Stability of erodible beds and Displacement of a 2D/ 3D dune in a shear flow

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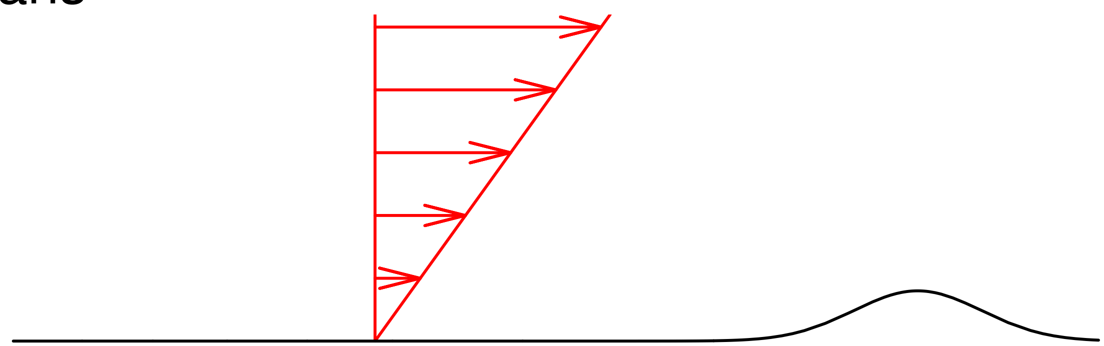
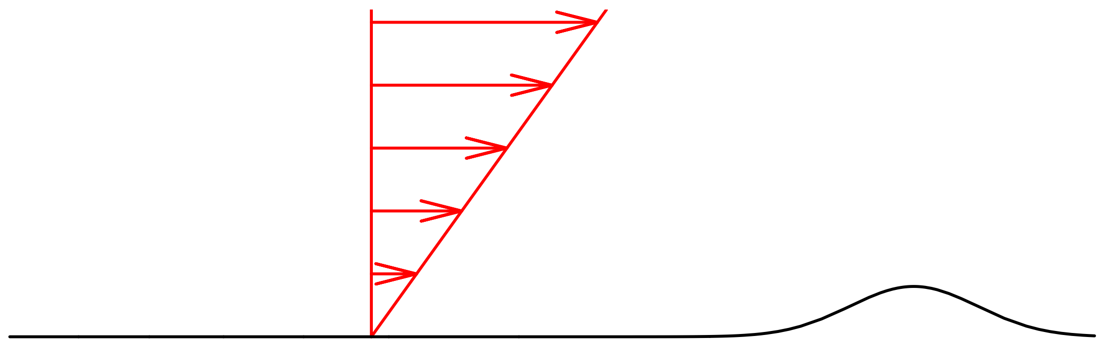
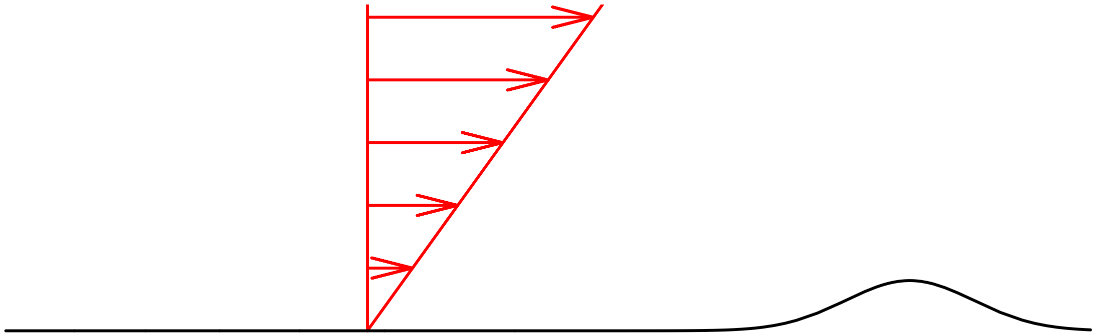




- fluid / soil interaction



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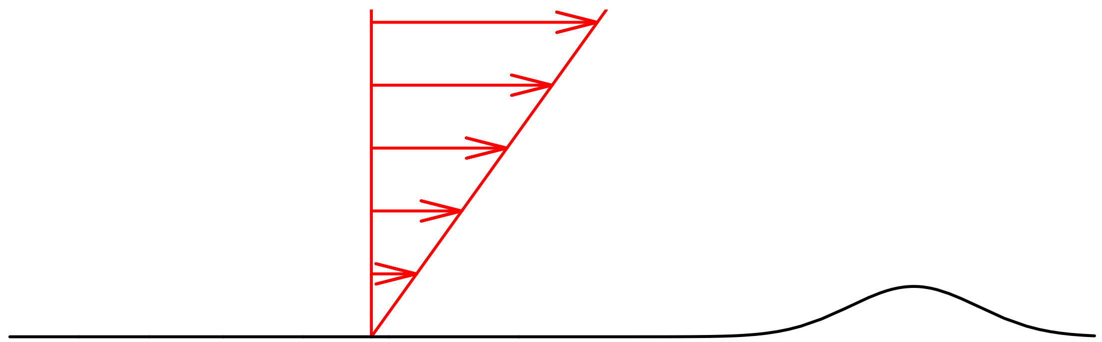
- complex problem



- fluid / soil interaction

- complex problem

- very strong simplifications:
  - basic shear flow
  - steady laminar 2D flow
  - simple linear flux/ shear stress relations

But comparison between linear/ non linear computations in 2D
3D linear

Contents

- Flux/ Shear stress relations
- Double Deck equations: pure shear flow, (erodible / solid bed)
- 3D Double deck, (erodible bed)
- Special details when $Re = \infty$ and $Re = 0$
- Conclusion,

The coupled problem

- for a given soil $f(x,t)$
- ...

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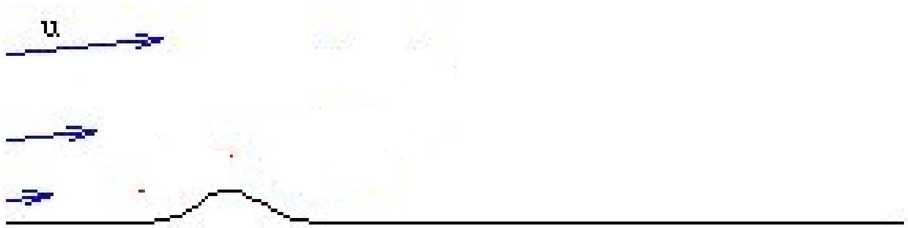
- for a given soil $f(x,t)$
- we have to compute the flow ($u(x,y,t)$).



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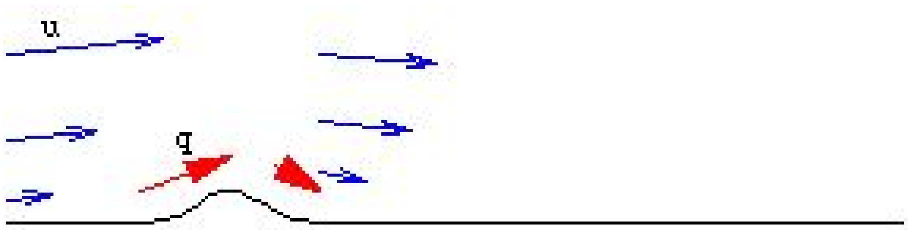

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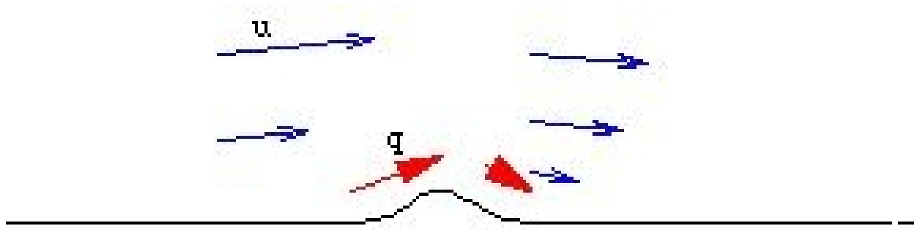

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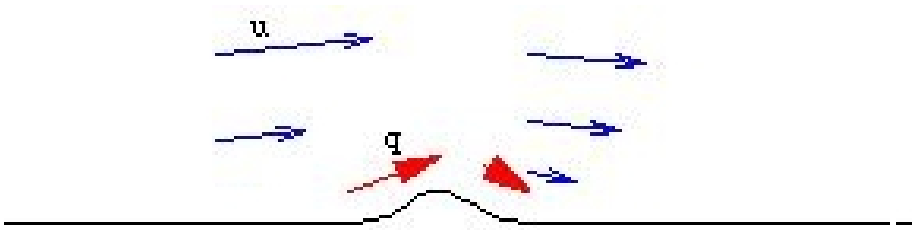

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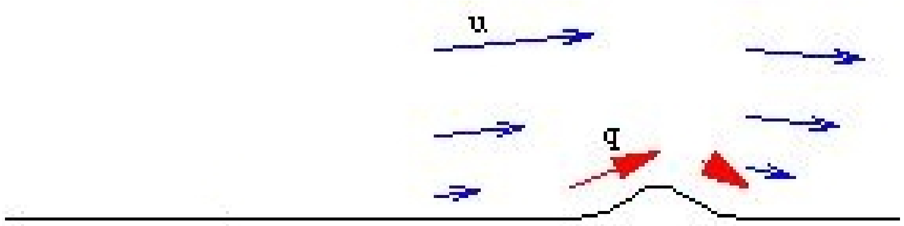

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The coupled problem

- for a given soil $f(x,t)$
- we have to compute the flow $(u(x,y,t))$.

- the flow erodes the soil.
- which changes the soil.
- *etc*

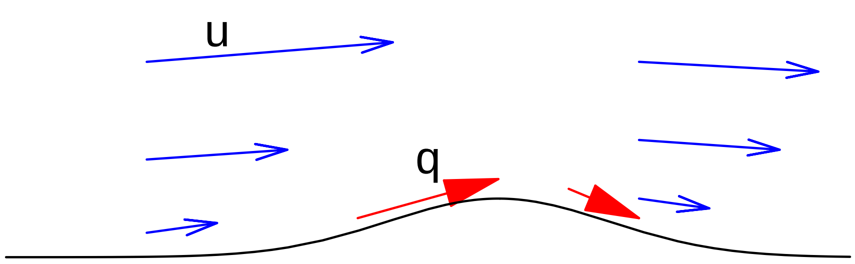
we aim to present a simple description for the flow and use simple model equations to describe the interaction.

The erodable bed

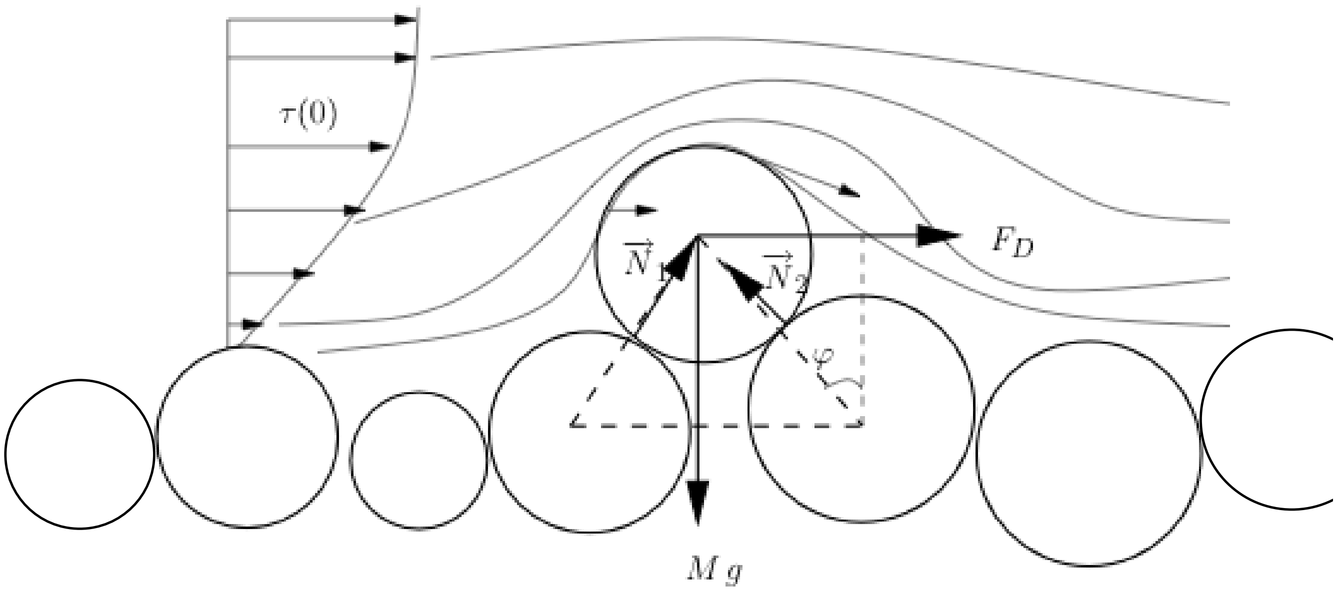
Mass conservation for the sediments:

$$\frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}.$$

Problem :
What is the relationship between q and the flow?
hint: the larger u the larger the erosion, the larger q
q seems to be proportional to the skin friction



Mass: Threshold, The Shield criteria



Les lois d'entraînement de M. Scipion Gras
sur les torrents des Alpes (Annales des ponts et Chaussées, 1857, 2^e semestre) résumées par du Boys 1879:

"*un caillou posé au fond d'un courant liquide, peut être déplacé par l'impulsion des filets qui le rencontrent : le mouvement aura lieu si la vitesse est supérieure à une certaine limite qu'il (S. Gras) nomme vitesse d'entraînement. Cette vitesse limite dépend de la densité, du volume et de la forme du caillou; elle dépend aussi de la densité du liquide et de la profondeur du courant.*"

Mass: Flux

In the literature one founds Charru /Izumi & Parker / Yang / Blondeau Du Boys

$$q_s = E\varpi(\tau - \tau_s)^a$$

if $(\tau - \tau_s) > 0$ then $\varpi(\tau - \tau_s) = (\tau - \tau_s)$ else $\varpi((\tau - \tau_s)) = 0$.

or with a slope correction for the threshold value:

$$\tau_s + \Lambda \frac{\partial f}{\partial x},$$

a, E coefficients, $a = 3/2$

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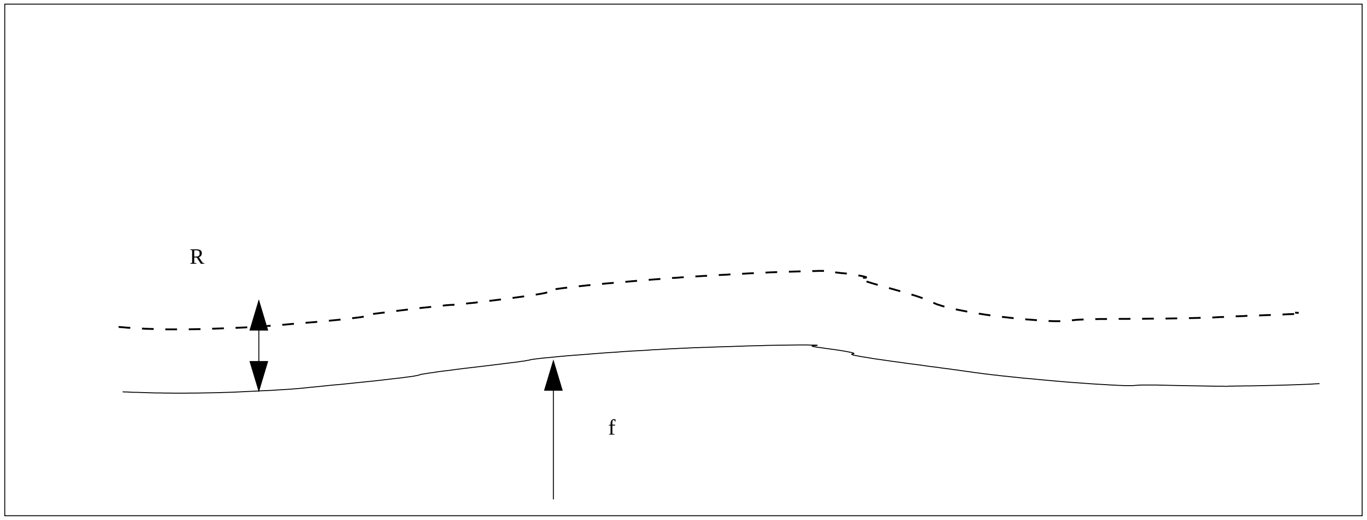
Mass: Law of conservation (Valance Langlois 2005)



$$\frac{\partial R}{\partial t} = \dots$$

$$\frac{\partial f}{\partial t} = \dots$$

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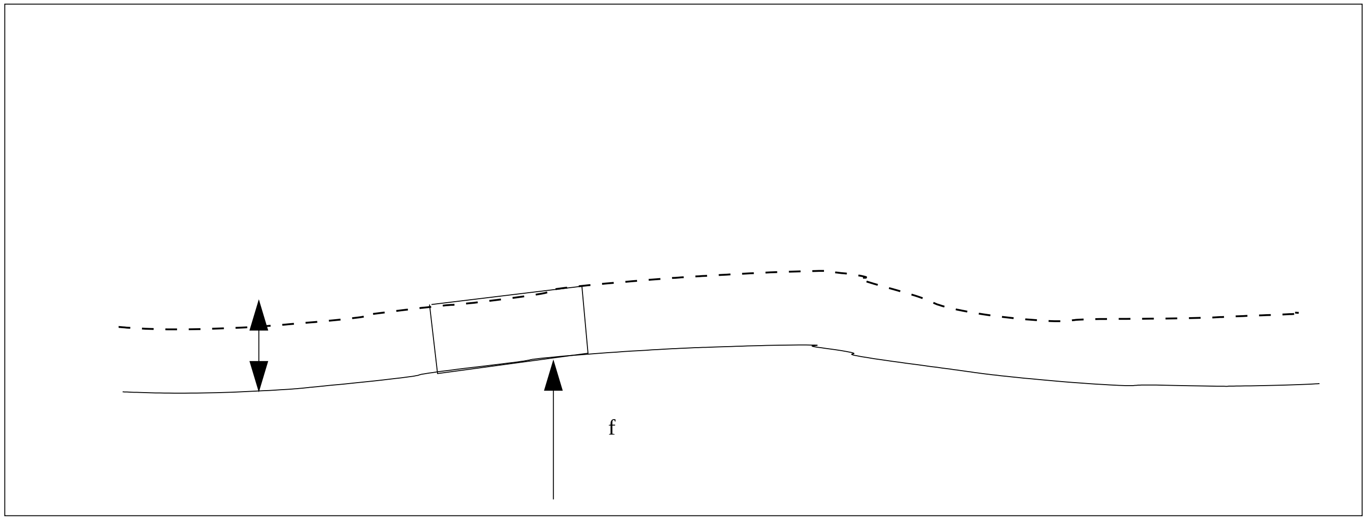

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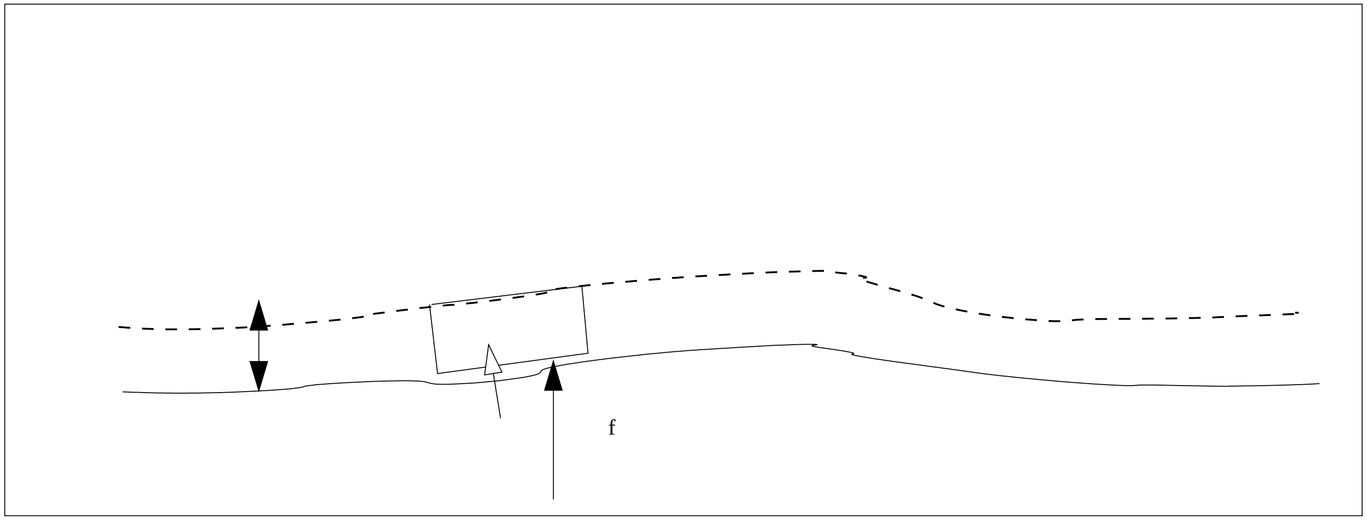

$$\frac{\partial R}{\partial t} = ... + \Gamma$$

$$\frac{\partial f}{\partial t} = +\Gamma$$

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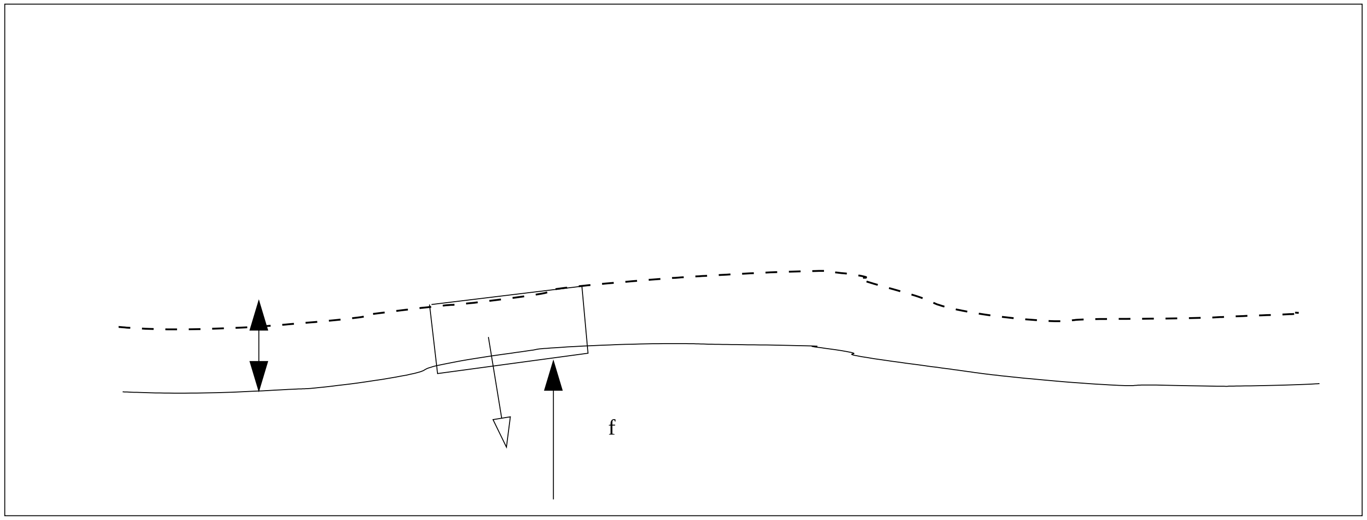

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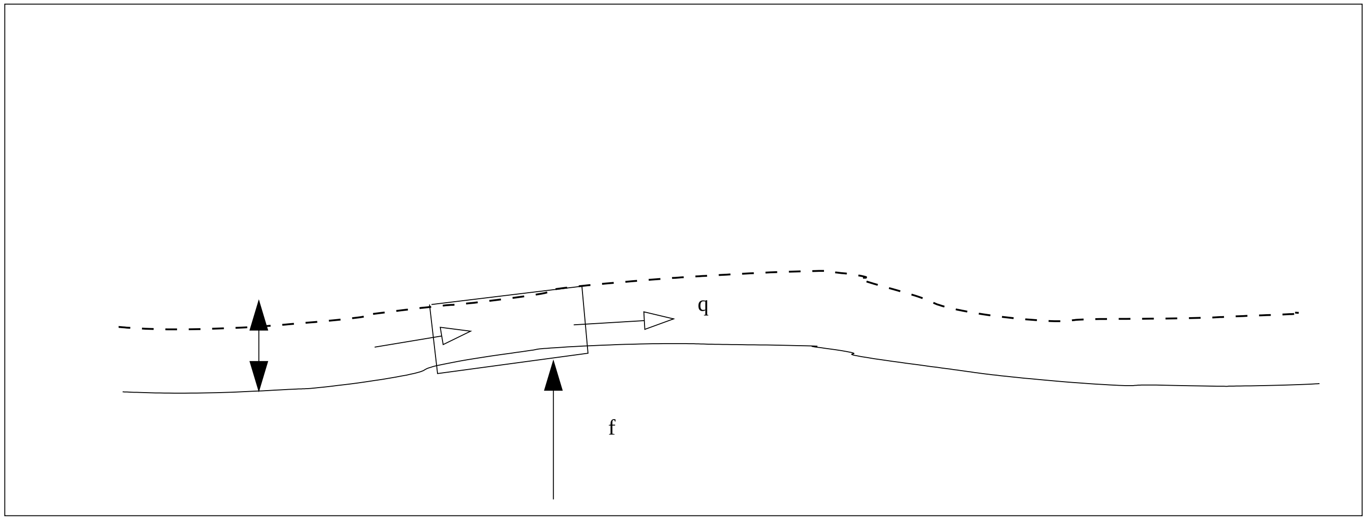
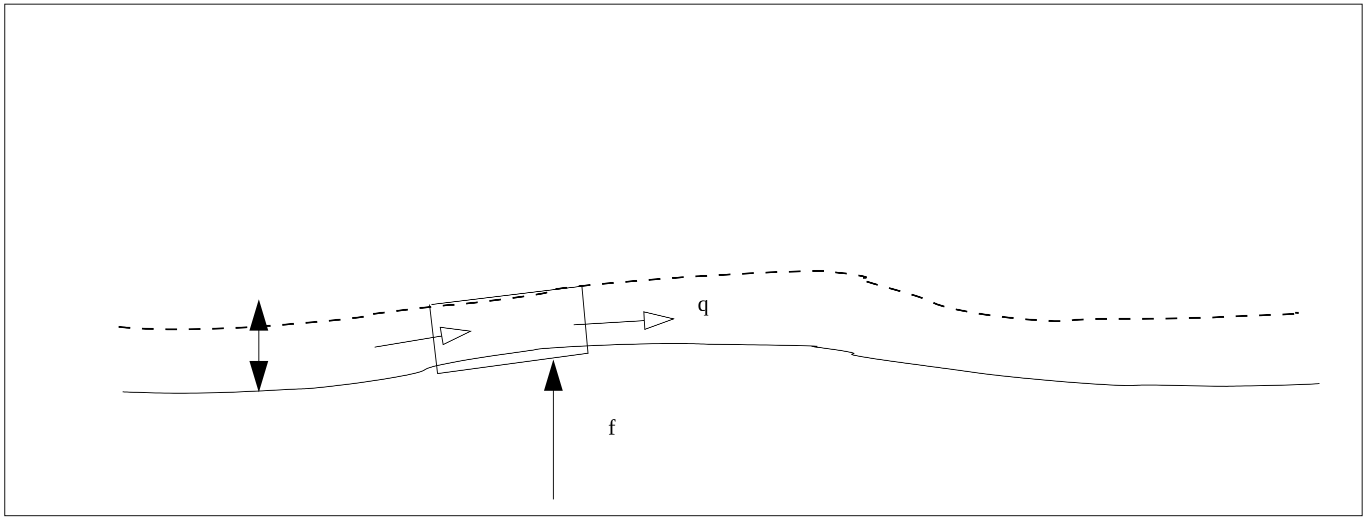


$$\frac{\partial R}{\partial t} = -\frac{\partial q}{\partial x} + \Gamma$$

$$\frac{\partial f}{\partial t} = -\Gamma$$

Mass: Law of conservation (Valance Langlois 2005)



$$\frac{\partial R}{\partial t} = -\frac{\partial q}{\partial x} + \Gamma \qquad \frac{\partial f}{\partial t} = -\Gamma$$

$$\Gamma \simeq (R_{sat} - R) \qquad R_{sat} \simeq (\tau - \tau_s)$$

Mass

$$q \simeq vR$$

Mass

$$q \simeq vR$$

Law of conservation (Charru Hinch)
nearly the same mechanism:

$$q = \tau R$$

$$R_{sat} \simeq (\tau - \tau_s) \qquad \text{so} \qquad q_{sat} \simeq (\tau - \tau_s)$$

and $\frac{\partial R}{\partial t} \simeq 0$

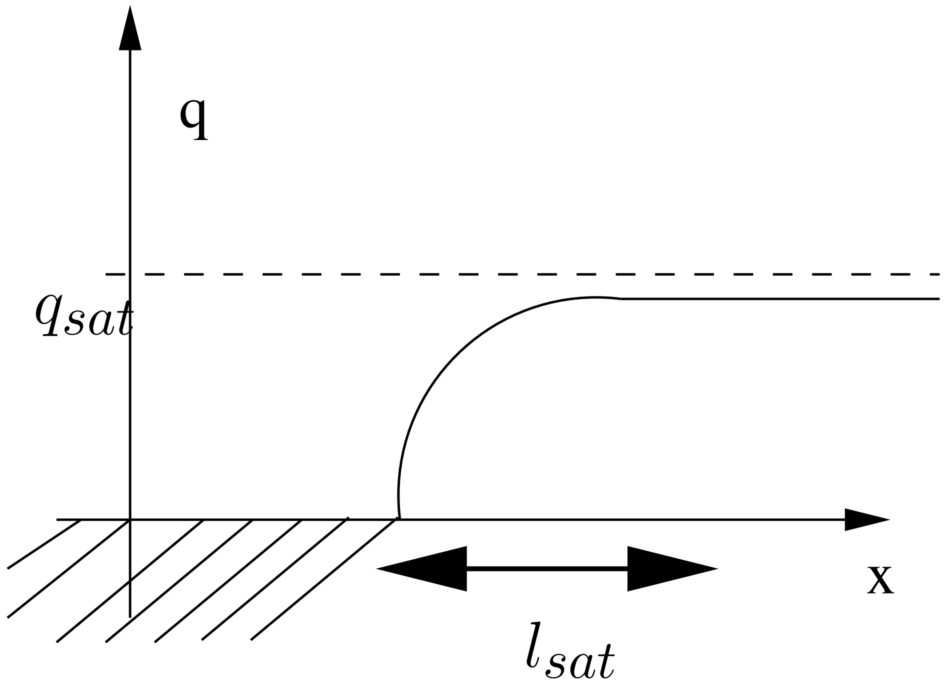
Mass

$$l_{sat}\frac{\partial q}{\partial x} + q = q_{sat}$$

$$\frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}$$

$$q_{sat} \simeq \varpi((\tau - \tau_s))$$

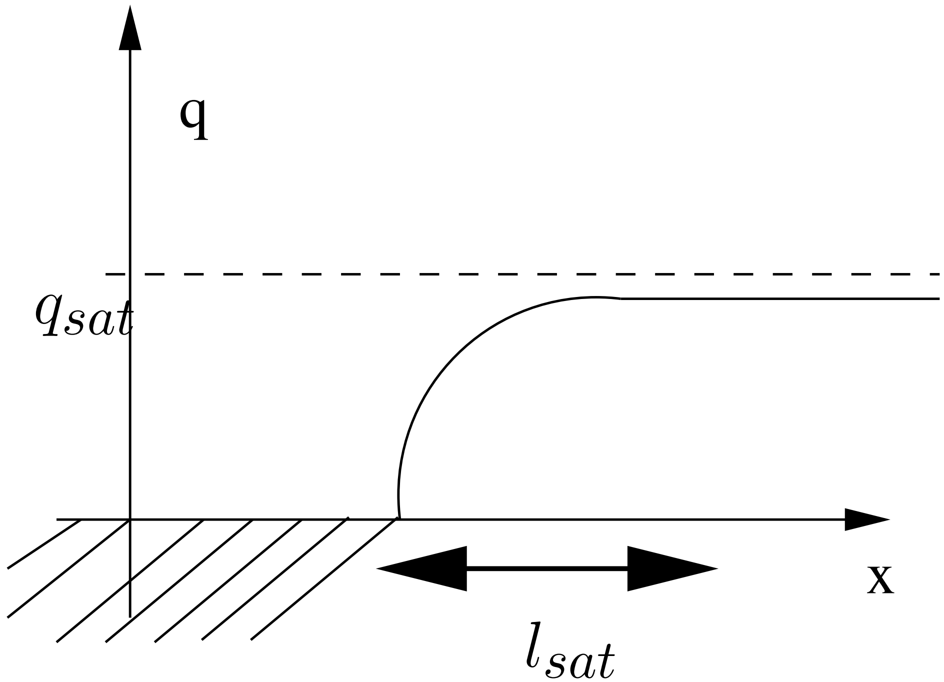
q_{sat}



Andreotti Claudin Douady (2002)

$$l_{sat}\frac{\partial q}{\partial x} + q = q_{sat}$$

q_{sat}



Du Boy (1879):

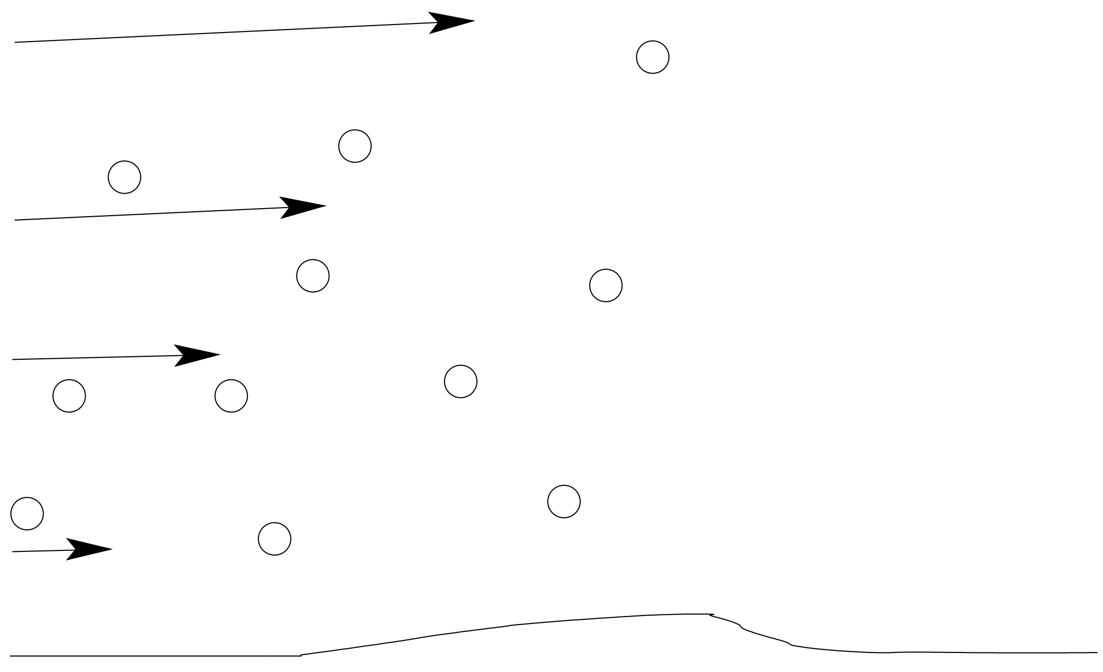
"*une fois une certaine quantité de matières en mouvement sur le fond du lit, la vitesse des filets liquides devient trop faible pour entraîner davantage : le cours d'eau est alors saturé. Un cours d'eau non saturé tend à le devenir en entraînant une partie des matériaux qui composent son lit, et en choisissant de préférence les plus petits.*"

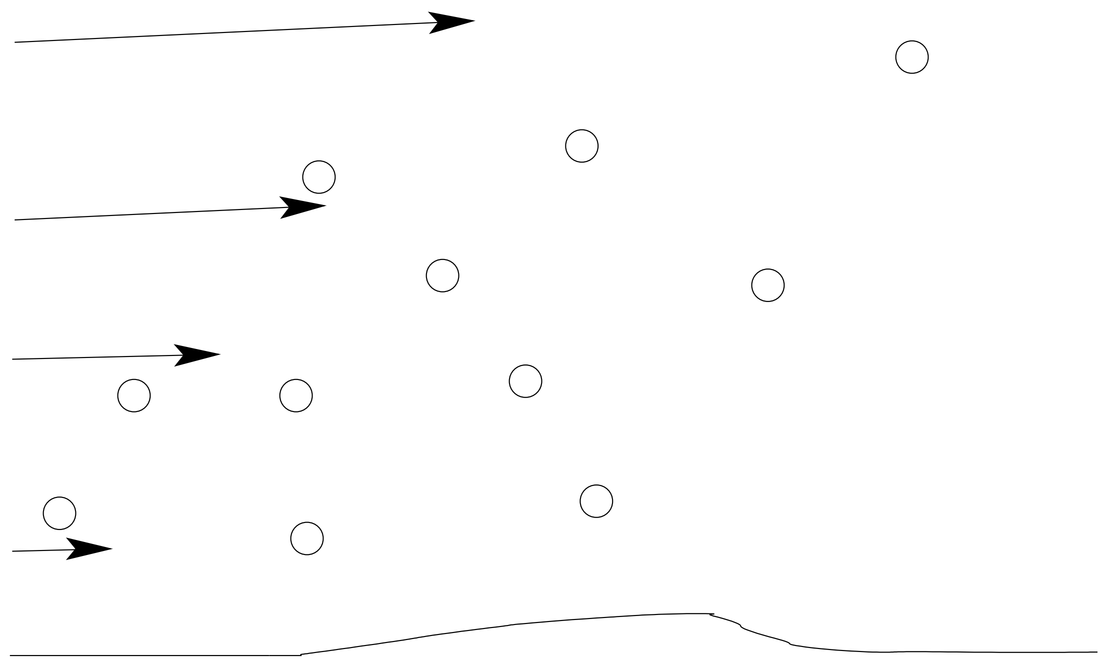
Mass transport

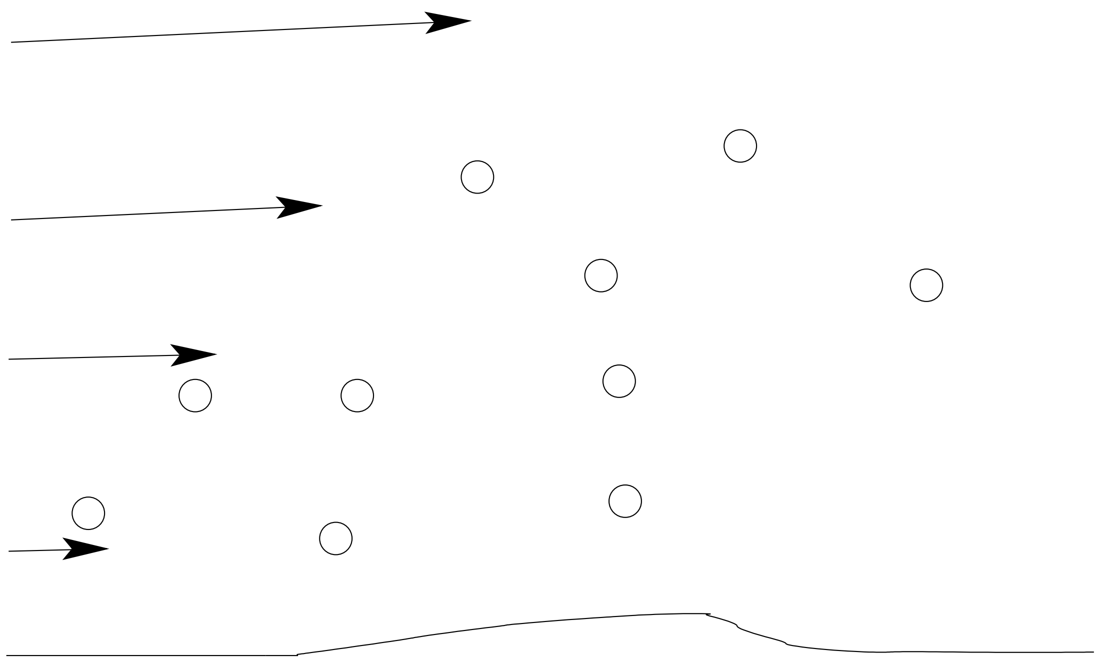
An other point of view:

