

A PERIODIC OPTIMAL CONTROL PROBLEM FOR A BILINEAR HARMONIC OSCILLATOR, WITH APPLICATIONS IN QUANTUM AND CLASSICAL PHYSICS*

ARNAUD LAZARUS[†] AND EMMANUEL TRÉLAT[‡]

Abstract. In this paper we study and solve a periodic optimal control problem for a bilinear harmonic oscillator (equivalently, for the Hill-type equation $\ddot{x}(t) + u(t)x(t) = 0$ with bounded, sign-indefinite stiffness) motivated by applications in quantum and classical physics. Although apparently simple, this optimal control problem is not easy to solve and we resort to various elaborated methods of optimal control theory. We finally show its relationships to two problems in physics: the computation of the ground state for 1D Schrödinger operators with a finite potential well, and the optimal dynamical Kapitza stabilization problem. We also briefly discuss the Floquet stability properties associated with the optimal periodic bang-bang control.

Key words. optimal control theory, applications in physics

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1. The optimal control problem. Given any $u_{\min}, u_{\max} \in \mathbb{R}$ such that $u_{\min} < 0 < u_{\max}$, given any $T > 0$, we consider the optimal control problem

$$\begin{aligned} (1.1a) \quad & \dot{x}(t) = y(t), \\ (1.1b) \quad & \dot{y}(t) = -u(t)x(t), \\ (1.1c) \quad & u_{\min} \leq u(t) \leq u_{\max}, \\ (1.1d) \quad & \min \int_0^T u(t) dt, \end{aligned}$$

where (1.1a), (1.1b), and (1.1c) are written for almost every $t \in [0, T]$, with the periodicity conditions and nontriviality constraint

$$\begin{aligned} (1.2a) \quad & x(0) = x(T), \quad y(0) = y(T), \\ (1.2b) \quad & x(0)^2 + y(0)^2 > 0. \end{aligned}$$

Eliminating the variable y from (1.1a)–(1.1b), the dynamics reduce to the scalar second-order equation

$$\ddot{x}(t) + u(t)x(t) = 0,$$

which is the equation of a harmonic oscillator with time-dependent stiffness (or spring constant) $u(t)$ and, more generally, a Hill-type equation with periodic coefficients (see, e.g., [1, 12, 18]). When u is constant and positive, the energy $E = \frac{1}{2}(y^2 + ux^2)$ is conserved and the phase portrait consists of ellipses; when u is constant and negative,

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[†]Sorbonne Université, CNRS, Institut Jean Le Rond d'Alembert, F-75005 Paris, France, also affiliated with Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (arnaud.lazarus@sorbonne-universite.fr).

[‡]Sorbonne Université, Université Paris Cité, CNRS, Inria, Laboratoire Jacques-Louis Lions, LJLL, F-75005 Paris, France (emmanuel.trelat@sorbonne-universite.fr).

the conserved quantity becomes $x^2 - \frac{y^2}{|u|}$ and the phase portrait consists of hyperbolas. This elementary harmonic-oscillator viewpoint will be used repeatedly to interpret the geometric objects (ellipses/hyperbolas) appearing in the sequel. Moreover, optimal control problems with periodic boundary conditions belong to the classical framework of optimal periodic control; we refer to [4, 6] for general developments.

The (nonclosed) constraint (1.2b) ensures nontriviality of optimal solutions, if they exist. Indeed, if we remove the constraint (1.2b), then obviously the unique optimal trajectory is $x(\cdot) = y(\cdot) = 0$, $u(\cdot) = u_{\min}$, and the optimal value is $Tu_{\min}$.

THEOREM 1.1. *Let $T > 0$ be arbitrary. There exists a unique optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of the optimal control problem (1.1) and (1.2), satisfying*

$$(1.3) \quad x(0) = x(T) = 1, \quad y(0) = y(T) = 0,$$

and

$$(1.4) \quad 0 < x(t) \leq 1 \quad \forall t \in [0, T].$$

Extending $(x(\cdot), y(\cdot), u(\cdot))$ to the whole $\mathbb{R}$ by T -periodicity, any other optimal solution of (1.1) and (1.2) is given by $(\mu x(\cdot + \delta), \mu y(\cdot + \delta), u(\cdot + \delta))$ for some $\mu \neq 0$ and $\delta \in \mathbb{R}$ (i.e., homothety and shifting in time).

The optimal control $u(\cdot)$ is bang-bang with two switchings:

$$(1.5) \quad u(t) = \begin{cases} u_{\max} & \text{if } 0 \leq t < t_1, \\ u_{\min} & \text{if } t_1 < t < T - t_1, \\ u_{\max} & \text{if } T - t_1 < t \leq T, \end{cases}$$

where the switching time $t_1 \in (0, \frac{1}{\omega_{\max}} \text{Arctan}(\frac{\omega_{\min}}{\omega_{\max}}))$ is characterized by the bijective relation

$$(1.6) \quad T = \frac{1}{\omega_{\min}} \ln \left(\frac{1 + \frac{\omega_{\max}}{\omega_{\min}} \tan(\omega_{\max} t_1)}{1 - \frac{\omega_{\max}}{\omega_{\min}} \tan(\omega_{\max} t_1)} \right) + 2t_1$$

(T is an increasing function of t_1) with

$$(1.7) \quad \omega_{\min} = \sqrt{-u_{\min}}, \quad \omega_{\max} = \sqrt{u_{\max}}.$$

The optimal trajectory $(x(\cdot), y(\cdot))$ is symmetric with respect to the x -axis and is homeomorphic to a clockwise circle (see Figure 1), entirely contained in the half-plane $x > 0$, consisting of the concatenation of an arc of ellipse and of an arc of hyperbola. Precisely, we have

$$(1.8) \quad x(t) = \begin{cases} \cos(\omega_{\max} t) & \text{if } 0 \leq t \leq t_1, \\ \frac{1}{2} \left(\cos(\omega_{\max} t_1) - \frac{\omega_{\max}}{\omega_{\min}} \sin(\omega_{\max} t_1) \right) e^{\omega_{\min}(t-t_1)} \\ \quad + \frac{1}{2} \left(\cos(\omega_{\max} t_1) + \frac{\omega_{\max}}{\omega_{\min}} \sin(\omega_{\max} t_1) \right) e^{-\omega_{\min}(t-t_1)} & \text{if } t_1 \leq t \leq T - t_1, \\ \cos(\omega_{\max}(T - t)) & \text{if } T - t_1 \leq t \leq T. \end{cases}$$

The corresponding velocity is obtained from (1.1a), namely $y(t) = \dot{x}(t)$ for all $t \in [0, T]$ (hence $y(t) = -\omega_{\max} \sin(\omega_{\max} t)$ on $[0, t_1]$, $y(t) = \omega_{\max} \sin(\omega_{\max}(T - t))$ on $[T - t_1, T]$, and on $[t_1, T - t_1]$ it is given by differentiating the exponential expression in (1.8)).

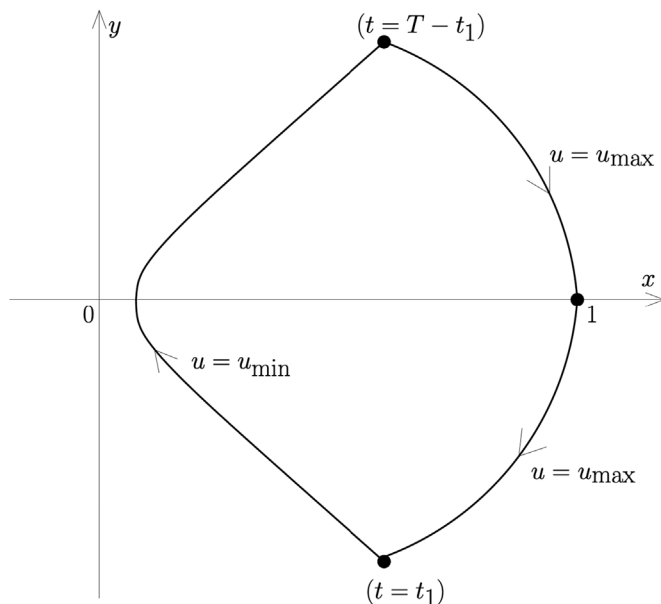


FIG. 1. Optimal trajectory (Theorem 1.1).

The cost of the optimal trajectory is

$$\begin{aligned}
 (1.9) \quad \int_0^T u(t) dt &= 2t_1 u_{\max} + (T - 2t_1) u_{\min} \\
 &= -\omega_{\min} \ln \left(\frac{1 + \frac{\omega_{\max}}{\omega_{\min}} \tan(\omega_{\max} t_1)}{1 - \frac{\omega_{\max}}{\omega_{\min}} \tan(\omega_{\max} t_1)} \right) + 2\omega_{\max}^2 t_1 < 0.
 \end{aligned}$$

It is always negative.

Remark 1.2. The optimal trajectory starts (and ends) at $(1, 0)$ with a vertical tangent. On $[0, t_1] \cup [T - t_1, T]$, the curve $t \mapsto (x(t), y(t))$ follows an ellipse of equation

$$x^2 + \frac{y^2}{\omega_{\max}^2} = 1.$$

For $u = u_{\max}$ constant, this ellipse is the level set $y^2 + u_{\max} x^2 = u_{\max}$ of the conserved energy of the harmonic oscillator.

The value $x(0) = x(T) = 1$ is the maximal value of $x(t)$ as $t \in [0, T]$.

On $[t_1, T - t_1]$, the curve $t \mapsto (x(t), y(t))$ follows a hyperbola of equation

$$x^2 - \frac{y^2}{\omega_{\min}^2} = c(t_1),$$

where $c(t_1) = \cos^2(\omega_{\max} t_1) (1 - \frac{\omega_{\max}^2}{\omega_{\min}^2} \tan^2(\omega_{\max} t_1)) > 0$. We have $y(\frac{T}{2}) = 0$, and $x(\frac{T}{2}) = \sqrt{c(t_1)}$ is the minimal value of $x(t)$. For $u = u_{\min}$ constant, these hyperbolas are the level sets $x^2 - \frac{y^2}{\omega_{\min}^2} = \text{cst}$ of the conserved quantity associated with the inverted oscillator.

