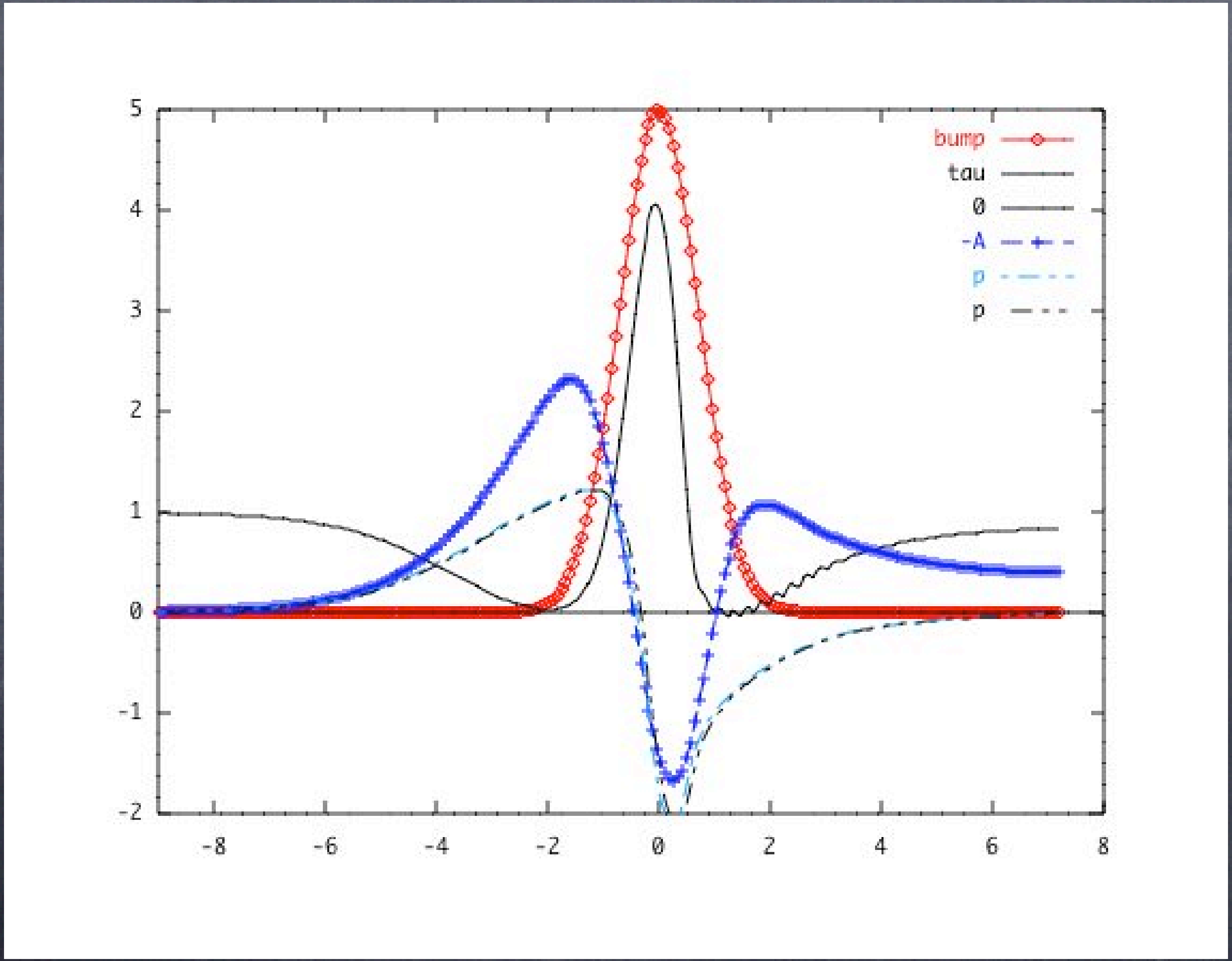
small separation bubble

incompressible $p = \frac{1}{\pi}\int \frac{\frac{dA(\xi)}{dx}}{x-\xi}d\xi$



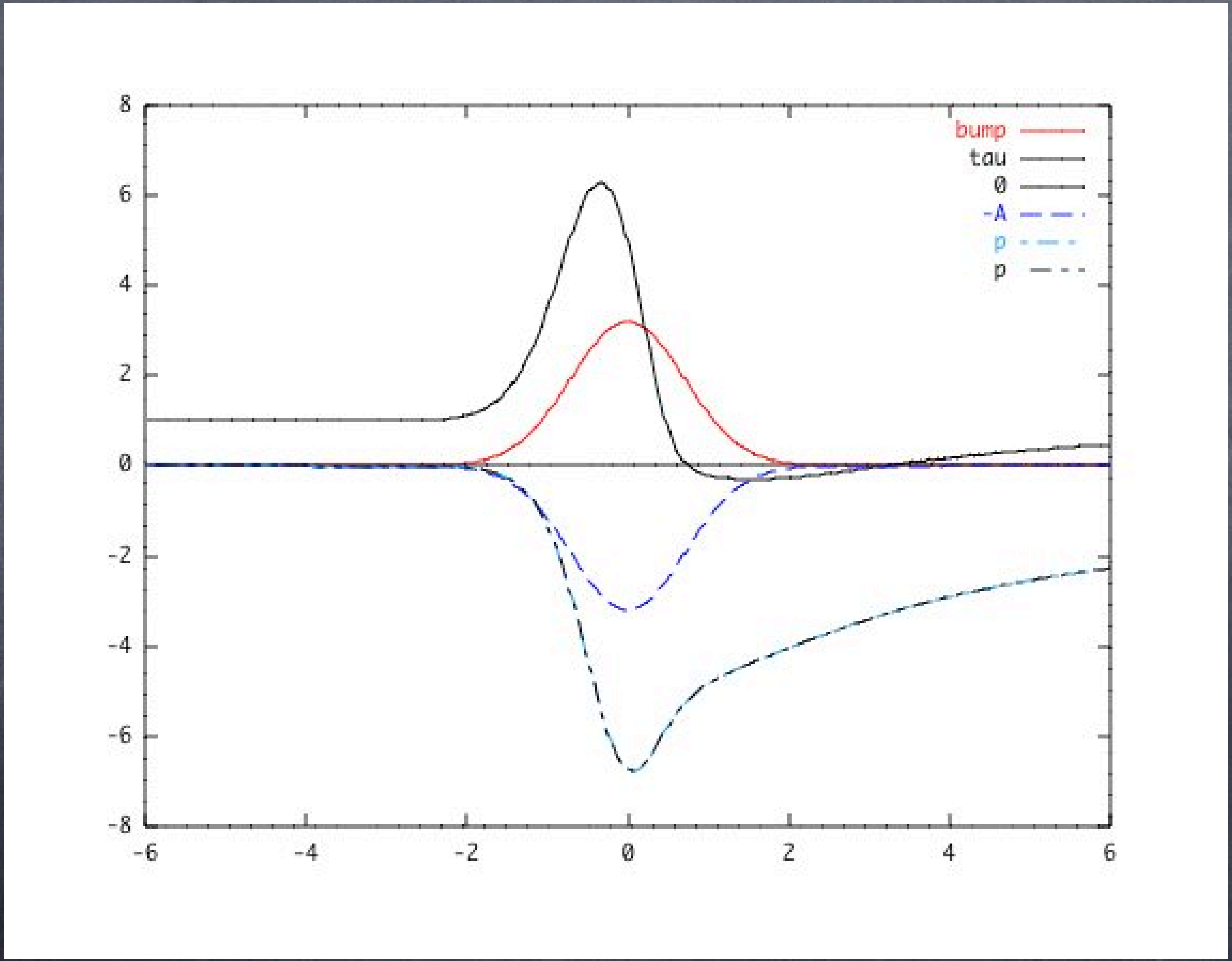
supersonic

$$p = \frac{-dA}{dx}$$