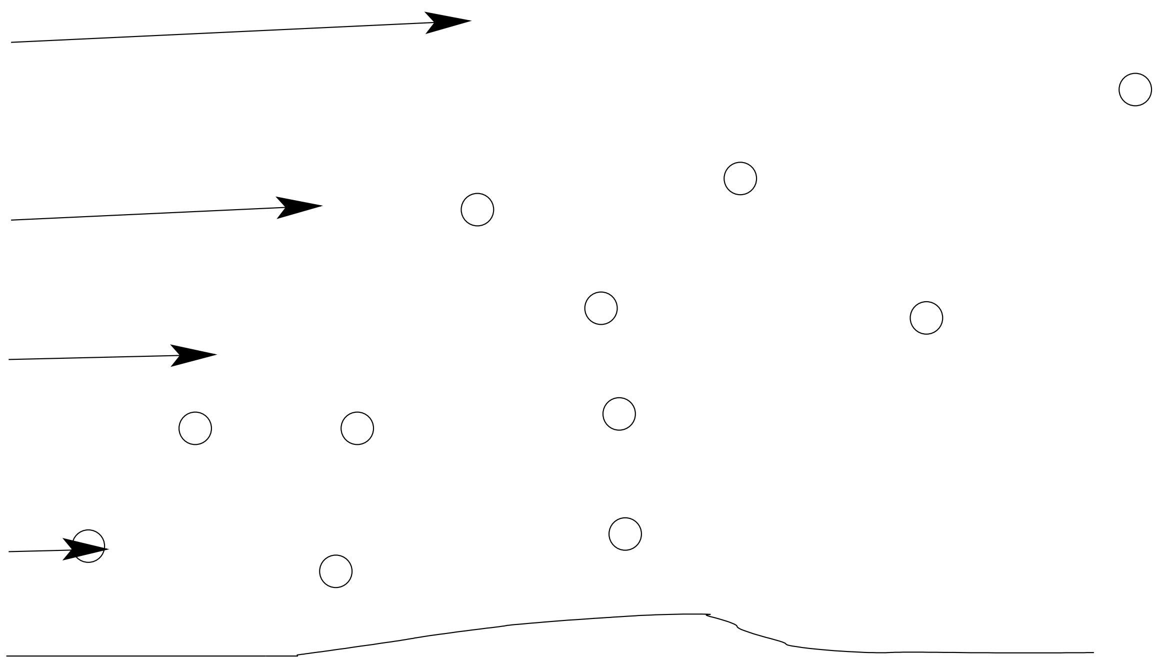
Mass transport

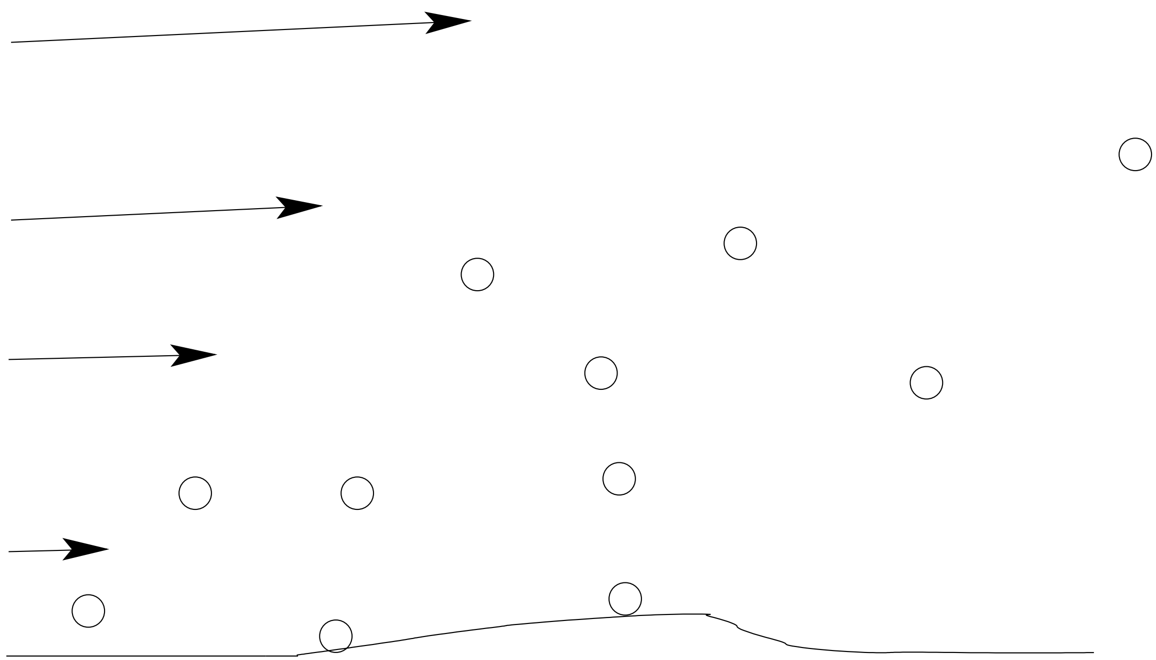
An other point of view:
Convection
Sedimentation
Diffusion

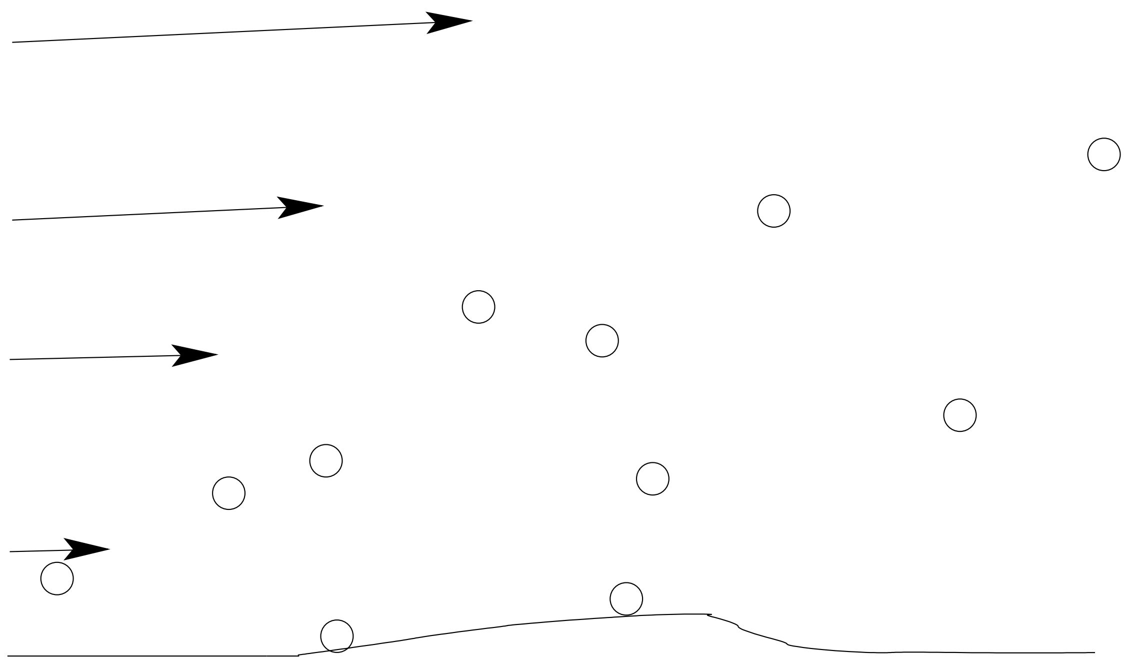


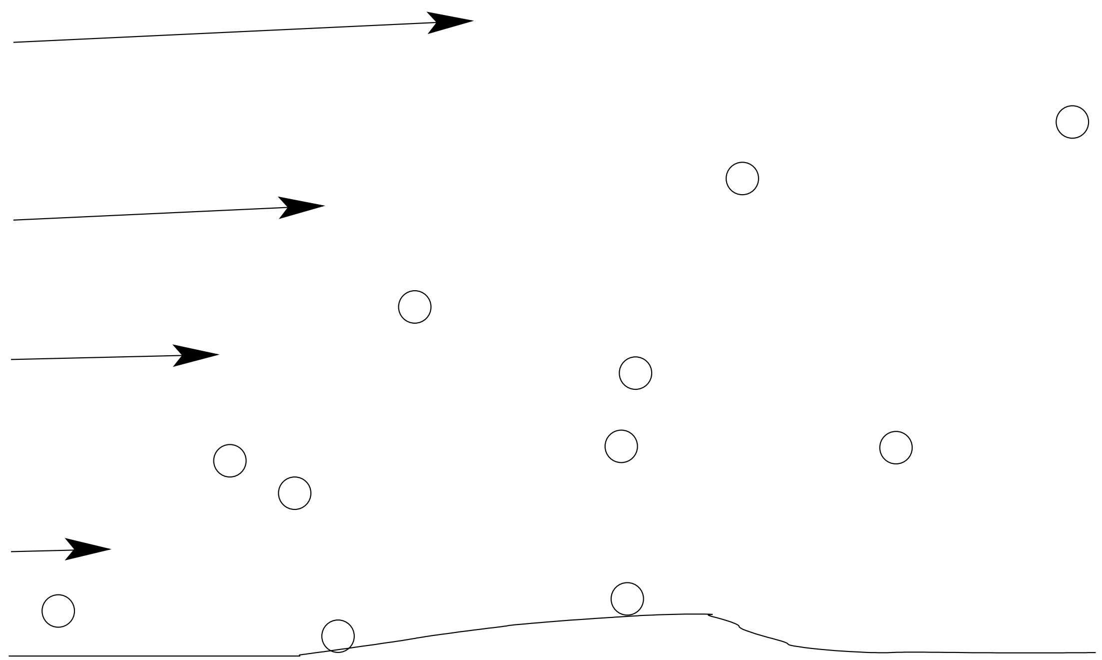


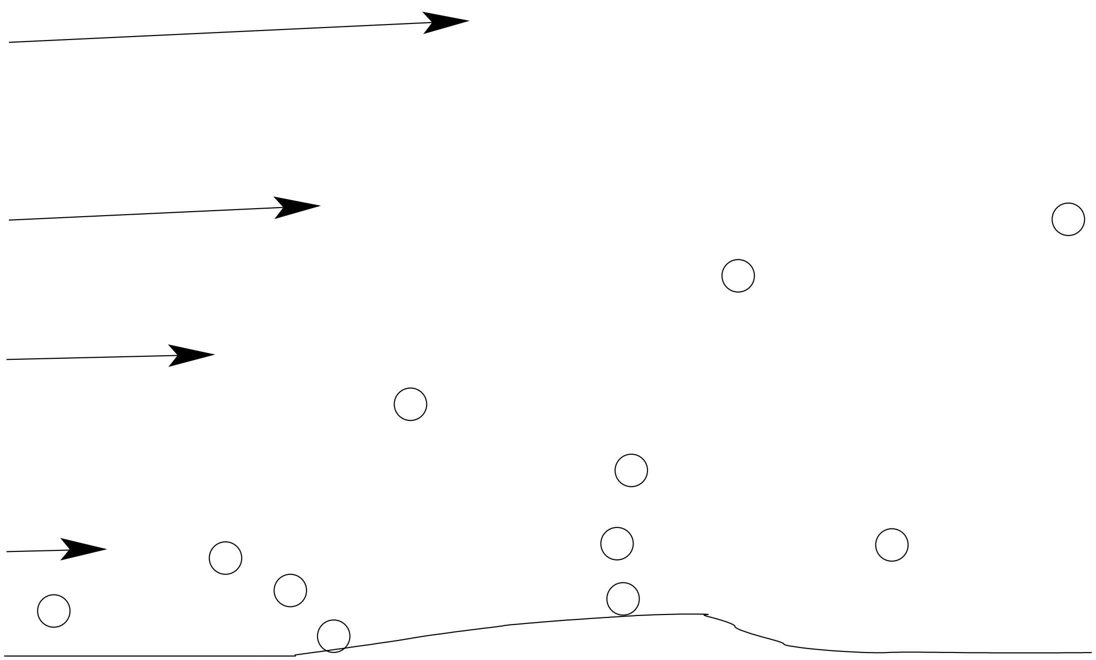


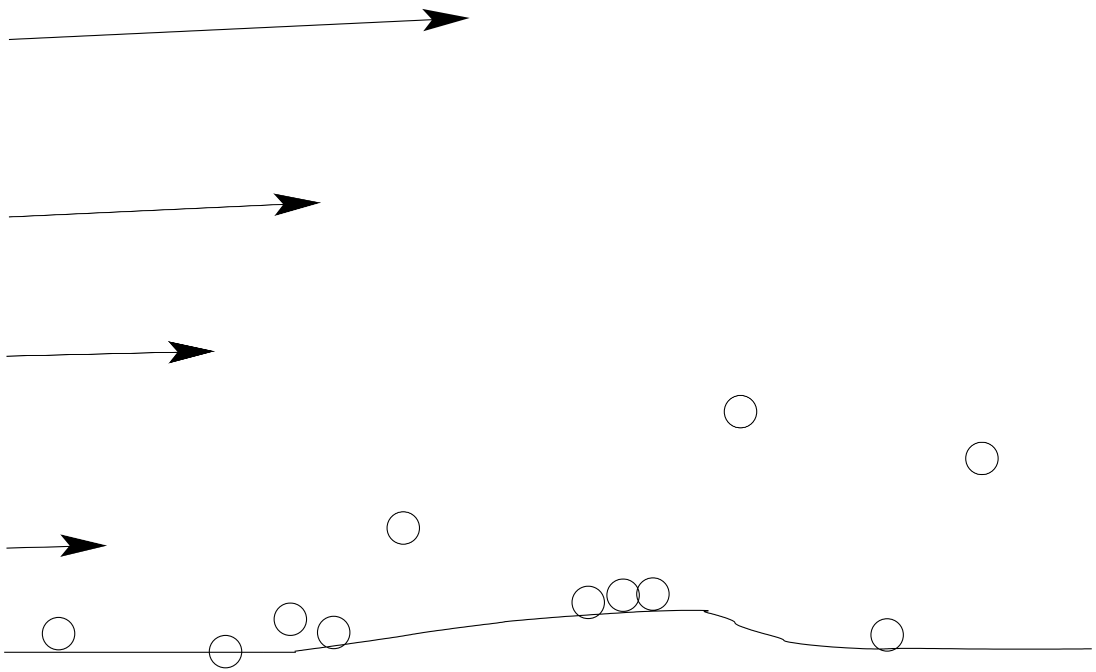


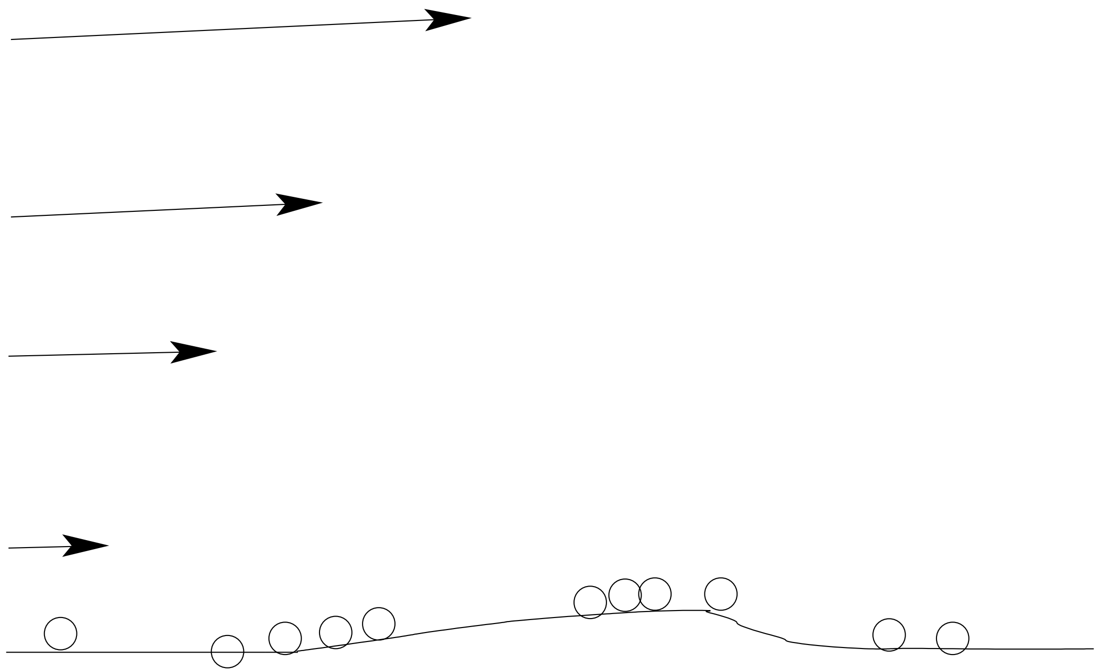


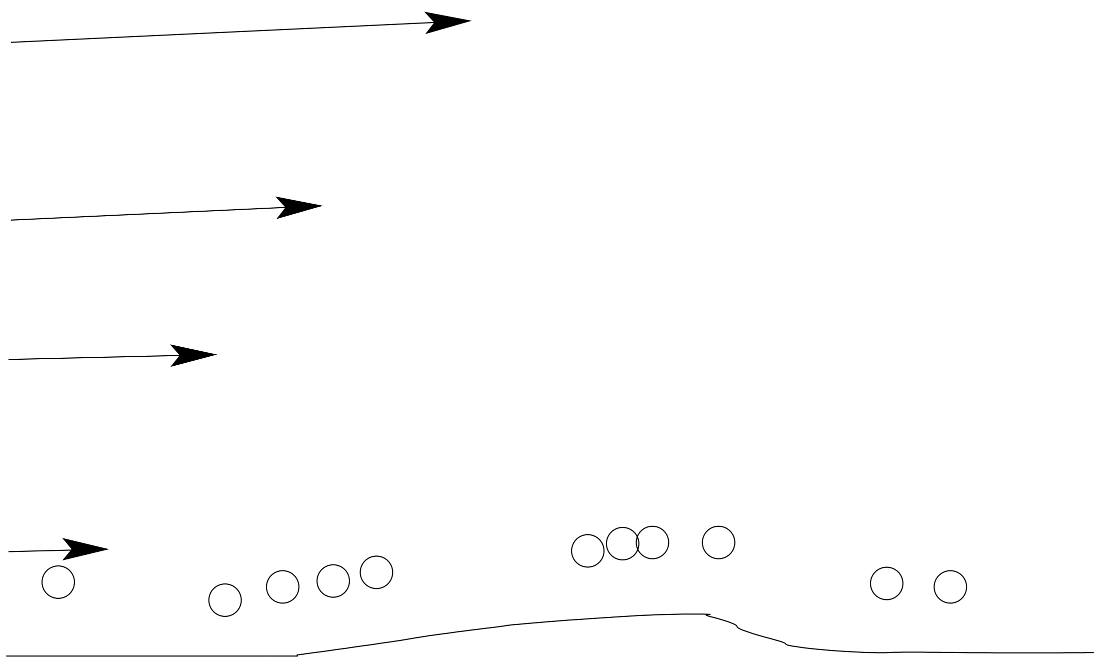












Velocity of the sediments:

$u_p =$
$v_p =$

Velocity of the sediments:

$u_p = u$
$v_p = v$
Convection

Velocity of the sediments:

$u_p = u$
$v_p = v - V_f$
Sedimentation

Velocity of the sediments:

$u_p = u - D\frac{\partial c}{\partial x}$
$v_p = v - V_f - D\frac{\partial c}{\partial y}$
Diffusion

Mass conservation of the sediments:

local form;

$$\frac{\partial cu}{\partial x} + \frac{\partial c(v - V_f)}{\partial y} = \frac{\partial c}{\partial x} D \frac{\partial c}{\partial x} + \frac{\partial}{\partial y} D \frac{\partial c}{\partial y}$$

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integral form: $\int_0^\infty cudy = q$, ...

$$\frac{\partial q}{\partial x} + c(x,0)(V_f) = -D\frac{\partial c}{\partial y}(x,0)$$

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if $\frac{\partial \tilde{u}}{\partial \tilde{y}}|_0 > \tau_s$ then $-\frac{\partial \tilde{c}}{\partial \tilde{y}}|_0 = \beta(\frac{\partial \tilde{u}}{\partial \tilde{y}}|_0 - \tau_s)^\gamma$, else $-\frac{\partial \tilde{c}}{\partial \tilde{y}}|_0 = 0$.

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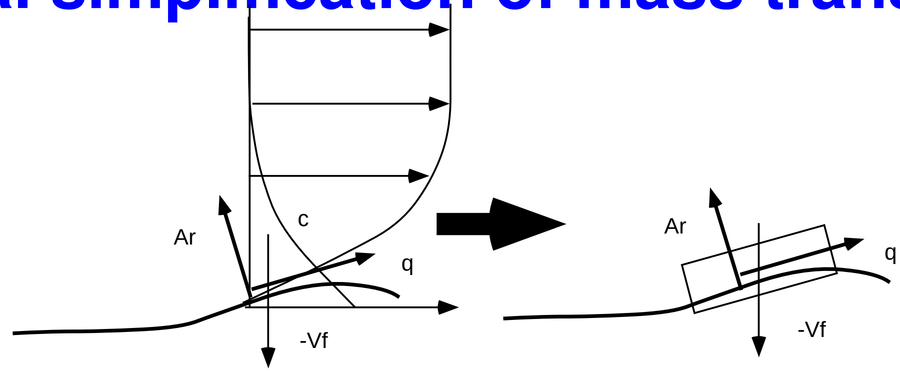
i.e.:

$$\frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}.$$

Brivois 2005/ Lagrée 2000, 2003

Final simplification of mass transport



Sauerman, Kroy, Hermann 01/ Andreotti Claudin Douady 02/ Lagrée 00/03
Valance Langlois 05 Hinch Charru (subm) Kouakou Lagrée (subm)

$$l_s \frac{\partial}{\partial x} q + q = (\varpi(\tau - \tau_s - \Lambda \frac{\partial f}{\partial x})^\gamma).$$

- total flux of convected sediments q (left figure).
- threshold effect τ_s
- slope effect $\Lambda \frac{\partial f}{\partial x}$
- $\varpi(x) = x$ if $x > 0$ (else 0), γ, l_s ...

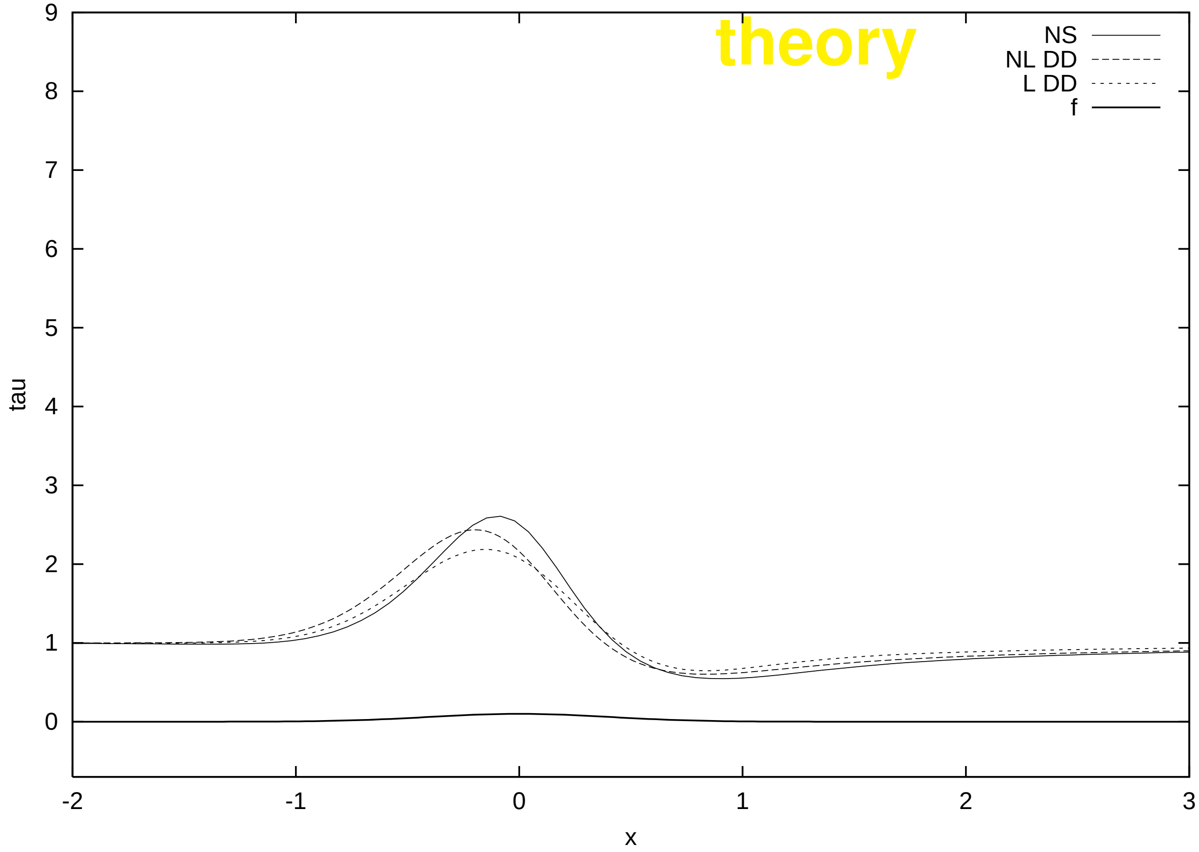
The fluid

Numerical resolution of Navier Stokes equations.
In real applications: viscosity changed... turbulence...

here we will present some severe simplifications:

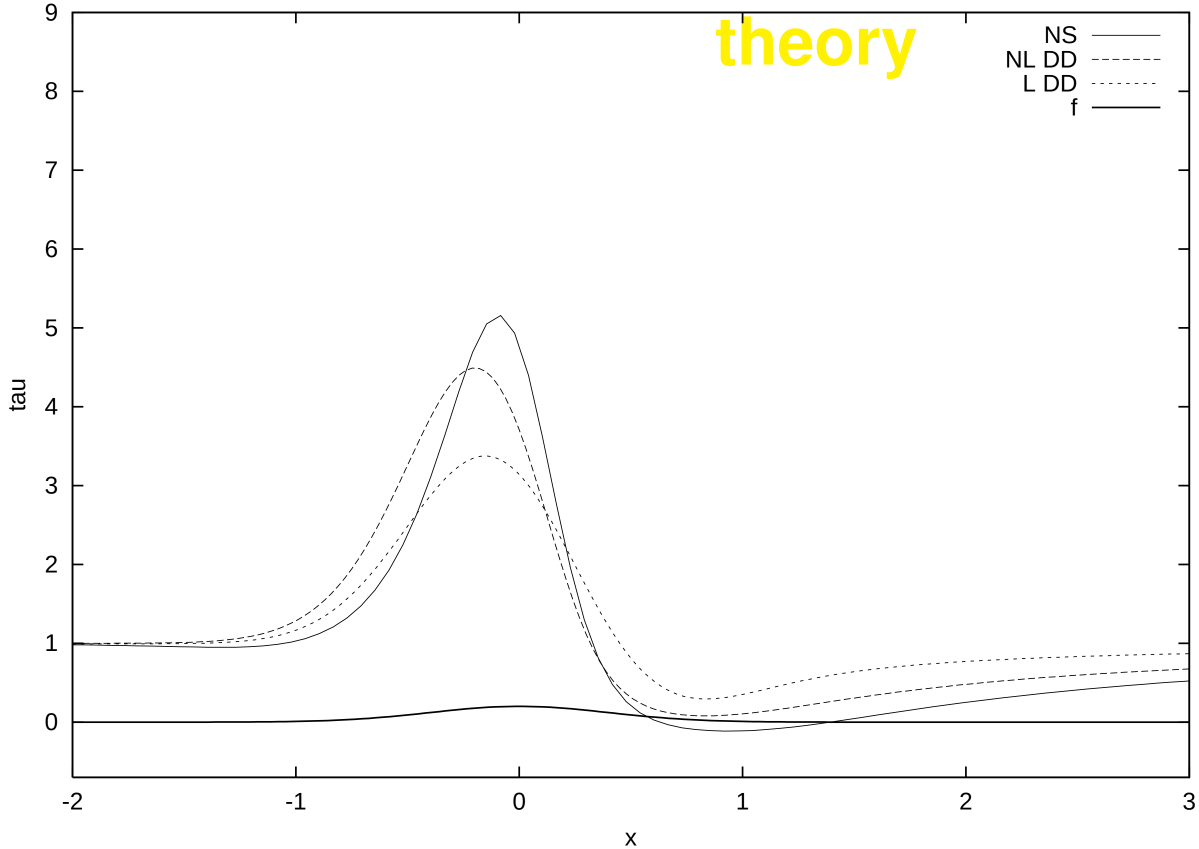
• Steady flow
• Asymptotic solution of N.S.: laminar viscous theory at $Re = \infty$
Triple Deck Stewartson 69/ Neiland 69 (in fact Double Deck Smith 80)
In fact Fowler 01
• Linearized solutions

Asymptotic solution of the flow over a bump; double deck theory



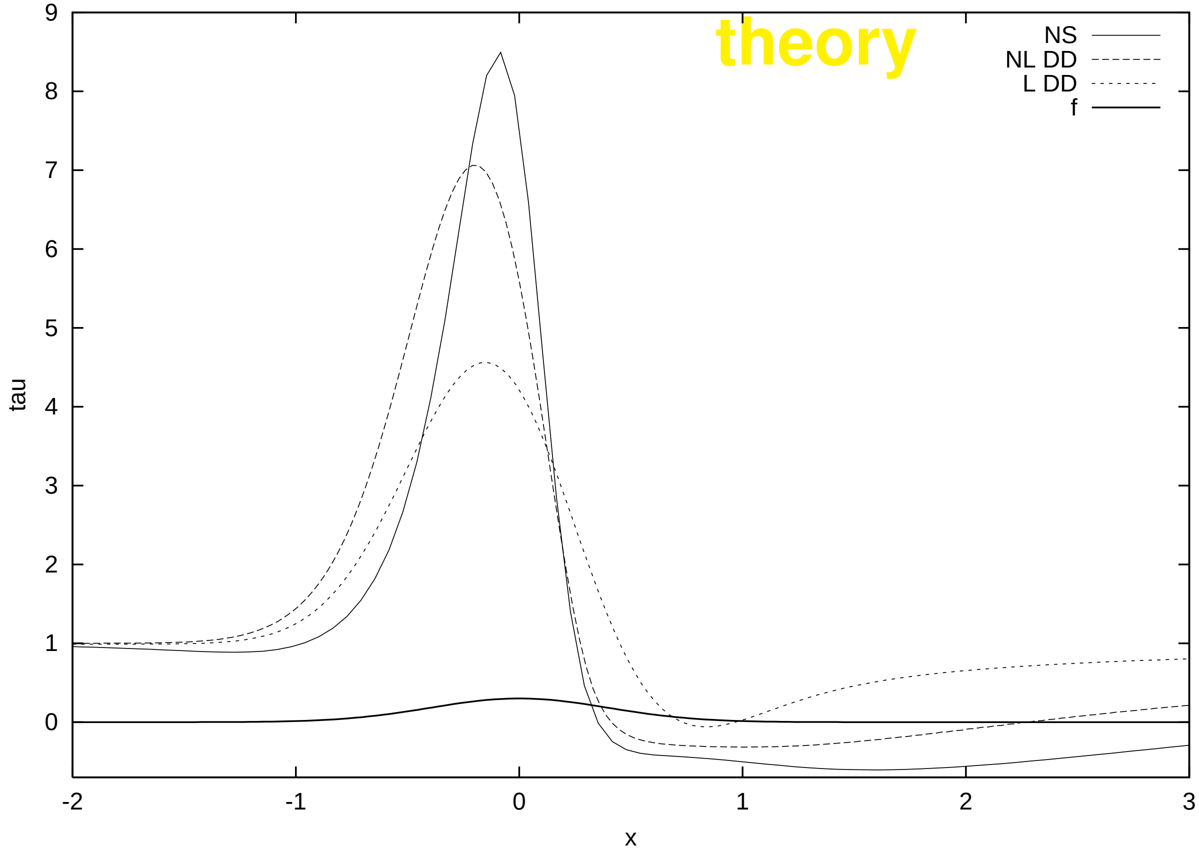
$h = 0.1,\ Re = 1000$

Asymptotic solution of the flow over a bump; double deck theory



$h = 0.2,\ Re = 1000$

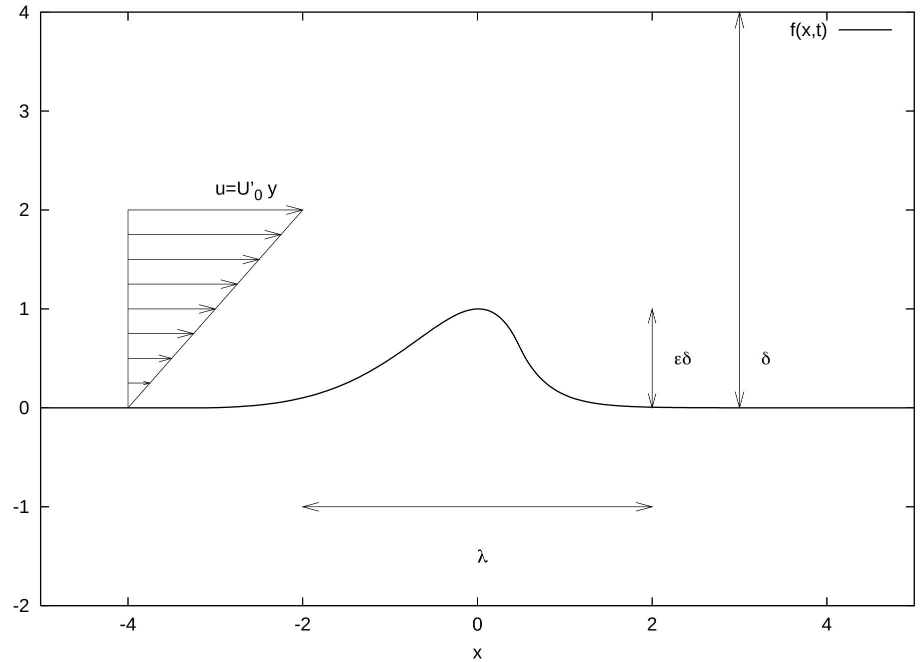
Asymptotic solution of the flow over a bump; double deck theory



$h = 0.3,\ Re = 1000$

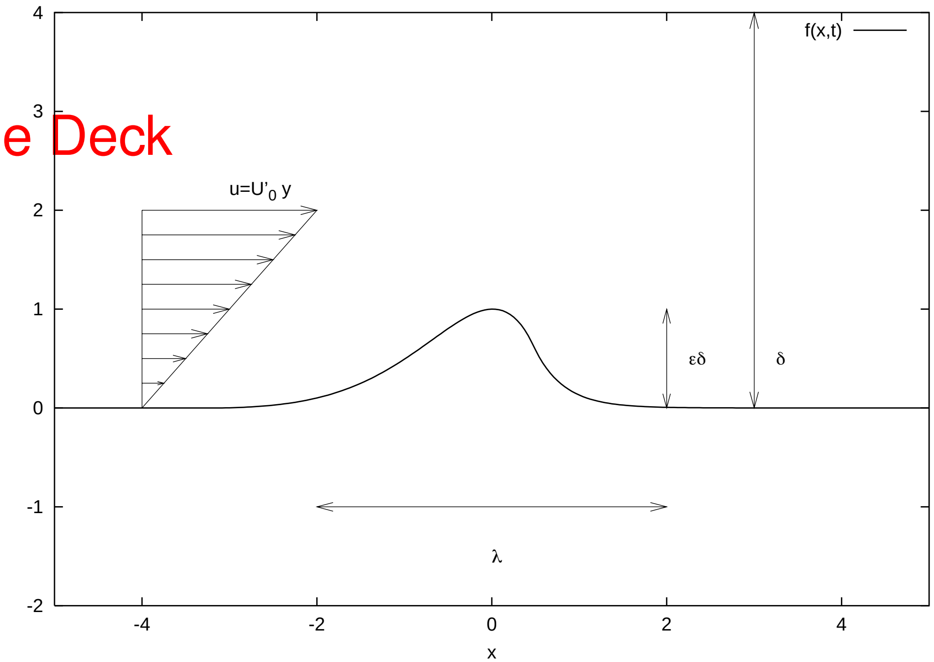
Asymptotic solution of the flow over a bump; double deck theory

We guess that viscous effects are important near the wall
Perturbation of a shear flow



$Re = \infty$, Triple/Double Deck

$$u \simeq U_0' \varepsilon \delta$$



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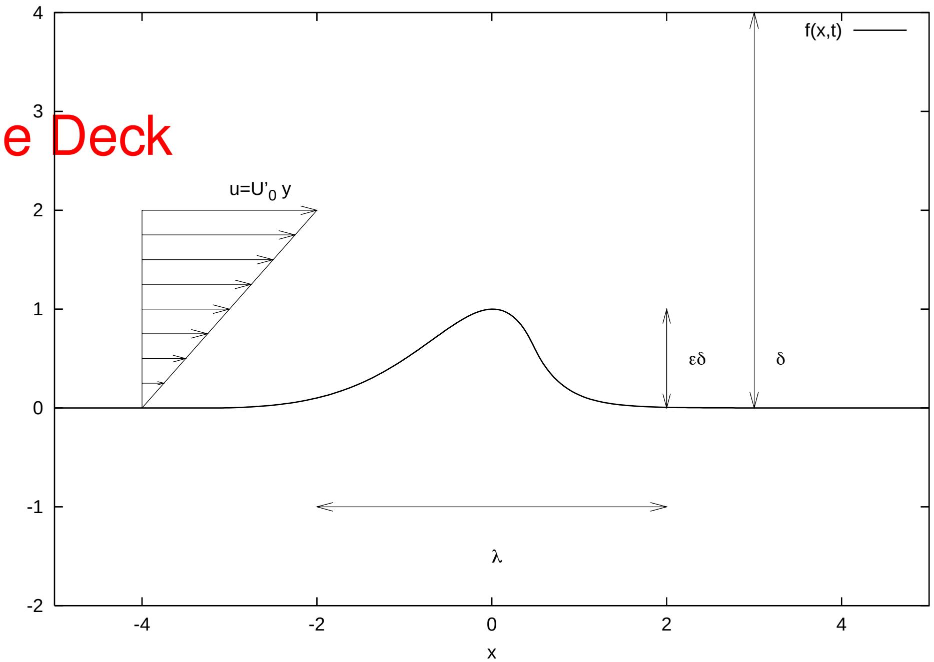

$$u \frac{\partial u}{\partial x} \simeq \nu \frac{\partial^2 u}{\partial y^2}$$

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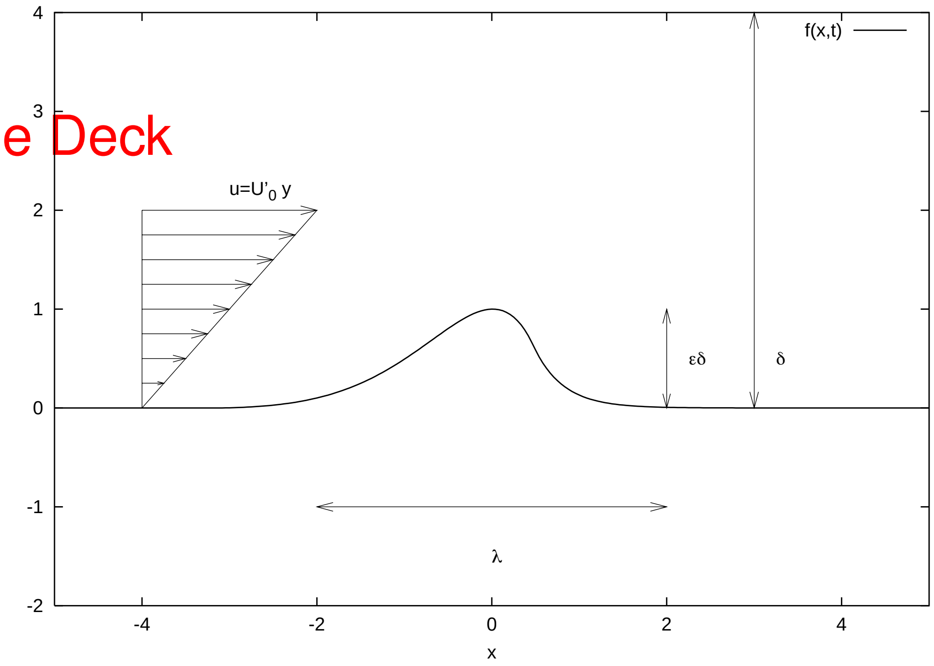
$$u \simeq U_0' \varepsilon \delta$$

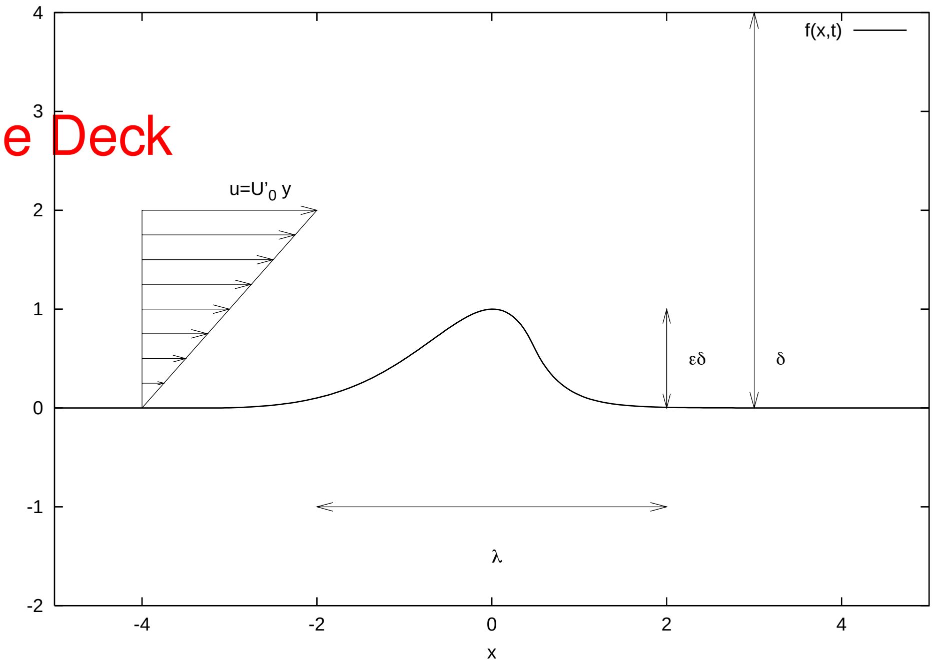


$$u \frac{\partial u}{\partial x} \simeq \nu \frac{\partial^2 u}{\partial y^2}$$

$$(U_0' \varepsilon \delta) \frac{U_0' \varepsilon \delta}{\lambda} \simeq \nu \frac{U_0' \varepsilon \delta}{\varepsilon^2 \delta^2}$$

$Re = \infty$, Triple/Double Deck

$$u \simeq U_0' \varepsilon \delta$$

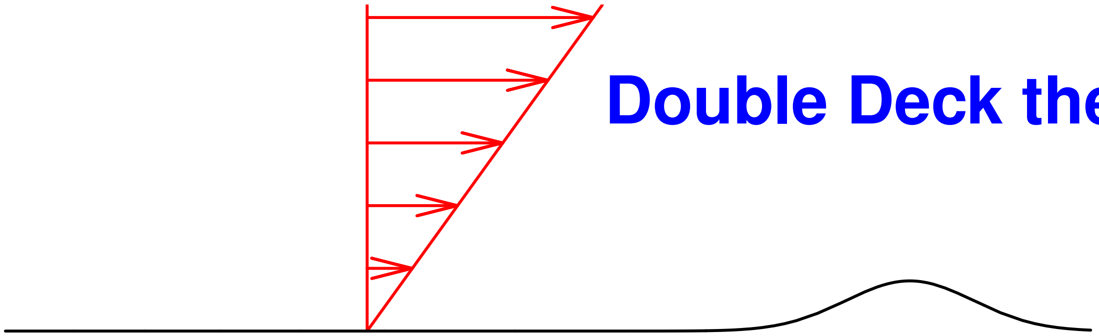


$$u \frac{\partial u}{\partial x} \simeq \nu \frac{\partial^2 u}{\partial y^2}$$

$$(U_0' \varepsilon \delta) \frac{U_0' \varepsilon \delta}{\lambda} \simeq \nu \frac{U_0' \varepsilon \delta}{\varepsilon^2 \delta^2}$$

$$\lambda = \varepsilon^3 \left(\frac{U_0' \delta^3}{\nu}\right)$$

so $\varepsilon = \lambda^{1/3} Re^{-1/3}$, with $Re = U_0' \delta^2 / \nu$.