The relation (1.6) gives T as a function of t_1 , which is increasing (see Lemma 2.7 further). In particular, the map $t_1 \mapsto T(t_1)$ is one-to-one and can be inverted, so that the switching time t_1 can equivalently be viewed as a function of the horizon T . When $T \rightarrow +\infty$, we have

- $t_1 \rightarrow \frac{1}{\omega_{\max}} \operatorname{Arctan}\left(\frac{\omega_{\min}}{\omega_{\max}}\right)$.
- $x(t_1) = \cos(\omega_{\max} t_1) \rightarrow \frac{\omega_{\max}}{\sqrt{\omega_{\min}^2 + \omega_{\max}^2}} = \sqrt{\frac{u_{\max}}{u_{\max} - u_{\min}}}$.
- $y(t_1) = -\omega_{\max} \sin(\omega_{\max} t_1) \rightarrow -\frac{\omega_{\min} \omega_{\max}}{\sqrt{\omega_{\min}^2 + \omega_{\max}^2}} = -\sqrt{\frac{-u_{\min} u_{\max}}{u_{\max} - u_{\min}}}$.
- The curve $t \mapsto (x(t), y(t))$, restricted to $[t_1, \frac{T}{2}]$ (along which $u(t) = u_{\min}$), converges to the segment joining the point $(\sqrt{\frac{u_{\max}}{u_{\max} - u_{\min}}}, -\sqrt{\frac{-u_{\min} u_{\max}}{u_{\max} - u_{\min}}})$ to the point $(0, 0)$. Moreover, for $t \in [t_1, \frac{T}{2}]$, we have

$$x(t) \sim \frac{\omega_{\max}}{\sqrt{\omega_{\min}^2 + \omega_{\max}^2}} \exp\left(\frac{\omega_{\min}}{\omega_{\max}} \operatorname{Arctan}\left(\frac{\omega_{\min}}{\omega_{\max}}\right)\right) \left(e^{\omega_{\min}(t-T)} + e^{-\omega_{\min}t}\right).$$

In particular

$$x\left(\frac{T}{2}\right) \sim \frac{2\omega_{\max}}{\sqrt{\omega_{\min}^2 + \omega_{\max}^2}} \exp\left(\frac{\omega_{\min}}{\omega_{\max}} \operatorname{Arctan}\left(\frac{\omega_{\min}}{\omega_{\max}}\right) - \omega_{\min} \frac{T}{2}\right).$$

- The cost of the trajectory satisfies $\int_0^T u(t) dt \sim Tu_{\min} < 0$.

Actually, when $T \rightarrow +\infty$, the curve $(x(\cdot), y(\cdot))$ converges to a “teardrop,” symmetric with respect to the x -axis, consisting of a “V” (two segments) rotated by $-\pi/2$, with edge at the origin, completed with an arc of ellipse. We refer to section 3.3 for more comments on this manifestation of the well known turnpike phenomenon.

2. Proof of Theorem 1.1.

2.1. Preliminaries and first reduction of the problem.

Existence of an optimal solution. We claim that there exists at least one optimal trajectory $(x(\cdot), y(\cdot), u(\cdot))$ solution of (1.1) and (1.2).

Indeed, we preliminary note that the (constant) trajectory defined by $x(t) = 1$, $y(t) = 0$ and $u(t) = 0$ for every $t \in [0, T]$ is a solution of (1.1a) and (1.1b) and (1.1c) and (1.2a) and (1.2b), hence the set of admissible trajectories is nonempty. Now, the existence of an optimal solution follows from standard existence results (see [16, Theorem 2.9] or see [3, 11]), noting that the control system (1.1a) and (1.1b) is control-affine, that the cost functional (1.1d) is convex, and that the controls are bounded (by (1.1c)).

Preliminary remarks. We start with some easy remarks.

LEMMA 2.1. *Given any optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of (1.1) and (1.2), and setting $q(\cdot) = (x(\cdot), y(\cdot))$:*

- (A₁) *We have $x(t)^2 + y(t)^2 > 0$ for every $t \in [0, T]$.*
- (A₂) *Extending $(x(\cdot), y(\cdot), u(\cdot))$ to the whole $\mathbb{R}$ by T -periodicity, for any $\delta \neq 0$, $(x(\cdot + \delta), y(\cdot + \delta), u(\cdot + \delta))$ (translation in time) is also an optimal solution of (1.1) and (1.2).*
- (A₃) *For any $\mu \neq 0$, $(\mu x(\cdot), \mu y(\cdot), u(\cdot))$ (homothety on $q(\cdot)$) is also an optimal solution of (1.1) and (1.2).*
- (A₄) *$x(\cdot)$ is nontrivial, $\dot{x}(\cdot)$ is continuous on $[0, T]$ and $\dot{x}(0) = \dot{x}(T)$.*

Proof. If $(x(\cdot), y(\cdot))$ passes through $(0, 0)$ then it must remain at $(0, 0)$ for every time, by Cauchy uniqueness, which contradicts (1.2b). This gives (A₁). Now, (A₂)

is obtained by using $(\mathbf{A}_1)$, and $(\mathbf{A}_3)$ is obvious. It remains to establish $(\mathbf{A}_4)$. By contradiction, if $x(\cdot) = 0$, then (1.1a) implies that $y(\cdot) = 0$, contradicting (1.2b). Hence $x(\cdot)$ is nontrivial. The second part of $(\mathbf{A}_4)$ follows from the facts that $\dot{x}(t) = y(t)$ by (1.1a) and that $y(\cdot)$ is continuous and T -periodic by (2.1). $\square$

Remark 2.2. According to $(\mathbf{A}_4)$, $x(\cdot)$ is C^1 and T -periodic. In contrast, $y(\cdot)$ is not C^1 on $[0, T]$. Indeed, $\dot{y}(t) = -u(x)x(t)$ by (1.1b) and u is not continuous, as it will be proved further.

First reduction of the problem. Combining $(\mathbf{A}_2)$, $(\mathbf{A}_3)$ and $(\mathbf{A}_4)$ of Lemma 2.1, without loss of generality we can replace (1.2a) and (1.2b) by

$$(2.1) \quad x(0) = x(T) = 1, \quad y(0) = y(T).$$

Hence, in what follows we consider the optimal control problem (1.1) and (2.1).

Note that, at this step, $y(0) = y(T)$ is let free. We will prove further that any solution of the optimal control problem (1.1) and (2.1) satisfies $y(0) = y(T) = 0$, i.e., (1.3) is satisfied. Further, we will also perform a second reduction to arrive at the state constraint $x(t) \leq 1$ (i.e., half of (1.4)) and establish the other half of (1.4).

The proof goes in several steps, by first applying the Pontryagin maximum principle and then establishing various properties. It turns out that the proof is far from being easy and does not follow straightforwardly from the Pontryagin maximum principle, as one could suspect at the first glance. This difficulty is probably due to the existence of too many symmetries and geometric transforms, that make the extremal equations, in some sense, somewhat degenerate. We will even have to resort to the Stokes theorem applied with a nonclassical one-differential form (different from the more classical clock form), thus employing arguments that are of a global nature. This is not so common in the study of optimal control problems.

2.2. Application of the Pontryagin maximum principle. The Hamiltonian of the optimal control problem (1.1) is

$$H(x, y, p_x, p_y, p^0, u) = p_x y - p_y x u + p^0 u.$$

Given any optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of (1.1) and (2.1) on $[0, T]$, by the Pontryagin maximum principle (see [11, 14, 16]), there exist an absolutely continuous *adjoint vector* $(p_x(\cdot), p_y(\cdot))$ on $[0, T]$ and $p^0 \leq 0$ such that $(p_x(\cdot), p_y(\cdot), p^0) \neq (0, 0, 0)$ and

$$(2.2a) \quad \dot{p}_x(t) = u(t)p_y(t)$$

$$(2.2b) \quad \dot{p}_y(t) = -p_x(t)$$

and

$$(2.3) \quad H(x(t), y(t), p_x(t), p_y(t), p^0, u(t)) = \max_{u_{\min} \leq v \leq u_{\max}} H(x(t), y(t), p_x(t), p_y(t), p^0, v)$$

for almost every $t \in [0, T]$. Defining the (absolutely continuous) *switching function* by

$$(2.4) \quad \varphi(t) = -p_y(t)x(t) + p^0 \quad \forall t \in [0, T],$$

the maximization condition (2.3) gives

$$\varphi(t)u(t) = \max_{u_{\min} \leq v \leq u_{\max}} \varphi(t)v$$

for almost every $t \in [0, T]$, which yields

$$(2.5) \quad u(t) = \begin{cases} u_{\min} & \text{if } \varphi(t) < 0 \\ u_{\max} & \text{if } \varphi(t) > 0 \end{cases}$$

for almost every $t \in [0, T]$. At this step, we state nothing on the closed subset I of $[0, T]$ where the continuous function φ vanishes identically. This set could be of positive measure and have a complicated structure. Actually, we will prove further in Lemma 2.10 (section 2.7) that I is of Lebesgue measure zero and thus (2.5) is enough to fully describe the optimal control almost everywhere. As a consequence of Lemma 2.10, since φ is continuous, the optimal control $u(\cdot)$ is *bang-bang*, i.e., the time interval $[0, T]$ is a countable union of open intervals along which either $u(t) = u_{\min}$ or $u(t) = u_{\max}$. But this result is far from being obvious and to prove we will first establish a number of other results.

For now, let us first finish to apply the Pontryagin maximum principle, which also gives the following additional information. The maximized Hamiltonian defined by

$$(2.6) \quad H_1(x(t), y(t), p_x(t), p_y(t), p^0) = p_x(t)y(t) + \max_{u_{\min} \leq v \leq u_{\max}} \varphi(t)v$$

for every $t \in [0, T]$ is constant on $[0, T]$. Note that, by the maximization condition (2.3), the optimal control $u(t)$ realizes the maximum in (2.6) for a.e. t , and thus $H_1(x(t), y(t), p_x(t), p_y(t), p^0) = H(x(t), y(t), p_x(t), p_y(t), p^0, u(t))$ for a.e. $t \in [0, T]$. Moreover, by the *transversality conditions* of the Pontryagin maximum principle, the periodicity condition $y(0) = y(T)$ (whose value is left free) of (2.1) implies that

$$(2.7) \quad p_y(0) = p_y(T)$$

(see [16, Section 2.2.3]), i.e., that $p_y(\cdot)$ is T -periodic. Actually, we are going to see that $p_x(\cdot)$ is T -periodic as well (see Lemma 2.4 further).