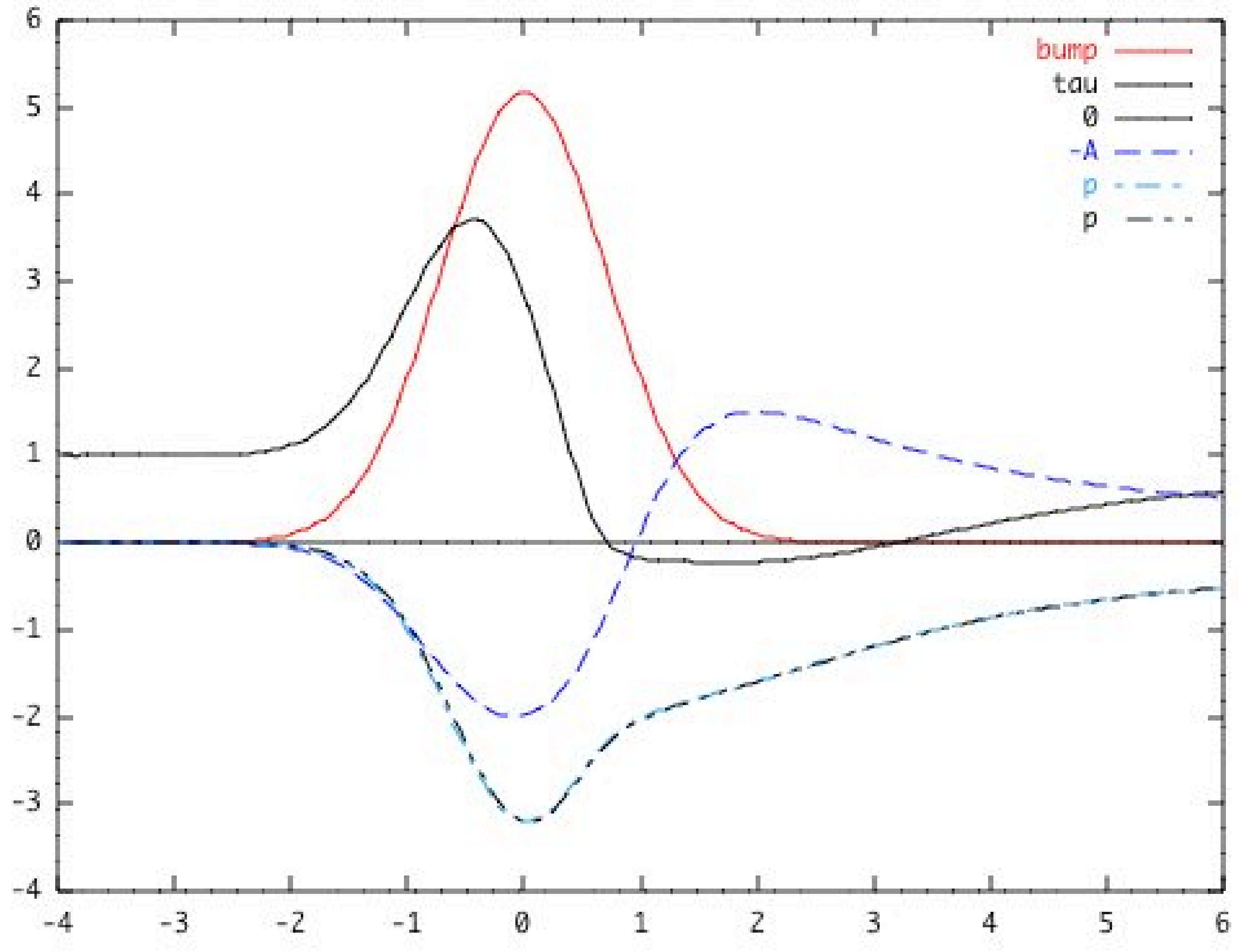
shear flow

$A = 0$



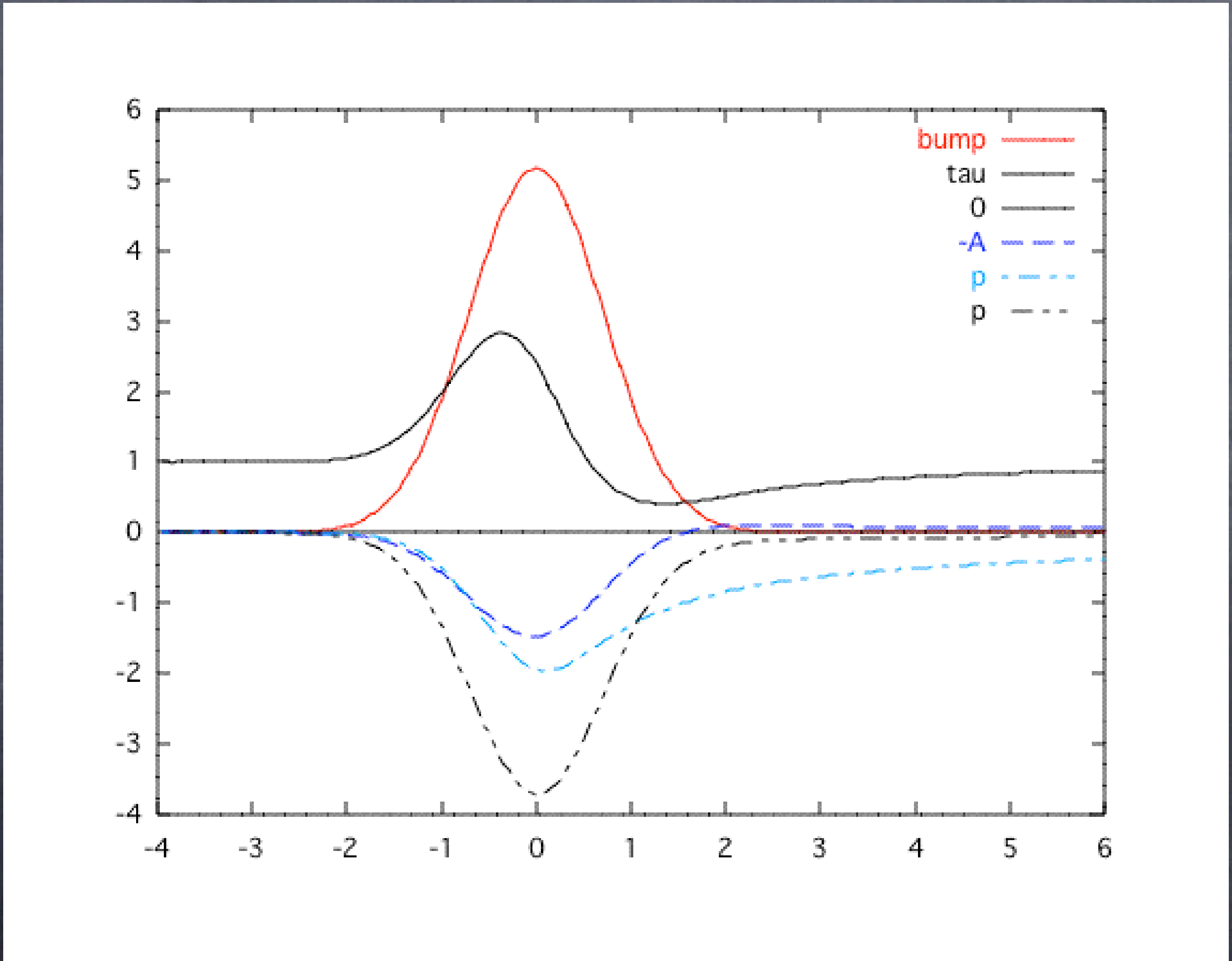
subcritical

$p = A$



subcritical

$p = A$



conclusion of this hydrodynamic part

IBL : strong interaction between the boundary layer and the ideal fluid