Double Deck theory

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0, \qquad u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = -\frac{d}{dx}p + \frac{\partial^2}{\partial y^2}u.$$

Boundary conditions: no slip condition: $u(x, y = f(x)) = 0, \qquad v(x, y = f(x)) = 0,$
matching with the shear flow ($y \to \infty$)

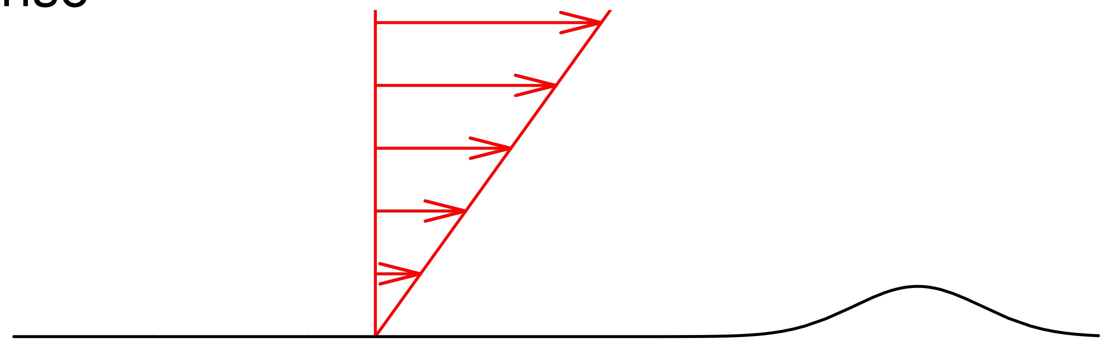
$$\lim_{y\to\infty} u(x, y) = U_S'(0)y.$$

upstream:

$$u(x \to -\infty, y) = U_S'(0)y, \qquad v(x \to -\infty, y) = 0.$$

Asymptotic solution of the flow over a bump; double deck theory

Viscous effects are important near the wall
Perturbation of a shear flow
Non linear resolution (with flow separation) possible
But first we linearise



Linearizing the equations: We look at a linearized solution: $u = y + \alpha u_1$, $v = \alpha v_1$, $p = \alpha p_1$ with $\alpha << 1$.

$$
\begin{aligned}
\frac{\partial}{\partial x} u_1 + \frac{\partial}{\partial y} v_1 &= 0, \\
y \frac{\partial}{\partial x} u_1 + v_1 &= -\frac{\partial}{\partial x} p_1 + \frac{\partial^2}{\partial y^2} u_1,
\end{aligned}
$$

with boundary conditions:
$u_1 = v_1 = 0$ in $y = f(x, z)$,
$y \to \infty$, $u_1 = +f(x, z)$,
$x \to -\infty$, $u_1 = 0$, $v_1 = 0$. Looking at solutions in Fourier space.

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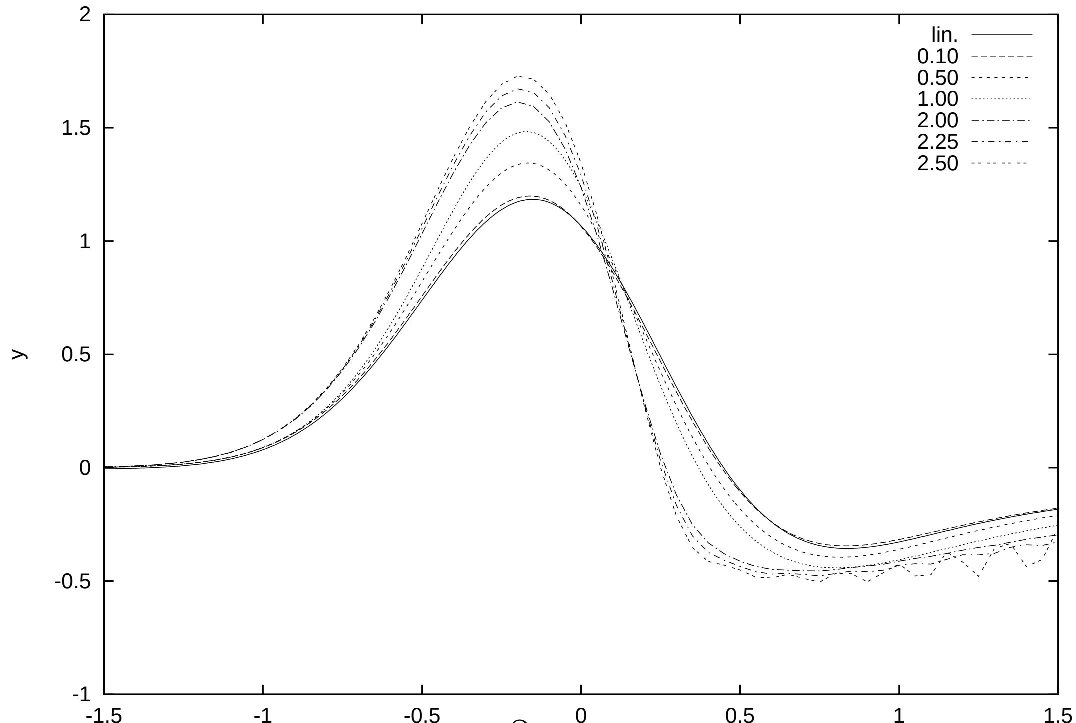
$y \rightarrow \infty$, $u_1 = +f(x, z)$,
$x \rightarrow -\infty$, $u_1 = 0$, $v_1 = 0$. Looking at solutions in Fourier space.

After some algebra:

$$\frac{\partial u}{\partial y}|_0 = 1 + \alpha FT^{-1}[(3Ai(0))(-ik)^{1/3}FT[f]] + O(\alpha^2).$$

Asymptotic solution of the flow over a bump; Linear/ Non Linear double deck theory



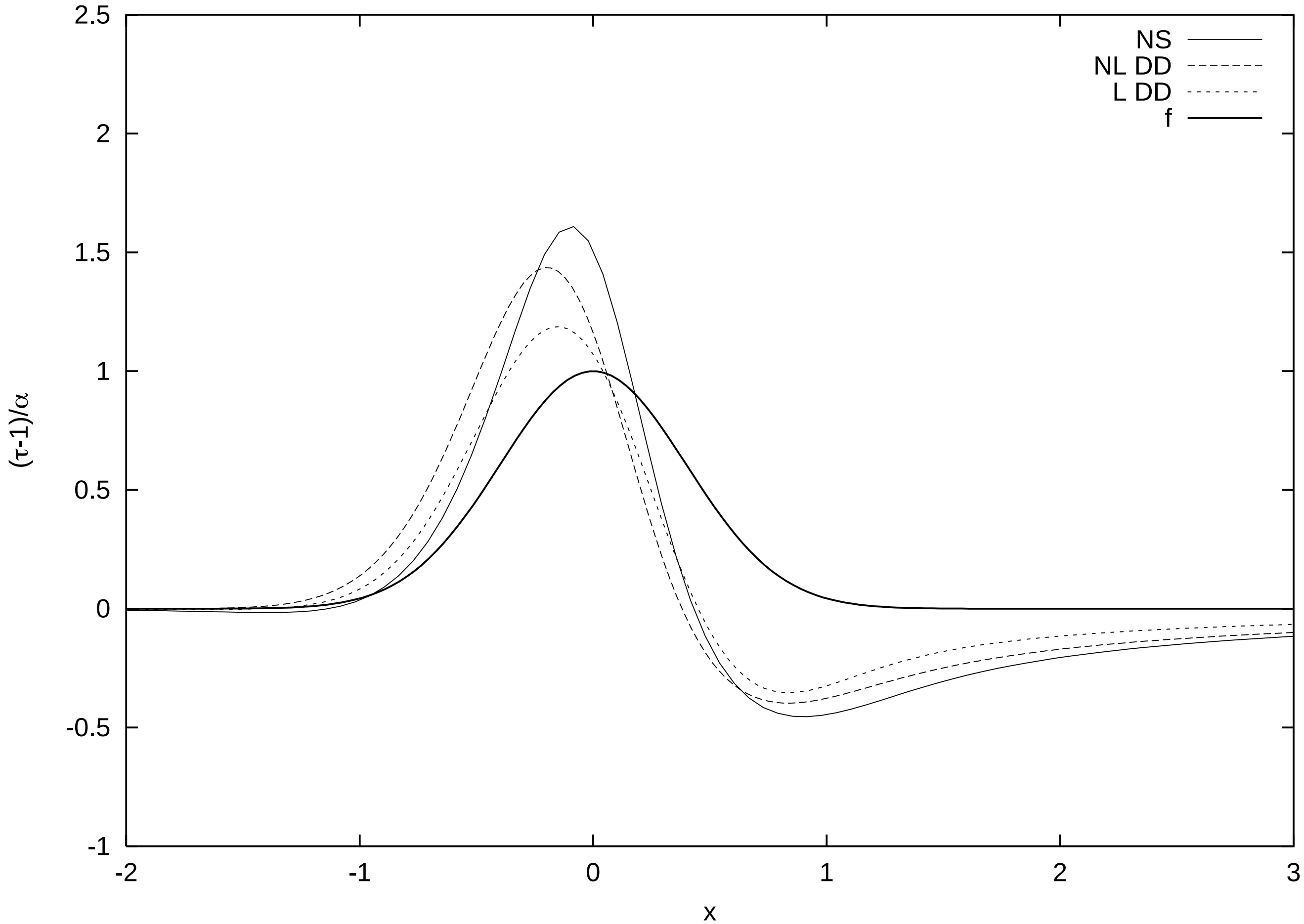
Reduced wall shear $(\frac{\partial u}{\partial y}_0 - 1)/\alpha$
function of x

for the bump $\alpha e^{-\pi x^2}$
with $\alpha = 0.10, \alpha = 0.5, \alpha = 1.0,$
$\alpha = 2, \alpha = 2.25, \alpha = 2.50.$
The plain curve (lin.") is the linear prediction , other curves come from the non linear numerical solution.

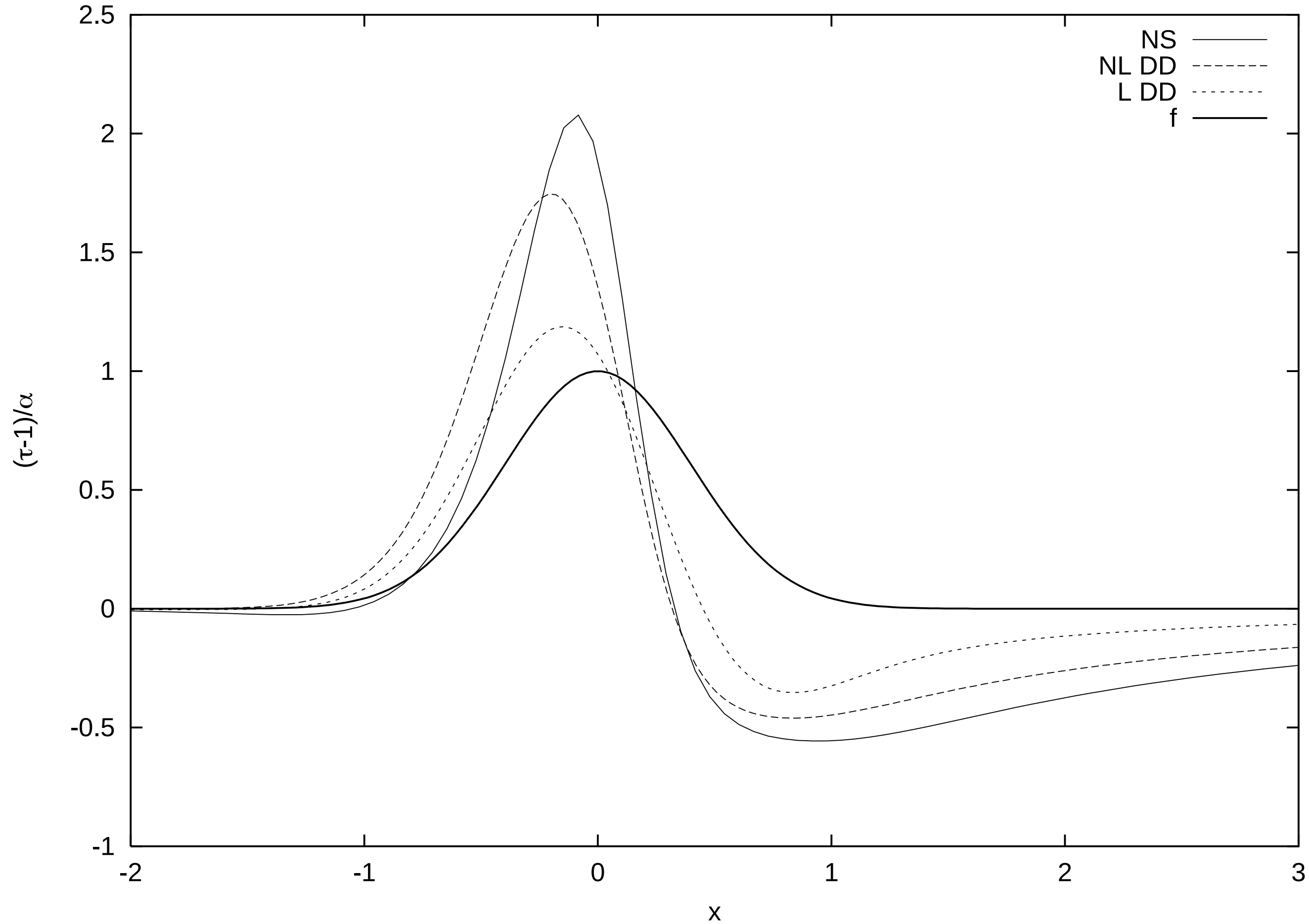
Notice the numerical oscillations in the case of

separated flow (separation is for $\alpha > 2.1$)

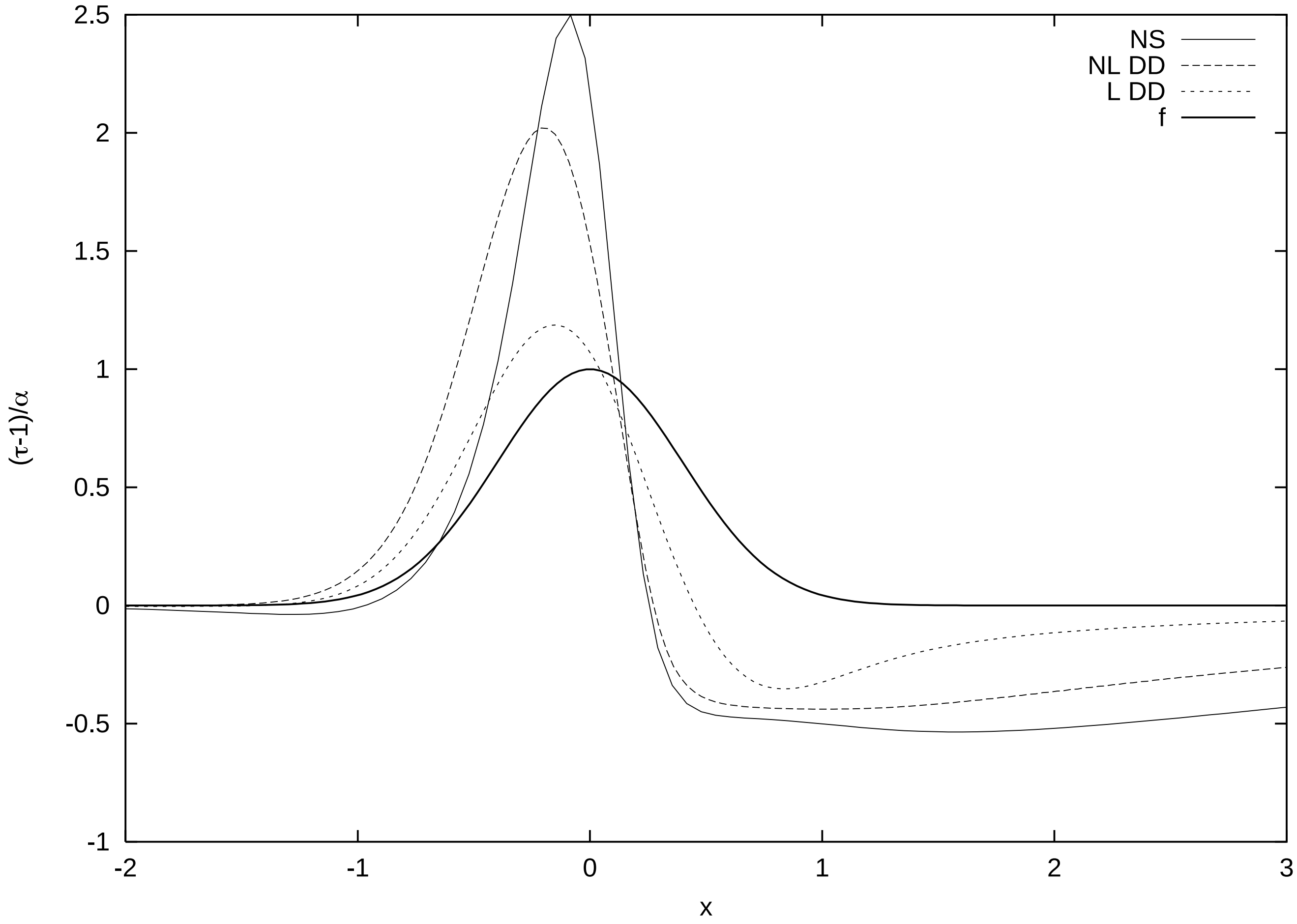
Asymptotic solution of the flow over a bump; double deck theory



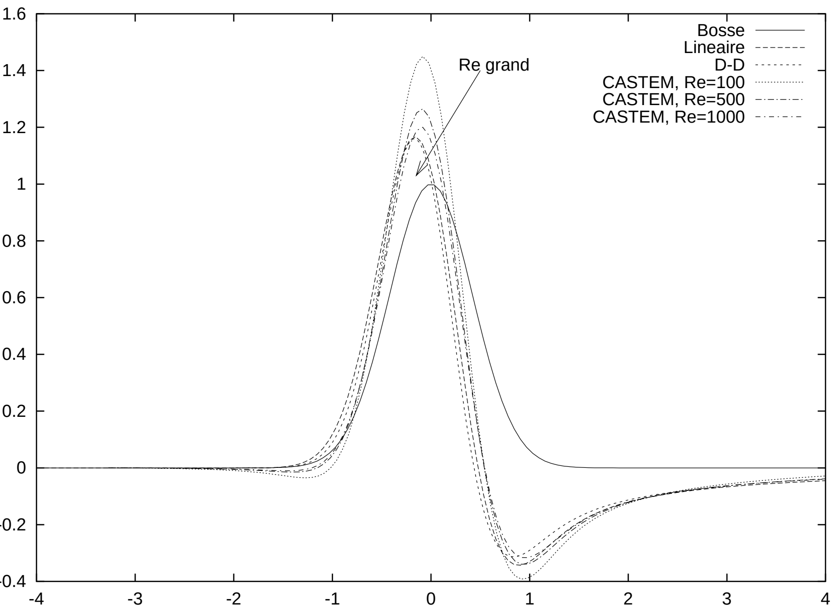
Asymptotic solution of the flow over a bump; double deck theory

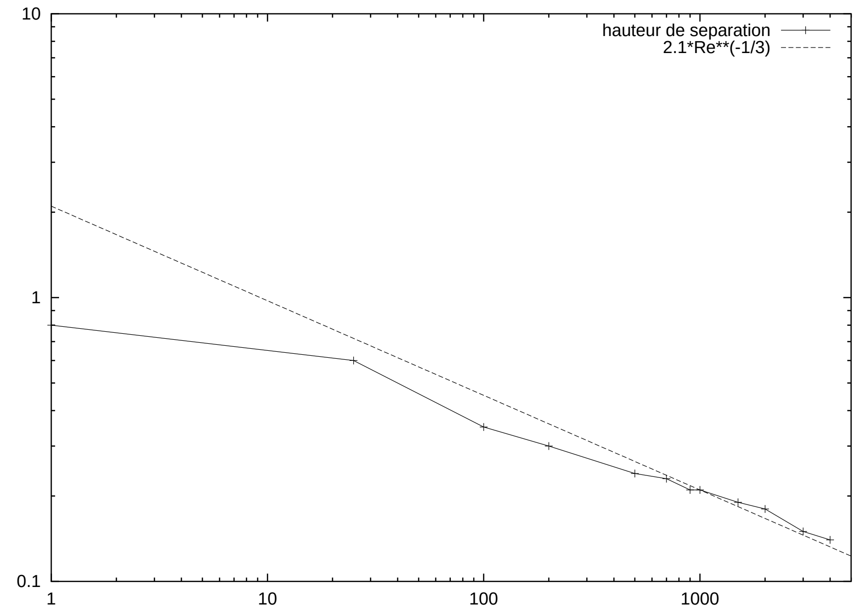


Asymptotic solution of the flow over a bump; double deck theory



Comparison with Navier Stokes

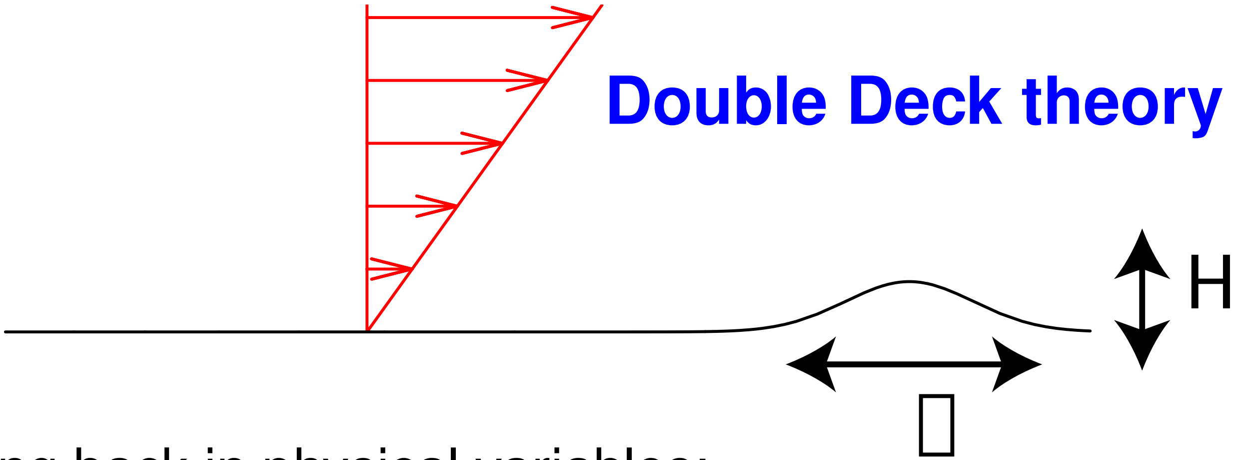




good!
Re increasing
α fixed.

conclusion: Perturbation of shear flow is in advance compared to the bump crest.

Double Deck theory



Going back in physical variables:
bump of length of order λ and of height of order $H << \delta$:

$$\tau = \mu U_0'(\bar{U}_S'(1 + (\frac{U_0'}{\nu\lambda})^{1/3} H\tilde{c})), \text{with } \tilde{c} = FT^{-1}[FT[\tilde{f}]3Ai(0)(-(i2\pi\tilde{k})\bar{U}_S')^{1/3}]$$

function of time $\bar{U}_S'$ is a number of order one.
$(\frac{U_0'}{\nu\lambda})^{1/3} H <= 1$

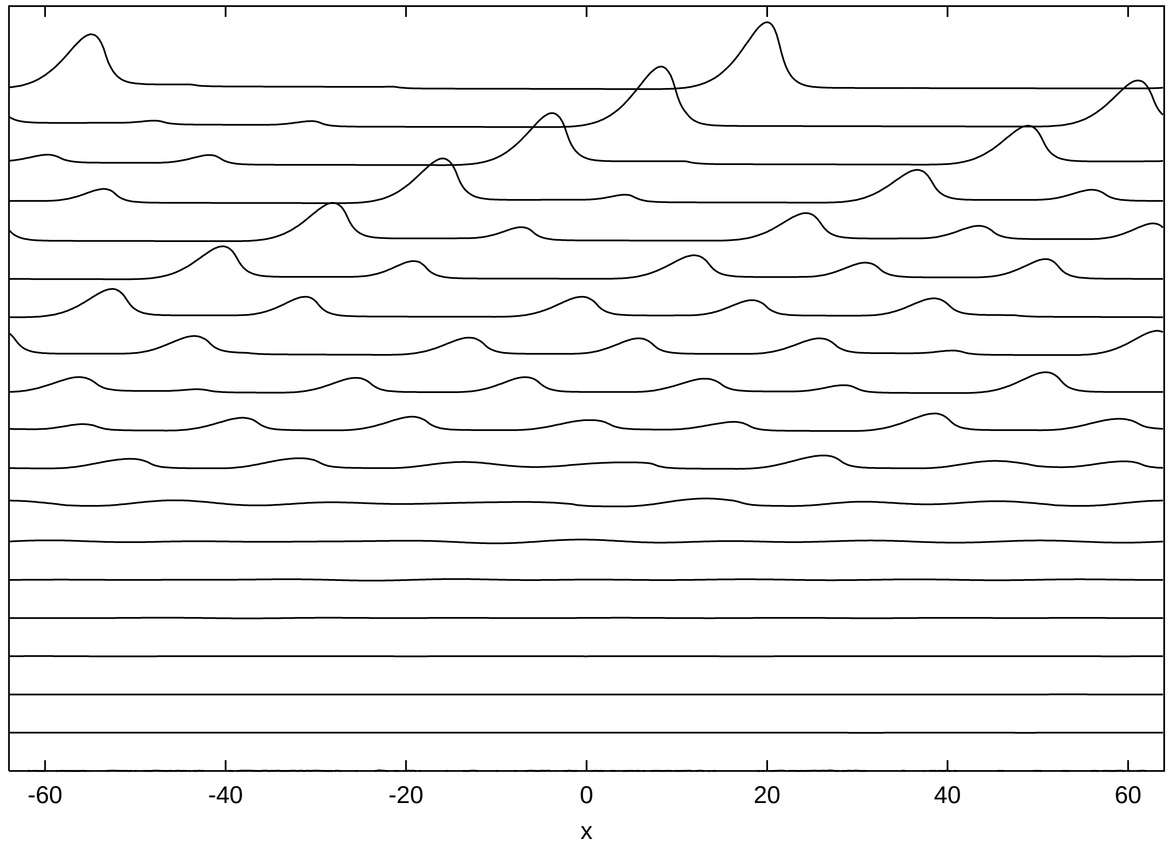
Completely erodible soil

Solution of

$$\tau = TF^{-1}[(3Ai(0))(-ik)^{1/3}TF[f]]$$

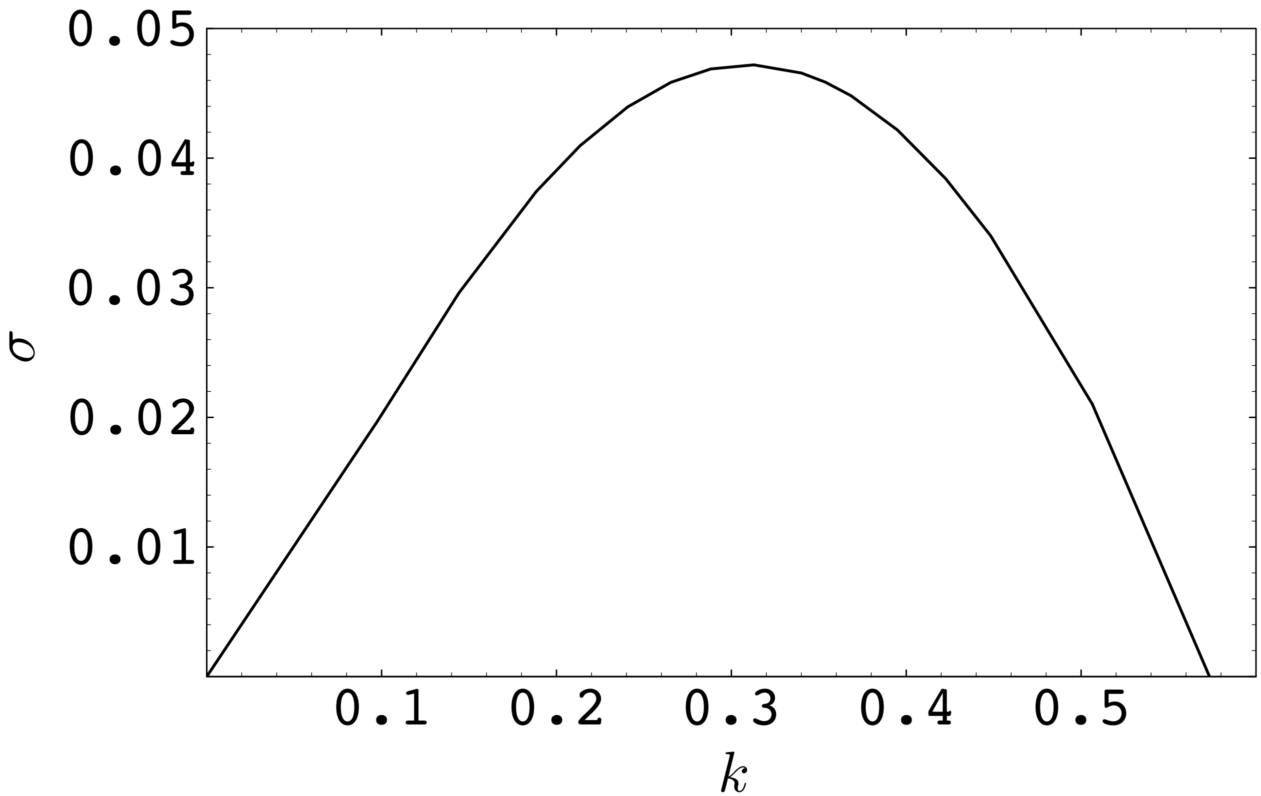
$$l_s \frac{\partial q}{\partial x} + q = \varpi(\tau - \tau_s)$$

$$\frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}$$


-60
-40
-20
0
20
40
60
x

Linear stability

up to now $U_0' = 1$



Linear stability

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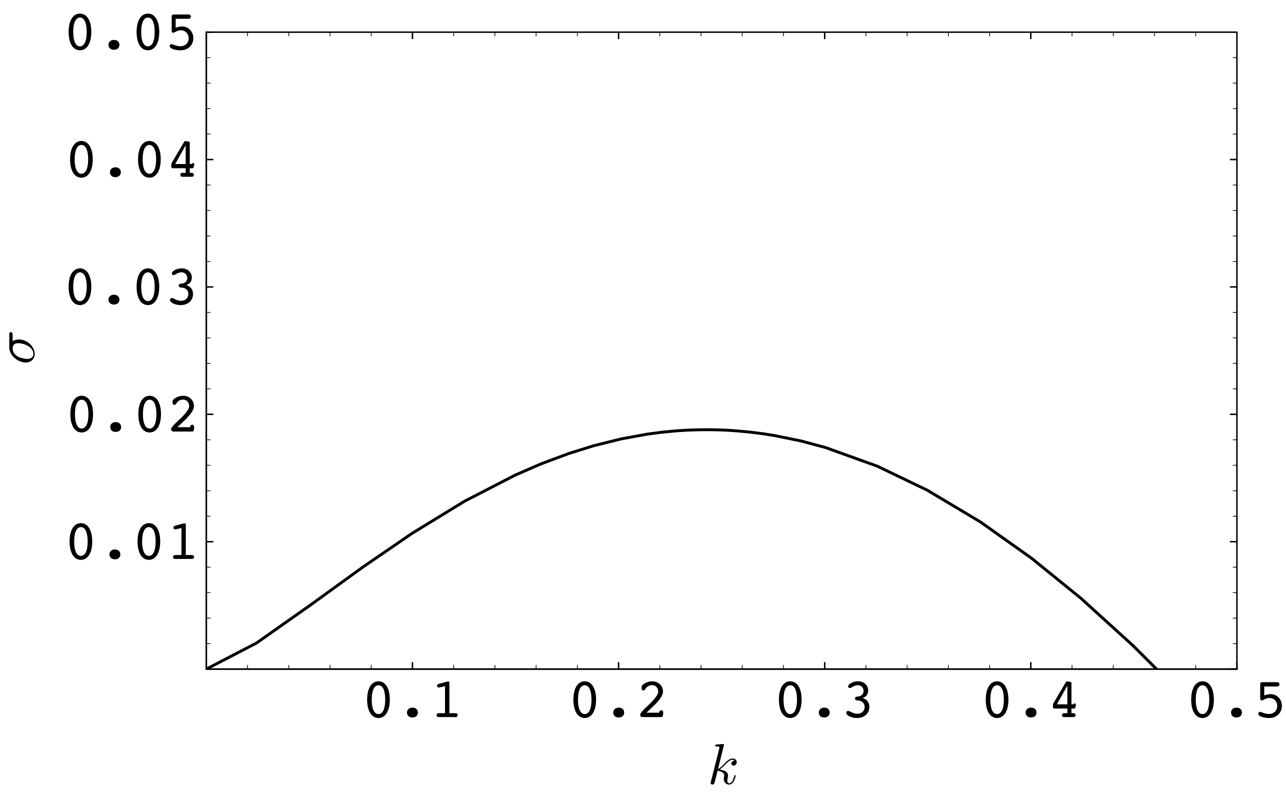

Interpretation AB effect

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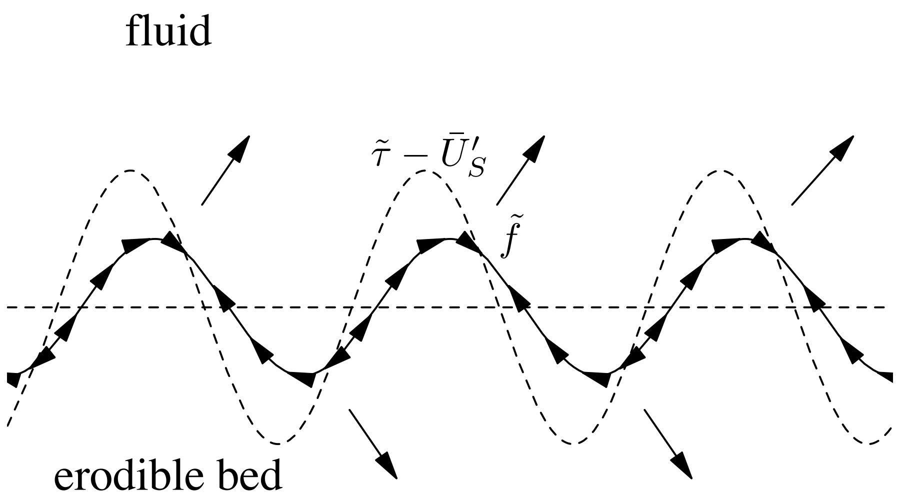
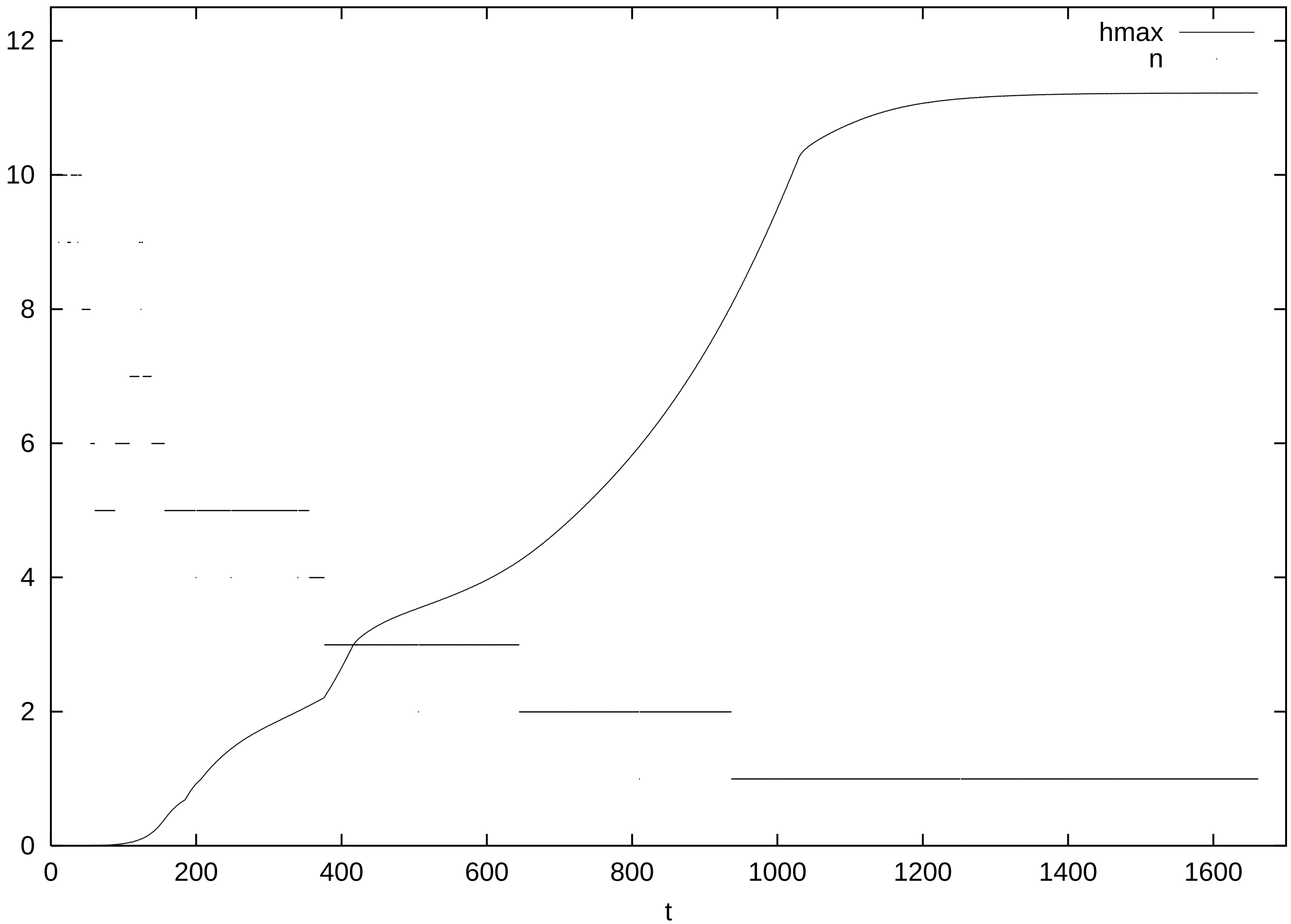


Figure 4: A wavy profile (bold line, $\tilde{f}$) has a perturbation of skin friction (dashed line, $\tilde{\tau} - \bar{U}_S'$) in advance of phase. When it is positive, the matter is moved down stream (small arrows on the profile), when is is negative, it is in opposite direction. The result is an increase of the wave and a displacement in the stream direction (large inclined arrows).


hmax
n
12
10
8
6
4
2
0
0
200
400
600
800
1000
1200
1400
1600
t

Completely erodible soil

example of runs:
animation 1,
animation 2 (length $*2$).
animation 2 (circular cuve).
always coarsening, finally there is only one bump in the "box".

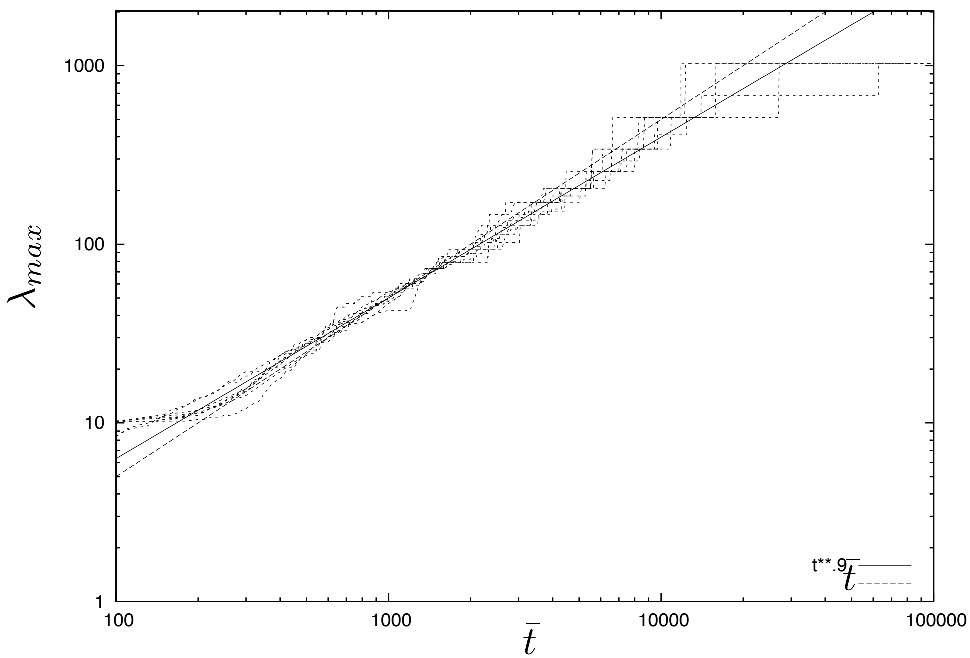


Figure 11: Constant shear, the wave length of the structure scales with a power between $\bar{t}^{0.9}$ and $\bar{t}$.

Linear stability

here $U_0' = cos(\bar{t})$

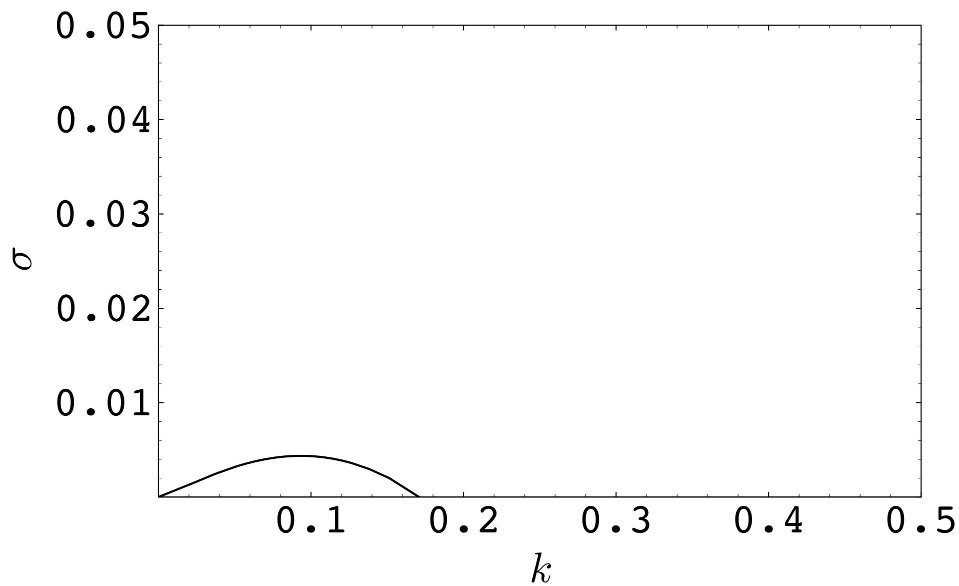


Figure 6: Amplification factor function of wave number. Averaged oscillating case, $\tilde{l}_K = 1$ case (6and 28).