Recall that $(x(\cdot), y(\cdot), p_x(\cdot), p_y(\cdot), p^0, u(\cdot))$ is called an *extremal lift* of the optimal trajectory $(x(\cdot), y(\cdot), u(\cdot))$. The triple $(p_x(T), p_y(T), p^0)$ is defined up to scaling. The extremal is said to be *normal* if $p^0 \neq 0$, and in this case it is usual to normalize it so that $p^0 = -1$. It is said to be *abnormal* if $p^0 = 0$.

We next exploit the various conditions given by the Pontryagin maximum principle.

2.3. First properties. We start with an easy lemma.

LEMMA 2.3. *Given any optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of (1.1) and (2.1) and given any extremal lift $(x(\cdot), y(\cdot), p_x(\cdot), p_y(\cdot), p^0, u(\cdot))$ of it, the function $t \mapsto p_x(t)x(t) + p_y(t)y(t)$ is constant on $[0, T]$.*

Proof. The result obviously follows from the fact that

$$\frac{d}{dt}(p_x(t)x(t) + p_y(t)y(t)) = 0,$$

which is inferred from (1.1a), (1.1b), (2.2a) and (2.2b). $\square$

Note that Lemma 2.3 is valid independently on the constraints on the initial and final conditions.

LEMMA 2.4. *Given any optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of (1.1) and (2.1) and given any extremal lift $(x(\cdot), y(\cdot), p_x(\cdot), p_y(\cdot), p^0, u(\cdot))$ of it, we have*

$$p_x(0) = p_x(T), \quad p_y(0) = p_y(T).$$

In other words, the adjoint vector is T -periodic.

Proof. We infer from Lemma 2.3 that

$$p_x(0)x(0) + p_y(0)y(0) = p_x(T)x(T) + p_y(T)y(T).$$

Since $x(0) = x(T) = 1$ (by (2.1)) and $p_y(0) = p_y(T)$ (by (2.7)), the conclusion follows. $\square$

2.4. Second reduction of the problem.

LEMMA 2.5. *Shifting in time and using a homothety if necessary, without loss of generality, we can assume that any optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of (1.1) and (2.1) satisfies $y(0) = y(T) = 0$ and $x(t) \leq 1$ for every $t \in [0, T]$.*

Proof. Let $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{u}(\cdot))$ be an optimal solution of (1.1) and (2.1). Since $\tilde{x}(0) = \tilde{x}(T) = 1$, $\tilde{x}(\cdot)$ takes positive values. By continuity and compactness, let $t_1 \in [0, T]$ be such that

$$\tilde{x}(t_1) = \max_{t \in [0, T]} \tilde{x}(t).$$

Extending $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{u}(\cdot))$ by T -periodicity and setting

$$x^{t_1}(\cdot) = \tilde{x}(\cdot + t_1), \quad y^{t_1}(\cdot) = \tilde{y}(\cdot + t_1), \quad u^{t_1}(\cdot) = \tilde{u}(\cdot + t_1),$$

(translation in time), the triple $(x^{t_1}(\cdot), y^{t_1}(\cdot), u^{t_1}(\cdot))$ is an optimal solution of (1.1) and (1.2) (see **(A₂)** in Lemma 2.1). By **(A₁)** in Lemma 2.1, we have $x^{t_1}(0)^2 + y^{t_1}(0)^2 > 0$. Now, we set

$$\mu = \frac{1}{\sqrt{x^{t_1}(0)^2 + y^{t_1}(0)^2}}$$

and we define

$$x(\cdot) = \mu x^{t_1}(\cdot), \quad y(\cdot) = \mu y^{t_1}(\cdot), \quad u(\cdot) = u^{t_1}(\cdot).$$

By **(A₃)** in Lemma 2.1, $(x(\cdot), y(\cdot), u(\cdot))$ is an optimal solution of (1.1) and (1.2) satisfying, by construction, $x(0)^2 + y(0)^2 = 1$. Moreover, since $\tilde{x}(t_1) = \max_{t \in [0, T]} \tilde{x}(t)$ we have $x(0) = \mu \tilde{x}(t_1) = \max_{t \in [0, T]} x(t)$. Extending $x(\cdot)$ by T -periodicity to $\mathbb{R}$, the point $t = 0$ is a local maximizer of x , and since $x(\cdot)$ is C^1 at $t = 0$ (by **(A₄)** in Lemma 2.1), we infer that $\dot{x}(0) = 0$, hence $y(0) = y(T) = 0$ by (1.1a). The lemma is proved. $\square$

At this step of our analysis, thanks to Lemma 2.5, in what follows we consider the optimal control problem (1.1) with the state constraint

$$(2.8) \quad x(t) \leq 1 \quad \forall t \in [0, T],$$

and with the terminal conditions (1.3) (i.e., with respect to (2.1), we have moreover $y(0) = y(T) = 0$). We have not obtained yet that any optimal solution of that problem satisfies also $x(t) > 0$ for every $t \in [0, T]$, i.e., (1.4). This will be established in Lemma 2.9 in section 2.6.

By the previous results and in particular by Lemma 2.4, given any optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of (1.1) and (1.3) and (2.8) and given any extremal lift $(x(\cdot), y(\cdot), p_x(\cdot), p_y(\cdot), p^0, u(\cdot))$ of it, we have

$$(2.9) \quad p_x(t)x(t) + p_y(t)y(t) = \text{Cst} = p_x(0) \quad \forall t \in [0, T],$$

$$(2.10) \quad p_x(0) = p_x(T), \quad p_y(0) = p_y(T).$$

Moreover, recalling that the maximized Hamiltonian H_1 is defined by (2.6) and is constant along any extremal, we have

$$(2.11) \quad H_1(x(t), y(t), p_x(t), p_y(t), p^0) = \text{Cst} \geq 0,$$

and we denote by H_1 this constant (which depends on the extremal). Indeed, taking $t = 0$ and noting that $y(0) = 0$, we have

$$(2.12) \quad H_1 = \max_{u_{\min} \leq v \leq u_{\max}} \varphi(0)v = \begin{cases} \varphi(0)u_{\max} & \text{if } \varphi(0) > 0, \\ \varphi(0)u_{\min} & \text{if } \varphi(0) < 0, \\ 0 & \text{if } \varphi(0) = 0. \end{cases}$$

2.5. Analysis of the periodic trajectory defined in Theorem 1.1. In this section, we consider the trajectory associated with the control u defined by (1.5) and starting at $(1, 0)$ (we will prove further that this is the actual optimal trajectory of the optimal control problem (1.1) and (1.3)).

So, we temporarily forget the optimal control problem. We only consider the control system (1.1a) and (1.1b) and (1.1c), with the initial condition $(x(0), y(0)) = (1, 0)$.

Let $T > 0$ and let $t_1 \in (0, \frac{1}{\omega_{\max}} \text{Arctan}(\frac{\omega_{\min}}{\omega_{\max}}))$ be arbitrary, and let u be the control defined by (1.5), i.e., $u(t) = u_{\max}$ if $0 < t < t_1$, $u(t) = u_{\min}$ if $t_1 < t < T - t_1$ and $u(t) = u_{\max}$ if $T - t_1 < t < T$. This control, which satisfies the control constraint (1.1c), generates a unique trajectory $(x(\cdot), y(\cdot))$ solution of (1.1a) and (1.1b) such that $(x(0), y(0)) = (1, 0)$.

PROPOSITION 2.6. *Given any $T > 0$, there exists a unique choice of a time $t_1 \in (0, \frac{1}{\omega_{\max}} \text{Arctan}(\frac{\omega_{\min}}{\omega_{\max}}))$ such that the trajectory $(x(\cdot), y(\cdot))$ defined above is T -periodic, i.e., satisfies $x(T) = x(0) = 1$ and $y(T) = y(0) = 0$. In turn, this trajectory satisfies the equalities (1.6), (1.8), (1.9) stated in Theorem 1.1, and the various properties stated in Remark 1.2. It is drawn on Figure 1.*

Proof. When $t \in (0, t_1)$, we have $u(t) = u_{\max} = \omega_{\max}^2$ (see (1.7)) and integrating (1.1a) with $x(0) = 1$ yields $x(t) = \cos(\omega_{\max}t)$, i.e., the first part of (1.8). Along this interval, the curve $(x(\cdot), y(\cdot))$ follows the ellipse of equation $x^2 + \frac{y^2}{\omega_{\max}^2} = 1$. Note that, since $\tan(\omega_{\max}t_1) < \frac{\omega_{\min}}{\omega_{\max}}$, we have $-\omega_{\max} \sin(\omega_{\max}t_1) > -\omega_{\min} \cos(\omega_{\max}t_1)$.

When $t \in (t_1, T - t_1)$, we have $u(t) = u_{\min} = -\omega_{\min}^2$ and integrating (1.1a) with $x(t_1) = \cos(\omega_{\max}t_1)$ yields

$$x(t) = Ae^{\omega_{\min}t} + Be^{-\omega_{\min}t}.$$

Since $x(t_1) = \cos(\omega_{\max}t_1)$ and $\dot{x}(t_1) = -\omega_{\max} \sin(\omega_{\max}t_1)$, we infer that

$$A = \frac{1}{2} \left(\cos(\omega_{\max}t_1) - \frac{\omega_{\max}}{\omega_{\min}} \sin(\omega_{\max}t_1) \right) e^{-\omega_{\min}t_1},$$

$$B = \frac{1}{2} \left(\cos(\omega_{\max}t_1) + \frac{\omega_{\max}}{\omega_{\min}} \sin(\omega_{\max}t_1) \right) e^{\omega_{\min}t_1}.$$

This gives the second part of (1.8). Note that $A > 0$. There, the curve $(x(\cdot), y(\cdot))$ follows a hyperbola of equation

$$x^2 - \frac{y^2}{\omega_{\min}^2} = c(t_1) = \cos^2(\omega_{\max} t_1) \left(1 - \frac{\omega_{\max}^2}{\omega_{\min}^2} \tan^2(\omega_{\max} t_1) \right).$$

Let us prove that there exists a unique choice of $t_1 \in (0, \frac{1}{\omega_{\max}} \text{Arctan}(\frac{\omega_{\min}}{\omega_{\max}}))$ such that this curve crosses the x -axis exactly at time $\frac{T}{2}$, i.e., such that $y(\frac{T}{2}) = 0$. Because the control is symmetric with respect to $T/2$ and the dynamics are invariant under the time-reversal symmetry $(t, y) \mapsto (T - t, -y)$, this midpoint condition is equivalent to the periodicity constraints $x(T) = x(0)$ and $y(T) = y(0)$, by the reflection argument used below. The equality $y(\frac{T}{2}) = 0$ is satisfied if and only if $Ae^{\omega_{\min} \frac{T}{2}} - Be^{-\omega_{\min} \frac{T}{2}} = 0$, i.e., $e^{\omega_{\min} T} = \frac{B}{A}$, which leads to (1.6) giving T as a function of t_1 . Hence, the claim is true if we can prove that the function $t_1 \mapsto T(t_1)$ is bijective. This indeed follows from the lemma below.