thanks to the displacement thickness

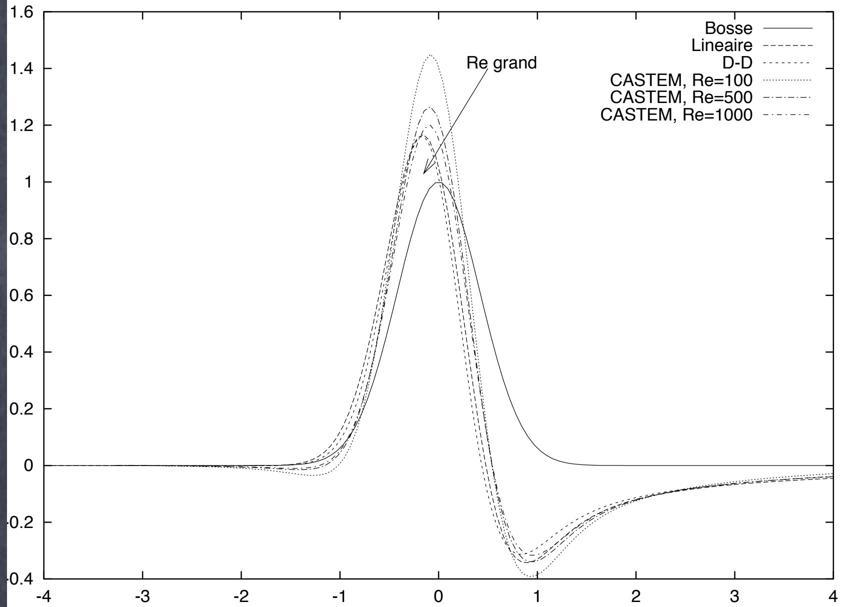
Triple Deck : rigorous asymptotical justification

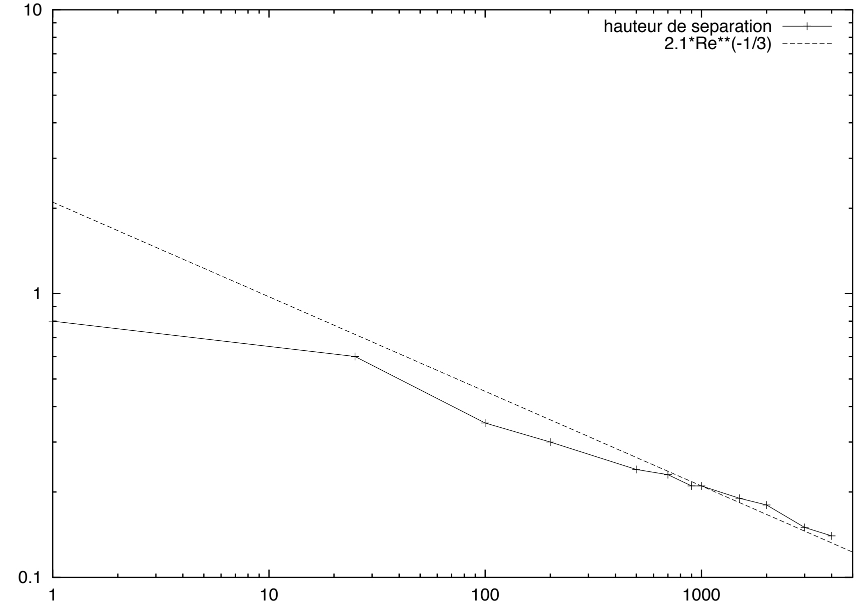
IBL: laminar or turbulent

Application

we use the proposed values of functions "A(k) and B(k)" to solve the case of the shear flow (i.e. the triple deck case A=0)

Comparison with Navier Stokes





good!
Re increasing
α fixed.

conclusion: Perturbation of shear flow is in advance compared to the bump crest.