Interpretation AB effect

here $U_0' = cos(\bar{t})$

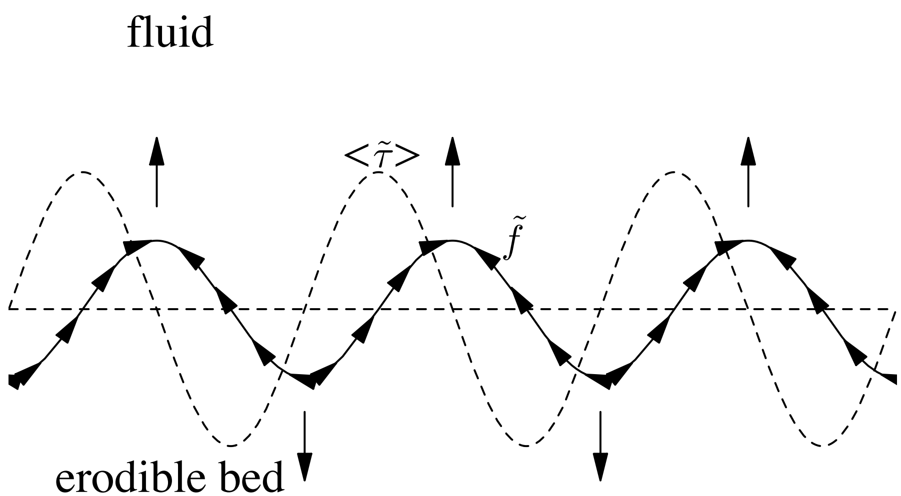


Figure 7: A wavy profile (bold line, $\tilde{f}$) has a mean perturbation of skin friction (dashed line, $< \tilde{\tau} >$) out of phase. When $< \tilde{\tau} >$ is positive, the matter is moved from left to right (small arrows on the profile), when it is negative, it is in opposite direction. The result is an increase of the wave without displacement (large vertical arrows).

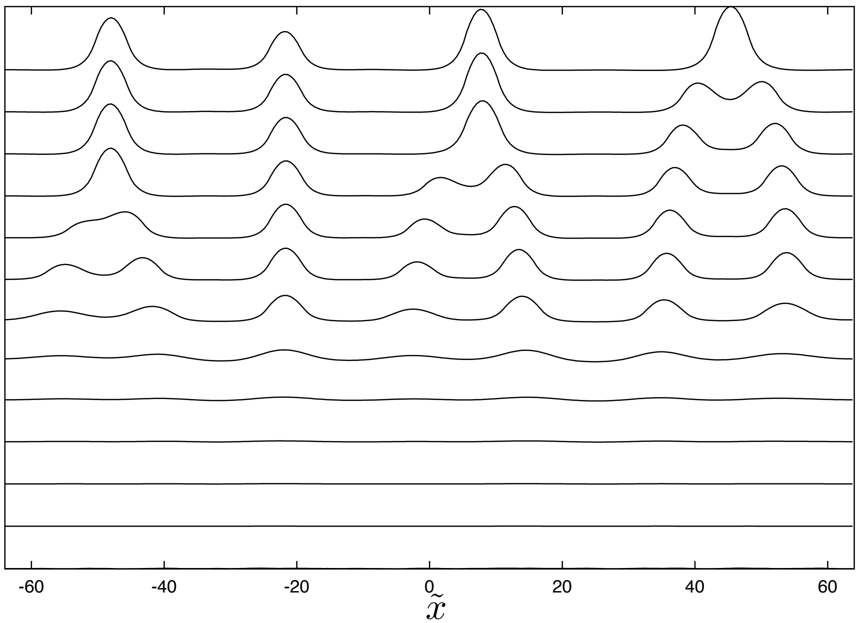


Figure 13: Oscillating régime with (22), spatio temporal diagram, time increases from bottom to top. Ripples growth from a random noise and merge two by two.

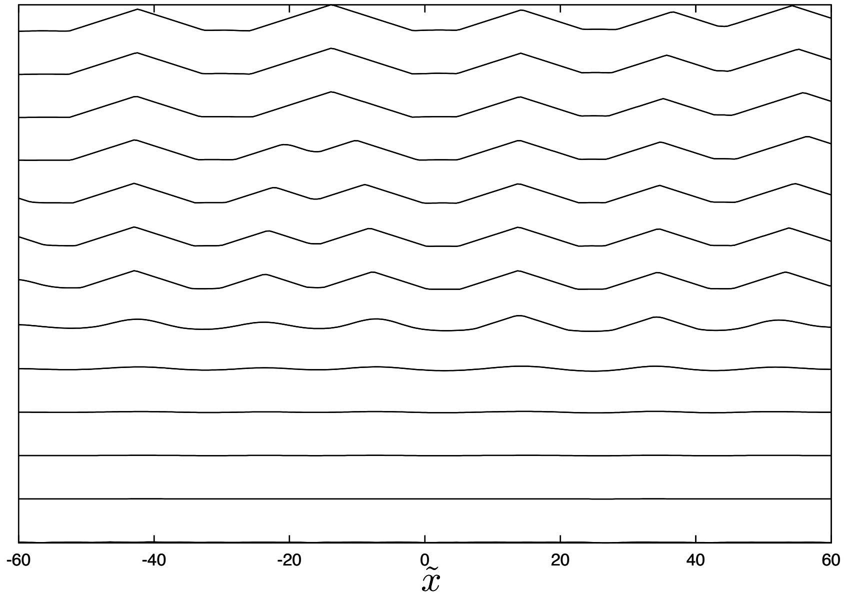


Figure 12: Oscillating régime with (22) and slope limitation $V = 1$, $\frac{1}{\mu} = 0.05$, spatio-temporal diagram, time increases from bottom to top
.

Completely erodible soil

example of runs:
animation (circular cuve, “avalanche“).
animation (circular cuve).

always coarsening,

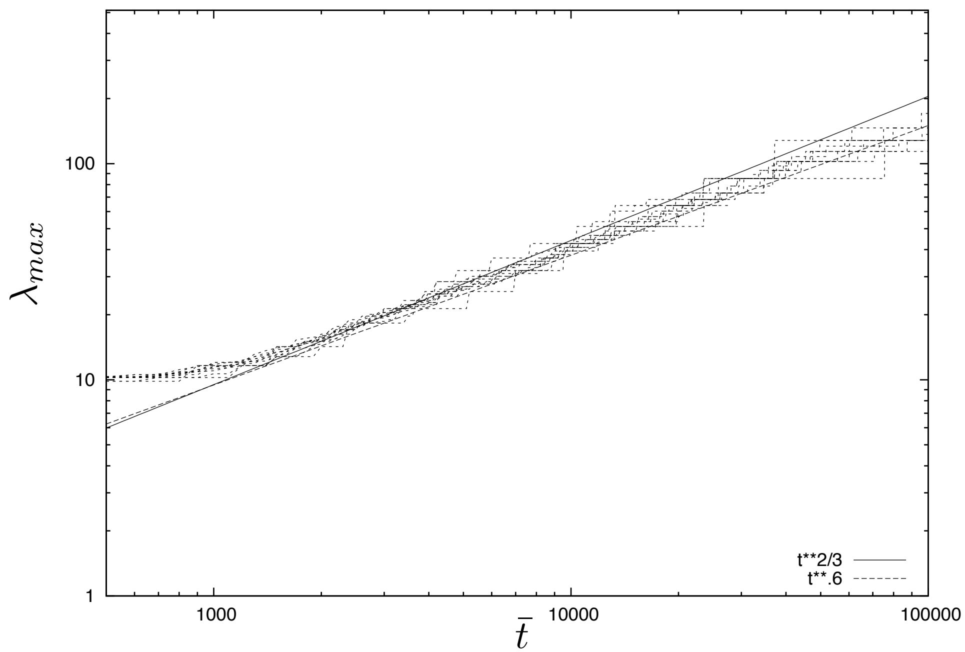
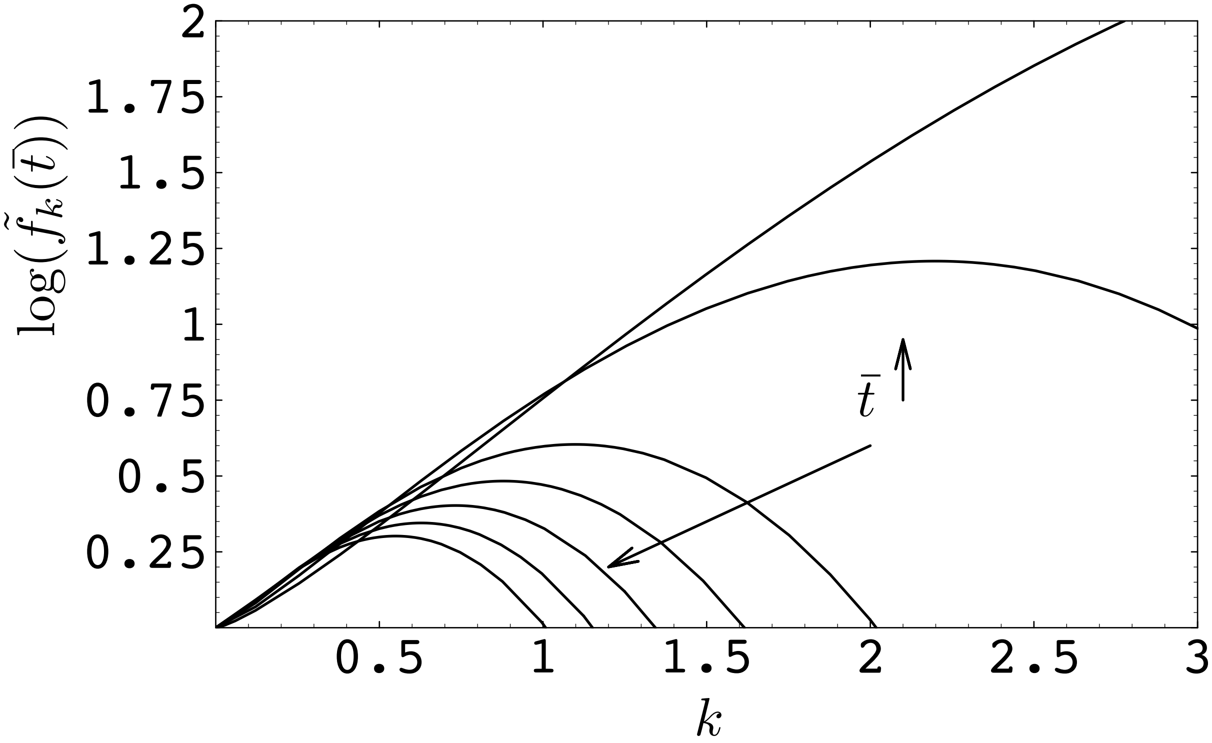


Figure 15: Oscillating shear, the wave length of the structure scales with a power law between $\bar{t}^{0.6}$ and $\bar{t}^{2/3}$.

Linear stability

here $U_0' = \bar{t}^{-1/2}$



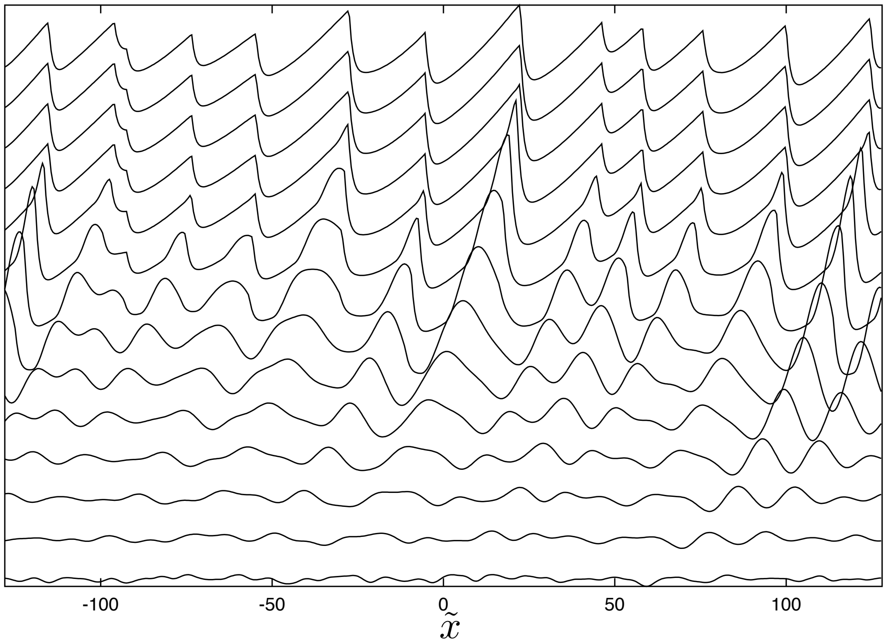


Figure 16: Decelerated case with (23) $l_s = 1$. Spatio- temporal diagram, time increases from bottom to top. There is a final steady bed because the shear stress is under the threshold.

animation

animation (circular cuve).

Some experimental crude comparisons

Betat, Kruelle, Frette, and Rehberg (2002):
the most unstable wave length is about $9cm$, $\sigma^* = 3\,10^{-3}s^{-1}$

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$$\lambda^* \simeq 15cm.$$

$$\sigma^* = 1.2\,10^{-3}s^{-1},$$

which is is the order of magnitude of the experimental value.

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which is is the order of magnitude of the experimental value.
σ^* increases with $(U_0')^3$ the shear, observed
λ^* increases with $U_0'^{-1}$, not observed

Some experimental crude comparisons

Oscillating case, experimentally Rousseaux Stegner Wesfreid (2004)
$\lambda_{initial} \simeq 0.5cm$.

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good trend

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good trend

Rousseau et al tried to fit: $\lambda_{max} \propto Log(t)$ (Cahn-Hillard)

$$\lambda_{max} \propto t^{2/3}$$

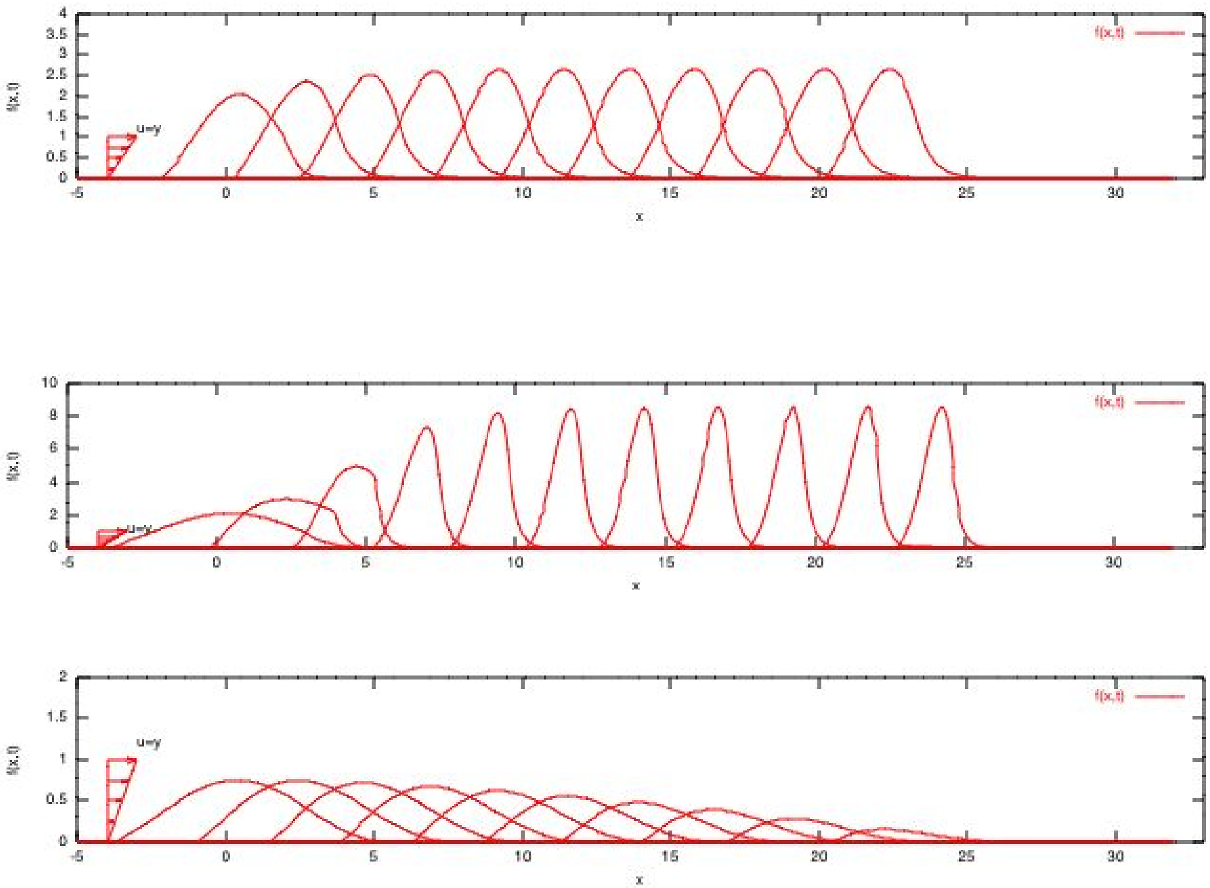
Displacement of a ”dune” in a shear flow: rigid soil

Solution of

$$\tau = TF^{-1}[(3Ai(0))(-ik)^{1/3}TF[f]]$$
$$\frac{\partial q}{\partial x} + Vq = V\varpi(\tau - \tau_s)$$
$$\frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}$$

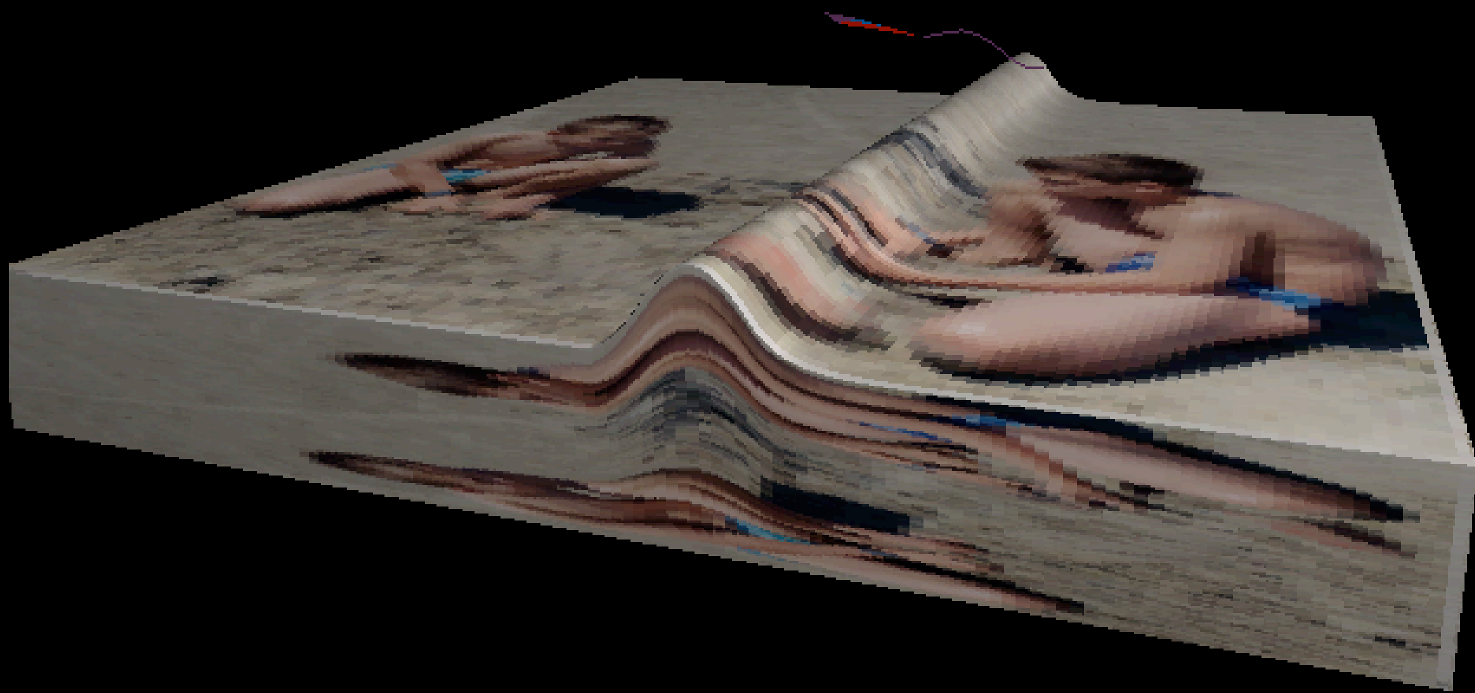
implementation of the fact that f cannot be negative.

Example of displacement of a ”dune”

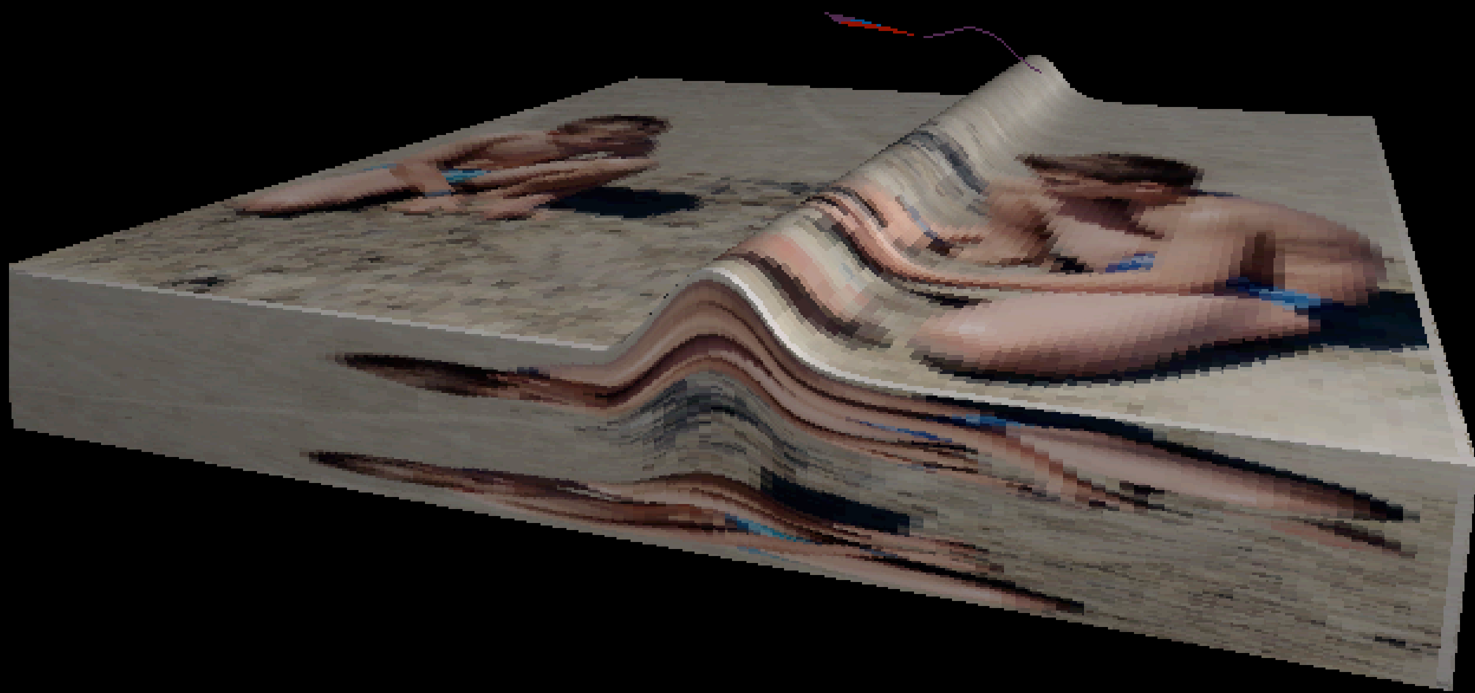


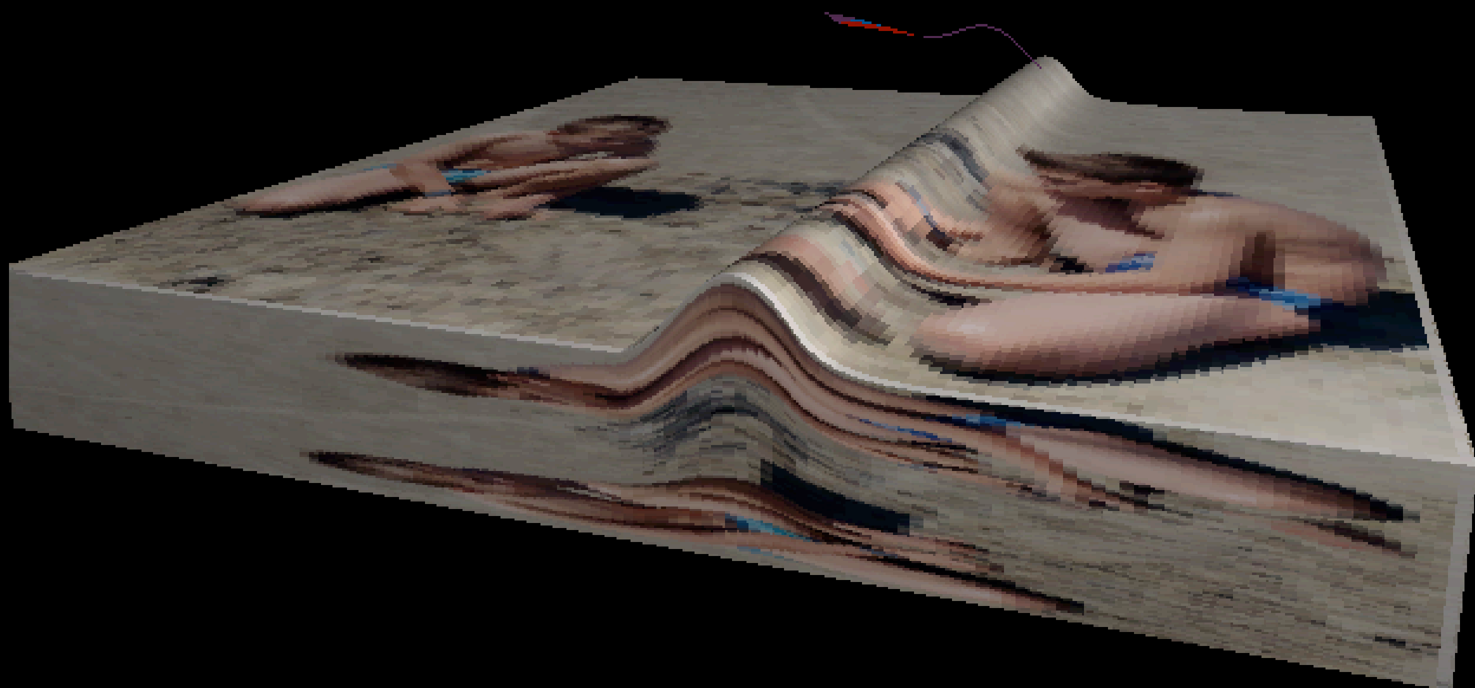
animation

Example

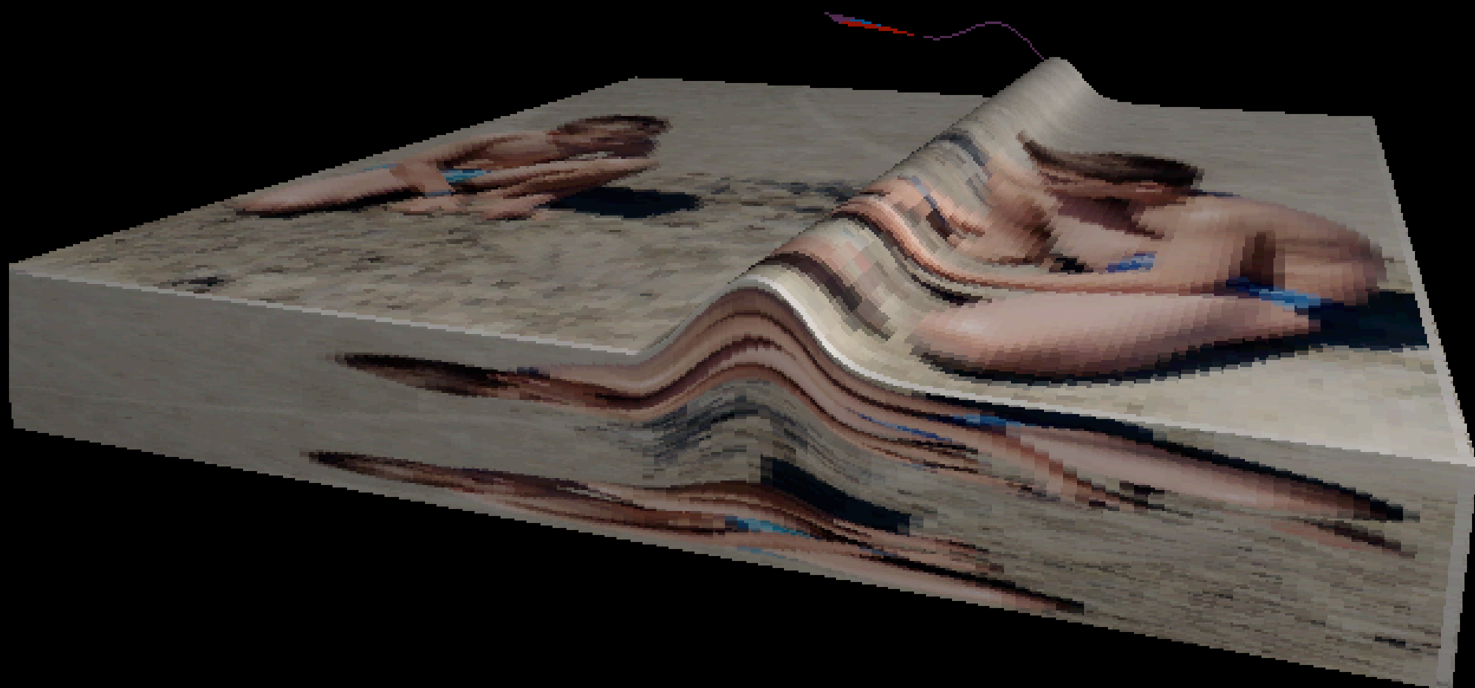


Rennes ... ...

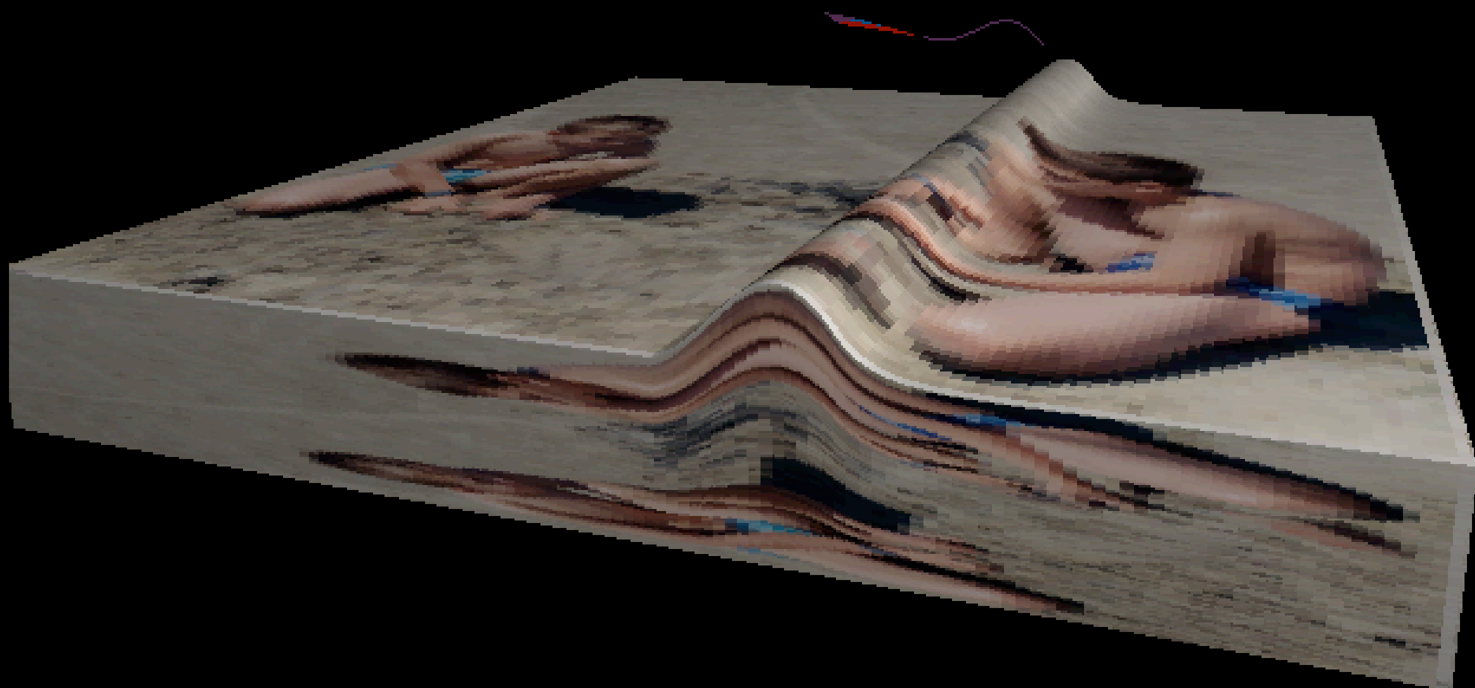




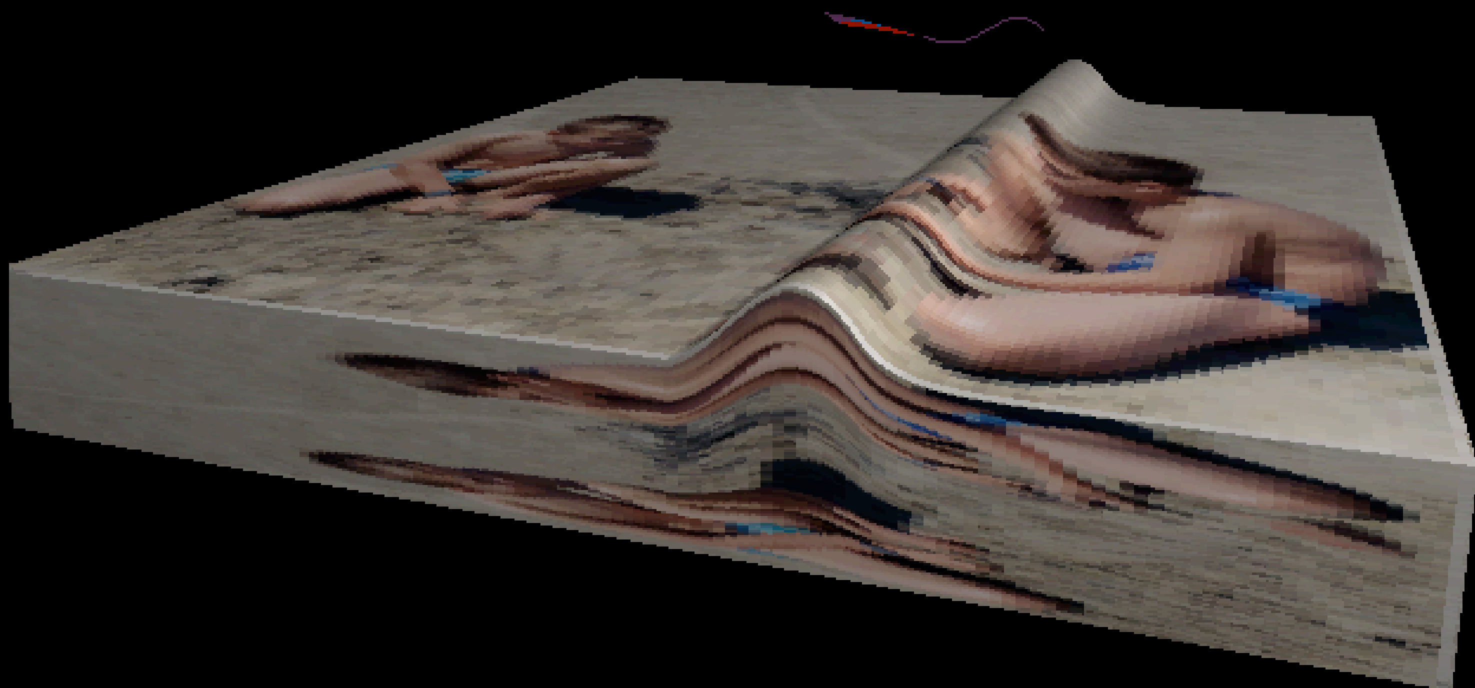
Rennes ... ...



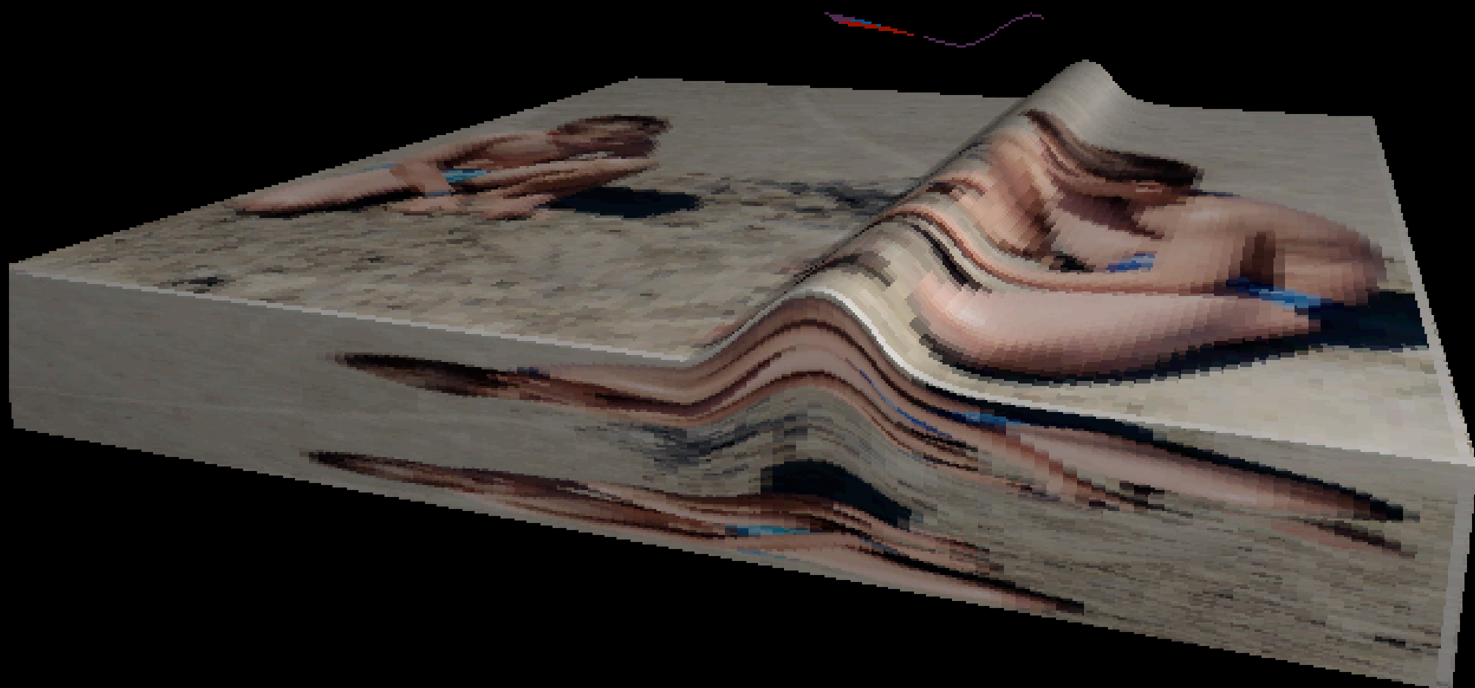
Rennes ... ...