LEMMA 2.7. *The function $t_1 \mapsto T(t_1)$, defined by (1.6), is increasing.*

Proof of Lemma 2.7. A computation shows that

$$\frac{dT}{dt_1} = 2 \left(1 + \frac{\omega_{\max}^2}{\omega_{\min}^2} \right) \frac{1}{1 - \frac{\omega_{\max}^2}{\omega_{\min}^2} \tan^2(\omega_{\max} t_1)}$$

and this quantity is positive since $\frac{\omega_{\max}}{\omega_{\min}} \tan(\omega_{\max} t_1) < 1$, because t_1 belongs to the interval $(0, \frac{1}{\omega_{\max}} \text{Arctan}(\frac{\omega_{\min}}{\omega_{\max}}))$. $\square$

Therefore, at this step we have proved that there is a unique choice for t_1 such that $y(\frac{T}{2}) = 0$. Then, completing the construction of the curve on $[\frac{T}{2}, T] = [\frac{T}{2}, T - t_1] \cup [T - t_1, T]$ gives a trajectory that is symmetric with respect to the x -axis. This proves that $x(T) = x(0) = 1$ and $y(T) = y(0) = 0$.

The cost $\int_0^T u(t) dt$ of that trajectory is equal to (1.9). It is however a nontrivial fact that this cost is always negative. This fact follows from the fact that

$$\begin{aligned} \frac{d}{dt_1} \left(-\omega_{\min} \ln \left(\frac{1 + \frac{\omega_{\max}}{\omega_{\min}} \tan(\omega_{\max} t_1)}{1 - \frac{\omega_{\max}}{\omega_{\min}} \tan(\omega_{\max} t_1)} \right) + 2\omega_{\max}^2 t_1 \right) \\ = -2 \frac{\omega_{\max}^2}{\omega_{\min}^2} (\omega_{\min}^2 + \omega_{\max}^2) \frac{\tan^2(\omega_{\max} t_1)}{1 - \frac{\omega_{\max}^2}{\omega_{\min}^2} \tan^2(\omega_{\max} t_1)} \end{aligned}$$

and this quantity is negative since $\frac{\omega_{\max}}{\omega_{\min}} \tan(\omega_{\max} t_1) < 1$, because t_1 belongs to the interval $(0, \frac{1}{\omega_{\max}} \text{Arctan}(\frac{\omega_{\min}}{\omega_{\max}}))$. Therefore the function that is differentiated above is decreasing, and since it is equal to 0 when $t_1 = 0$, it is always negative on the open interval $(0, \frac{1}{\omega_{\max}} \text{Arctan}(\frac{\omega_{\min}}{\omega_{\max}}))$.

The various limits stated in Remark 1.2 are now easily checked. To get the equivalent of $x(t)$ on $[t_1, \frac{T}{2}]$, we note that $A = Be^{-\omega_{\min} T}$ and thus $x(t) = B(e^{\omega_{\min}(t-T)} + e^{-\omega_{\min} t})$, and we compute the limit of B as $T \rightarrow +\infty$. $\square$

Again, at this step we do not know yet that the trajectory $(x(\cdot), y(\cdot), u(\cdot))$ defined above is the optimal solution of (1.1) and (2.1). But, at least, it is an admissible solution, i.e., it satisfies (1.1a) and (1.1b) and (1.1c) and $x(0) = x(T) = 1$ and $y(0) = y(T) = 0$.

COROLLARY 2.8. *Let s_1, s_2, T be arbitrary real numbers such that $0 \leq s_1 < s_2 \leq T$, and let $\beta > 0$ be arbitrary. There exists a trajectory $(x_{s_1, s_2}(\cdot), y_{s_1, s_2}(\cdot), u_{s_1, s_2}(\cdot))$, solution of (1.1a) and (1.1b) and (1.1c) on $[s_1, s_2]$, such that $x_{s_1, s_2}(s_1) = x_{s_1, s_2}(s_2) = \beta$ and $y_{s_1, s_2}(s_1) = y_{s_1, s_2}(s_2) = 0$, of negative cost, i.e.,*

$$(2.13) \quad \int_{s_1}^{s_2} u_{s_1, s_2}(t) dt < 0,$$

and such that $x_{s_1, s_2}(t) \leq \beta$ for every $t \in [s_1, s_2]$.

Proof. We apply Proposition 2.6 with $T = s_2 - s_1$, then we shift in time to get a trajectory $(x_{s_1, s_2}(\cdot), y_{s_1, s_2}(\cdot), u_{s_1, s_2}(\cdot))$ solution of (1.1a) and (1.1b) and (1.1c) on $[s_1, s_2]$, with $x_{s_1, s_2}(s_1) = x_{s_1, s_2}(s_2) = 1$ and $y_{s_1, s_2}(s_1) = y_{s_1, s_2}(s_2) = 0$. Then, using a homothety argument (like in **(A₃)** in Lemma 2.1), we modify the trajectory (but not the control) so that $x_{s_1, s_2}(s_1) = x_{s_1, s_2}(s_2) = \beta$ and $y_{s_1, s_2}(s_1) = y_{s_1, s_2}(s_2) = 0$. The computations done in the proof of Proposition 2.6, in order to prove that the cost (1.9) is negative, show that (2.13) holds true. $\square$

Corollary 2.8 shows that we can always create a periodic trajectory, with an arbitrarily small period, solution of (1.1a) and (1.1b) and (1.1c) and making a loop from $(\beta, 0)$ to $(\beta, 0)$, for any $\beta > 0$, with a negative cost. This nontrivial fact will be useful further.

2.6. Optimal trajectories are contained in $0 < x \leq 1$.

LEMMA 2.9. *Any optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of (1.1) and (1.3) and (2.8) satisfies $0 < x(t) \leq 1$ for every $t \in [0, T]$. Hence, the optimal control problem (1.1) and (1.3) and (1.4) is equivalent to the optimal control problem (1.1) and (1.3) and (2.8).*

Proof. Let $(x(\cdot), y(\cdot), u(\cdot))$ be an optimal solution of (1.1) and (1.3). We already know that $x(t) \leq 1$ for every $t \in [0, T]$. By continuity and compactness, we can define

$$x_{\min} = \min_{t \in [0, T]} x(t) = x(t_{\min})$$

for some $t_{\min} \in (0, T)$. Since $\dot{x}(t) = y(t)$, we must have $y(t_{\min}) = 0$. We are going to prove that $x_{\min} > 0$.

Let us consider the family of hyperbolas

$$\mathcal{H}_c = \left\{ (x, y) \in \mathbb{R}^2 \mid x > 0, \quad x^2 - \frac{y^2}{\omega_{\min}^2} = c \right\}$$

indexed by $c > 0$. Recall that, when $c = c(t_1) = \cos^2(\omega_{\max} t_1) (1 - \frac{\omega_{\max}^2}{\omega_{\min}^2} \tan^2(\omega_{\max} t_1))$ (which is positive), the hyperbola $\mathcal{H}_c$ contains the trajectory constructed in Proposition 2.6, drawn on Figure 1, restricted to $[t_1, T - t_1]$. But now we consider the whole family of hyperbolas $(\mathcal{H}_c)_{c>0}$. We make two useful remarks:

- (H₁) The hyperbolas $\mathcal{H}_c$ are invariant under the dynamics (1.1a) and (1.1b) with $u = u_{\min}$. More precisely, for all $c > 0$, $x_0 > 0$ and $y_0 \in \mathbb{R}$ such that $(x_0, y_0) \in \mathcal{H}_c$, the unique solution of the control system (1.1a) and (1.1b) and (1.1c) with $u(t) = u_{\min}$ and $x(0) = x_0$, $y(0) = y_0$, satisfies $(x(t), y(t)) \in \mathcal{H}_c$ for every $t \in \mathbb{R}$.
- (H₂) Let $y_0 < 0$ and $y_1 > 0$ be arbitrary. Given any $T > 0$, there exist a unique $c > 0$ and a unique $x_0 > 0$ such that $(x_0, y_0) \in \mathcal{H}_c$ and such that the unique solution of the control system (1.1a) and (1.1b) and (1.1c) with $u(t) = u_{\min}$ and $x(0) = x_0$, $y(0) = y_0$, satisfies $y(T) = y_1$. Moreover, $c \rightarrow 0$ if $T \rightarrow +\infty$.

These preliminary remarks being done, to prove the lemma, we argue by contradiction. Let us assume that $x_{\min} \leq 0$.

The curve $(x(\cdot), y(\cdot))$ is periodic, contained in the strip $x_{\min} \leq x \leq 1$, and satisfies $(x(0), y(0)) = (x(T), y(T)) = (1, 0)$ and $(x(t_{\min}), y(t_{\min})) = (x_{\min}, 0)$. Moreover, since

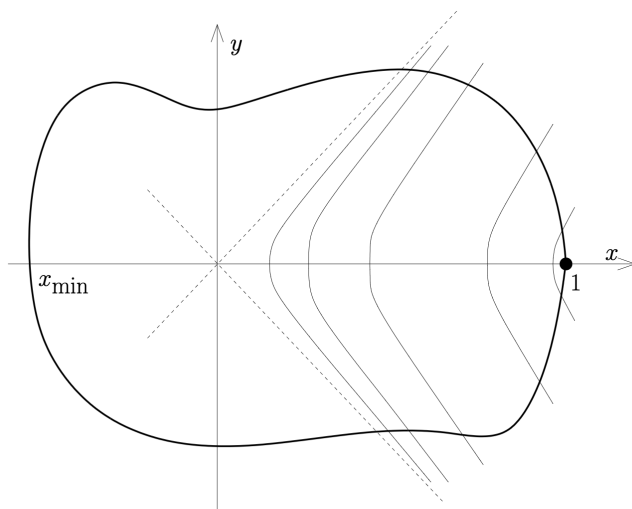


FIG. 2. Illustration of the curve intersecting hyperbolas.