The erodable bed: relations between q and u

$$\frac{\partial f}{\partial t} + \frac{\partial q}{\partial x} = 0$$

In the literature one founds Charru /Izumi & Parker / Yang / Blondeau

$$q_s = E\varpi(\tau - \tau_s)^a$$

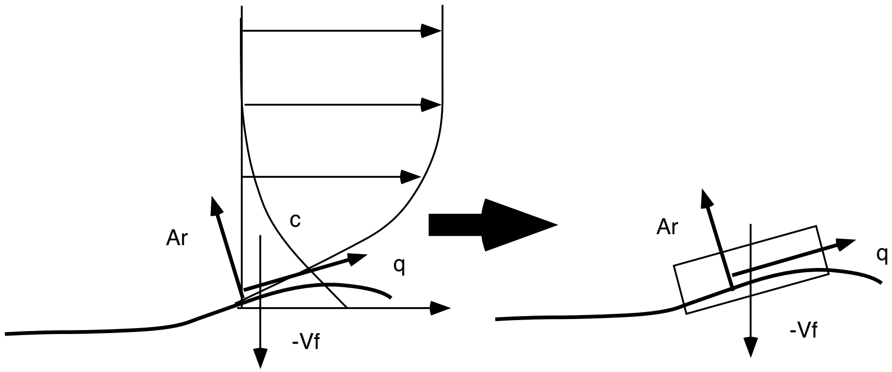
if $(\tau - \tau_s) > 0$ then $\varpi(\tau - \tau_s) = (\tau - \tau_s)$ else $\varpi((\tau - \tau_s)) = 0$.

or with a slope correction for the threshold value:

$$\tau_s + \Lambda \frac{\partial f}{\partial x},$$

a, E coefficients, $a = 3/2$

Other simplification of mass transport



Sauerman, Kroy, Hermann 01/ Andreotti Claudin Douady 02/ Lagrée 00/03

$$l_s \frac{\partial}{\partial x} q + q = (\varpi(\tau - \tau_s - \Lambda \frac{\partial f}{\partial x})^\gamma).$$

- total flux of convected sediments q (left figure).
- threshold effect τ_s
- slope effect $\Lambda \frac{\partial f}{\partial x}$
- $\varpi(x) = x$ if $x > 0$ (else 0), γ, l_s ...