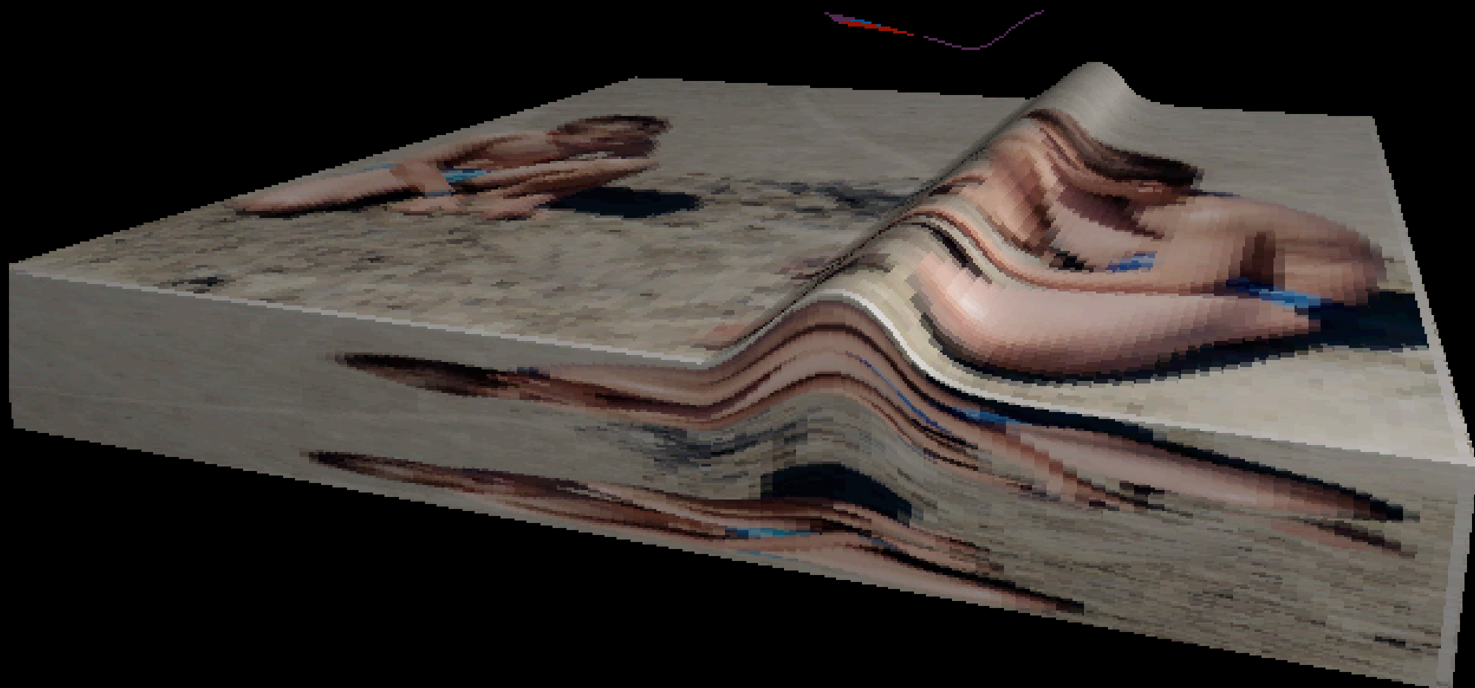


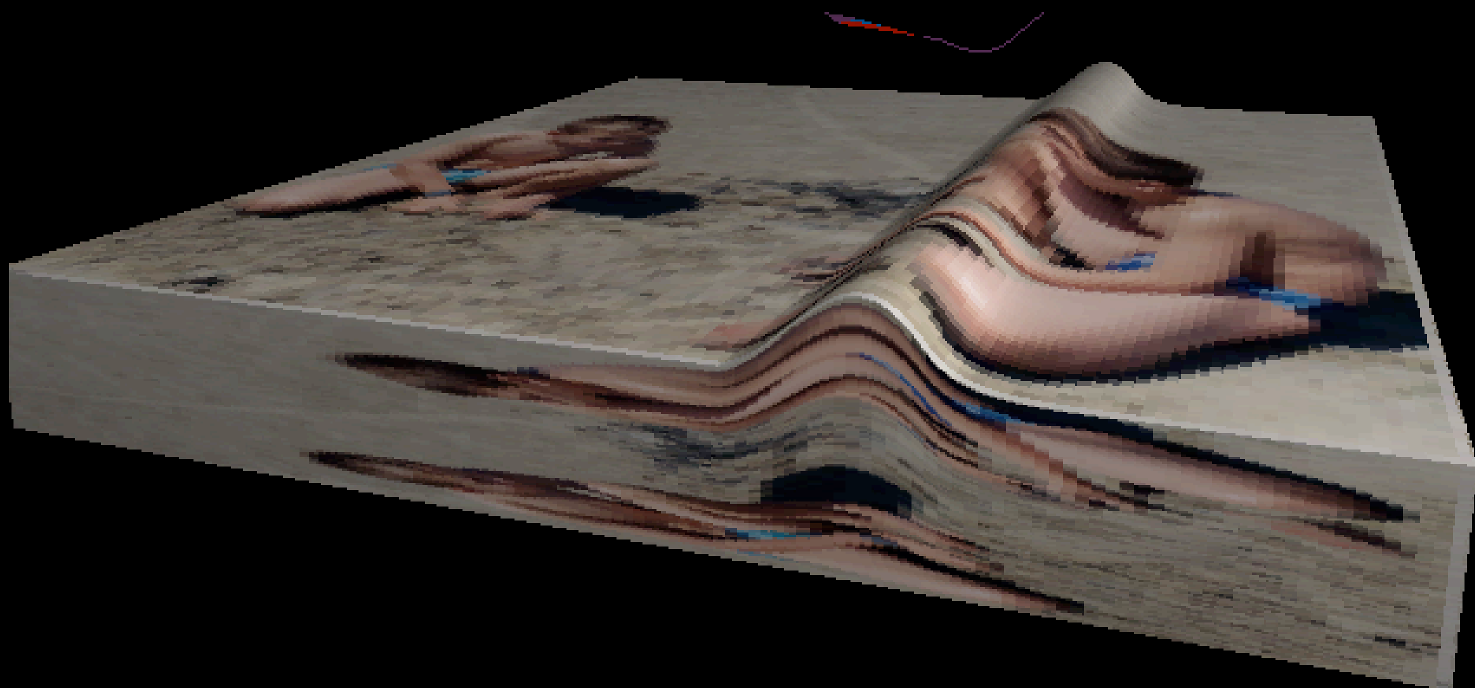
Rennes ... ...

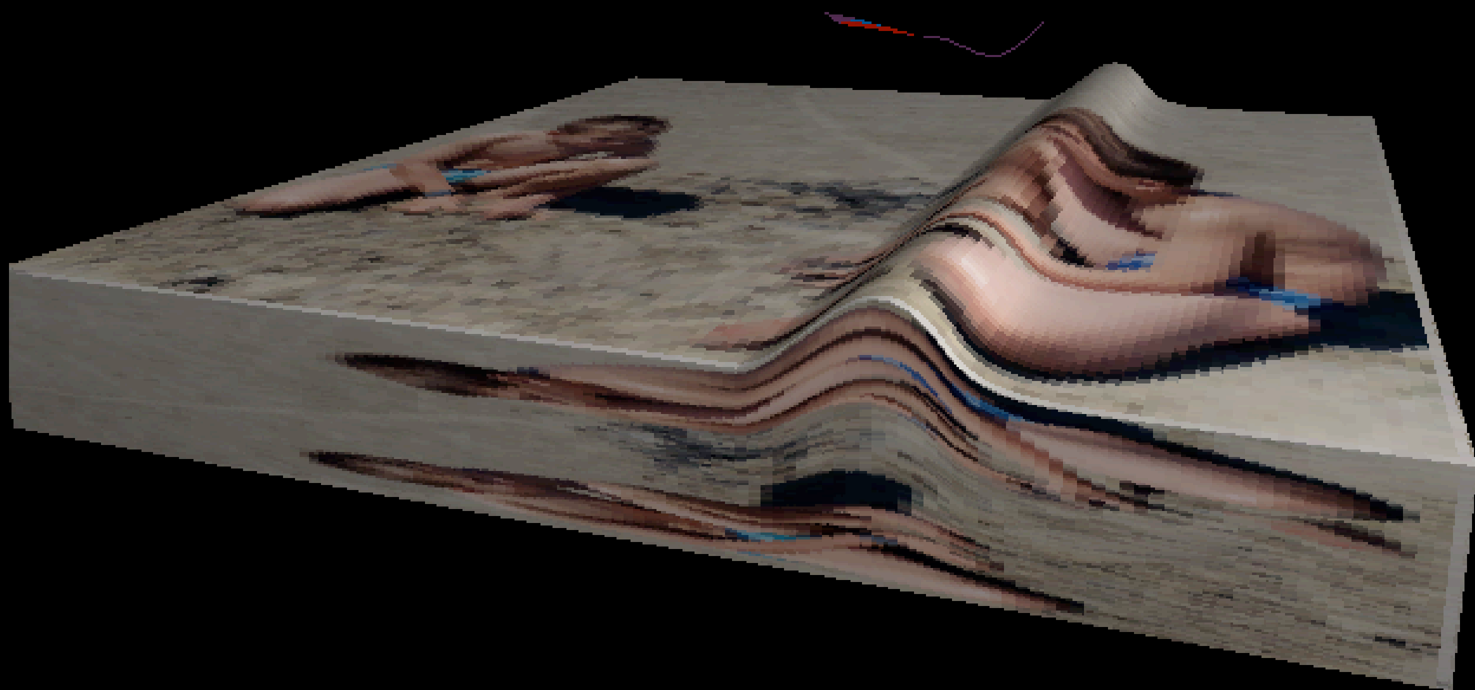


Rennes ... ...

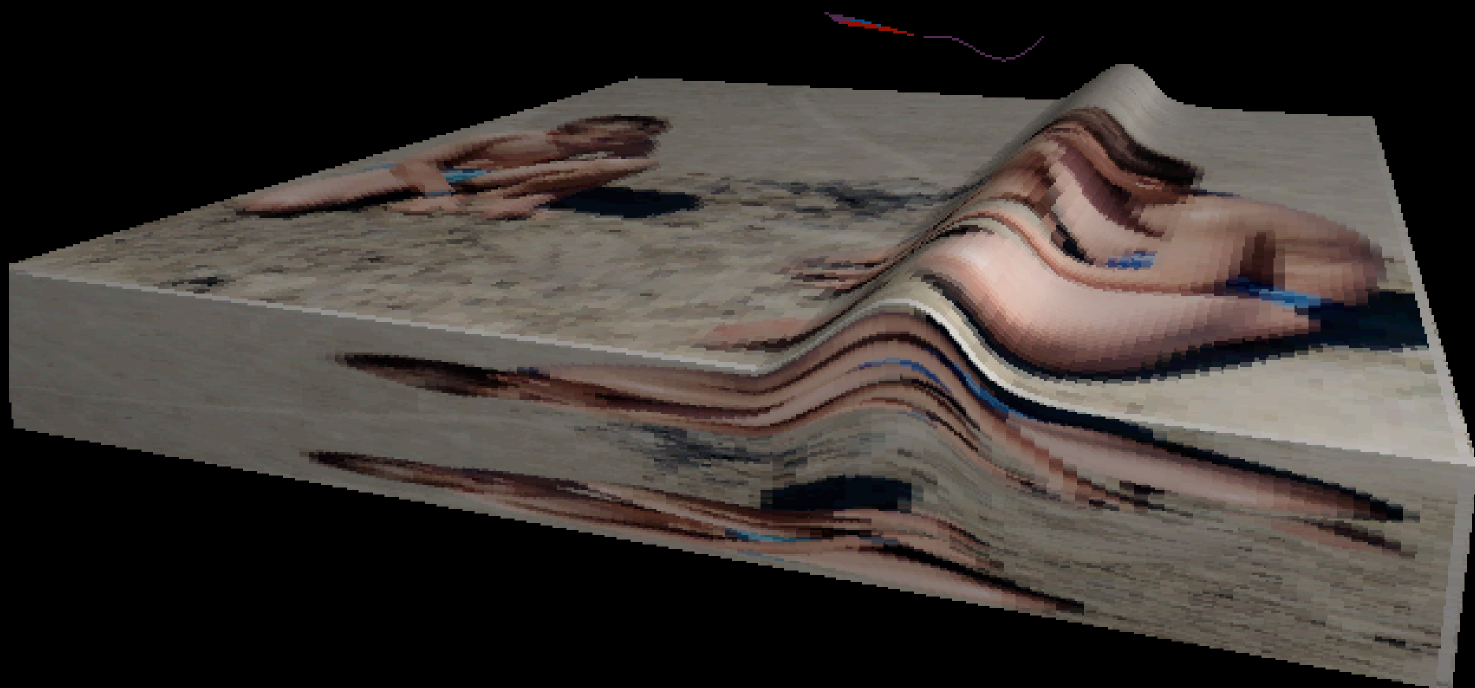


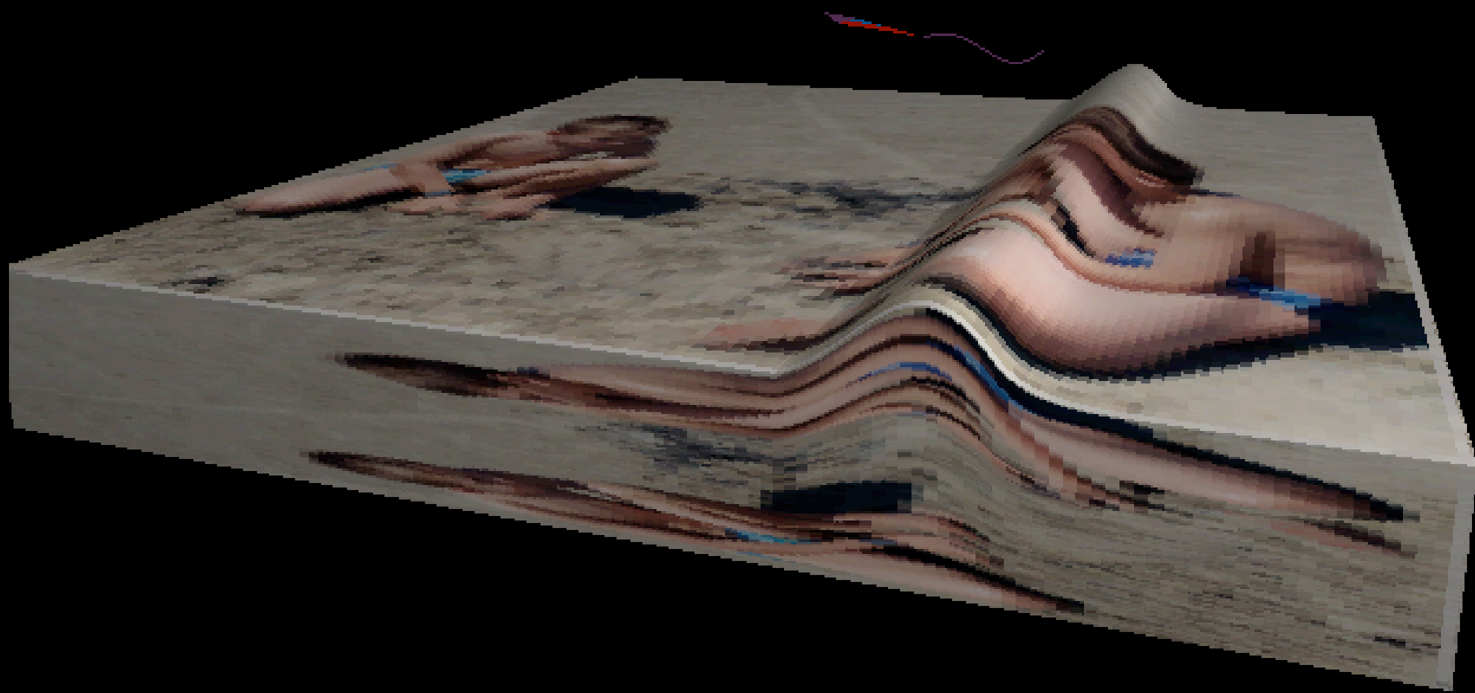




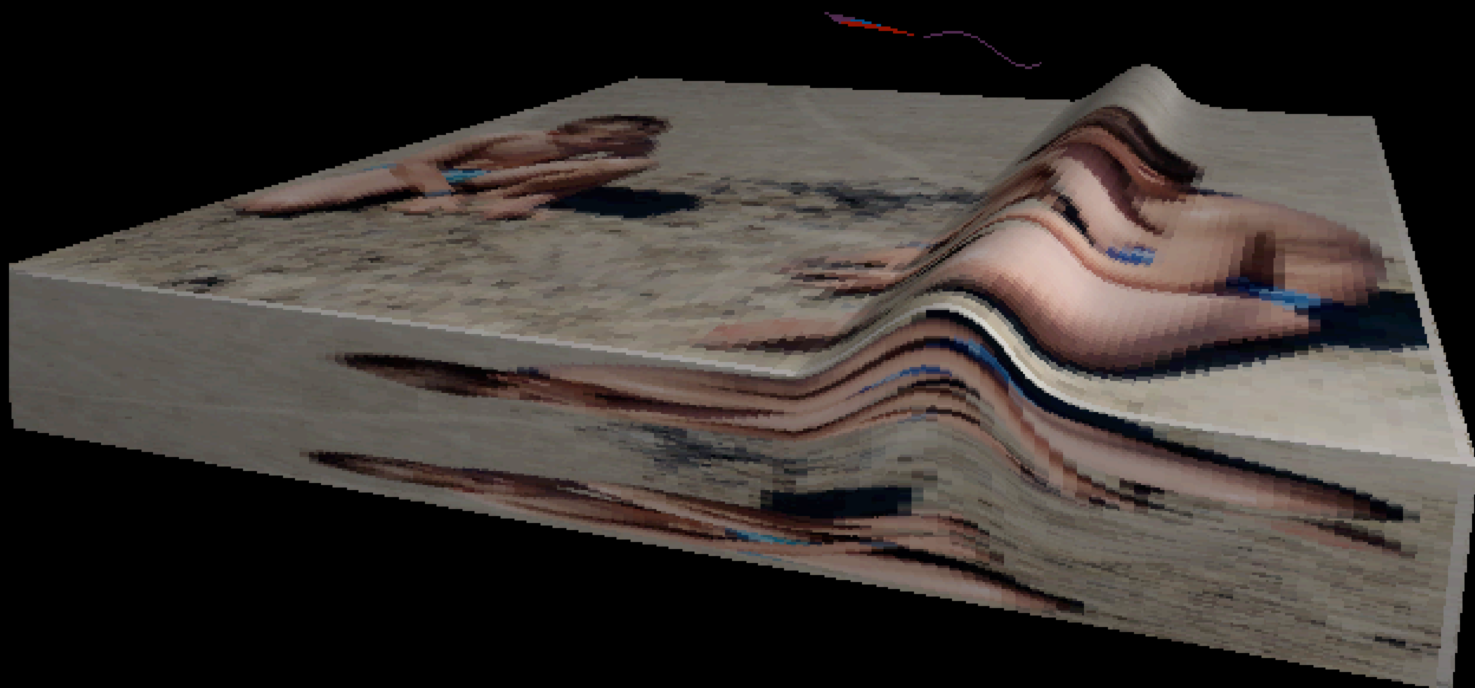


Rennes ... ...

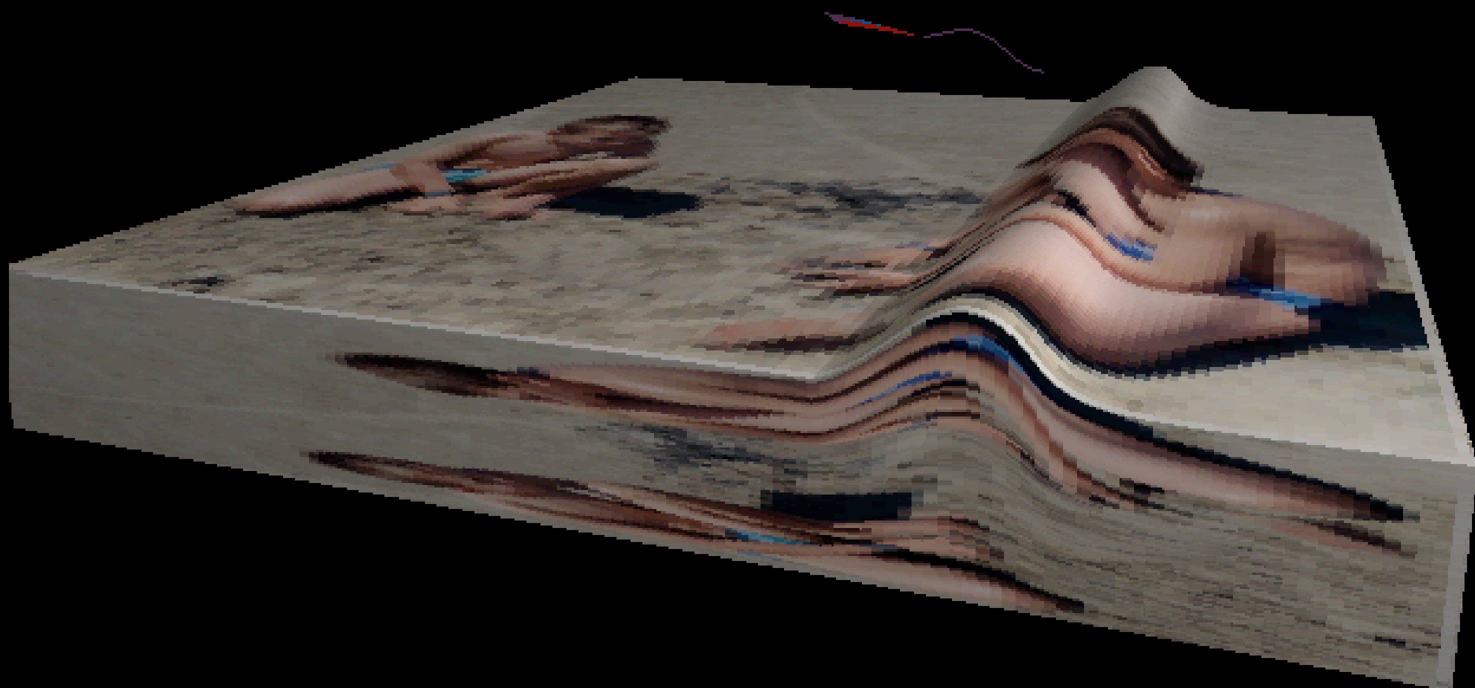




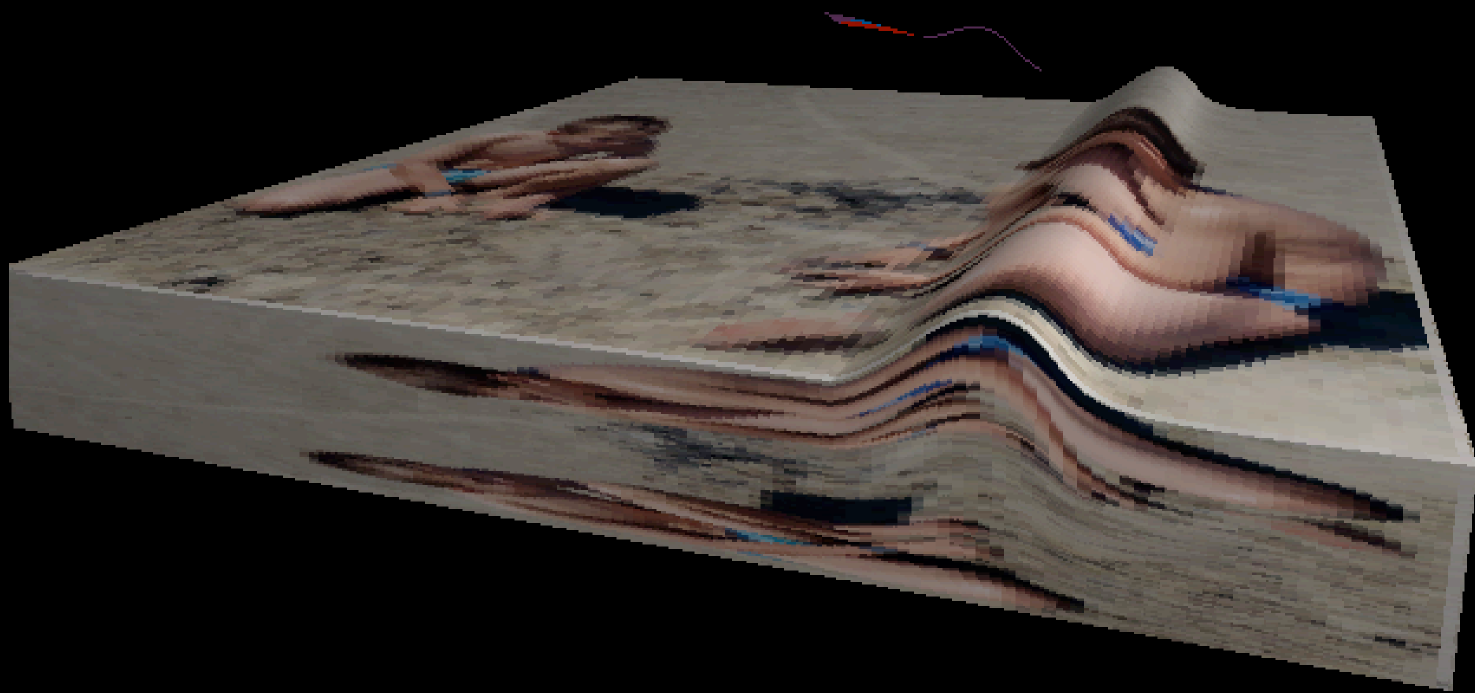
Rennes ... ...

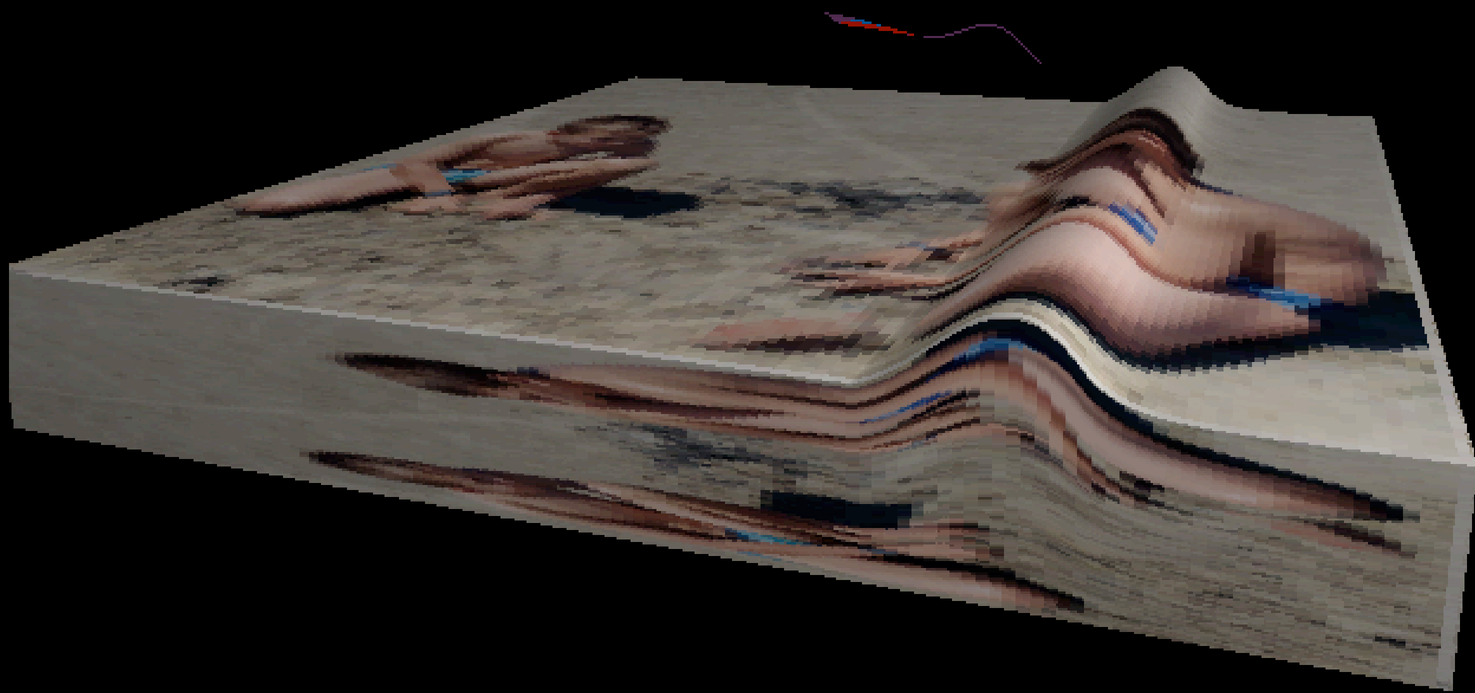


Rennes ... ...

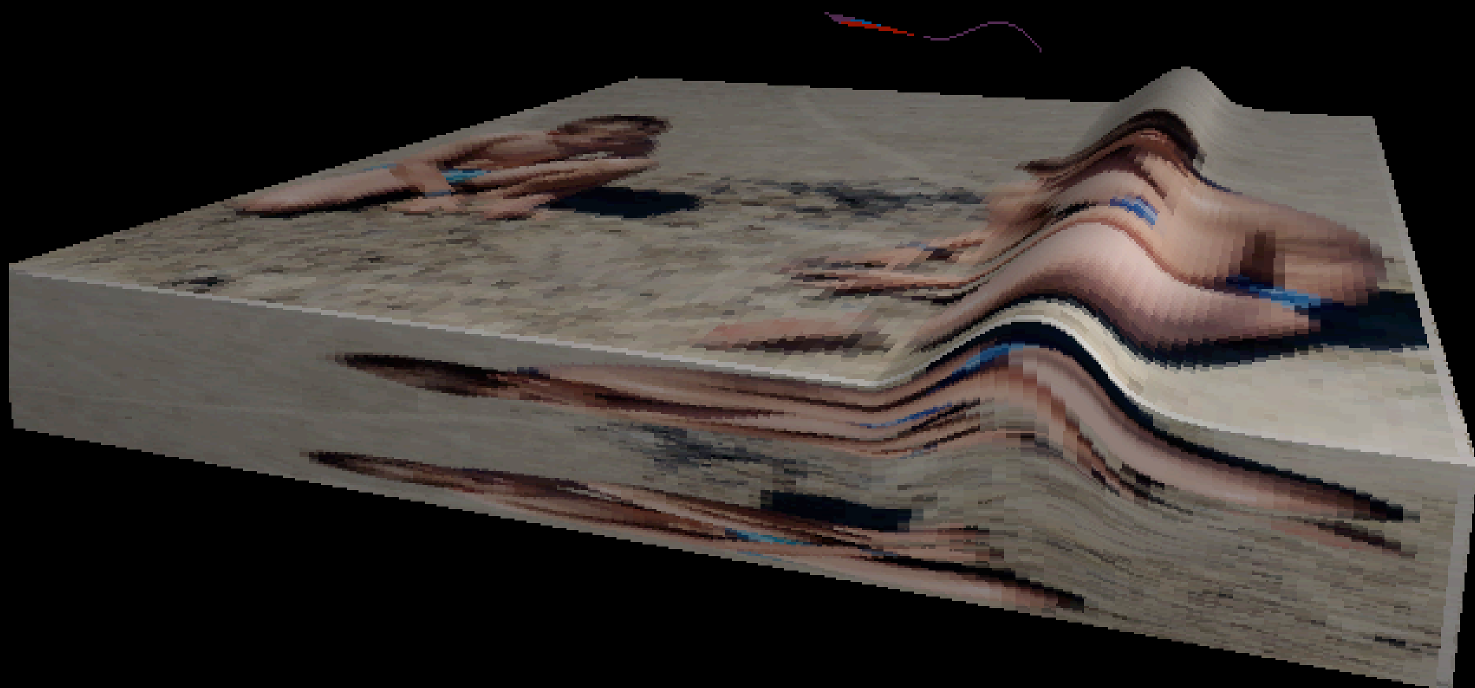


Rennes ... ...

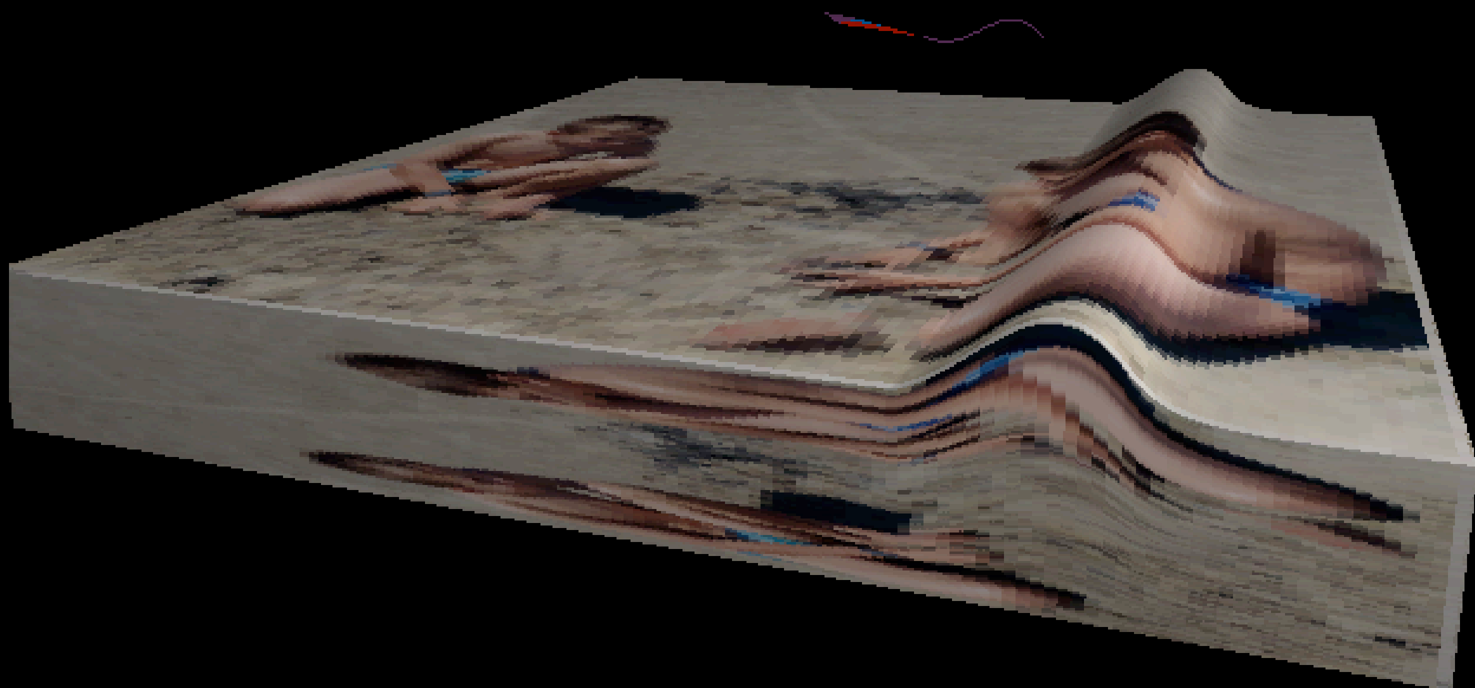




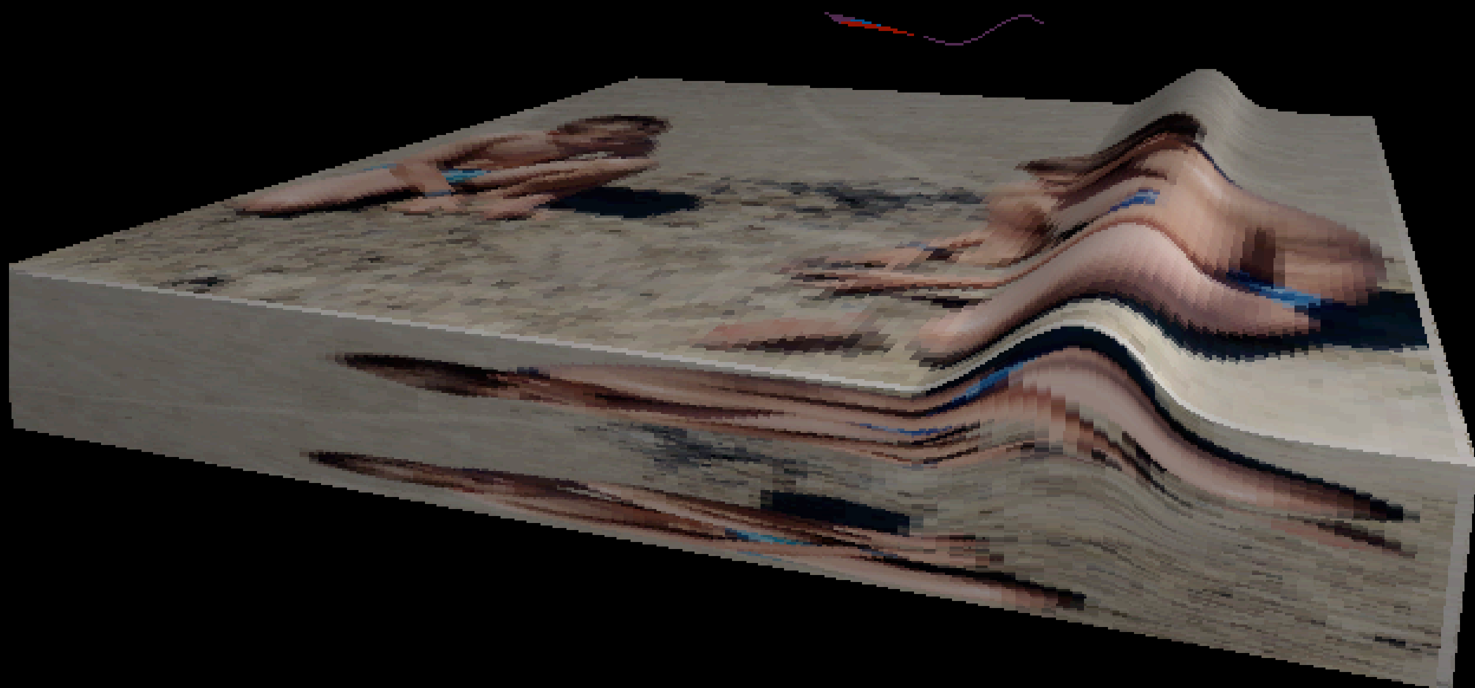
Rennes ... ...



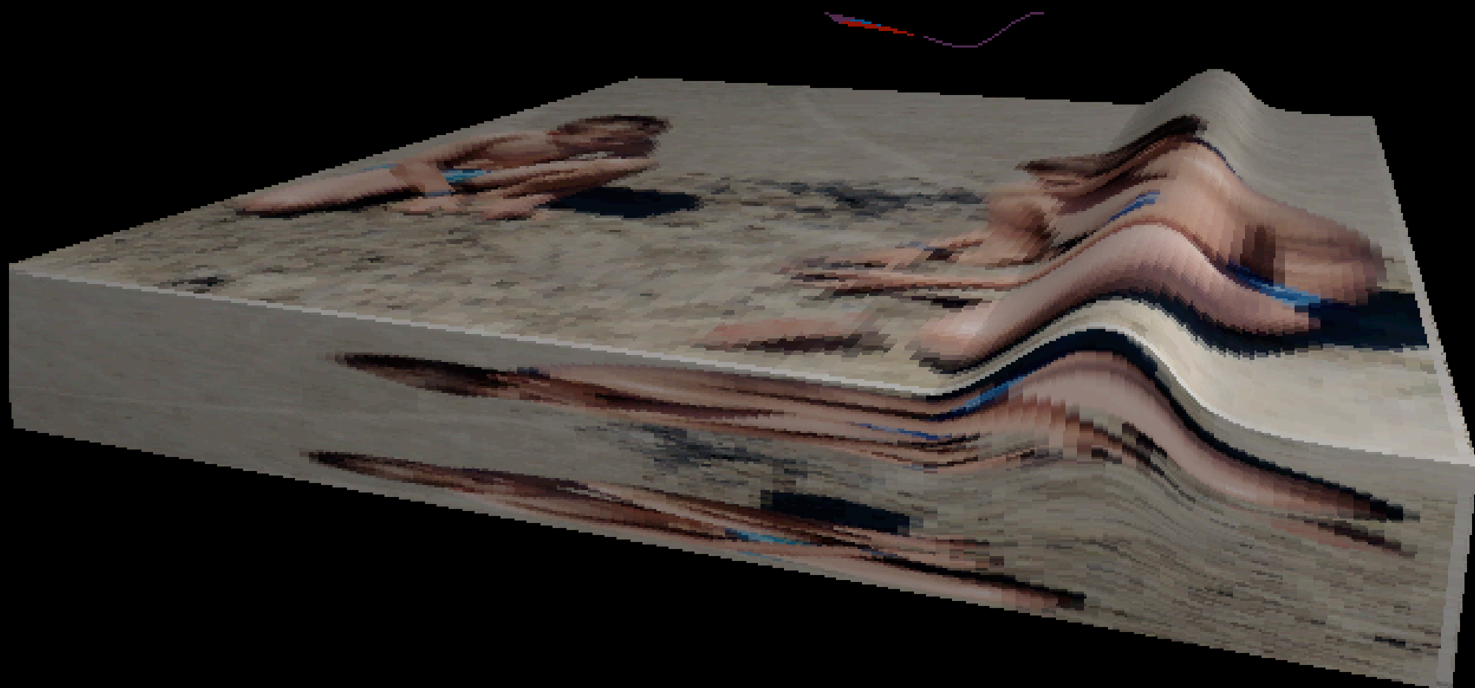
Rennes ... ...

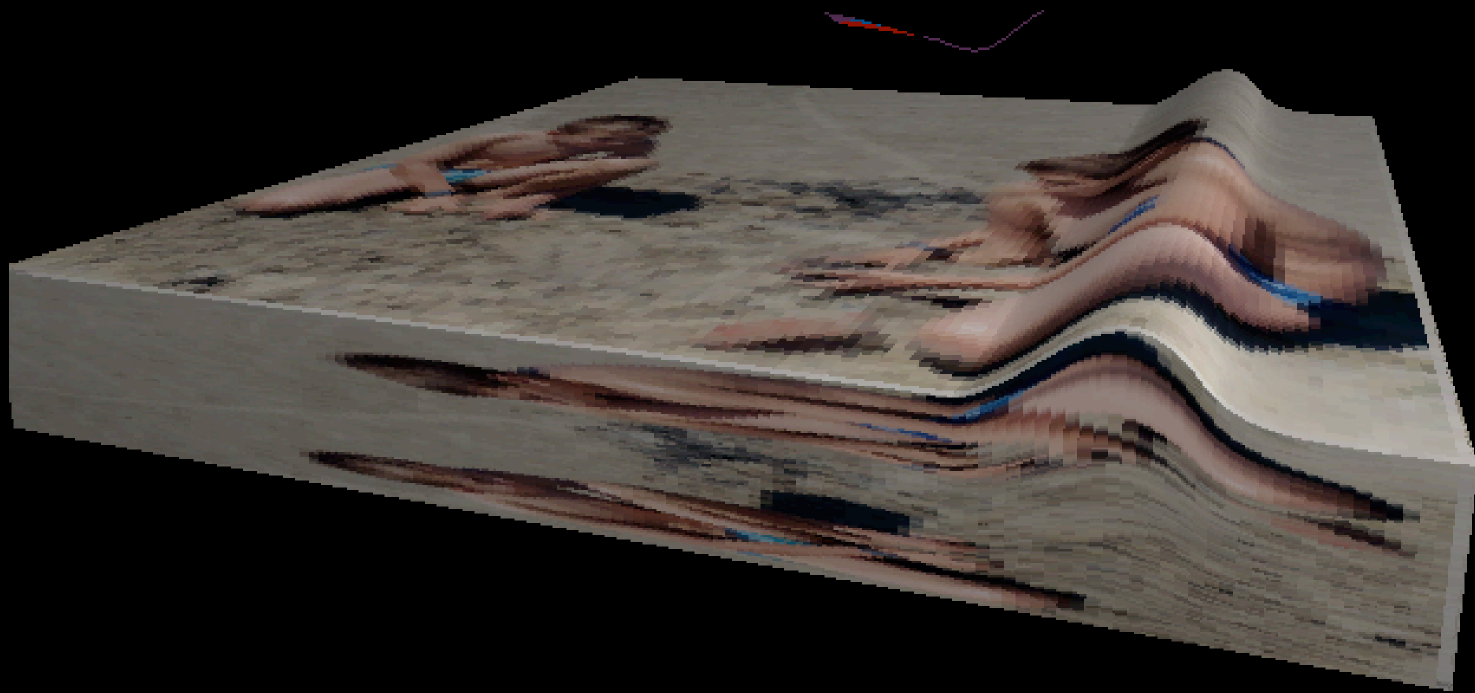


Rennes ... ...