$\dot{x}(t) = y(t)$, the function $t \mapsto x(t)$ is decreasing in the half-plane $y < 0$ and increasing in the half-plane $y > 0$. It turns clockwise as $t \in [0, T]$.

This curve intersects each hyperbola $\mathcal{H}_c$, for every $c \in (0, 1]$ (see Figure 2). For every $c \in (0, 1]$, choose two points of intersection $(x(s_1), y(s_1))$ and $(x(s_2), y(s_2))$ with $\mathcal{H}_c$ such that $x(s_1) > 0$, $y(s_1) < 0$ (and $0 < s_1 < t_{\min}$) and $x(s_2) > 0$, $y(s_2) > 0$ (and $t_{\min} < s_2 < T$). Then, by Remark **(H₁)**, the trajectory solution of the control system (1.1a) and (1.1b) and (1.1c) with $u(t) = u_{\min}$, starting at $(x(s_1), y(s_1))$, follows the hyperbola $\mathcal{H}_c$ and reaches $(x(s_2), y(s_2))$ in a time $\delta > 0$ that ranges continuously from 0 when $c = 1$ to $+\infty$ when $c \rightarrow 0$ (by Remark **(H₂)**). Moreover, this hyperbolic arc is obviously optimal for the cost $\int_0^\delta u$, since $u = u_{\min}$ along this arc. Therefore, there exists $c \in (0, 1]$ such that $\delta = s_2 - s_1$. Hence, we have obtained a trajectory consisting of $(x(\cdot), y(\cdot), u(\cdot))$ on $[0, s_1]$, then of the hyperbolic trajectory (with $u = u_{\min}$) on $[s_1, s_2]$ that arrives exactly at $(x(s_2), y(s_2))$ at time s_2 , then again of $(x(\cdot), y(\cdot), u(\cdot))$ on $[s_2, T]$.

This new trajectory, obtained by concatenation, has a lower cost than the initial trajectory $(x(\cdot), y(\cdot), u(\cdot))$, since

$$\int_0^{s_1} u(t) dt + (s_2 - s_1)u_{\min} + \int_{s_2}^T u(t) dt < \int_0^{s_1} u(t) dt + \int_{s_1}^{s_2} u(t) dt + \int_{s_2}^T u(t) dt$$

unless $u(t) = u_{\min}$ on $[s_1, s_2]$, which is impossible because otherwise the trajectory would follow the hyperbola along this interval and would not penetrate the region $x < 0$, while we have assumed that $x_{\min} < 0$. But then, we have reached a contradiction, because $(x(\cdot), y(\cdot), u(\cdot))$ is optimal. The lemma is proved. $\square$

2.7. Optimal controls are bang-bang. As said in section 2.2, the maximization condition of the Pontryagin maximum principle has led to (2.5), i.e., $u(t) = u_{\min}$ if $\varphi(t) < 0$ and $u(t) = u_{\max}$ if $\varphi(t) > 0$, where $\varphi(t) = -p_y(t)x(t) + p^0$ is the switching function defined by (2.4), but we concluded nothing on $u(t)$ on the closed set

$$(2.14) \quad I = \{t \in [0, T] \mid \varphi(t) = 0\},$$

where the continuous function φ vanishes identically. Of course, (2.5) is enough if I is of zero Lebesgue measure. However, it could be that I be of positive Lebesgue measure and have a complicated structure. The next lemma shows that, fortunately, this complicated situation does not occur. The proof is however not straightforward.

LEMMA 2.10. *Let $(x(\cdot), y(\cdot), u(\cdot))$ be any optimal solution of (1.1) and (1.3) and (1.4) and let I be defined by (2.14). Then I is of zero Lebesgue measure, and thus the optimal control $u(\cdot)$ is bang-bang, fully described by (2.5).*

Proof. In this proof, given any measurable function f on $[0, T]$ and any subset $J \subset [0, T]$ of positive Lebesgue measure, we write “ $f = 0$ a.e. on J ” to say that $f(t) = 0$ for almost every $t \in J$. We recall that, when f is absolutely continuous and thus almost everywhere differentiable, $f = 0$ a.e. on I implies that $\dot{f} = 0$ a.e. on I (see [15, Lemma p. 177] or [5, Theorem 6.3 p. 262 and Remark p. 264]). Note that, when $g = 0$ a.e. on I for some continuous function g then $g(t) = 0$ for every $t \in I$ of positive density in I .

The proof of the lemma goes by contradiction. Let us assume that the set I defined by (2.14) is of positive Lebesgue measure. We have $\varphi = 0$ on I , i.e.,

$$(2.15) \quad p_y x = p^0 \quad \text{on } I.$$

Since I is of positive Lebesgue measure, derivating two times (2.15) and using the dynamical equations (1.1a), (1.1b), (2.2a) and (2.2b), we obtain

$$(2.16) \quad p_x x = p_y y \quad \text{a.e. on } I,$$

$$(2.17) \quad -p^0 u = p_x y \quad \text{a.e. on } I.$$

There are two cases, depending on whether $p^0 = 0$ or not.

First case: $p^0 = 0$ (abnormal case). Then $p_x x = p_y y = p_y x = p_x y = 0$ a.e. on I . Note that: (1) we cannot have $x(t) = y(t) = 0$ for some $t \in [0, T]$ by **(A₁)** in Lemma 2.1; (2) we cannot have $p_x(t) = p_y(t) = 0$ for some $t \in [0, T]$, for otherwise $(p_x(t), p_y(t), p^0) = (0, 0, 0)$, which contradicts the nontriviality of this triple stated in the Pontryagin maximum principle. These two remarks and the four above cancellations a.e. on I lead to a contradiction, and thus this case does not occur.

Second case: $p^0 \neq 0$ (normal case). Since $(p_x(T), p_y(T), p^0)$ is defined up to scaling, we choose to normalize it, as usual (see [11, 14, 16]), so that $p^0 = -1$. From (2.15) and (2.17), we have then

$$(2.18) \quad p_y x = -1 \quad \text{on } I,$$

$$(2.19) \quad u = p_x y \quad \text{a.e. on } I.$$

Recalling that the maximized Hamiltonian H_1 defined by (2.6) is constant on $[0, T]$ along any extremal and that $H_1 \geq 0$ (see (2.11)), and denoting by H_1 this constant, we have $p_x y = H_1$ on I (because $\varphi = 0$ on I), and thus, by (2.19)

$$(2.20) \quad u = p_x y = H_1 \geq 0 \quad \text{a.e. on } I.$$

In particular, u is almost everywhere constant on I , and this constant is H_1 that is nonnegative. We are now going to prove that $H_1 = 0$.

By (2.9), we have $p_x(t)x(t) + p_y(t)y(t) = p_x(0)$ for every $t \in [0, T]$. Using (2.16), we infer that

$$(2.21) \quad p_x x = p_y y = \frac{1}{2} p_x(0) \quad \text{a.e. on } I.$$

Let us prove that $p_x(0) = 0$. By contradiction, if $p_x(0) \neq 0$, we consider a small interval $[s_1, s_2] \subset [0, T]$, with $s_1 < s_2$, such that $[s_1, s_2] \cap I$ has a positive measure, with $s_2 - s_1$ small enough so that, by continuity, x , y , p_x and p_y do not vanish on $[s_1, s_2]$. By Lemma 2.9, we have $x(t) > 0$, and by (2.18) we must have $p_y(t) < 0$ on $[s_1, s_2]$. By (2.20) we must have $\text{sign}(p_x(t)) = \text{sign}(y(t))$, and by (2.21), $\text{sign}(p_x(t)) = \text{sign}(p_x(0))$ and $\text{sign}(y(t)) = -\text{sign}(p_x(0))$, hence $y(t)$ and $p_x(t)$ have opposite signs. But this contradicts (2.20).

Therefore, $p_x(0) = 0$. Then, by (2.21), $p_x x = p_y y = 0$ a.e. on I , but since p_y and x cannot vanish by (2.18), it follows that $p_x = y = 0$ a.e. on I . Derivating $y = 0$ a.e. on I , we finally obtain $u = 0$ a.e. on I . This also proves that $H_1 = 0$.

Now, since $H_1 = 0$ is constant on the interval $[0, T]$, using (2.12) (or, taking $t = 0$ in (2.6) and noting that $y(0) = 0$), we must have $\varphi(0) = 0$. Since $\varphi(0) = -p_y(0)x(0) - 1$ and $x(0) = 1$, this implies that $p_y(0) = -1$.

At this step, we have thus obtained that $p_x(0) = 0$ and $p_y(0) = -1$.

Inspecting the adjoint differential equations (2.2a) and (2.2b), we observe that the triple $(-p_y(\cdot), p_x(\cdot), u(\cdot))$ is solution of (1.1a) and (1.1b), like the triple $(x(\cdot), y(\cdot), u(\cdot))$, with the same control $u(\cdot)$ and the same initial condition $(1, 0)$ (indeed, note that $(-p_y(0), p_x(0)) = (1, 0)$). By Cauchy uniqueness it follows that $p_x(t) = y(t)$ and $p_y(t) = -x(t)$ for every $t \in [0, T]$. But then, since $H_1 = 0$ is constant on $[0, T]$, using again (2.6) we infer that

$$y(t)^2 + \max_{u_{\min} \leq v \leq u_{\max}} \varphi(t)v = 0 \quad \forall t \in [0, T]$$

and since the above maximum is nonnegative, we must have $y(t)^2 = 0$ thus $y(t) = 0$ for every $t \in [0, T]$. Using (1.1a) and (1.1b) and $x(0) = 1$, this implies that $(x(t), y(t)) = (1, 0)$ for every $t \in [0, T]$ and $u(t) = 0$ for almost every $t \in [0, T]$. But this trivial solution is not optimal because its cost is equal to 0, while the loop trajectory constructed in Proposition 2.6 has a negative cost (in time T), thus does better. We have thus obtained a contradiction, and the lemma is proved. $\square$

Remark 2.11. As suggested by a referee, in the second half of the above proof, one may rule out singular arcs more directly by invoking the generalized Legendre–Clebsch (Kelley) necessary condition for scalar control-affine problems; see, e.g., [2, Chap. 5, Section 5.6.1, page 129]. To apply this condition, one may rewrite the Lagrange problem as a Mayer problem by adding the state $z' = u$ and minimizing $z(T)$. Recalling that the switching function is $\varphi(t) = p^0 - p_y(t)x(t)$, along a (normal) singular arc, one has $\varphi = \dot{\varphi} = \ddot{\varphi} = 0$, and a computation yields $\ddot{\varphi} = 2p_x y + 2u p_y x$, hence $\partial_u \ddot{\varphi} = 2p_y x$. Since $\varphi = 0$ implies $p_y x = p^0$ on the singular arc, we obtain $\partial_u \ddot{\varphi} = 2p^0$. In the normal case $p^0 = -1$, this gives $\partial_u \ddot{\varphi} = -2 < 0$, which contradicts the generalized Legendre–Clebsch (Kelley) condition (for a first-order singular extremal) in our convention.