first case

unstability of a bed in a steady shear flow

Interpretation AB effect

up to now $U'_0 = 1$

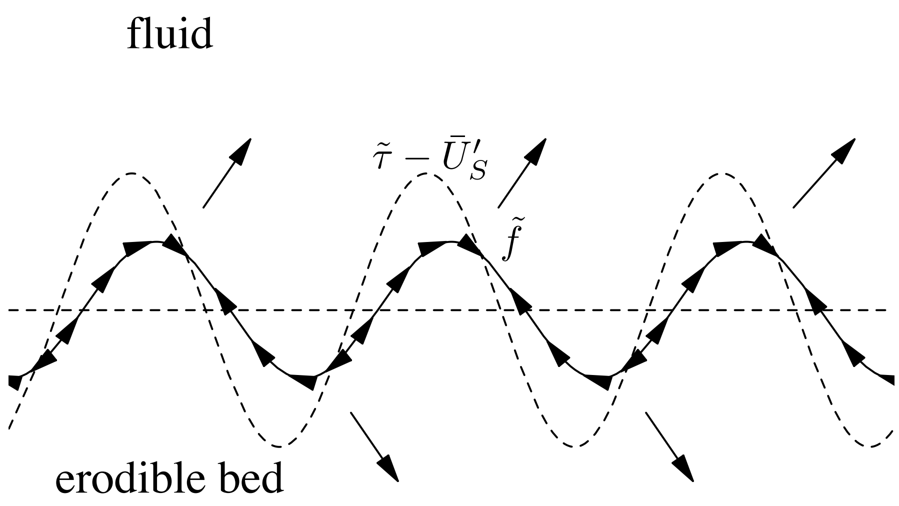
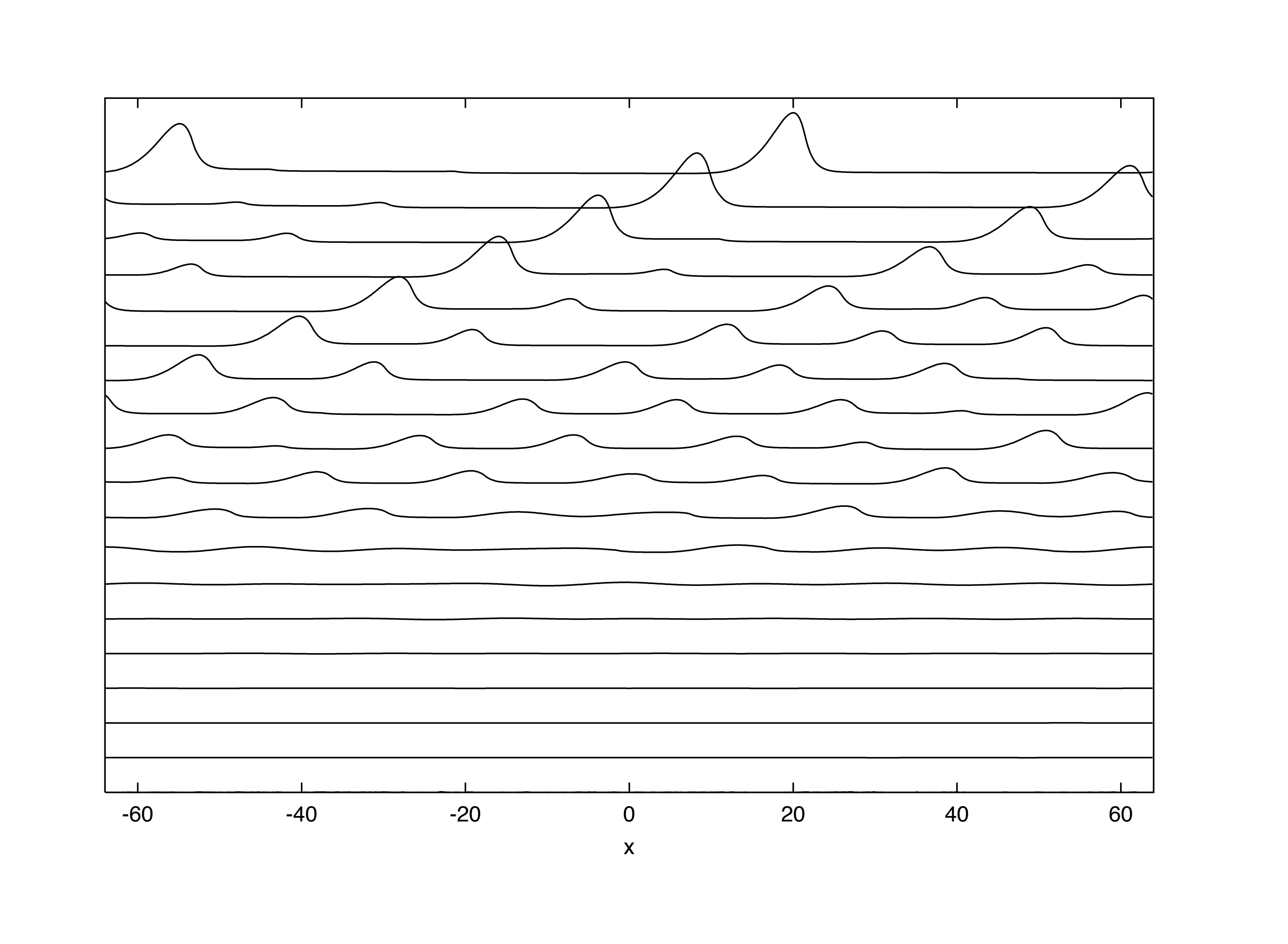


Figure 4: A wavy profile (bold line, $\tilde{f}$) has a perturbation of skin friction (dashed line, $\tilde{\tau} - \bar{U}'_S$) in advance of phase. When it is positive, the matter is moved down stream (small arrows on the profile), when is is negative, it is in opposite direction. The result is an increase of the wave and a displacement in the stream direction (large inclined arrows).

Linear instability of a bed in a shear flow



numerical resolution of the long time evolution

there is coarsening

second case

unstability of a bed in an oscillating shear flow

oscillating case

Interpretation AB effect

here $U_0' = cos(\bar{t})$

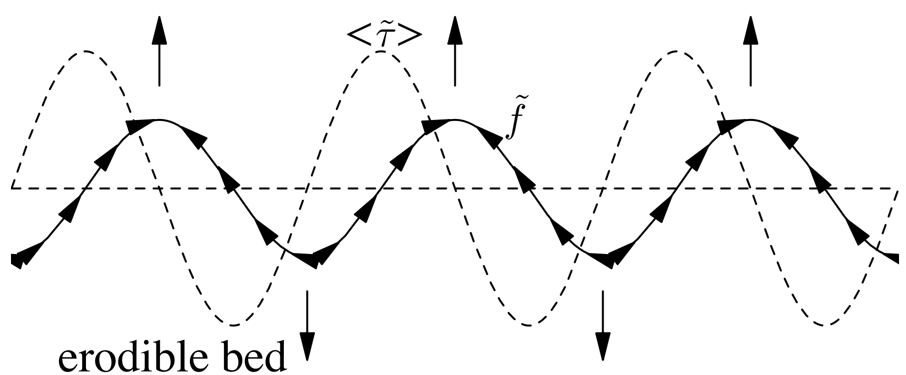


Figure 7: A wavy profile (bold line, $\tilde{f}$) has a mean perturbation of skin friction (dashed line, $< \tilde{\tau} >$) out of phase. When $< \tilde{\tau} >$ is positive, the matter is moved from left to right (small arrows on the profile), when it is negative, it is in opposite direction. The result is an increase of the wave without displacement (large vertical arrows).