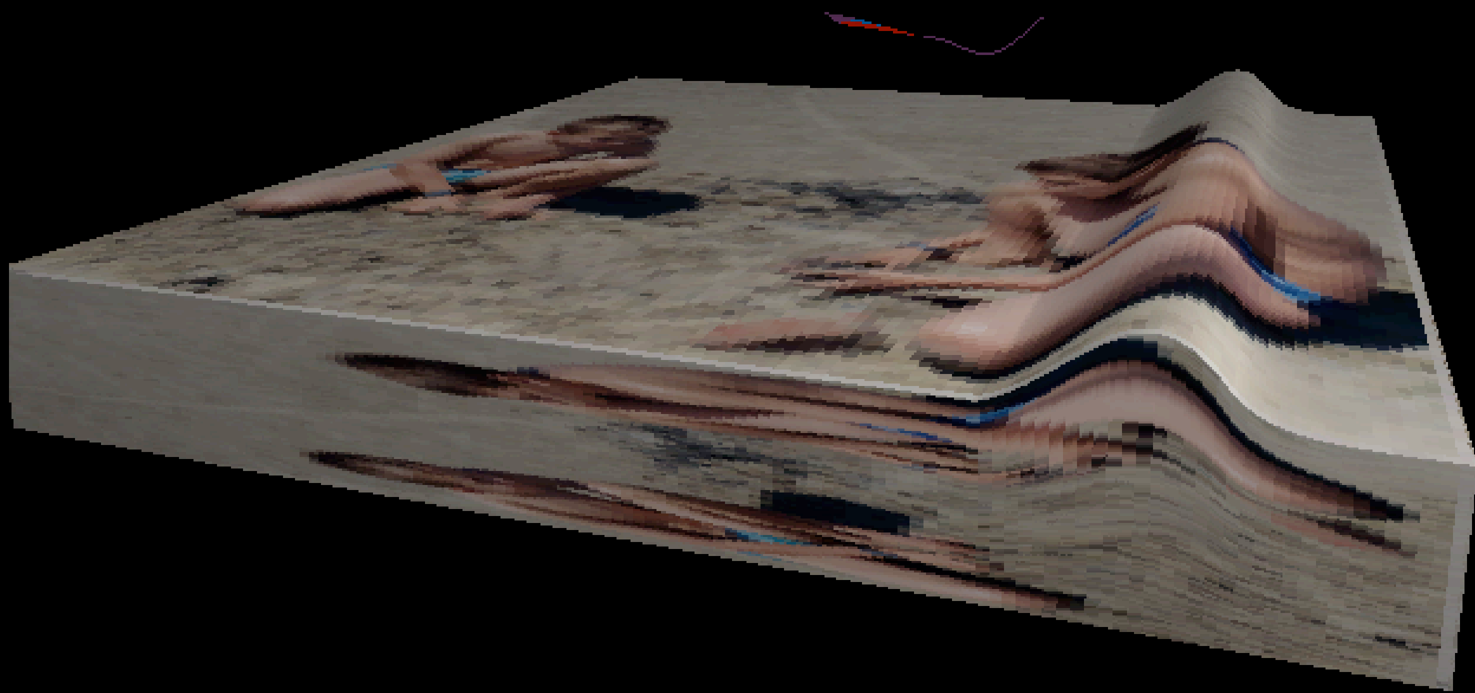


Rennes ... ...

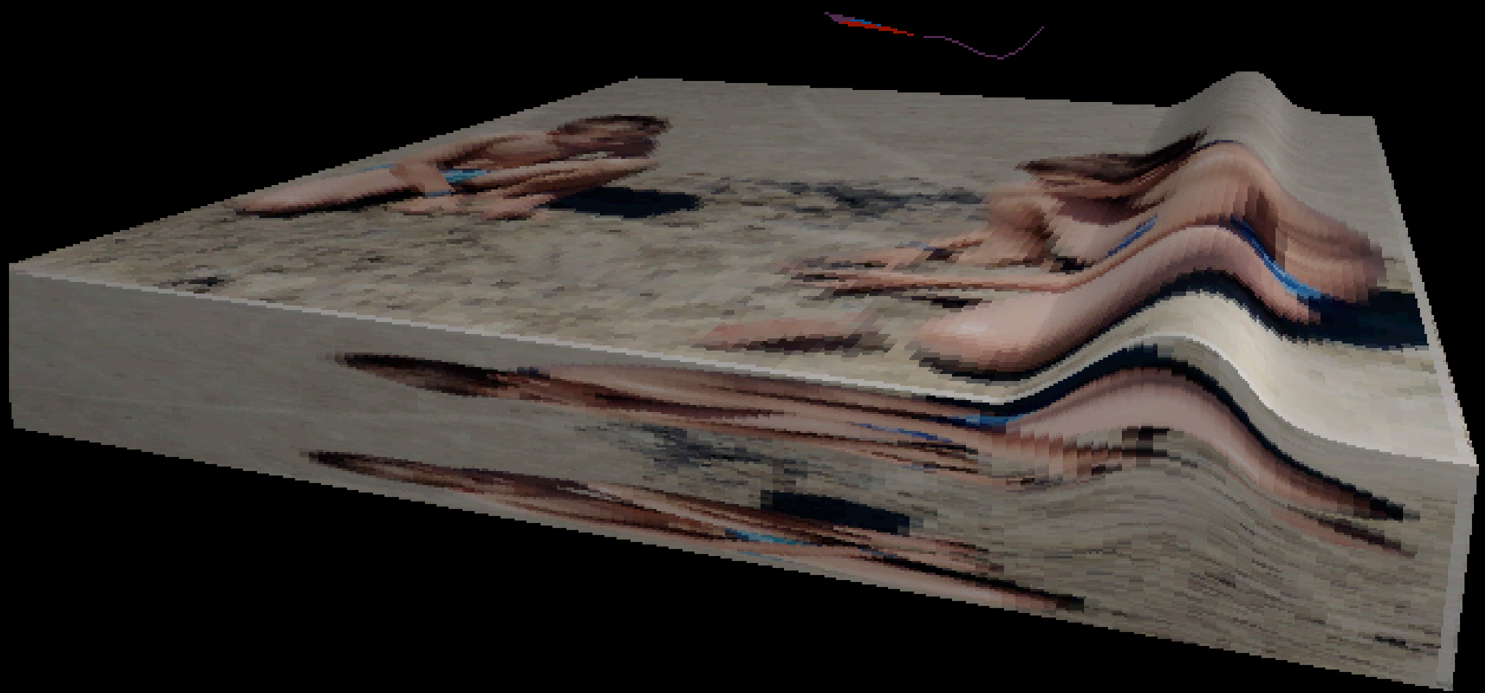




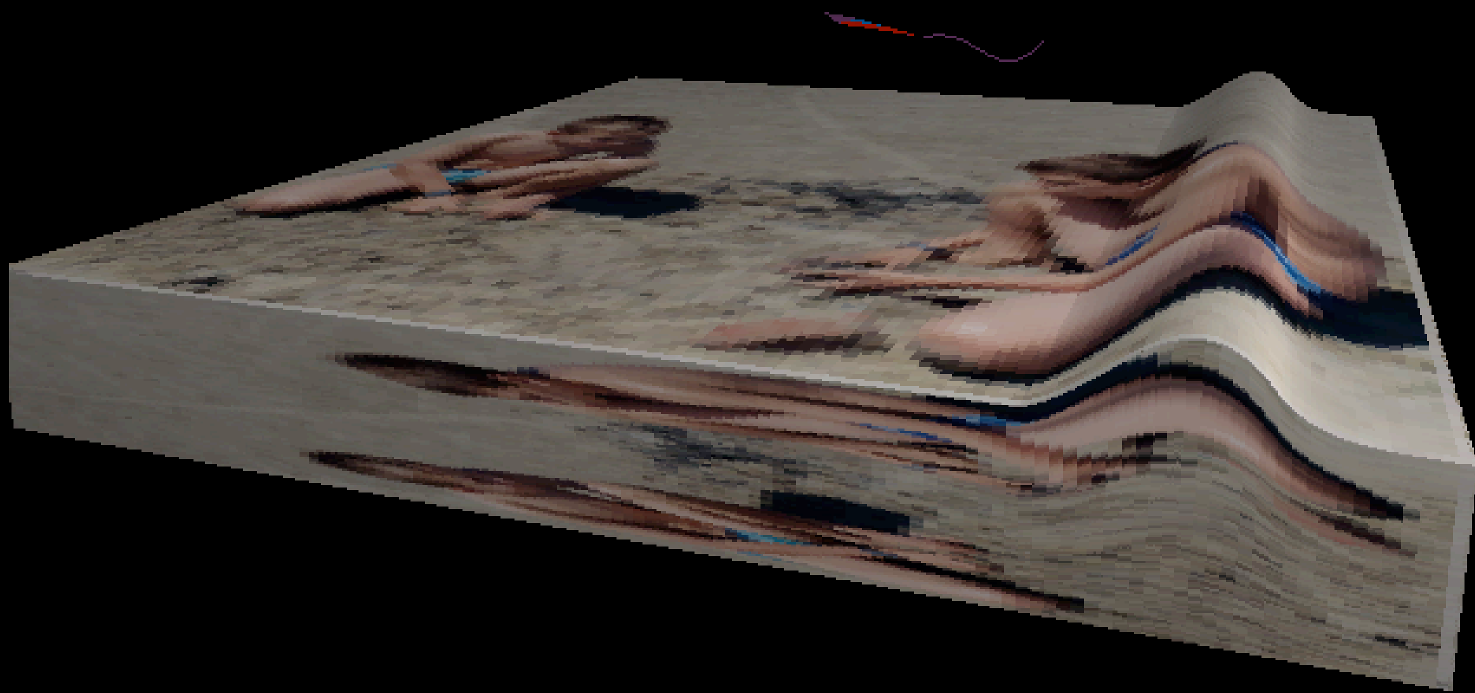
Rennes ... ...



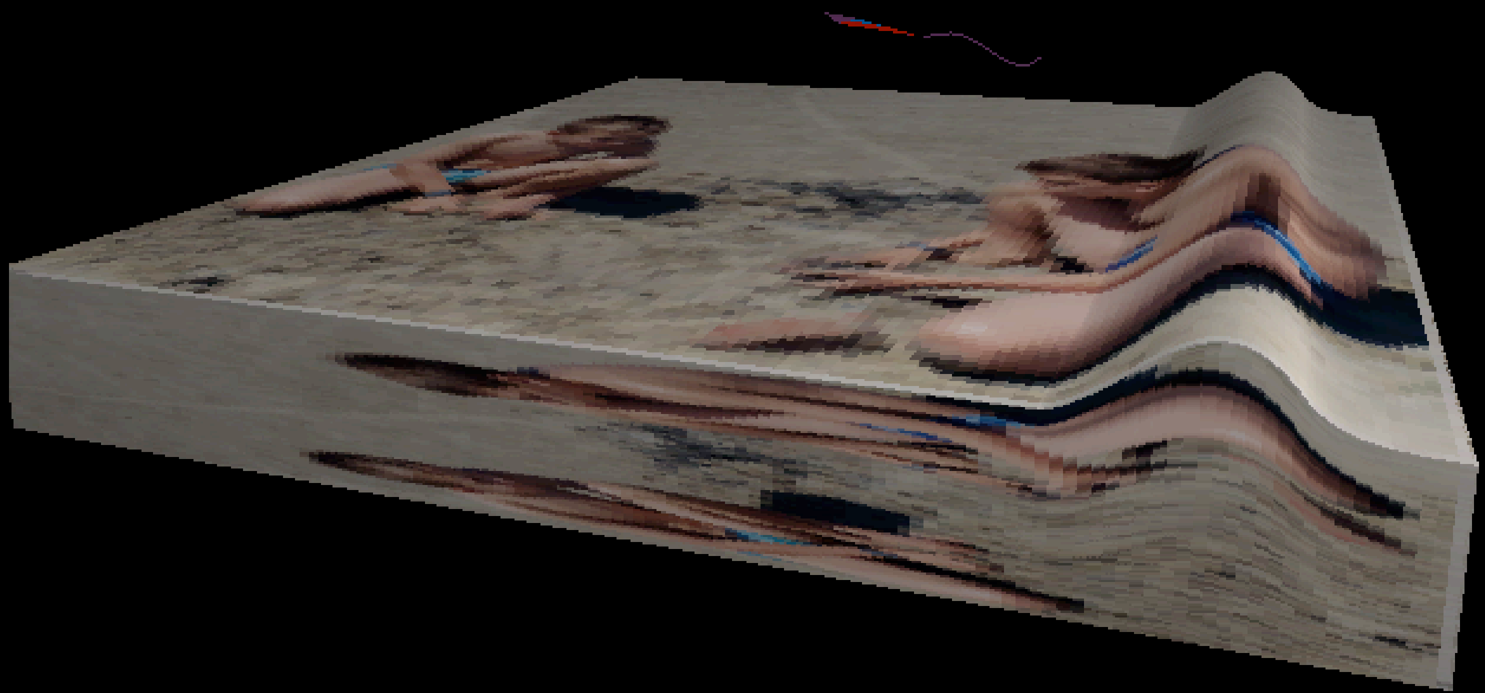
Rennes ... ...



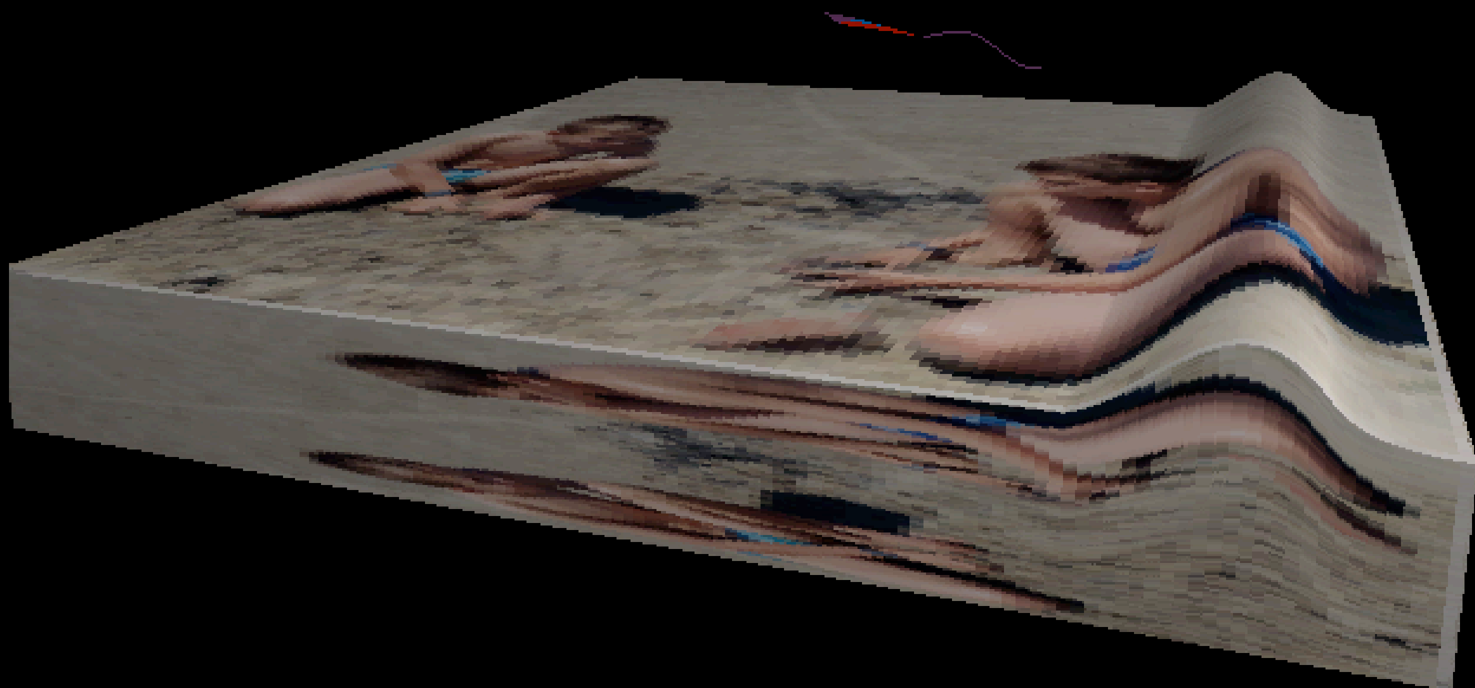
Rennes ... ...



Rennes ... ...



Rennes ... ...



Rennes ... ...

Self Similarity

rescaling $x = Lx^*$, we have $f = L^{1/3} f^*$ so that τ is invariant

$$\tau = L^{-1/3} L^{1/3} TF^{-1}[(3Ai(0))(-ik^*)^{1/3} TF[f^*]] = \tau^*$$

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$$q = q^*$$

$\int f dx = m$ so $L^{4/3} = m$ with $\int f^* dx^* = 1$

$$(\frac{l_s}{L})\frac{\partial q^*}{\partial x^*} + q^* = \varpi(\tau^* - \tau_s)$$

Self Similarity

rescaling $x = Lx^*$, we have $f = L^{1/3} f^*$ so that τ is invariant

$$\tau = L^{-1/3} L^{1/3} TF^{-1}[(3Ai(0))(-ik^*)^{1/3} TF[f^*]] = \tau^*$$

$$q = q^*$$

$\int f dx = m$ so $L^{4/3} = m$ with $\int f^* dx^* = 1$

$$(\frac{l_s}{L})\frac{\partial q^*}{\partial x^*} + q^* = \varpi(\tau^* - \tau_s)$$

$$\frac{\partial f^*}{\partial t^*} = -\frac{\partial q^*}{\partial x^*}$$

$t = L^{4/3} t^*$ and $c = L^{-1/3} c^*$ so $c = m^{-1/4} c^*$

Self Similarity

rescaling $x = Lx^*$, we have $f = L^{1/3} f^*$ so that τ is invariant

$$\tau = L^{-1/3} L^{1/3} TF^{-1}[(3Ai(0))(-ik^*)^{1/3} TF[f^*]] = \tau^*$$

$$q = q^*$$

$\int f dx = m$ so $L^{4/3} = m$ with $\int f^* dx^* = 1$

$$(\frac{l_s}{L})\frac{\partial q^*}{\partial x^*} + q^* = \varpi(\tau^* - \tau_s)$$

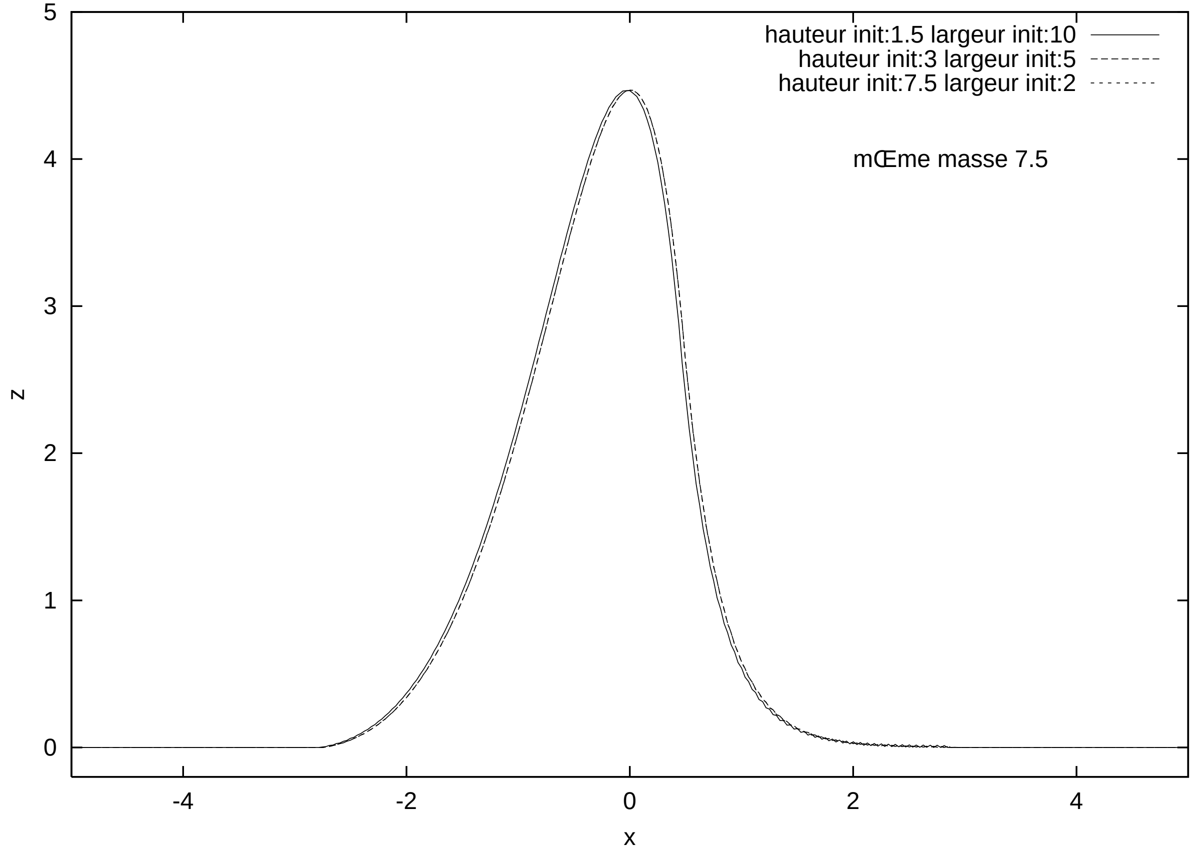
$$\frac{\partial f^*}{\partial t^*} = -\frac{\partial q^*}{\partial x^*}$$

$t = L^{4/3} t^*$ and $c = L^{-1/3} c^*$ so $c = m^{-1/4} c^*$

$1/c$ proportional to $m^{1/4}$ and function $l_s^{-1} m^{3/4}$

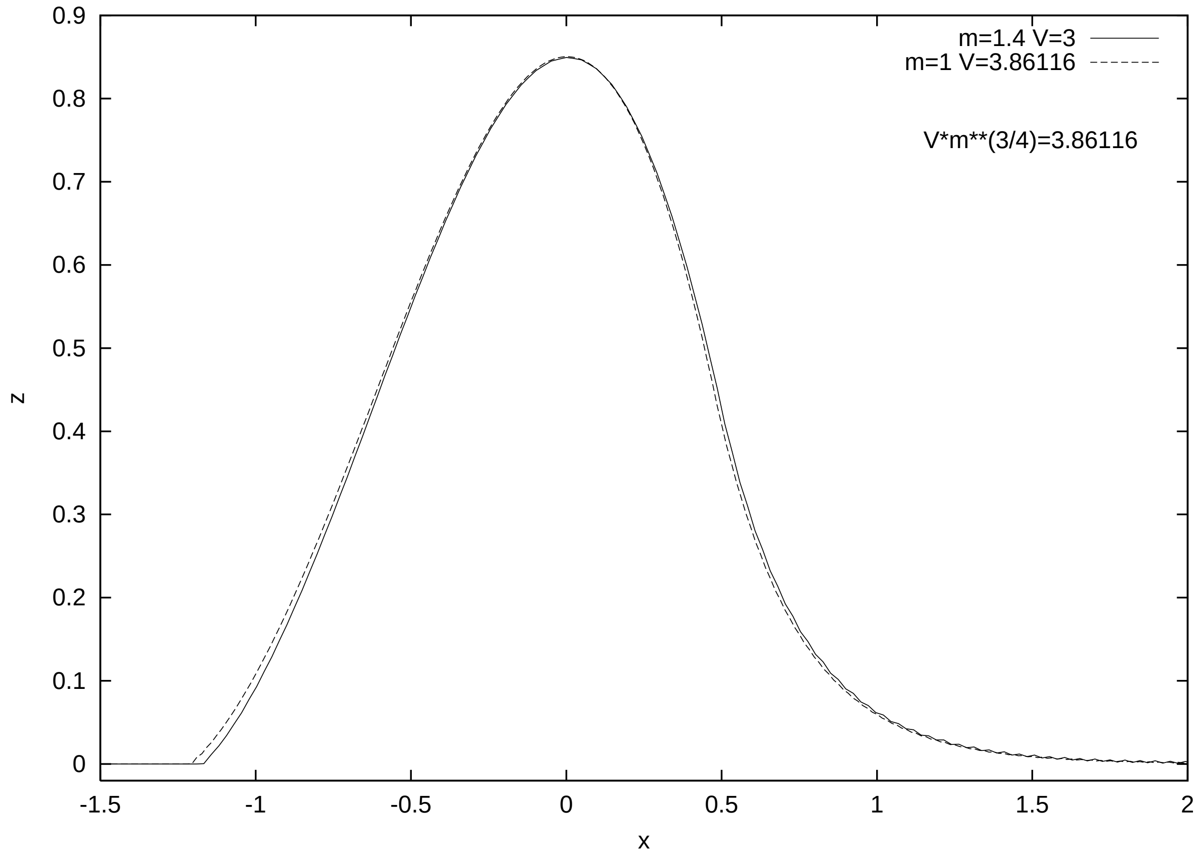
Self Similarity

Self Similarity



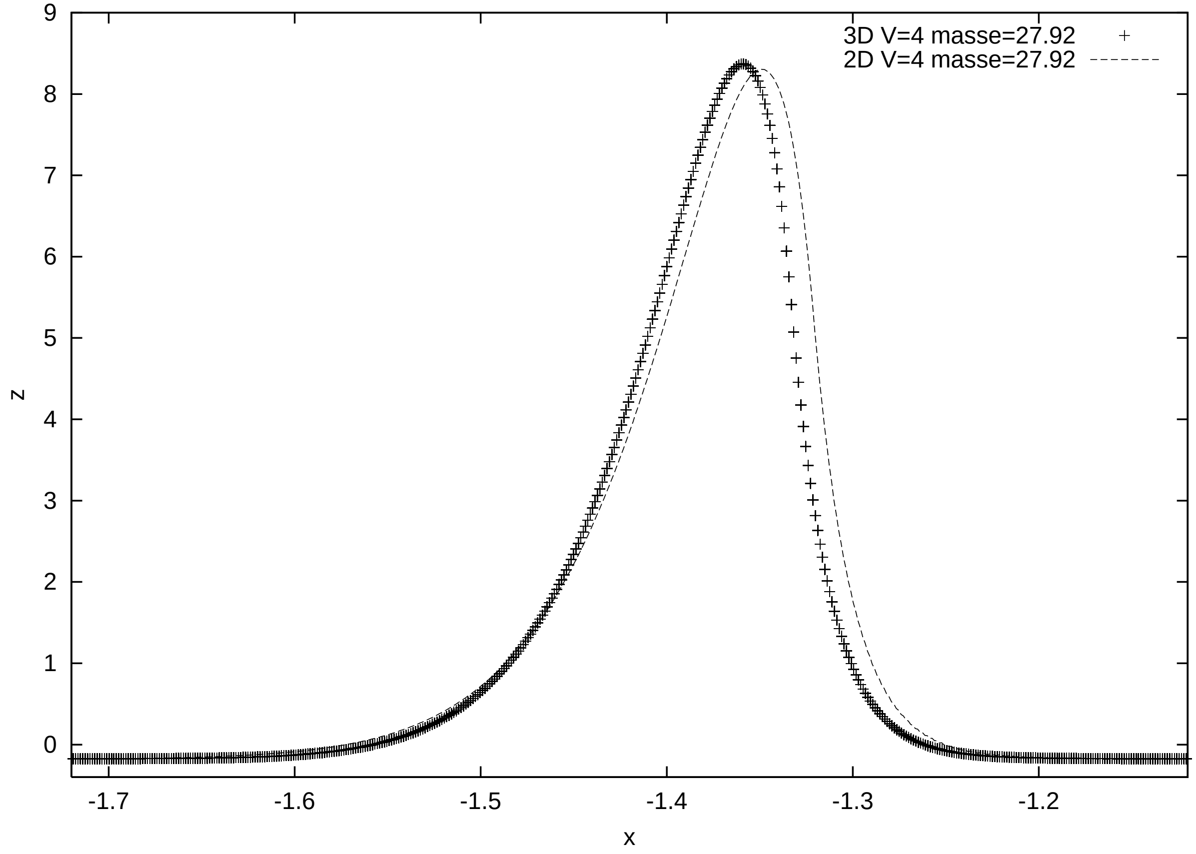
two different initial bumps of same m lead to the same final state

Self Similarity



two cases of same $l_s^{-1} m^{3/4}$.

Self Similarity



comparing the 2D non erodible code to the 3D code (in 2D!)

Self Similarity

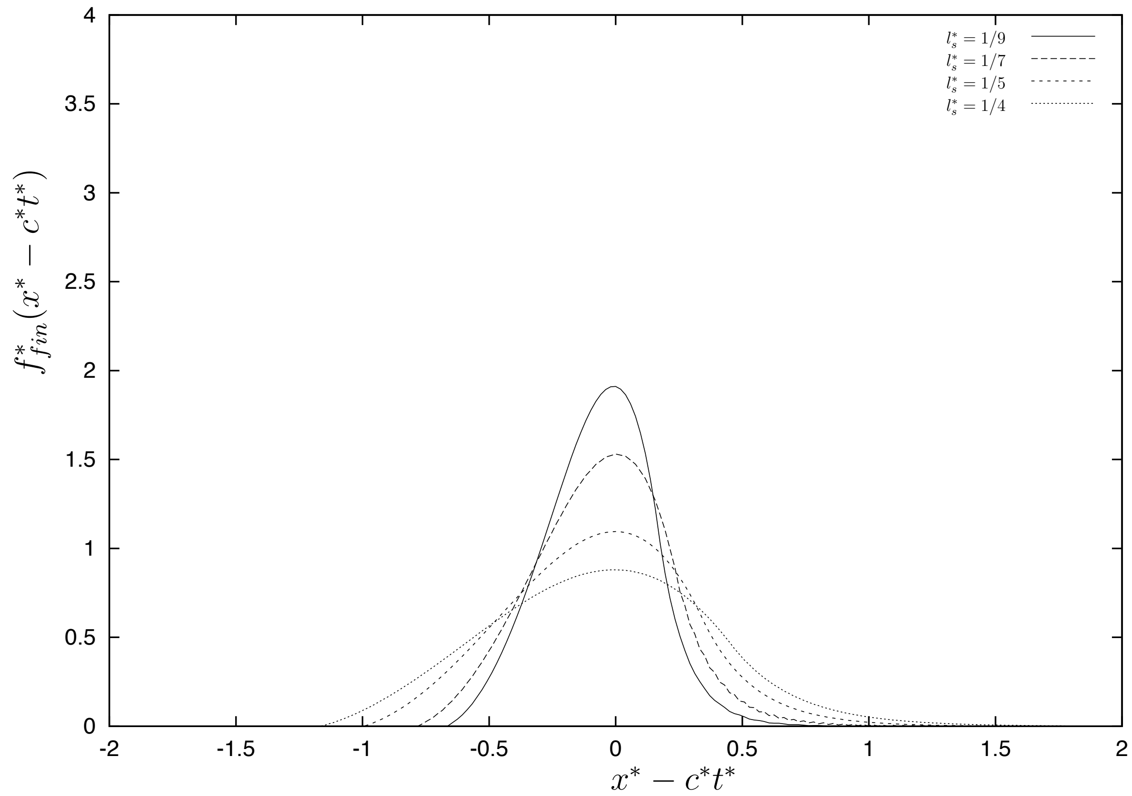


Fig. 8. "Dunes" of unit mass with $l_s^* = 1/4, 1/5, 1/7, 1/9$ ($\tau_s = 0.9$). The smaller l_s^* is, the thinner and higher the "dune" is.

selfsimilarity, unit mass $m = 1$, different $l_s^{-1} m^{3/4}$.

Self Similarity

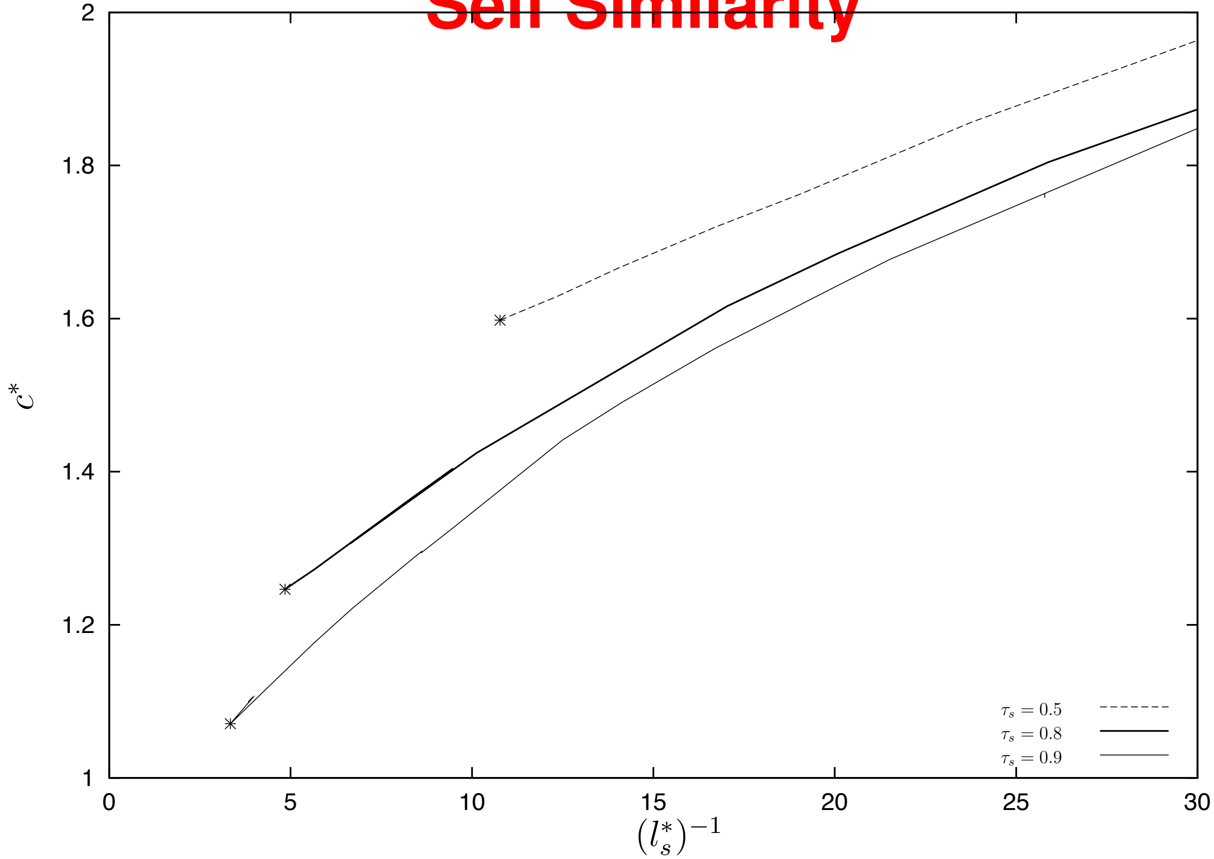


Fig. 7. The selfsimilar relation between the mass m, the inverse of the saturation length $l_s^* = l_s m^{-3/4}$, and the velocity $c^* = cm^{1/4}$ of the "dune" for three values of the threshold: $\tau_s = 0.9$, 0.8, and 0.5. For a fixed threshold, there is a maximal value of the saturation length l_s^* over which there is no solution.

$c\,m^{1/4}$ as function of $l_s^{-1} m^{3/4}$.

Linear / Non linear comparison

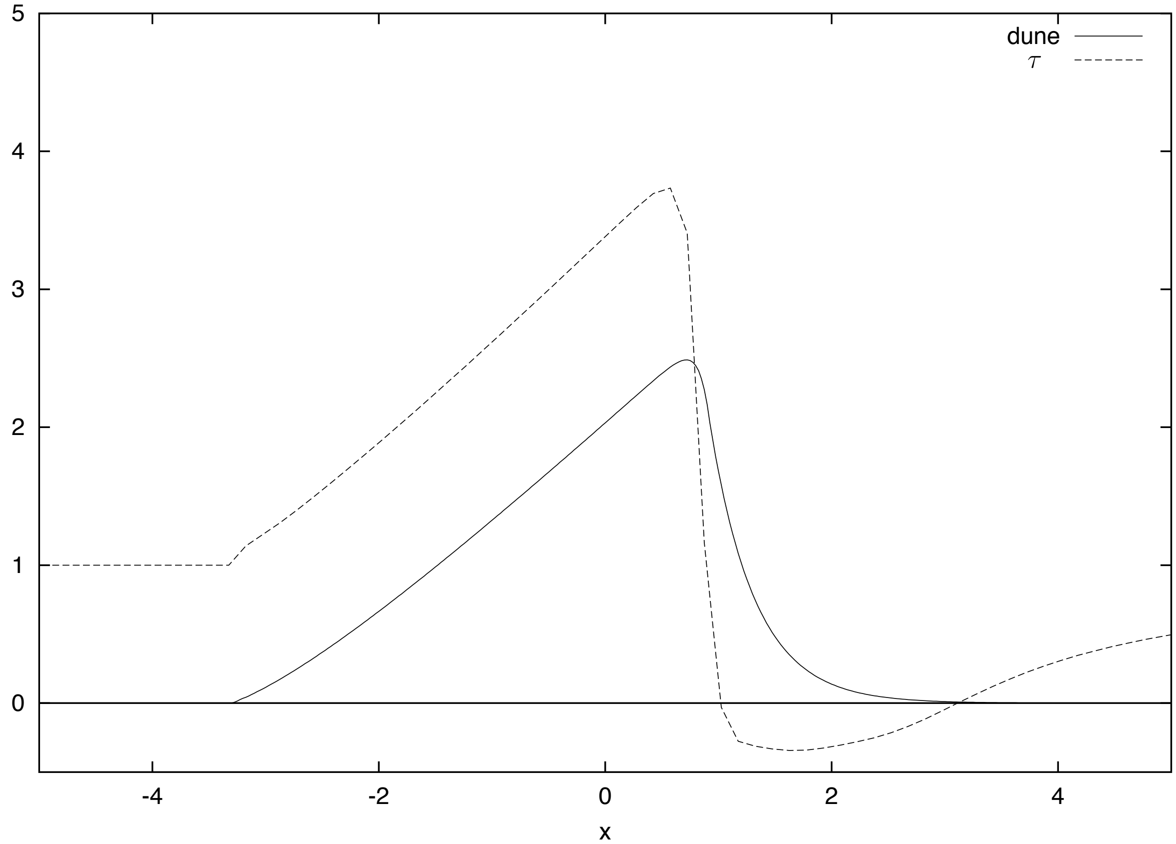


Fig. 6. An example of a non-linear final moving "dune" solution ($\tau_s = 0.9$, $1/l_s = 2.5$, $m = 6$). The weather side is nearly flat. The skin friction is represented; it is negative in the lee side: there is boundary layer separation.