2.8. Uniqueness of the optimal trajectory.

PROPOSITION 2.12. *The curve constructed in Proposition 2.6 is the unique optimal solution of (1.1) and (1.3) and (1.4).*

Proof. Let $(x(\cdot), y(\cdot), u(\cdot))$ be an optimal solution of (1.1) and (1.3) and (1.4). By the proof of Lemma 2.9, we have $0 < x_{\min} \leq x(t) \leq 1$ for every $t \in [0, T]$, where $x_{\min} > 0$ is the minimal value of $x(t)$. Besides, as a consequence of Lemma 2.10, the time interval $[0, T]$ is a countable union of open intervals along which either $u(t) = u_{\min}$, and then the curve $(x(\cdot), y(\cdot))$ follows clockwise an arc of hyperbola (according to the computations done in the proof of Proposition 2.6), or $u(t) = u_{\max}$, and then

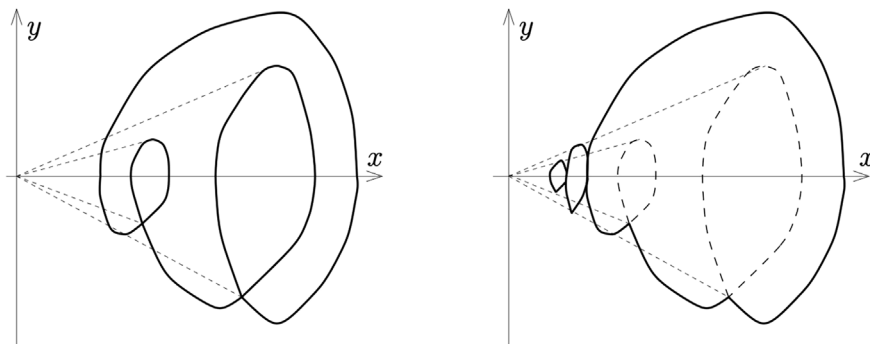


FIG. 3. *Surgery. On the left figure, the initial curve (in plain). On the right figure, periodic arcs have been cut and homothetized to the left.*

the curve $(x(\cdot), y(\cdot))$ follows clockwise an arc of ellipse. Moreover, since $\dot{x} = y$ and $\dot{y} = -ux$, the function $t \mapsto x(t)$ is decreasing in the half-plane $y < 0$, and increasing in the half-plane $y > 0$. The curve $(x(\cdot), y(\cdot))$ turns clockwise and can cross the x -axis only with a vertical tangent.

The structure of the control $u(\cdot)$ may be complicated, though: a priori, $u(\cdot)$ might switch an infinite number of times, but the set $\{t \in [0, T] \mid \varphi(t) = 0\}$ is of measure zero.

Like in Lemma 2.9, we denote by $x_{\min}$ the minimal value of $x(t)$ over all possible $t \in [0, T]$, and let $t_{\min} \in (0, T)$ be such that $x(t_{\min}) = x_{\min}$. The function $t \mapsto x(t)$ may fail to be decreasing on $[0, t_{\min}]$: it may happen that the curve $(x(\cdot), y(\cdot))$ crosses the x -axis at some time $\bar{t} \in (0, t_{\min})$, i.e., with $x_{\min} < x(\bar{t}) < 1$ and $y(\bar{t}) = 0$, then penetrates in the region $y > 0$ (where $t \mapsto x(t)$ is increasing) and comes back later in the region $y < 0$ (see Figure 3 on the left).

The proof goes by contradiction. Assuming that $(x(\cdot), y(\cdot), u(\cdot))$ differs from the trajectory constructed in Proposition 2.6, we are going to build a new trajectory with the same terminal conditions, having a lower cost, thus reaching a contradiction. Without loss of generality, we assume that the trajectories differ in the half-plane $y \leq 0$.

Let us write the interval $[0, t_{\min}]$ as the countable union of some intervals I_k and J_p (all disjoint two by two), for k and p ranging in some countable set, such that

- on each open interval I_k , x is decreasing and $y \leq 0$,
- on each closed interval J_p , the curve $(x(\cdot), y(\cdot))$ is periodic, i.e., $x(\min J_p) = x(\max J_p)$ and $y(\min J_p) = y(\max J_p) \leq 0$ (it makes one or several loops),

as illustrated on Figure 3, on the left.

The proof now goes in three steps.

Step 1. We first make some “surgery,” in order to build another curve having the same cost (and thus, being optimal as well): take an arbitrary periodic arc, parametrized on an interval J_p , cut it from the original curve, apply to it a homothety (from the origin) so as to glue it to the left of the curve; then repeat this operation for all such periodic arcs index by p (this can be done in any order, and there may be an infinite countable number of such arcs). By **(A₃)** in Lemma 2.1, this homothety does not affect the control along J_p and thus does not change its cost.

Now, we reparametrize the newly obtained curve (of which the minimal value $\tilde{x}_{\min}$ of the x -component is now lower than $x_{\min}$), by shifting in time and concatenating,

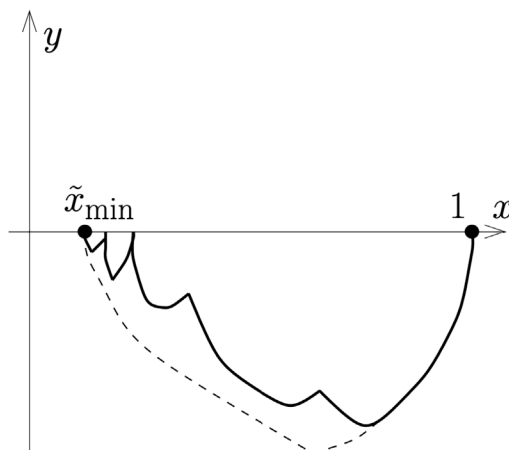


FIG. 4. In plain, the curve $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ restricted to $[0, \tilde{t}_{\min}]$, obtained after surgery. In dashed, the comparison trajectory $(\hat{x}(\cdot), \hat{y}(\cdot))$ used in Step 2, namely the restriction to $y \leq 0$ of the optimal loop of Proposition 2.6 with period $2\hat{t}_{\min}$ such that $(x(\hat{t}_{\min}), y(\hat{t}_{\min})) = (\tilde{x}_{\min}, 0)$.

so that the new trajectory, denoted $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{u}(\cdot))$, is still defined on $[0, T]$ and now consists of a countable number of successive arcs as follows: there exists $\tilde{t}_{\min} \in (0, t_{\min}]$ such that, on $[0, \tilde{t}_{\min}]$, the function $t \mapsto \tilde{x}(t)$ is decreasing, with $\tilde{x}(\tilde{t}_{\min}) = \tilde{x}_{\min}$ that is the minimal value of $\tilde{x}(t)$ over all possible $t \in [0, T]$. The part of $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ that is contained in the half-plane $y \leq 0$ is drawn on Figure 4. On the interval $[\tilde{t}_{\min}, t_{\min}]$, we keep the pieces of the homothetized arcs contained in $y \geq 0$ until we reach the point $(x_{\min}, 0)$, and then we set $(\tilde{x}(t), \tilde{y}(t), \tilde{u}(t)) = (x(t), y(t), u(t))$ on the time interval $[t_{\min}, T]$, this piece steering $(x_{\min}, 0)$ to $(1, 0)$ (see Figure 3 on the right, in the half-plane $y \geq 0$). The cost of the new trajectory $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{u}(\cdot))$ is the same as the initial one $(x(\cdot), y(\cdot), u(\cdot))$, i.e.,

$$(2.22) \quad \int_0^T \tilde{u}(t) dt = \int_0^T u(t) dt.$$

In particular, the triple $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{u}(\cdot))$ is an optimal solution of (1.1) and (2.8) and (1.3), like $(x(\cdot), y(\cdot), u(\cdot))$. Actually, the control $\tilde{u}(\cdot)$ is a kind of rearrangement of the initial control $u(\cdot)$, obtained by shifting in time some subintervals and rearranging them differently, without changing the time interval $[0, T]$. Moreover, $(\dot{\tilde{x}}(t), \dot{\tilde{y}}(t)) \neq (0, 0)$ for almost every $t \in [0, T]$, and even, for every $t \in [0, T]$ except maybe on a countable subset of $[0, T]$ thanks to the monotonicity of $\tilde{x}(\cdot)$. This shows that $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ is a Jordan curve, what we are going to use in the second step.

Note also that, since we have assumed (by contradiction) that $(x(\cdot), y(\cdot), u(\cdot))$ differs from the trajectory constructed in Proposition 2.6, the same holds true for the new trajectory $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{u}(\cdot))$.

Step 2. As a second step, based on $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{u}(\cdot))$, we are now going to construct a new trajectory $(\hat{x}(\cdot), \hat{y}(\cdot), \hat{u}(\cdot))$ defined on $[0, T]$, with a (strictly) lower cost, thus reaching a contradiction.