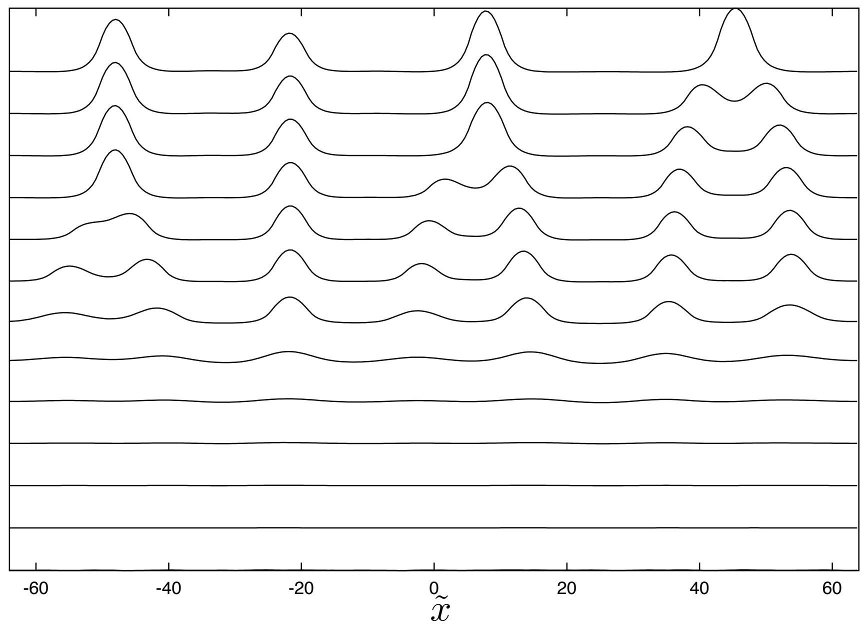


Figure 13: Oscillating régime with (22), spatio temporal diagram, time increases from bottom to top. Ripples growth from a random noise and merge two by two.