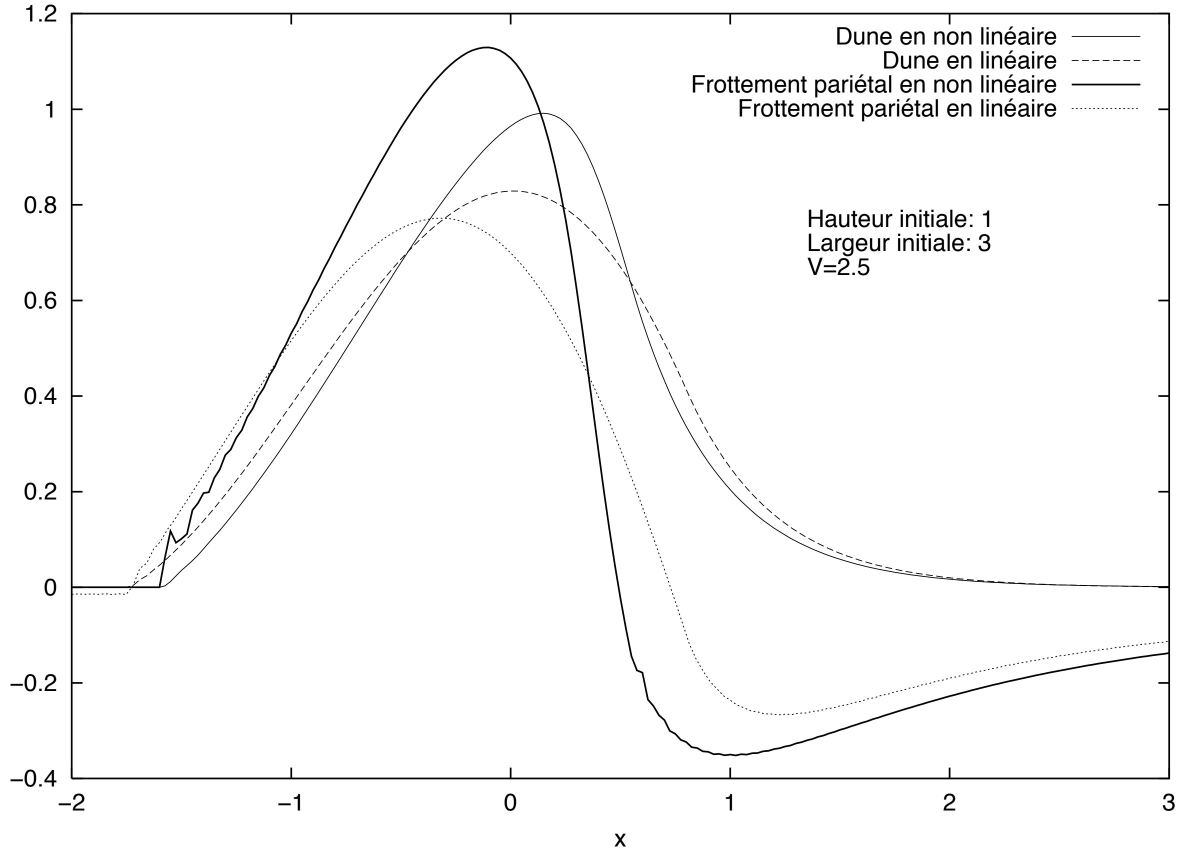


Abbildung 1: Comparing the skin fricition perturbation $(\tau - 1)$ and the "dunes" in the linear and non linear cases, here with separation

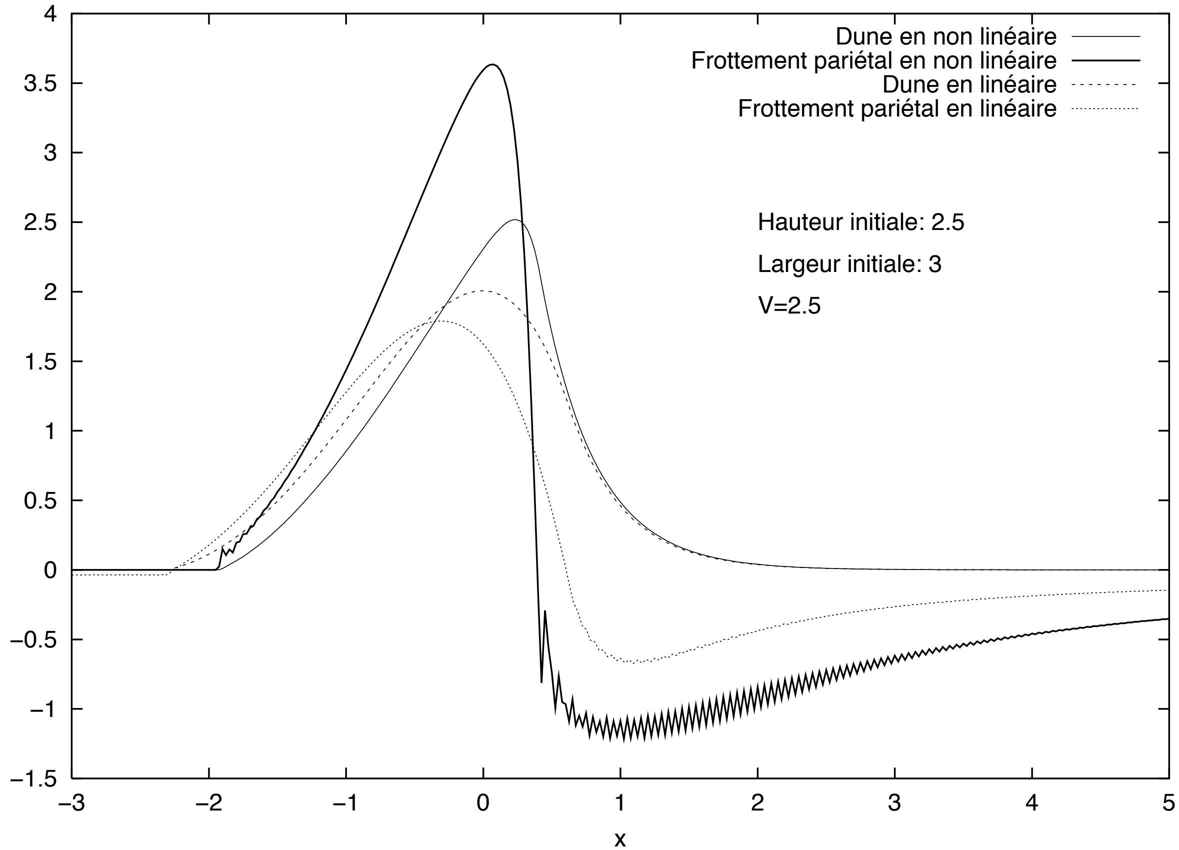


Abbildung 2: Comparing the skin fricition perturbation $(\tau - 1)$ and the ”dunes” in the linear and non linear cases , **animation**

final shapes lin/ non lin

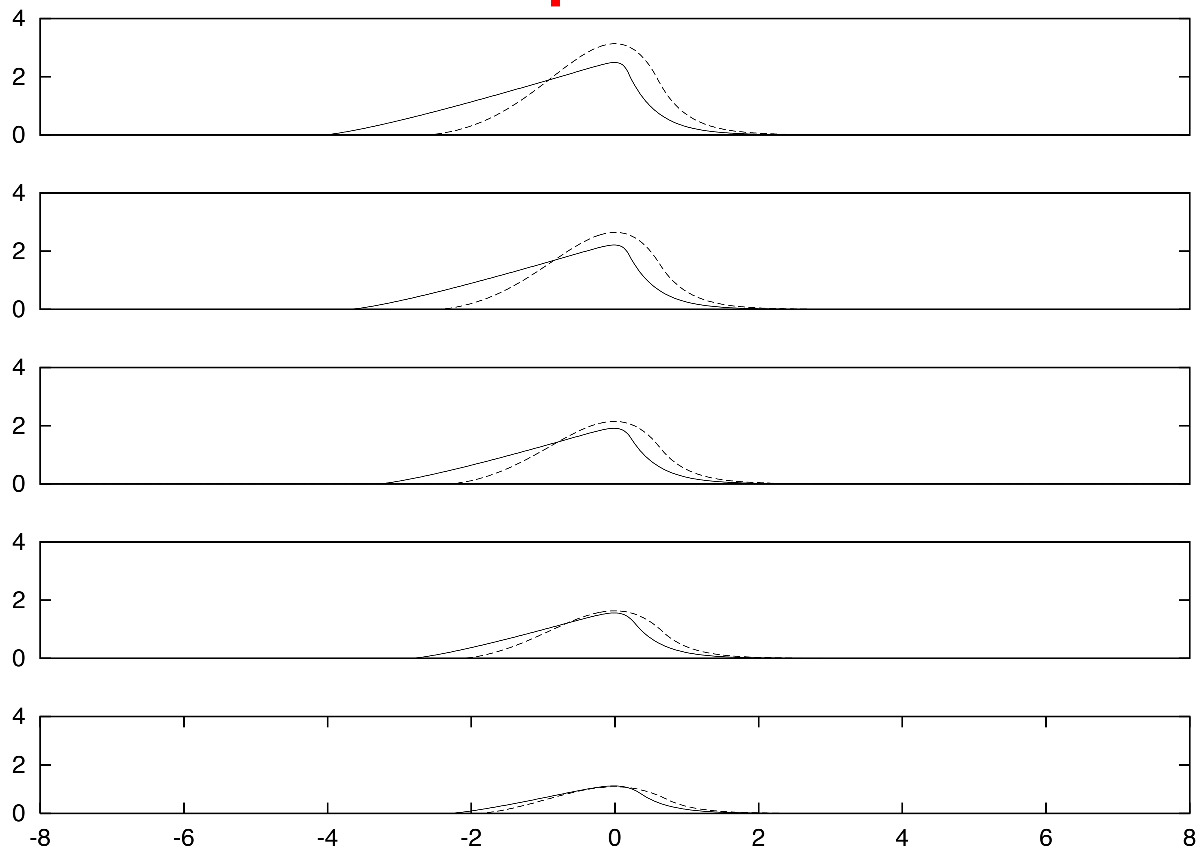
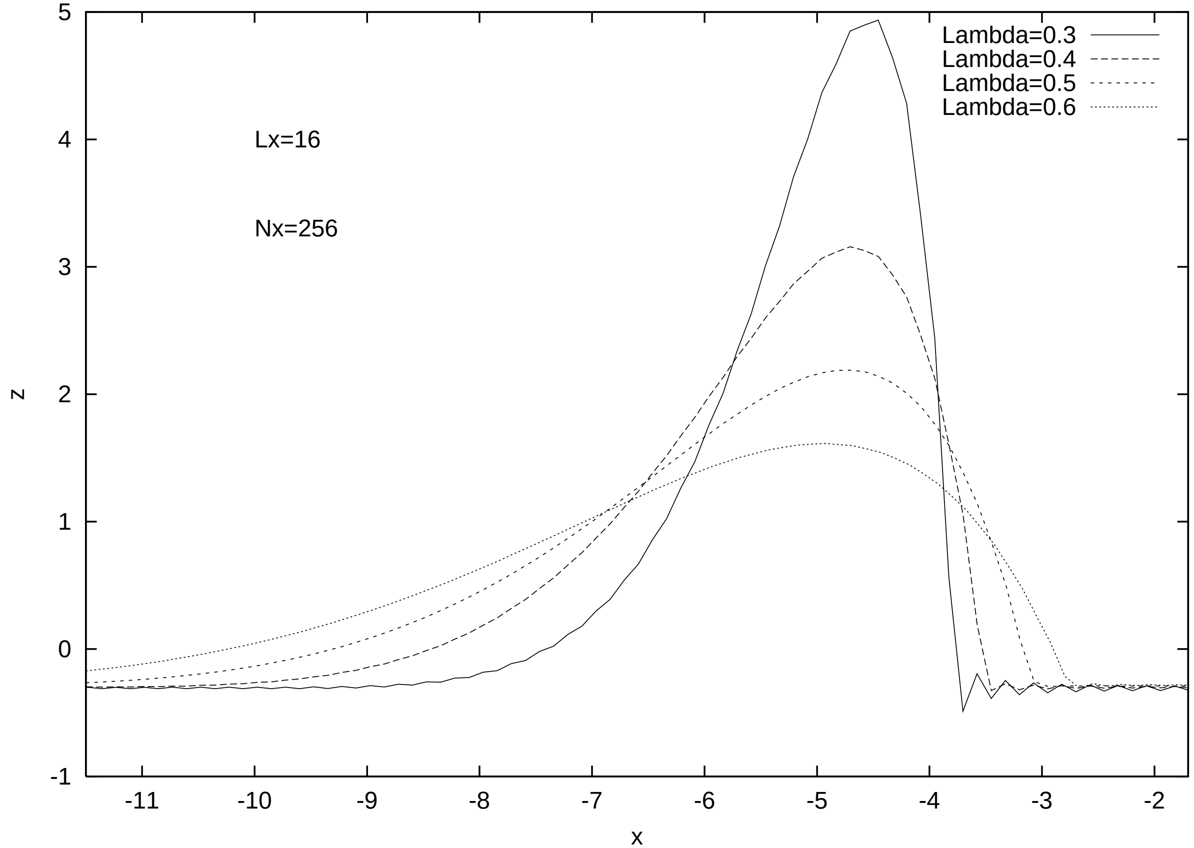


Fig. 5. The non-linear final moving "dune" solution $f_{fin}(x - ct)$ is represented with solid lines, the linear solution is represented with dashed lines, and $\tau_s = 0.9$, $1/l_s = 2.5$, $m = 2, 3, 4, 5$ (bottom curve to top curve).

q proportional only to skin friction

$q = \tau - \tau_s - \Lambda \frac{\partial f}{\partial x}$ new similarity $\Lambda m^{-1/4}$, $c = m^{-1/4}$



Influence of Λ linear case $x = m^{3/4}x^*$ and $f = m^{1/4}f^*$ seems to be no $q = \tau - \tau_s$ solution.

Movement of a 3D Bump in a shear flow

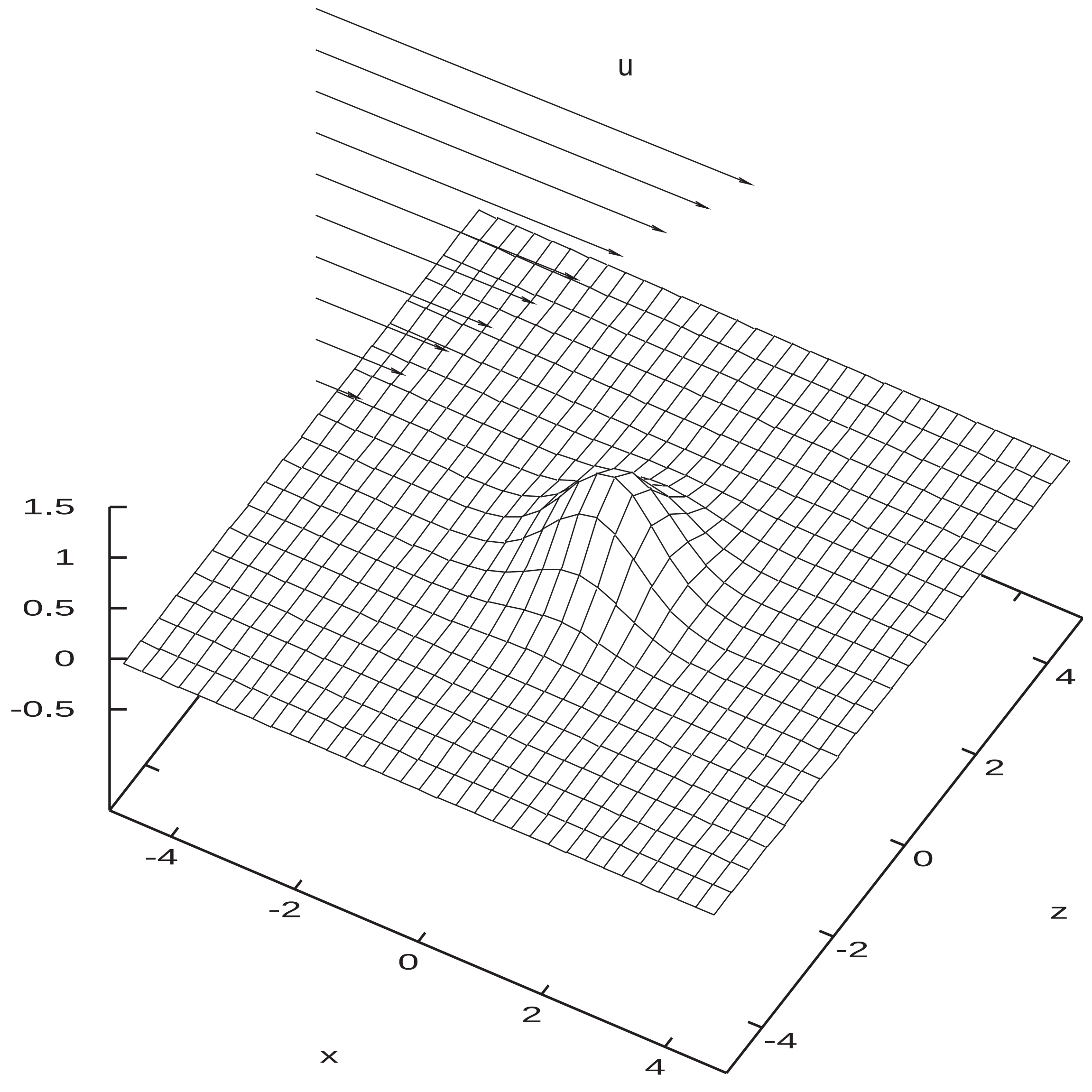


Abbildung 3: A bump in a shear flow

We look at a linearized solution:

$$u = y + au_1,\ v = av_1,\ w = aw_1,\ p = ap_1 \text{ with } a << 1.$$

The system becomes:

$$\begin{aligned}
\frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial y}v_1 + \frac{\partial}{\partial z}w_1 &= 0, \\
y\frac{\partial}{\partial x}u_1 + v_1 &= -\frac{\partial}{\partial x}p_1 + \frac{\partial^2}{\partial y^2}u_1, \\
y\frac{\partial}{\partial x}w_1 &= -\frac{\partial}{\partial z}p_1 + \frac{\partial^2}{\partial y^2}w_1,
\end{aligned}$$

with boundary conditions:

$u_1 = v_1 = w_1 = 0$ in $y = f(x, z)$,

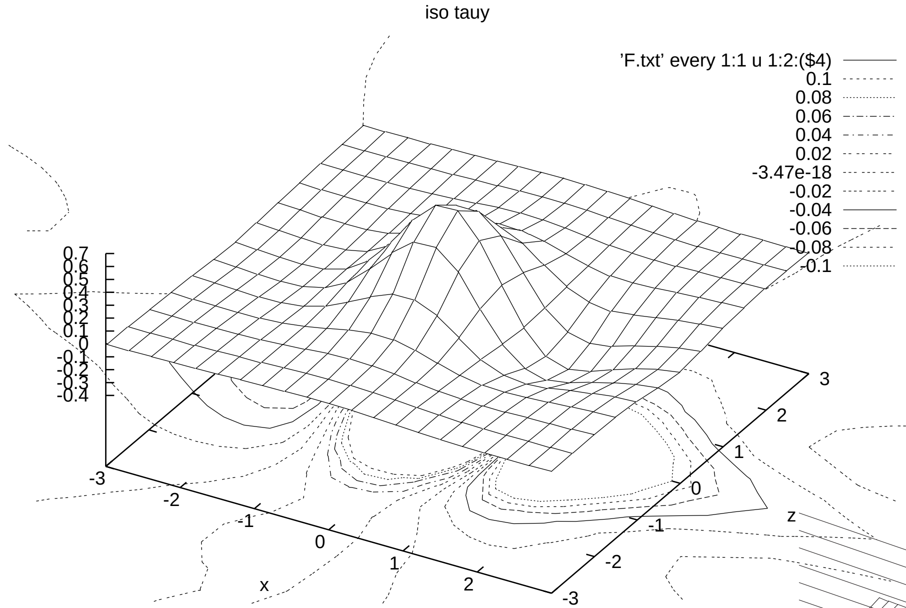
$y \rightarrow \infty$, $u_1 = +f(x, z)$, $w_1 = 0$

$x \rightarrow -\infty$, $u_1 = 0$, $v_1 = 0$, $w_1 = 0$.

Looking at solutions in Fourier space...

This finally gives the perturbation for the skin friction

$$\frac{d\hat{u}}{dy} = 3((-ik_x)^{1/3}Ai(0))k_x(1 - \frac{(-3Ai'(0))k_z^2}{9Ai(0)^2(k_x^2+k_z^2)})\hat{f} \qquad \frac{d\hat{w}}{dy} = 3((-ik_x)^{1/3}Ai(0))\frac{k_x}{k_z}\frac{(-3Ai'(0))k_z^2}{9Ai(0)^2(k_x^2+k_z^2)}\hat{f}$$



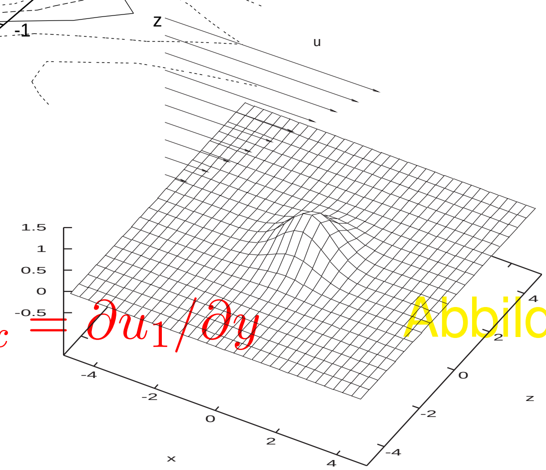


Abbildung 4: skin friction $\tau_x = \partial u_1/\partial y$

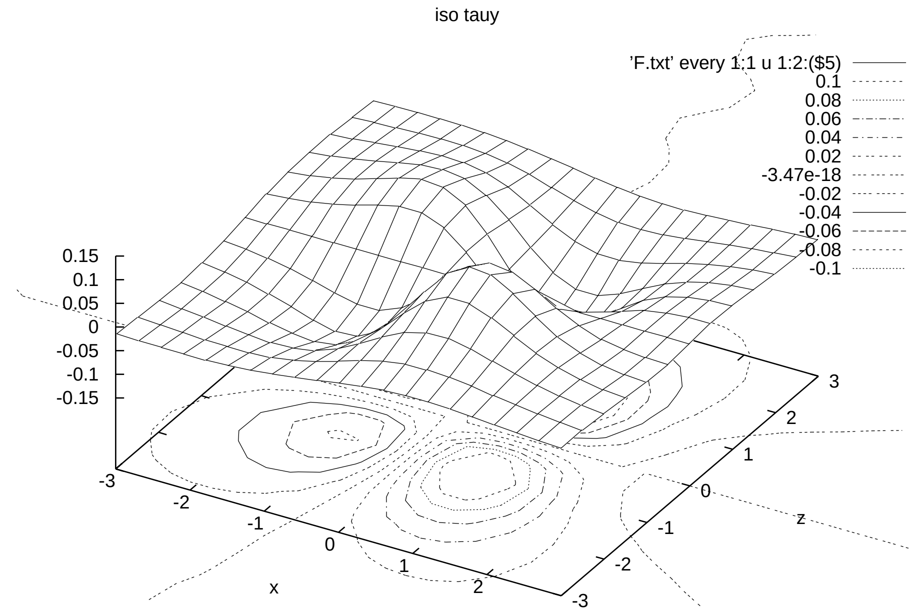
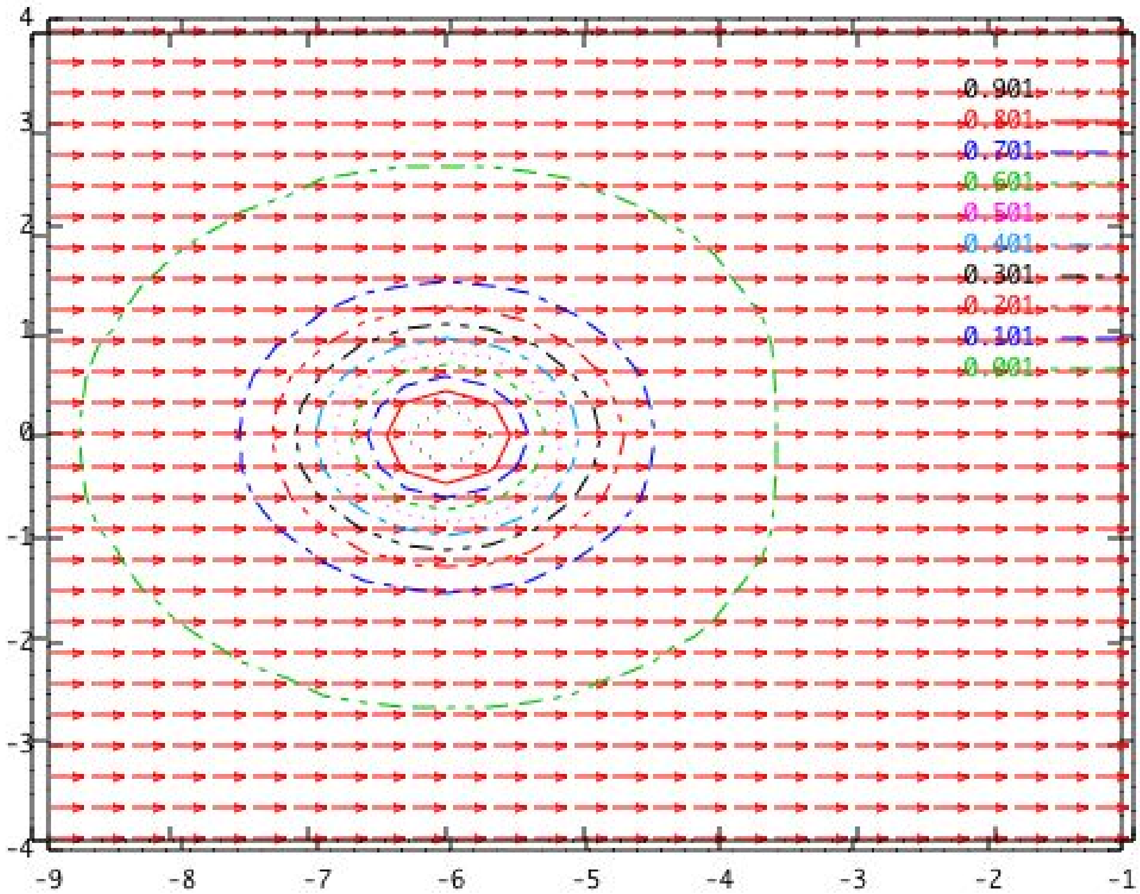
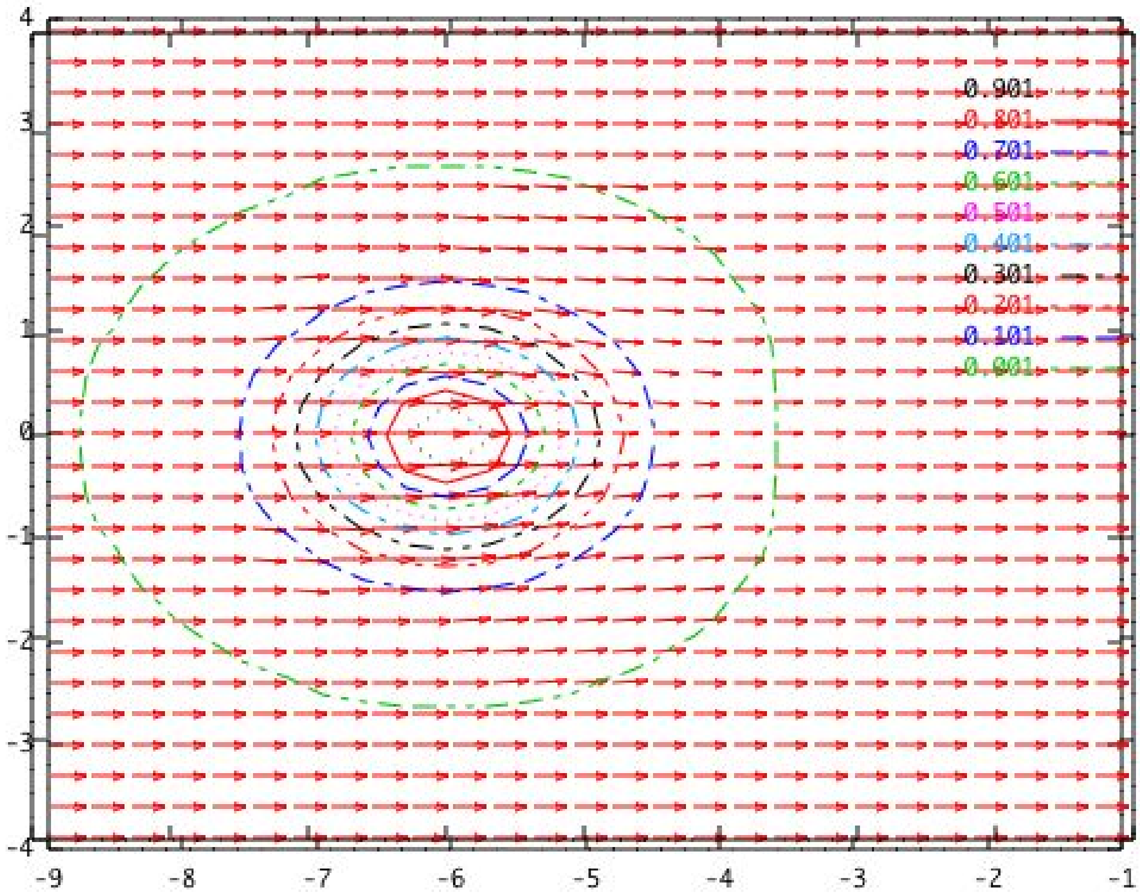


Abbildung 5: skin friction $\tau_y = \partial w_1/\partial y$

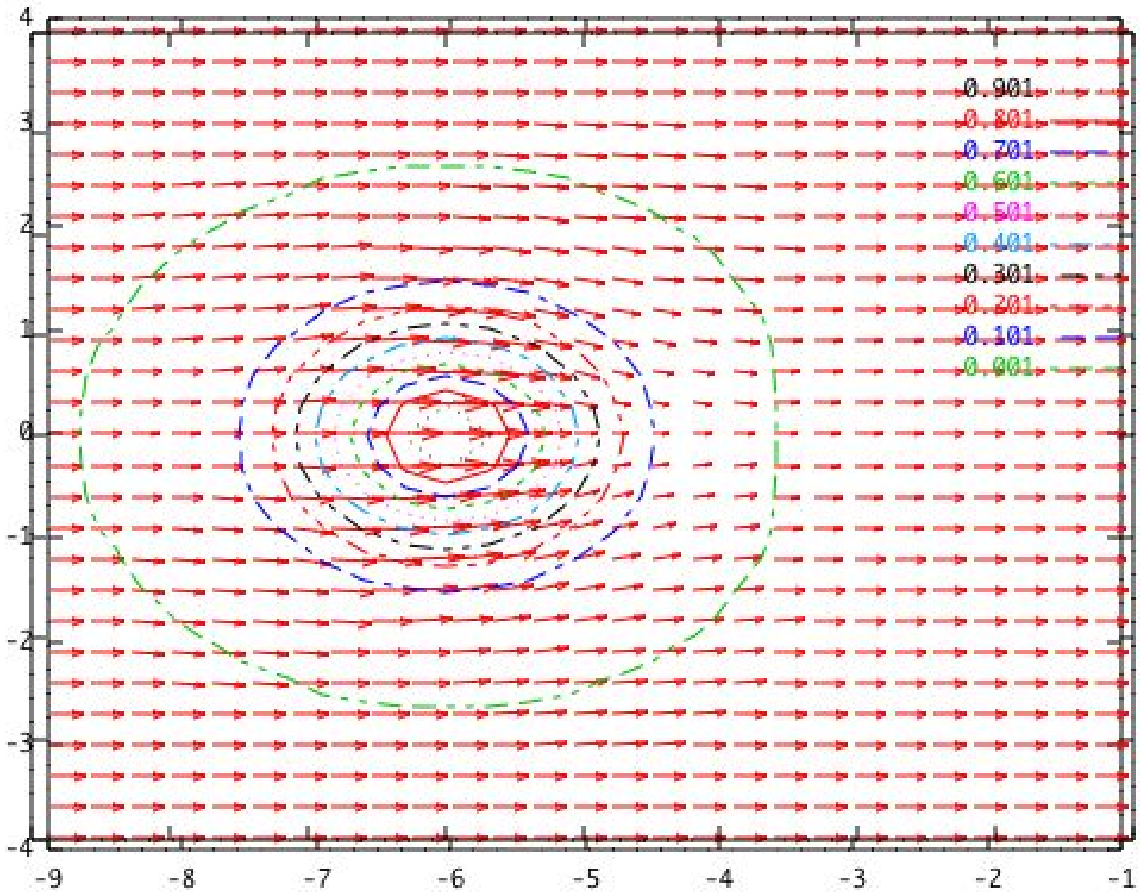
Skin friction on a 3D bump $\alpha = 0.0$



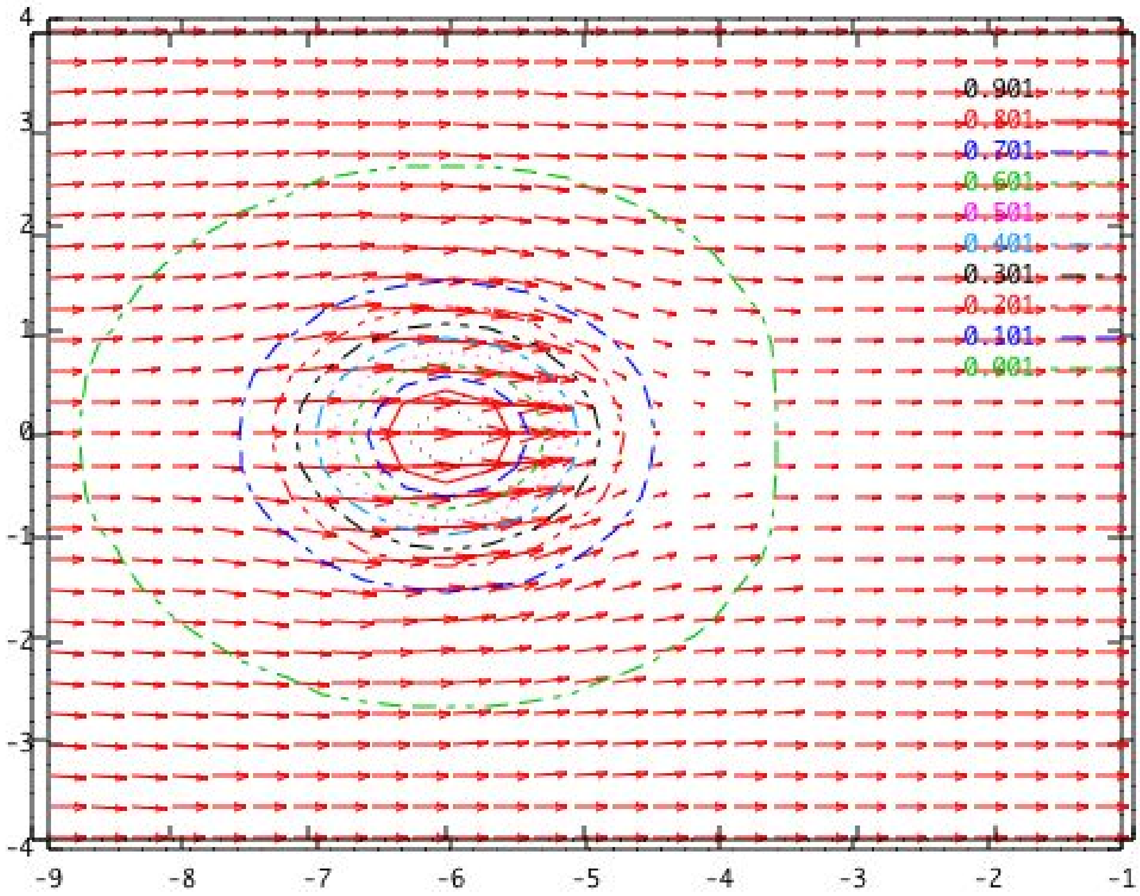
Skin friction on a 3D bump $\alpha = 0.1$



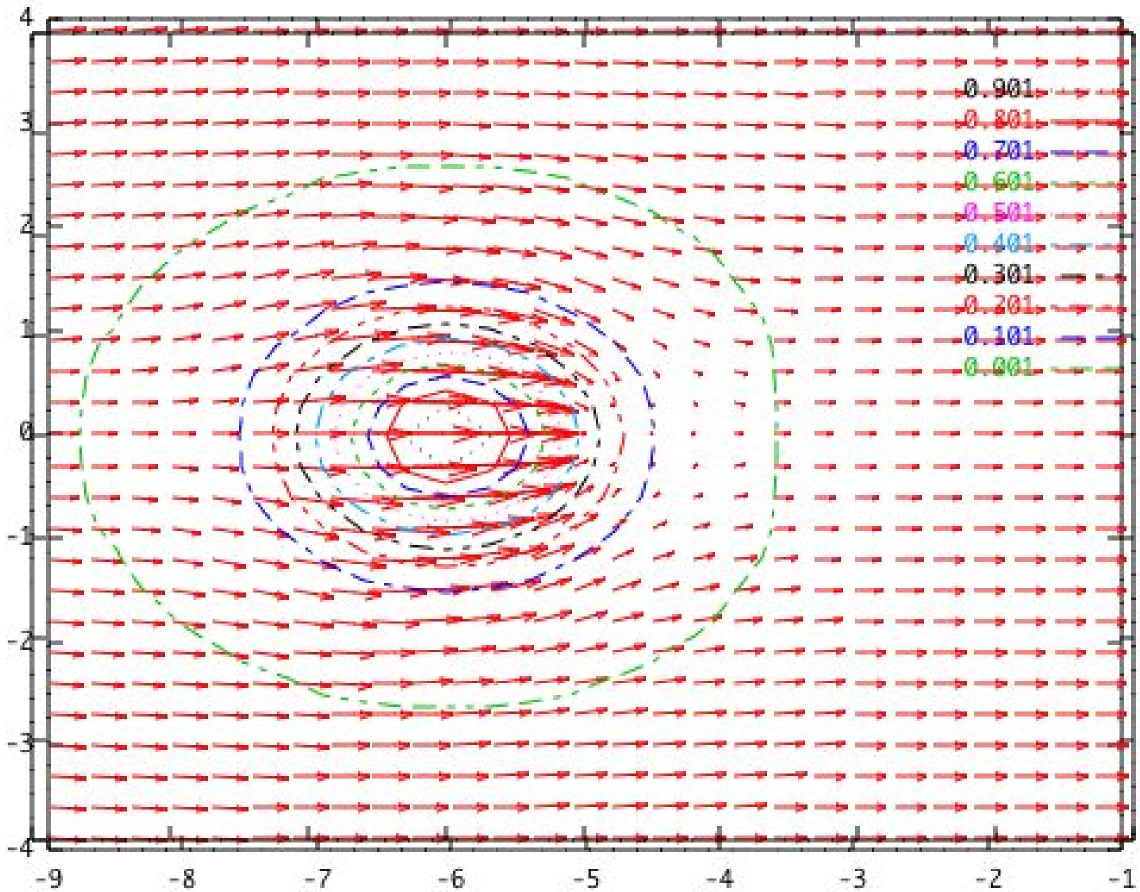
Skin friction on a 3D bump $\alpha = 0.2$



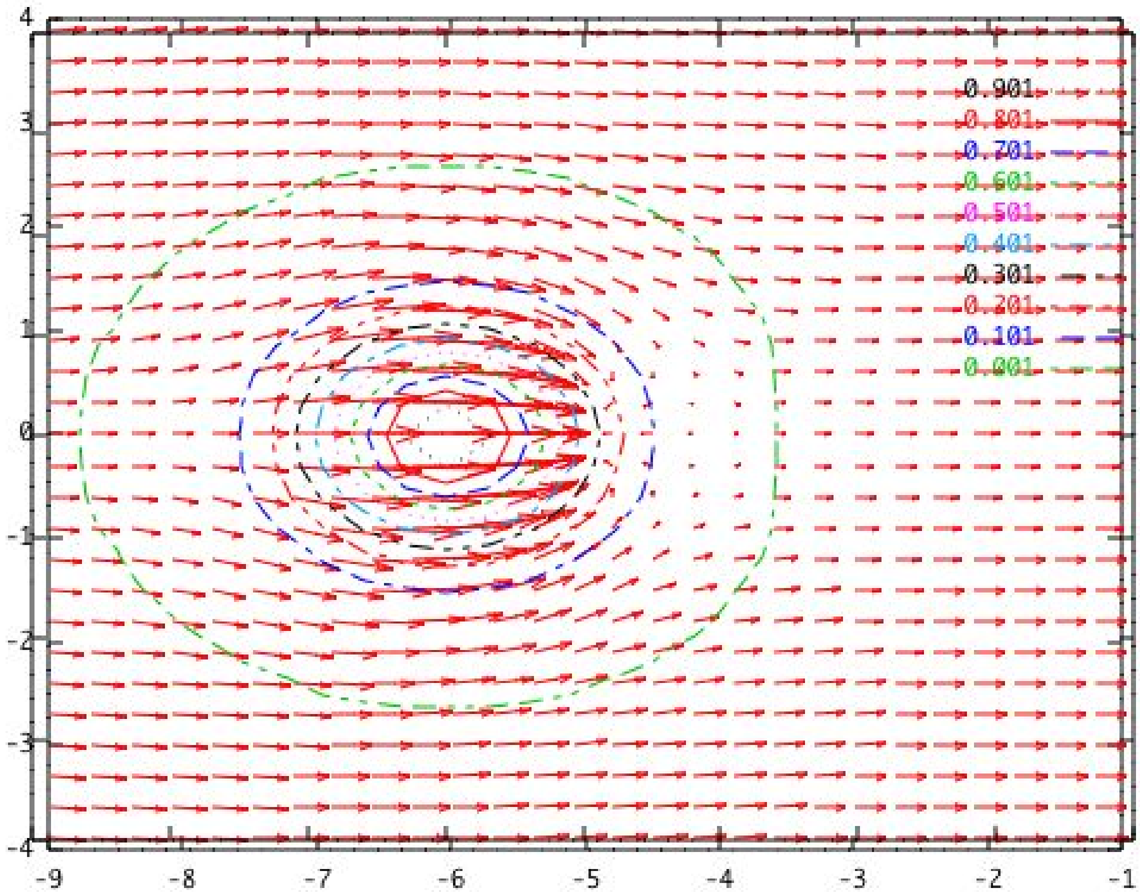
Skin friction on a 3D bump $\alpha = 0.3$



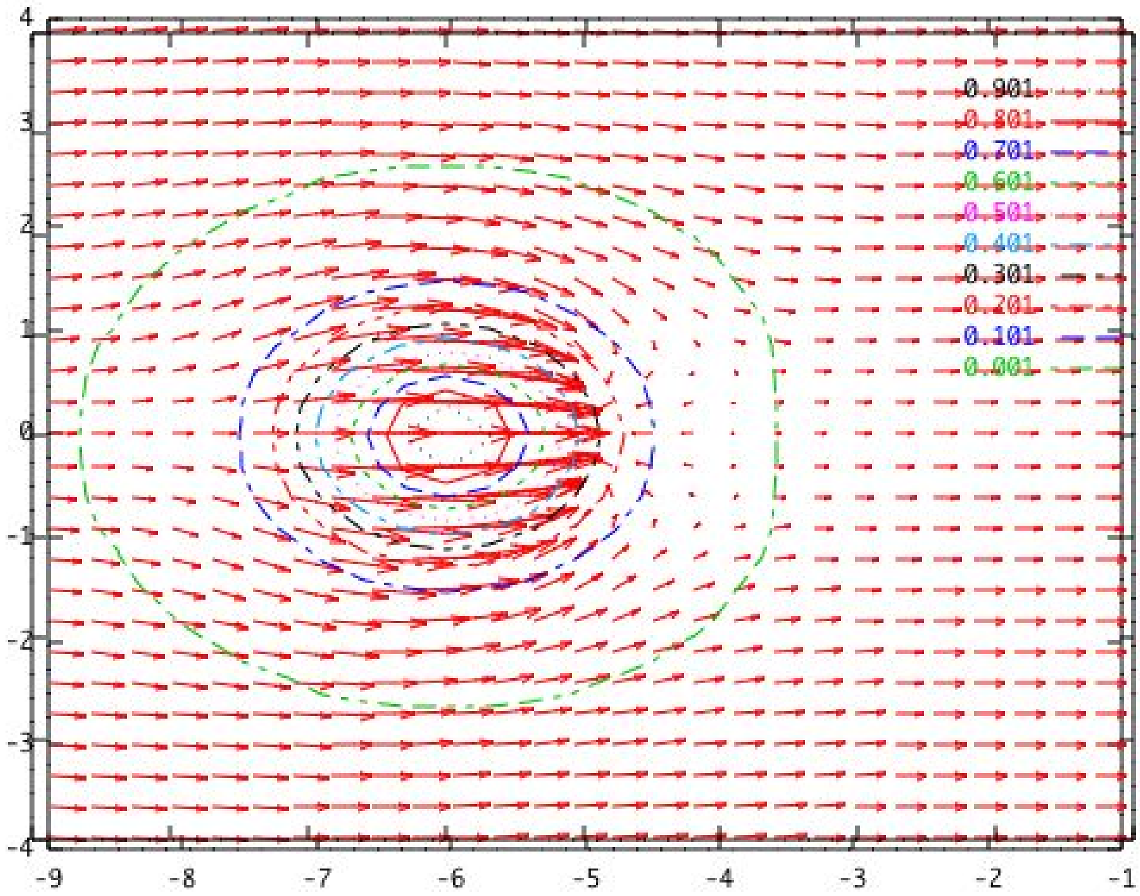
Skin friction on a 3D bump $\alpha = 0.4$