The strategy is completely different from the one used in Step 1. To explain it, we start by noting that the control system (1.1a) and (1.1b) is control-affine, written as

$$(2.23) \quad \dot{q}(t) = X(q(t)) + u(t)Y(q(t)),$$

where $q = (x, y)$ and the two vector fields X and Y on $\mathbb{R}^2$ are defined by

$$X(q) = y \frac{\partial}{\partial x} \quad \text{and} \quad Y(q) = -x \frac{\partial}{\partial y}.$$

Let α and β be the smooth differential one-forms on $(0, +\infty) \times (-\infty, 0)$ defined by

$$\alpha = \frac{dx}{y} \quad \text{and} \quad \beta = -\frac{dy}{x}.$$

Note that $d\alpha = \frac{dx \wedge dy}{y^2}$ and $d\beta = \frac{dx \wedge dy}{x^2}$. By definition of α and β , we have $\alpha(X) = 1$ and $\alpha(Y) = 0$, $\beta(X) = 0$ and $\beta(Y) = 1$. Hence $\alpha_{q(t)}(\dot{q}(t)) = 1$ and $\beta_{q(t)}(\dot{q}(t)) = u(t)$ for almost every t , for any given solution $q(\cdot)$ of (2.23) such that $y(t) \neq 0$ almost everywhere.

The use of one-differential forms, combined with the application of the Green–Riemann (Stokes) theorem, to get optimality properties for trajectories in the plane is known in optimal control (see [9], see also [2] where the one-form α is referred to as the clock form), but the above form β is nonclassical, up to our knowledge.

The reasoning goes as follows. First of all, we consider, in the half-plane $y \leq 0$, the trajectory constructed in Proposition 2.6, denoted $(\hat{x}(\cdot), \hat{y}(\cdot), \hat{u}(\cdot))$, that steers the control system (1.1a) and (1.1b) and (1.1c) from $(1, 0)$ to $(\tilde{x}_{\min}, 0)$ in time denoted by $\tilde{t}_{\min} > 0$ (and thus, it goes from $(1, 0)$ to $(1, 0)$ in time $2\tilde{t}_{\min}$). The curve $(\hat{x}(\cdot), \hat{y}(\cdot))$ is drawn in dashed on Figure 4.

Hence, we now have two trajectories steering the control system (1.1a) and (1.1b) and (1.1c) from $(1, 0)$ to $(\tilde{x}_{\min}, 0)$:

- $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{u}(\cdot))$ (in plain on Figure 4) does it in time $\tilde{t}_{\min}$,
- $(\hat{x}(\cdot), \hat{y}(\cdot), \hat{u}(\cdot))$ (in dashed on Figure 4) does it in time $\hat{t}_{\min}$.

Let us consider the curve Γ , making a counterclockwise loop from $(1, 0)$ to $(1, 0)$, by first following the curve $(\tilde{x}(\cdot), \tilde{y}(\cdot))$, going from $(1, 0)$ to $(\tilde{x}_{\min}, 0)$ in time $\tilde{t}_{\min}$, and then following backward in time the curve $(\hat{x}(\cdot), \hat{y}(\cdot))$, going from $(\tilde{x}_{\min}, 0)$ to $(1, 0)$ in time $\hat{t}_{\min}$. Since $(\hat{x}(t), \hat{y}(t)) \neq (0, 0)$ for almost every $t \in [0, \hat{t}_{\min}]$ and $(\tilde{x}(t), \tilde{y}(t)) \neq (0, 0)$ for almost every $t \in [0, \tilde{t}_{\min}]$, it follows that Γ is a Jordan curve. Applying the Green–Riemann theorem yields

$$\int_{\Gamma} \alpha = \int_{\Omega} d\alpha > 0 \quad \text{and} \quad \int_{\Gamma} \beta = \int_{\Omega} d\beta > 0$$

for $i = 1, 2$, because we have $d\omega_i > 0$ in both cases (one can note that $\int_{\Omega} d\alpha$ converges in spite of the singularity $1/y^2$ because we integrate on a cusp region). This gives

$$(2.24) \quad \tilde{t}_{\min} > \hat{t}_{\min} \quad \text{and} \quad \int_0^{\tilde{t}_{\min}} \tilde{u}(t) dt > \int_0^{\hat{t}_{\min}} \hat{u}(t) dt.$$

Step 3. Now comes the final construction, which raises a contradiction. Recall that, by the above construction, $0 < \hat{t}_{\min} < \tilde{t}_{\min} \leq t_{\min} < T$, $\tilde{x}_{\min} = \hat{x}(\hat{t}_{\min}) = \tilde{x}(\tilde{t}_{\min})$ and $x_{\min} = \tilde{x}(t_{\min}) = x(t_{\min})$. In particular, we set $\Delta t = \tilde{t}_{\min} - \hat{t}_{\min} > 0$. We consider the curve $(x_1(\cdot), y_1(\cdot))$ (of control $u_1(\cdot)$) that is defined as the concatenation of three pieces:

1. First, follow the curve $(\hat{x}(\cdot), \hat{y}(\cdot))$ from $(1, 0)$ to $(\tilde{x}_{\min}, 0)$ (contained in $y \leq 0$). This curve is defined on the time interval $[0, \hat{t}_{\min}]$.
2. Then, follow the curve $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ from $(\tilde{x}_{\min}, 0)$ to $(x_{\min}, 0)$ (contained in $\tilde{x}_{\min} \leq x \leq x_{\min}$, $y \geq 0$). This curve is defined on the time interval $[\tilde{t}_{\min}, t_{\min}]$, so we need to shift it in time, by advancing it by $\Delta t = \tilde{t}_{\min} - \hat{t}_{\min} > 0$.

3. Finally, follow the curve $(\tilde{x}(\cdot), \tilde{y}(\cdot)) = (x(\cdot), y(\cdot))$ from $(x_{\min}, 0)$ to $(1, 0)$ (contained in $y \geq 0$). This curve is defined on the time interval $[t_{\min}, T]$, so we need to shift it in time, by advancing it by $\Delta t = \hat{t}_{\min} - \hat{t}_{\min} > 0$.

Precisely, the control $u_1(\cdot)$ is defined by

$$u_1(t) = \begin{cases} \hat{u}(t) & \text{if } 0 < t < \hat{t}_{\min}, \\ \tilde{u}(t + \Delta t) & \text{if } \hat{t}_{\min} - \Delta t < t < t_{\min} - \Delta t, \\ \tilde{u}(t + \Delta t) = u(t + \Delta t) & \text{if } t_{\min} - \Delta t < t < T - \Delta t. \end{cases}$$

The trajectory $(x_1(\cdot), y_1(\cdot), u_1(\cdot))$ is a solution of (1.1a) and (1.1b) and (1.1c), satisfies $x_1(\cdot) \leq 1$ and steers the control system from $(1, 0)$ to $(1, 0)$ in time $T - \Delta t$. Using the strict inequality (2.24) and the equality (2.22), its cost is

$$\begin{aligned} \int_0^{T-\Delta t} u_1(t) dt &= \int_0^{\hat{t}_{\min}} \hat{u}(t) dt + \int_{\hat{t}_{\min}}^{t_{\min}} \tilde{u}(t) dt + \int_{t_{\min}}^T \tilde{u}(t) dt \\ &< \int_0^{\hat{t}_{\min}} \tilde{u}(t) dt + \int_{\hat{t}_{\min}}^{t_{\min}} \tilde{u}(t) dt + \int_{t_{\min}}^T \tilde{u}(t) dt = \int_0^T \tilde{u}(t) dt = \int_0^T u(t) dt, \end{aligned}$$

i.e., its cost is lower than the cost of the initial trajectory $(x(\cdot), y(\cdot), u(\cdot))$.

To get a trajectory defined on the same interval of time $[0, T]$, we finally extend the trajectory $(x_1(\cdot), y_1(\cdot), u_1(\cdot))$ to the interval $[0, T]$, by concatenating it with a small loop of negative cost on $[T - \Delta t, T]$. This can be done because, by Corollary 2.8, there exists a trajectory $(x_{\Delta t}(\cdot), y_{\Delta t}(\cdot), u_{\Delta t}(\cdot))$, solution of (1.1a) and (1.1b) and (1.1c) on $[T - \Delta t, T]$, such that $x_{\Delta t}(T - \Delta t) = x_{\Delta t}(T) = 1$ and $y_{\Delta t}(T - \Delta t) = y_{\Delta t}(T) = 0$, satisfying $x_{\Delta t}(t) \leq 1$ for every $t \in [T - \Delta t, T]$, and of negative cost $\int_{T-\Delta t}^T u_{\Delta t}(t) dt < 0$.

Then, concatenating $(x_1(\cdot), y_1(\cdot), u_1(\cdot))$ on $[0, T - \delta T]$ with $(x_{\Delta t}(\cdot), y_{\Delta t}(\cdot), u_{\Delta t}(\cdot))$ on $[T - \Delta t, T]$, we obtain a solution of (1.1a) and (1.1b) and (1.1c) on $[0, T]$, lying in $0 < x \leq 1$, steering the control system from $(1, 0)$ to $(1, 0)$ in time T , of cost (strictly) lower than $\int_0^T u(t) dt$. This raises a contradiction with the optimality of the trajectory $(x(\cdot), y(\cdot), u(\cdot))$.

The proposition is proved. $\square$

2.9. Conclusion: end of the proof of Theorem 1.1. In section 2.1, we have first reduced the initial optimal control problem (1.1) and (1.2) to the optimal control problem (1.1) and (2.1), noting that any other solution of (1.1) and (1.2) can be deduced from the one of (1.1) and (2.1) by simple geometric considerations. Then, in section 2.4, we have shown that the optimal control problem (1.1) and (2.1) is equivalent to the optimal control problem (1.1) and (1.3) and (2.8), which itself is equivalent, by Lemma 2.9 in section 2.6, to the optimal control problem (1.1) and (1.3) and (1.4). Finally, we have proved in Proposition 2.12 (in section 2.8) that the optimal control problem (1.1) and (1.3) and (1.4) has a unique solution, which is the one constructed in Proposition 2.6. This proves Theorem 1.1.

3. Additional remarks.

3.1. Uniqueness of the adjoint.

LEMMA 3.1. *The unique optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of (1.1) and (1.2), described in Theorem 1.1, has a unique extremal lift $(x(\cdot), y(\cdot), p_x(\cdot), p_y(\cdot), p^0, u(\cdot))$ up to scaling on the adjoint triple, which is normal. Moreover, normalizing it so that $p^0 = -1$, and the adjoint vector is given by*

$$(p_x(t), p_y(t)) = p_y(0)(-y(t), x(t)) \quad \forall t \in [0, T],$$

and we have $p_y(0) < 0$.