numerical resolution of the long time evolution

there is coarsening

third case

movement of a "dune" in a steady shear flow

final shapes lin/ non lin

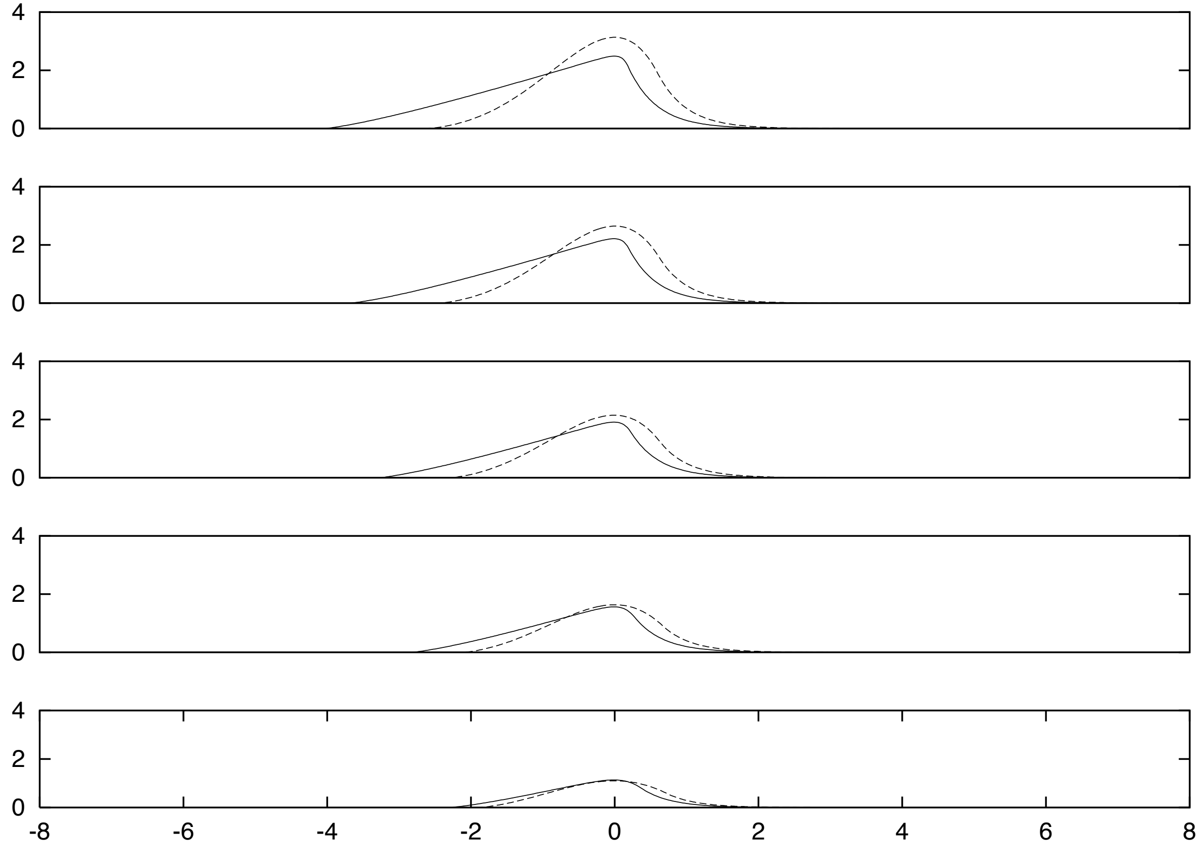


Fig. 5. The non-linear final moving "dune" solution $f_{fin}(x - ct)$ is represented with solid lines, the linear solution is represented with dashed lines, and $\tau_s = 0.9$, $1/l_s = 2.5$, $m = 2, 3, 4, 5$ (bottom curve to top curve).

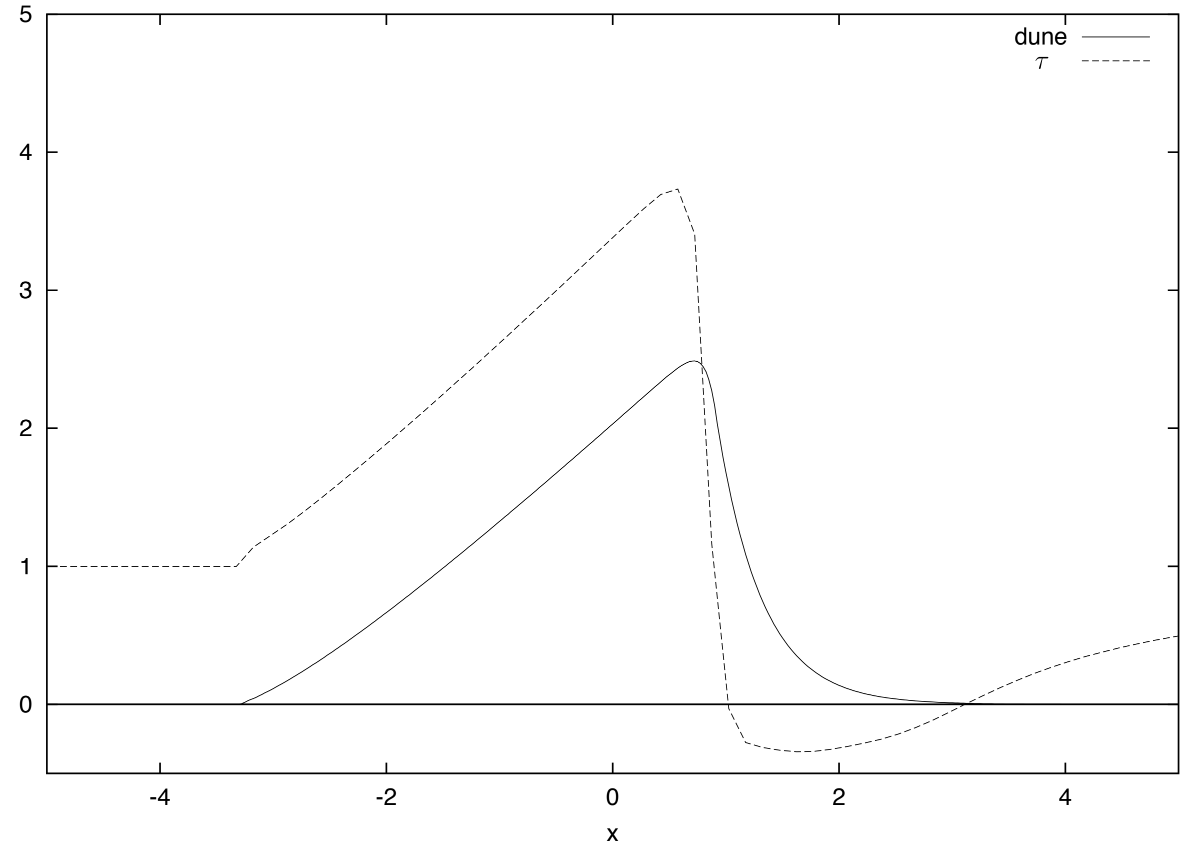


Fig. 6. An example of a non-linear final moving "dune" solution ($\tau_s = 0.9$, $1/l_s = 2.5$, $m = 6$). The weather side is nearly flat. The skin friction is represented; it is negative in the lee side: there is boundary layer separation.

conclusion

- a method to otain A(k)B(k) functions in laminar flows
- long time evolution shows coarsening
- ...