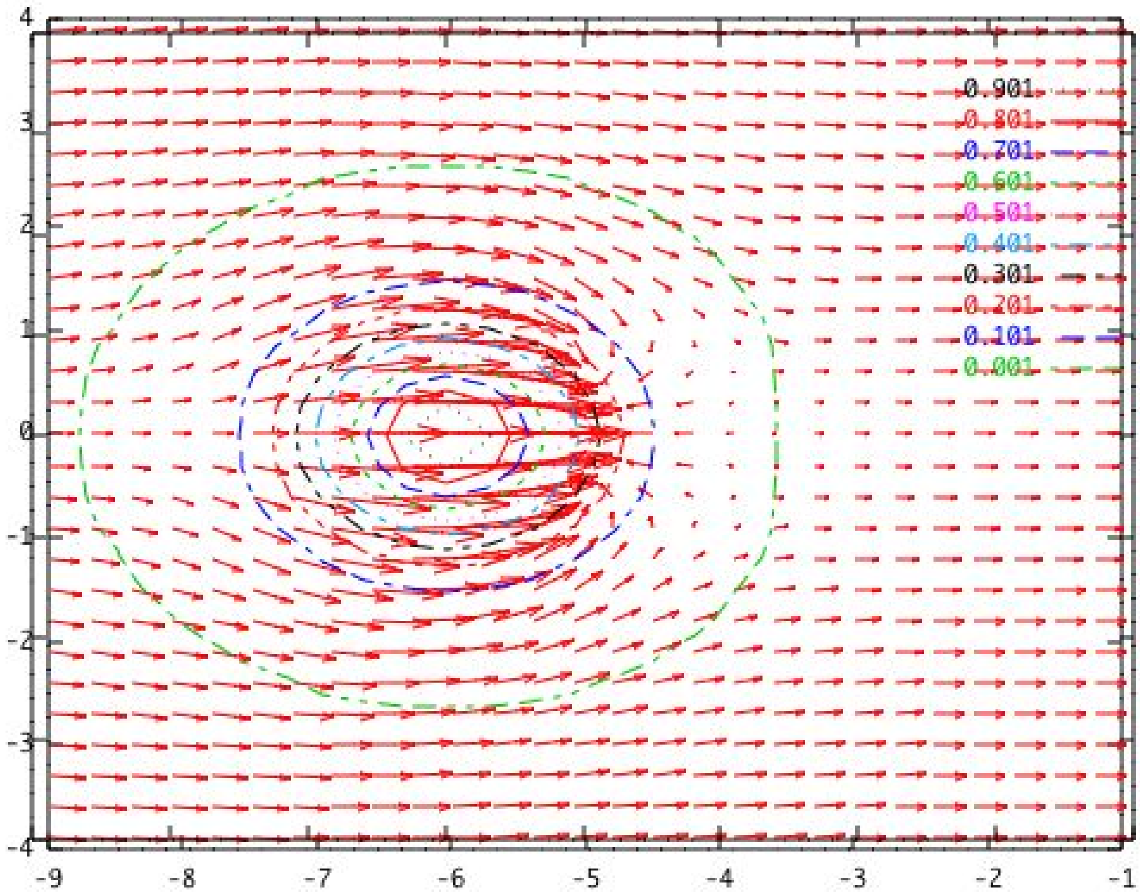
Skin friction on a 3D bump $\alpha = 0.5$



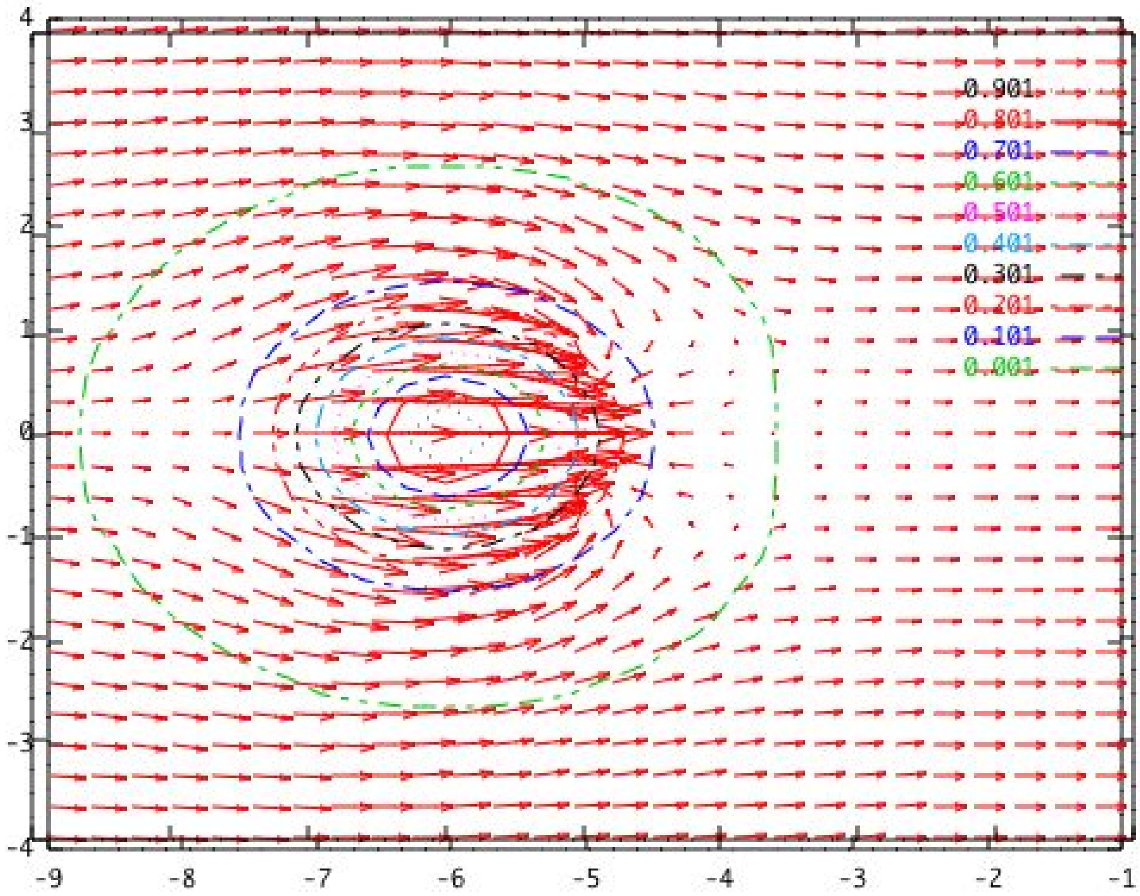
Skin friction on a 3D bump $\alpha = 0.6$



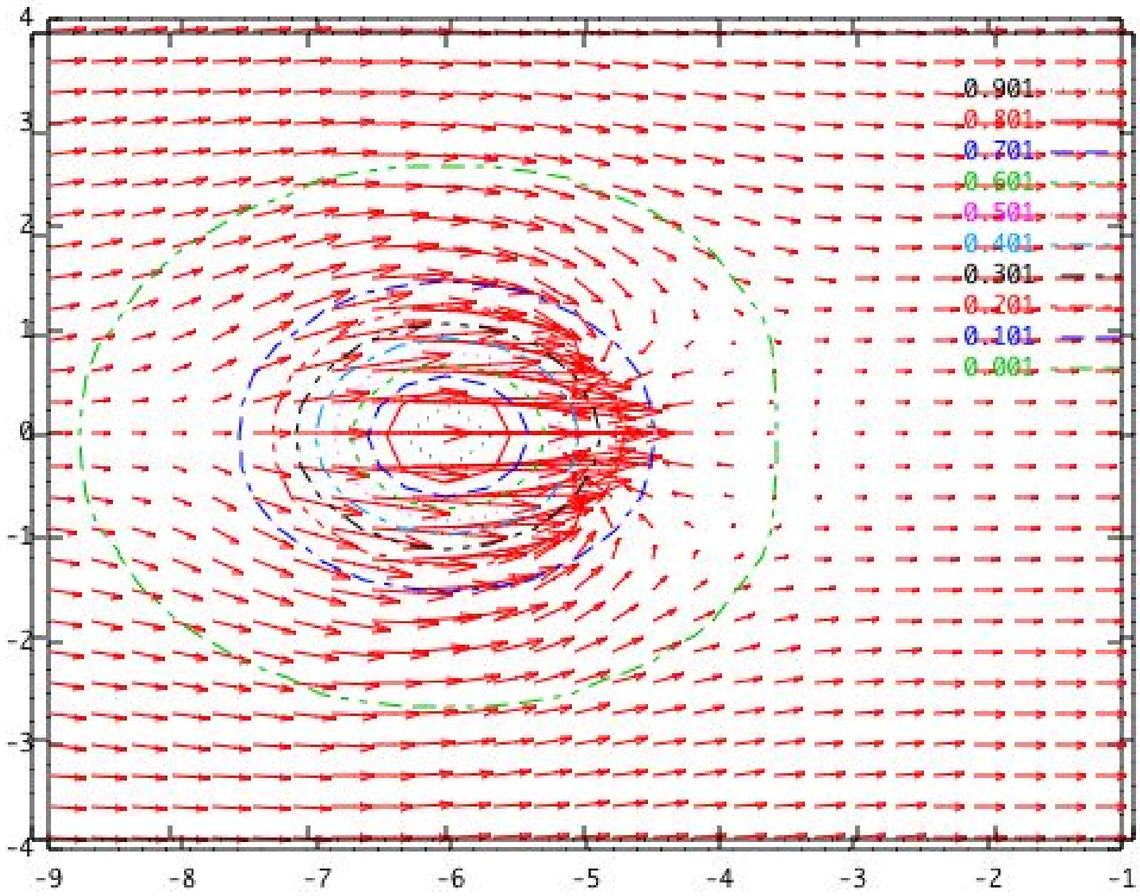
Skin friction on a 3D bump $\alpha = 0.7$



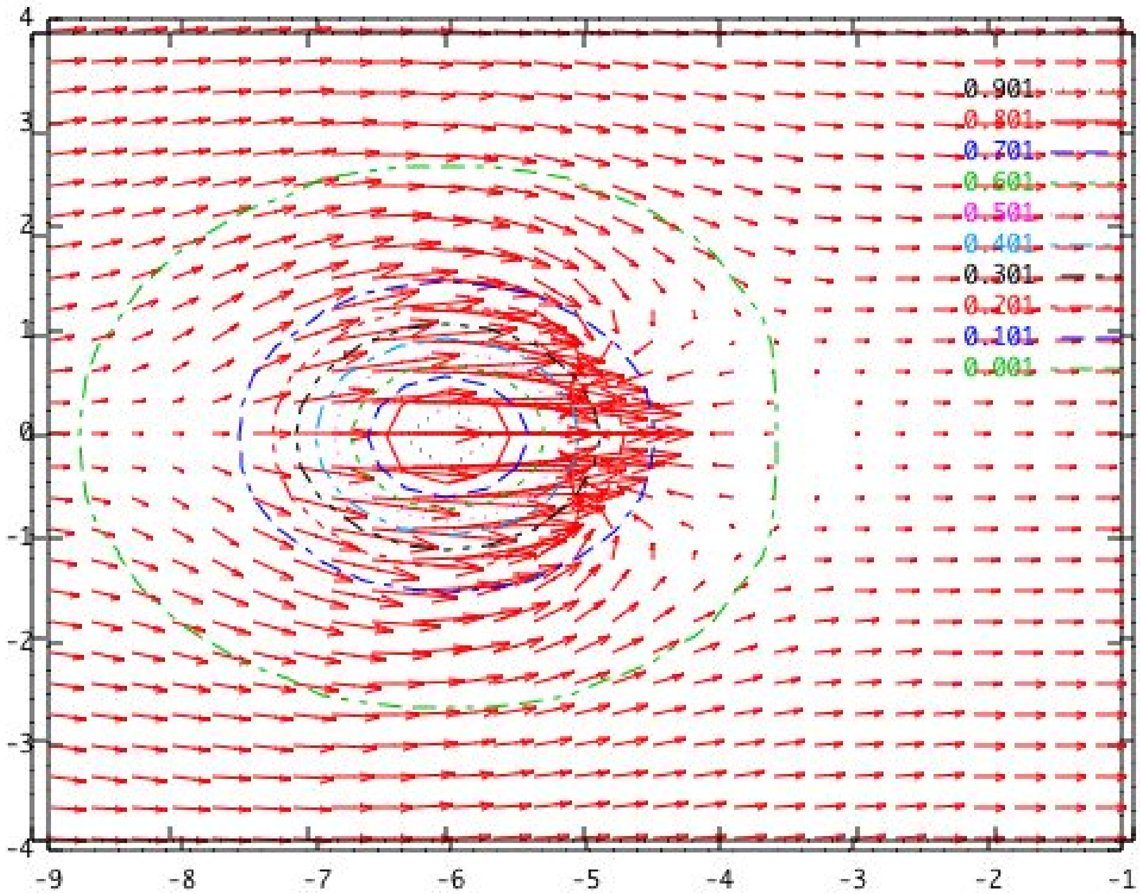
Skin friction on a 3D bump $\alpha = 0.8$



Skin friction on a 3D bump $\alpha = 0.9$



Skin friction on a 3D bump $\alpha = 1.0$



Example over an erodible bed

Solution of

$$\hat{\tau}_x = 3((-ik_x)^{1/3}Ai(0))k_x(1 - \frac{(-3Ai'(0))k_z^2}{9Ai(0)^2(k_x^2 + k_z^2))}\hat{f}$$

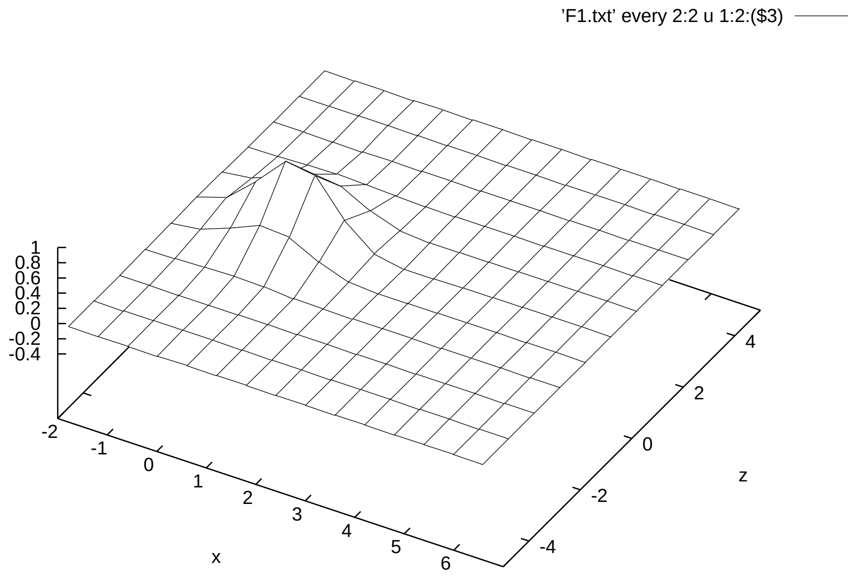
$$\hat{\tau}_y = 3((-ik_x)^{1/3}Ai(0))\frac{k_x(-3Ai'(0))k_z^2}{9Ai(0)^2k_z(k_x^2 + k_z^2)}$$

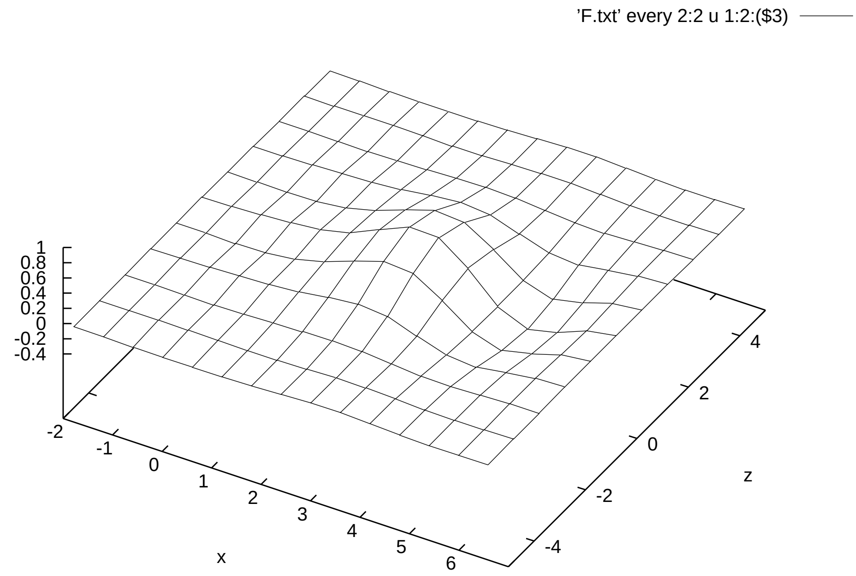
$$q_x = \tau_x - \Lambda\frac{\partial f}{\partial x}$$

$$q_y = \tau_y - \Lambda\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y}$$

example of resolution





animation

Abbildung 6: initial time

Abbildung 7: $t = 2.5$

Transport flux

We propose a 3D extension as:

$$l_s \frac{\partial \mathbf{q}}{\partial \mathbf{s}} + \mathbf{q} = \varpi(\tau - \tau_s \mathbf{e})$$

with $\mathbf{e} = \frac{\tau}{\tau}$ where s is counted in the direction of the streamlines near the soil: $\frac{\partial}{\partial \mathbf{s}} = \mathbf{e} \cdot \nabla$

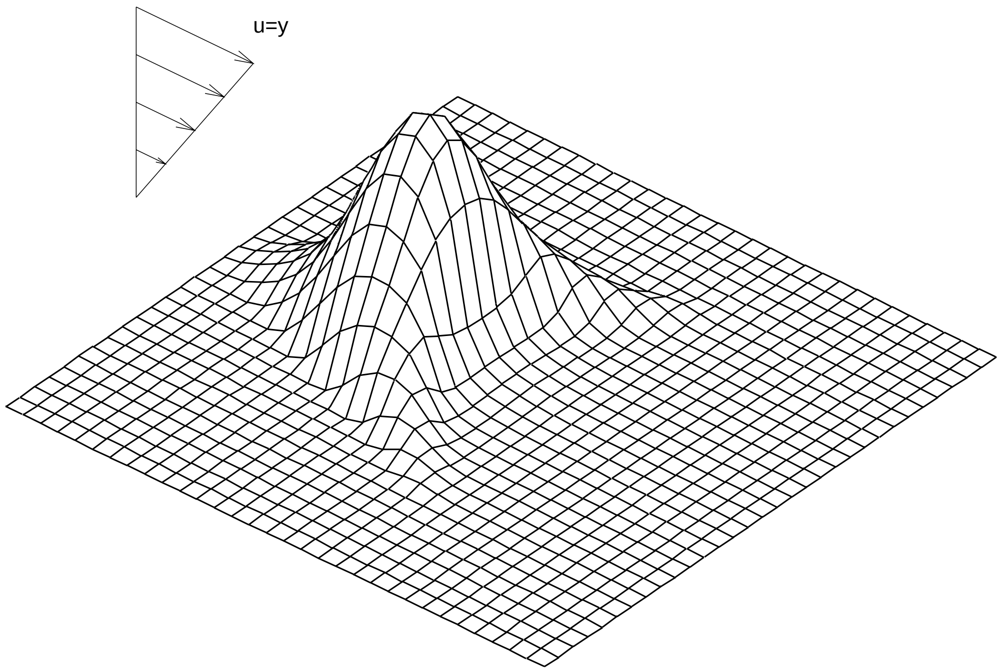
Small deflection of the bump: flow remains in x direction $\mathbf{s} = (x, 0)$: the saturated flux $q_{sat} = \varpi(\tau - \tau_s \mathbf{e})$ is in the direction of the skin friction.

$$l_s \frac{\partial q_x}{\partial x} + q_x = \varpi(\tau_x - \tau_s)$$

$$l_s \frac{\partial q_z}{\partial x} + q_z = \tau_z(\varpi(\tau_x - \tau_s))$$

note here we take $\mathbf{q_{sat}} = 0$ when $f <= 0$
we add an *ad hoc* extra diffusion term:

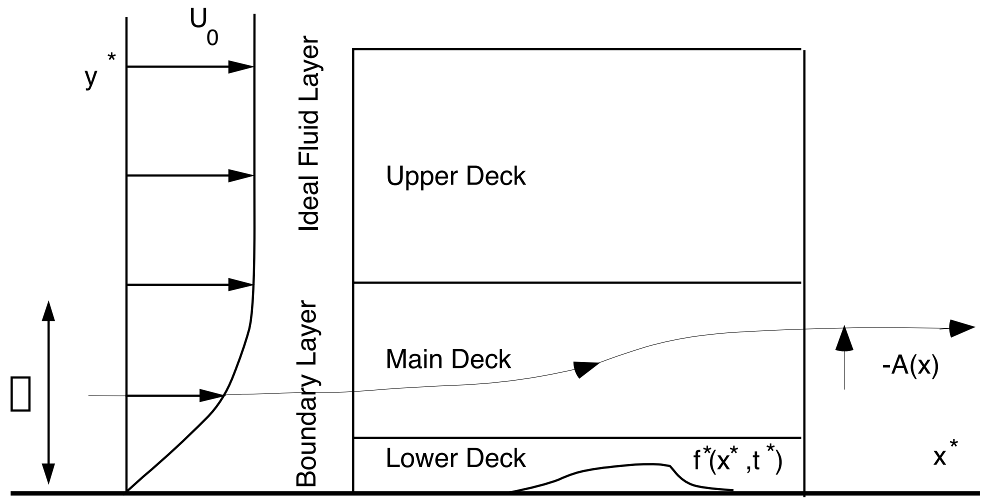
$$\frac{\partial f}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_z}{\partial z} + D(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2})$$



animation

Abbildung 8: A ”dune” in a shear flow,

Influence of the ideal fluid



$$FT[\tau] = \frac{(-ik)^{2/3}}{Ai'(0)} Ai(0) \frac{FT[f]}{\beta^* - 1/|k|}, \text{ with } \beta^* = (3Ai'(0))^{-1}(-ik)^{1/3}$$

Influence of the ideal fluid

$$FT[\tau] = \frac{(-ik)^{2/3}}{Ai'(0)} Ai(0) \frac{FT[f]}{\beta^* - 1/|k|},$$

with $\beta^* = (3Ai'(0))^{-1}(-ik)^{1/3}$

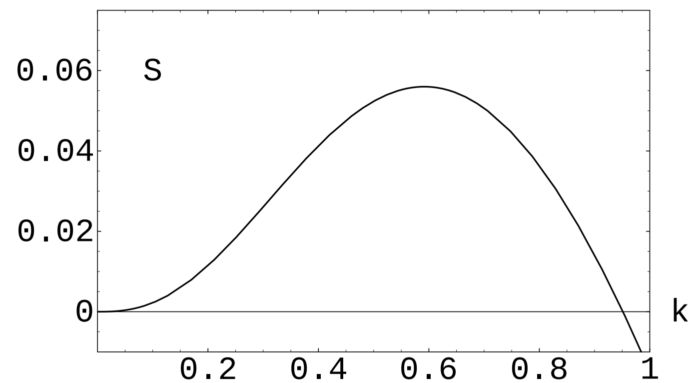
remember Hermann, Kroy, & Sauermann and Andreotti, Claudin & Doaudy:

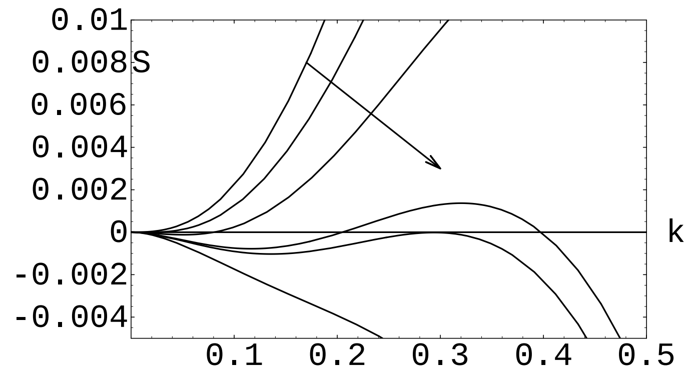
$$\tau = (\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f'}{x - \xi} d\xi) + Bf'$$

$$FT[\tau] = \frac{FT[f]}{|k|} + (-ik)B\, FT[f]$$

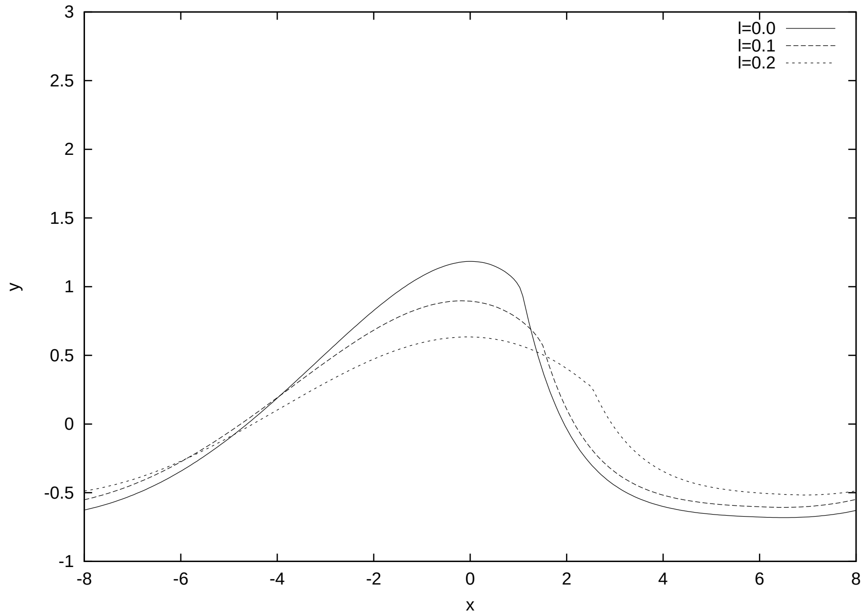
Stability analysis

- Infinite depth case (Hilbert case). The real part of σ for $\beta = l_s = \gamma = 1$ as function of the wave length k:
  - on the left figure $\Lambda = 0$: there is no slope effect
  - on the right figure, we focus on the small k which are amplified when $\Lambda = 0$, but are damped for $\Lambda > 0$ (following the arrow, from up to down $\Lambda = 0$, $\Lambda = 0.1$, $\Lambda = 0.2$, $\Lambda = 0.3$, $\Lambda = 0.316$ and $\Lambda = 0.4$).



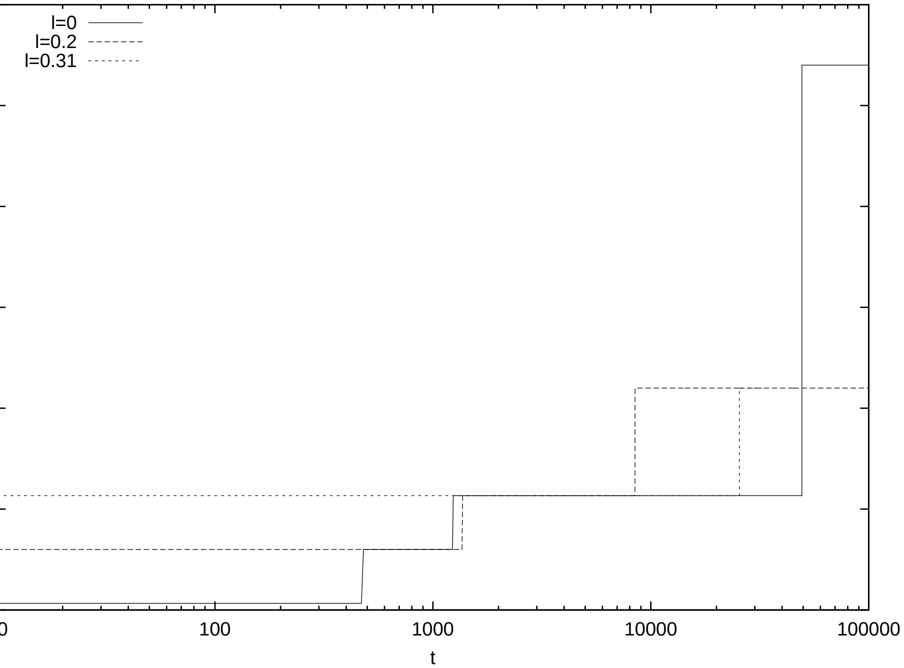


Slope effect: influence of Λ



Bump shape $t = 500$, (4 bumps coexist with $\beta = 1$, $\gamma = 1$, $l_s = 1$, $\tau_s = -0.05$), $\Lambda = 0$, $\Lambda = 0.1$ and $\Lambda = 0.2$ (the curves are shifted to place the maximum at the origin)

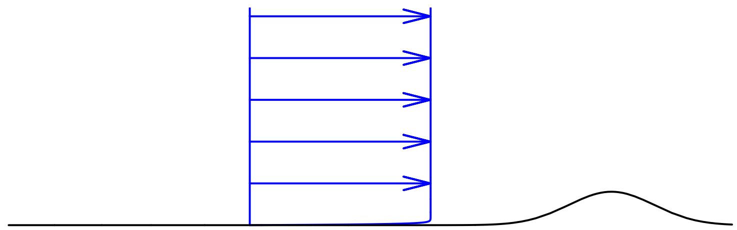
Coarsening process,Hilbert case



Examples of long time evolution of $2\pi/k$ the wave length value maximizing the bump spectrum (corresponding mostly to the number of bumps present in the domain). This is an infinite depth case for a domain of length $2L_x$. If $\Lambda = 0$, there is finally only one bump of size $2L_x$ (the largest possible). If $\Lambda < 0.316$, two bumps (of size L_x) are present, the larger are damped. If Λ is increased, there is no dune anymore as predicted by the linearized theory. Here $l_s = \beta = 1$, $L_x = 32$, $\tau_s = -0.25$. Notice that several bumps may live during a very long time: here in the case $\lambda = 0.31$, during a very long time ($10 < t < 25000$) three bumps are present.

animation

Comming back to ideal fluid: $Re = \infty$



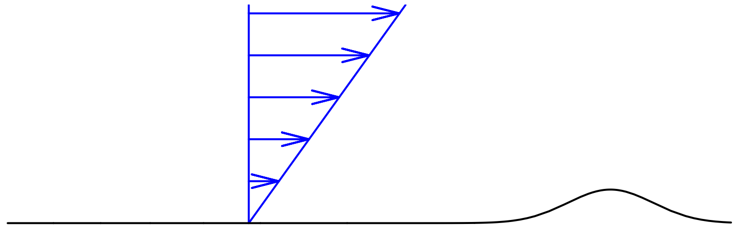
Uniform flow over a topography at large Reynolds number

Starting from an initial shape, the ideal fluid flow is computed (in the Small Perturbation Theory):

$$f(x,t) \text{ gives } u = (1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f'}{x-\xi} d\xi) \text{ in FT, it is: } FT[\tau] = \frac{FT[f]}{|k|}$$

This is known as a very good approximation
But problems arise in the decelerated region (we saw).

Second example: Basic case, at $Re = 0$

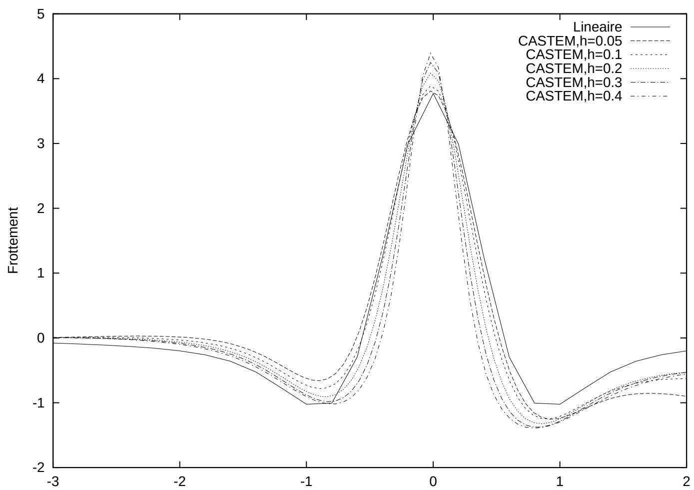


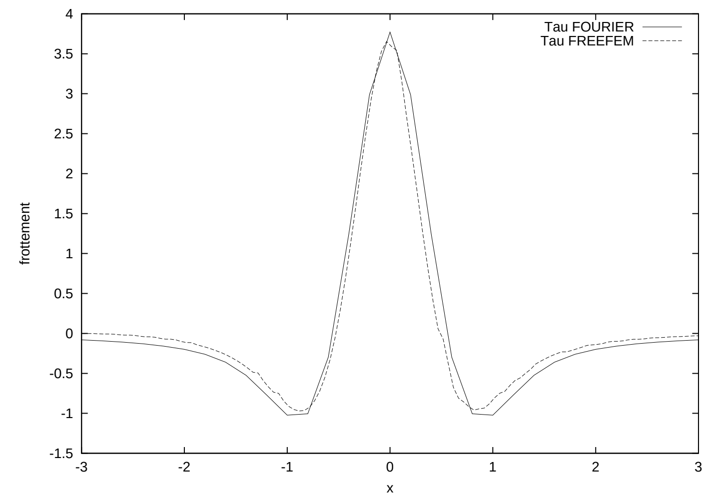
Shear flow over a topography $f(x, t)$ at small Reynolds number

Starting from an initial shape, the creeping flow is computed (in the Small Perturbation Theory), we obtain after some algebra:

$$f(x,t) \text{ gives } \tau = 1 + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{f'}{x-\xi} d\xi$$

perturbation of a shear flow $Re = 0$





L: flow over a gaussian bump, comparisons linear theory/ computations perturbation of skin friction computed with CASTEM $\frac{1}{h_0}\frac{\partial \overline{u}}{\partial \overline{y}}$ for $0.05 < h_0 < 0.4$ (bump size) and $Re = 1$

R: perturbation of skin friction computed with FreeFem.

Linking q and u

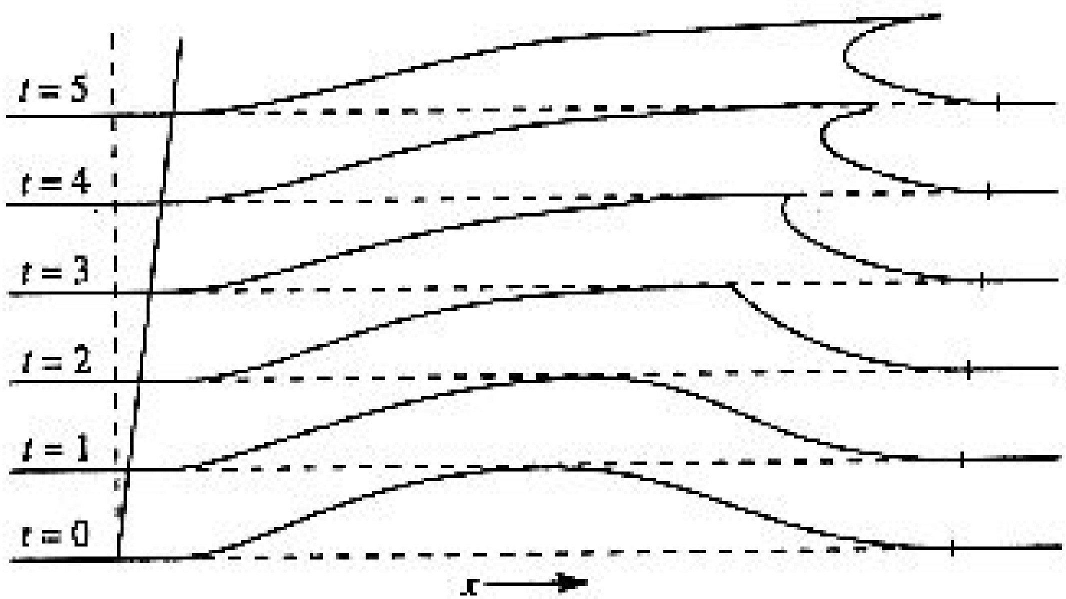
assuming that q is proportional to $u - 1$ or q proportional to $\tau - 1$ without threshold this gives the same relation in the two cases $(2!)$:

$$\frac{\partial f}{\partial t} = -\frac{1}{\pi}\frac{\partial}{\partial x}\int \frac{f'}{x-\xi}d\xi.$$

we recognize the linear Benjamin -Ono equation.

Supposed Evolution

The ideal fluid theory has been introduced by Exner.

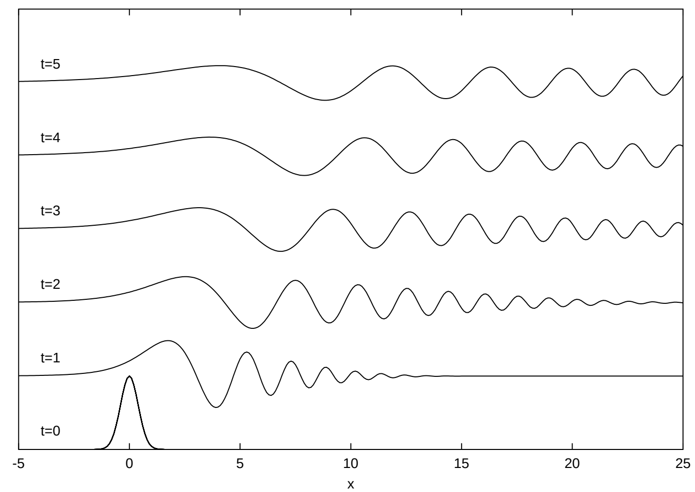


Issued from Yang (1995) reproduced from Exner (1925?).
”wave” inspiration in the dune evolution

Computed Evolution

Numerical resolution: finite differences, explicit
Tested on complete Benjamin - Ono: RHS+ $4f\partial f/\partial x$ gives the soliton $1/(1+x^2)$

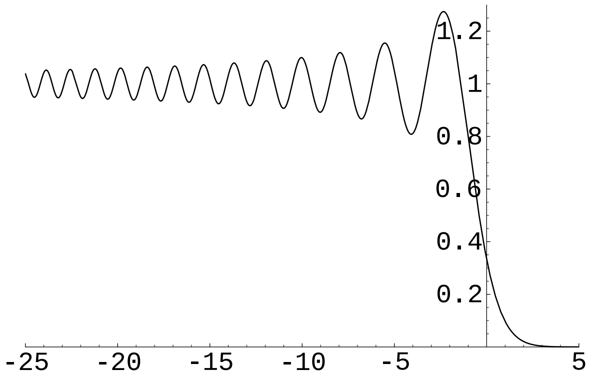
But here we observe the dispersion of the bump...

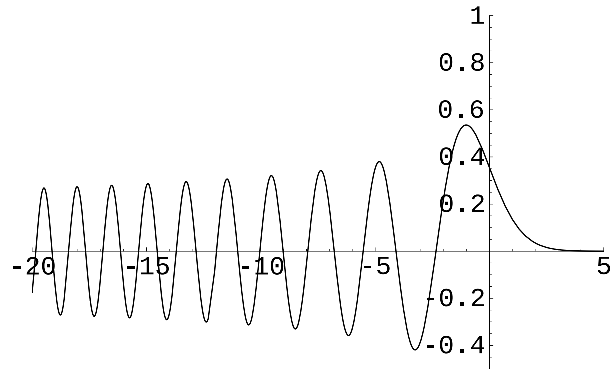


animation another animation

Remark: linear KDV equation

The linear KDV equation reads $\frac{\partial f}{\partial t} = \frac{\partial^3 f}{\partial x^3}$, with selfsimilar solutions, $\eta = x t^{-1/3}$:





"Mascaret" solution: $f(x,t) = \int_{3^{-1/3}\eta}^{\infty} Ai(\xi) d\xi$; Airy solution: $f(x,t) = t^{-1/3} Ai(\frac{\eta}{3^{1/3}})$.

animation

asymptotic solution of L.B.O.

L.B.O. $$\frac{\partial f}{\partial t} = -\frac{1}{\pi}\frac{\partial}{\partial x}\int \frac{f'}{x-\xi}d\xi.$$

Selfsimilar variable $\eta = xt^{-1/2}$, self similar solution $f(x,t) = t^{-1/2}\phi(xt^{-1/2})$.

In the Fourier space $exp(-ikx)$ gives, in the RHS, $-i|k|kexp(-ikx)$, so:

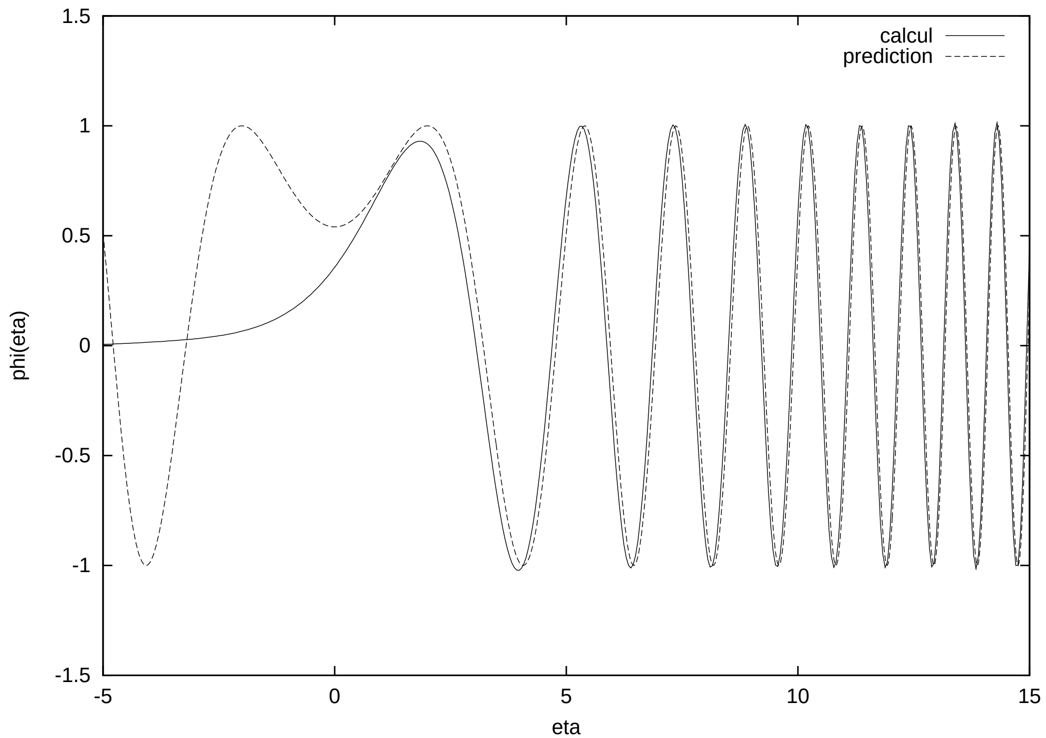
$$-\frac{1}{\pi}\frac{\partial}{\partial x}\int \frac{f'}{x-\xi}d\xi \simeq i\frac{\partial^2 f}{\partial x^2}$$

The self similar problem is approximated by:

$$\frac{-1}{2}(\phi(\eta) + \eta\phi'(\eta)) \simeq i\phi''(\eta).$$

whose exact solution is $\phi(\eta) = exp(i(\eta/2)^2)$

asymptotic solution of L.B.O.



Plot of the numerical solution $t^{1/2}f(x,t)$ function of $xt^{-1/2}$
the exact solution of the approximated problem $cos(1+(\eta/2)^2)$.

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- comprehension of the influence of the viscous boundary layer (destabilisation) versus the ideal fluid effect (dispersive).

Perspectives

- Application for a special case: Hele Shaw
- Turbulent integral Interacting Boundary Layer theory

springen,

Zuruck zur vorher angezeigten Seite.