Proof. As in the proof of Lemma 2.10 in section 2.7, by the adjoint differential equations (2.2a) and (2.2b), we observe that $(-p_y(\cdot), p_x(\cdot), u(\cdot))$ is solution of (1.1a) and (1.1b), like the triple $(x(\cdot), y(\cdot), u(\cdot))$, with the same control $u(\cdot)$ (thus, with the same cost) but not with the same initial condition. However, the pair $(-p_y(\cdot), p_x(\cdot))$ is T -periodic. By uniqueness of the solution of the optimal control problem (1.1) and (2.1), and using Lemma 2.1, we infer that $(-p_y(\cdot), p_x(\cdot))$ is equal to $(x(\cdot), y(\cdot))$ up to scaling and to shifting in time. But since both curves are generated by the same control $u(\cdot)$ and thus consist of following two arcs, an arc of ellipse and an arc of hyperbola (see Figure 1, it follows that the shift in time is zero and thus there exists $\lambda \neq 0$ such that $x(t) = -\lambda p_y(t)$ and $y(t) = \lambda p_x(t)$ for every $t \in [0, T]$. In particular, we must have $p_x(0) = 0$ and, using for instance the positive sign of H_1 , necessarily $\lambda > 0$ and $p_y(0) = -\frac{1}{\lambda} < 0$. The switching function is then $\varphi(t) = \frac{x(t)^2}{\lambda} + p^0$. Therefore $p^0 < 0$ (otherwise, if $p^0 = 0$ then $\varphi > 0$ and there would be no switching) and we can normalize the adjoint so that $p^0 = -1$. $\square$

3.2. Shooting method. Setting $\lambda = -1/p_y(0) > 0$ as in the proof of Lemma 3.1, we have $\lambda\varphi = x^2 - \lambda$, and we thus infer from (2.5) that

$$(3.1) \quad u(t) = \begin{cases} u_{\min} & \text{if } x(t)^2 < \lambda, \\ u_{\max} & \text{if } x(t)^2 > \lambda. \end{cases}$$

By the way, since $x(0) = 1$, we must have, actually, $0 < \lambda < 1$.

The expression (3.1) for the optimal control gives an alternative way to compute optimal trajectories by implementing the *shooting method*. Here, there is only one shooting parameter, that is $\lambda \in (0, 1)$ (and actually, $\lambda = -1/p_y(0)$ as said above), and this parameter must be tuned so that, when one integrates the differential equations (1.1a) and (1.1b), with initial condition $(x(0), y(0)) = (1, 0)$, one must have $x(T) = 1$ at time T .

This is the method that has been implemented in [10].

3.3. Turnpike phenomenon. As said in Remark 1.2, when $T \rightarrow +\infty$ the arc of the curve $t \mapsto (x(t), y(t))$ on $[t_1, \frac{T}{2}]$, along which $u = u_{\min}$, converges to the segment joining the point $(\sqrt{\frac{u_{\max}}{u_{\max} - u_{\min}}}, -\sqrt{\frac{u_{\min} u_{\max}}{u_{\max} - u_{\min}}})$ to the point $(0, 0)$. This is because the arc of hyperbola converges to a “V-shape” (with the “V” rotated by $-\pi/2$), as it can be seen on the numerical simulation reported on Figure 5.

The turnpike phenomenon refers to the following typical property enjoyed by optimal trajectories of a large classes of optimal control problems in large horizon of time: when T is large, the optimal trajectories (as well as their control and adjoint vector) tend to spend most of their time near a steady-state, itself being the optimal solution of a static optimization problem (see [17]).

This phenomenon is evident here: the optimal steady-state is the triple $(\bar{x}, \bar{y}, \bar{u}) = (0, 0, u_{\min})$. The turnpike phenomenon observed on Figure 5 is particularly intuitive for the optimal control problem (1.1) and (1.2): as said in section 1, when we relax the constraint (1.2b) then the optimal solution is the steady-state $(\bar{x}, \bar{y}, \bar{u}) = (0, 0, u_{\min})$. Under the nontriviality constraint (1.2b), optimal trajectories tend to “reproduce” as much as possible this optimal steady-state. This is the turnpike phenomenon.

In some sense, this observation “trivializes” the computation of optimal trajectories in large time: when $T \rightarrow +\infty$, the unique optimal trajectory passing through the point $(1, 0)$ consists of two pieces, one being an arc of ellipse with a shape of a “U” rotated by $\pi/2$, and the other being the union of two segments (“V-shape”), as

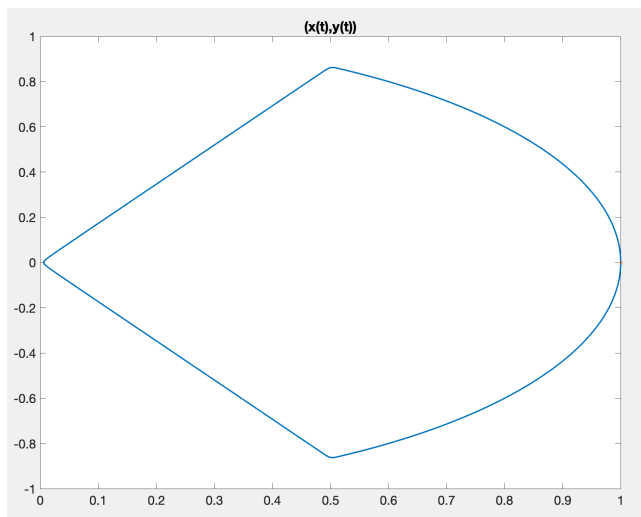


FIG. 5. Numerical simulation when T is large. Here: $u_{\min} = -3$, $u_{\max} = 1$, $T = 8$.

said at the beginning of this section. Said in other words, in large time the optimal trajectories look like “teardrops.”

3.4. Floquet stability of the optimal periodic trajectory. For a fixed (open-loop) control $u(\cdot)$, the controlled dynamics (1.1a)–(1.1b) form a linear Hamiltonian system with T -periodic coefficients whenever $u(t+T) = u(t)$. In terms of the scalar equation $\ddot{x}(t) + u(t)x(t) = 0$, this is a classical Hill equation, for which the stability of solutions is described by Floquet theory (see, e.g., [1, 12, 18]). Denoting by $\Phi(t)$ the fundamental matrix solution of

$$\dot{z}(t) = A(t)z(t), \quad A(t) = \begin{pmatrix} 0 & 1 \\ -u(t) & 0 \end{pmatrix},$$

and $\Phi(0) = I$, the monodromy matrix $M = \Phi(T)$ has determinant 1 (symplecticity), and its eigenvalues (λ_1, λ_2) are the Floquet multipliers. The system is (Lyapunov) stable if and only if $|\operatorname{tr} M| < 2$, unstable if $|\operatorname{tr} M| > 2$, and marginal on the boundary if $|\operatorname{tr} M| = 2$.

For the optimal periodic solution given by Theorem 1.1, the optimal control $u(\cdot)$ is T -periodic and piecewise constant, equal to $u_{\max}$ on $[0, t_1] \cup [T - t_1, T]$ and to $u_{\min}$ on $[t_1, T - t_1]$. The periodicity constraint $x(T) = x(0)$, $y(T) = y(0)$ means that M has an eigenvalue equal to 1 (indeed, $(x(0), y(0))$ is a fixed point of the flow over one period). Since $\det M = 1$, both Floquet multipliers are equal to 1, and therefore $\operatorname{tr} M = 2$. Consequently, the optimal periodic trajectory sits on the boundary between stable and unstable regions of the associated Hill equation. In the conservative model (1.1a) and (1.1b), this corresponds to marginal stability: perturbations remain bounded when $M = I$ and may grow at most linearly when M has a nontrivial Jordan block (as is typical on stability boundaries). In the experimental dynamical stabilization setting of section 4.2, small dissipation turns this marginal situation into genuine asymptotic stability, which is consistent with the observed robustness of the optimal modulation.

3.5. Locally optimal butterfly solutions. In Theorem 1.1 we have proved that any *globally* optimal solution of (1.1) and (1.2) is entirely contained in the half-plane $x > 0$ or $x < 0$.

In this section, we present a family of *locally* but not *globally* optimal solutions of (1.1) and (1.2), that we call “butterfly solutions” (because of their shape). Such solutions cross the y -axis.

So, let us now consider the optimal control problem (1.1) and (1.2), in which we add the constraint

$$(3.2) \quad \exists t \in [0, T] \mid x(t) = 0.$$

Using Lemma 2.3 and in particular homotheties and shifting in time, the optimal control problem (1.1) and (1.2) and (3.2) can be reduced to the optimal control problem (1.1) and (1.3) and (3.2), i.e., with $x(0) = x(T) = 1$, $y(0) = y(T) = 0$, under (3.2) and under the state constraint

$$(3.3) \quad -1 \leq x(t) \leq 1 \quad \forall t \in [0, T].$$

With similar arguments as the ones developed to prove Theorem 1.1, it can be proved that, given any $T \geq 2\pi$, the optimal control problem (1.1) and (1.3) and (3.2) and (3.3) has a unique solution $(x_b(\cdot), y_b(\cdot), u_b(\cdot))$ (the index “b” refers to “butterfly”), which is symmetric with respect to the x -axis and to the y -axis, and which can be explicitly described similarly to what was done in Theorem 1.1. The optimal curve consists of two arcs of an ellipse that are symmetric with respect to the y -axis, in-between of which there are two arcs of hyperbola that are symmetric with respect to the x -axis (see Figure 6).

For $T = 2\pi$, the butterfly trajectory is given by

$$x(t) = \cos(\omega_{\max} t), \quad y(t) = -\omega_{\max} \sin(\omega_{\max} t), \quad u(t) = u_{\max},$$

and the curve is the ellipse $x^2 + \frac{y^2}{\omega_{\max}^2} = 1$. Actually for $T < 2\pi$ there does not exist any solution crossing the y -axis, i.e., satisfying (3.2), so $2 = 2\pi$ is the minimal possible time for which butterfly solutions exist. Then, as $T \geq 2\pi$ grows, the ellipse “opens” at its minimal and maximal values of y , and one completes the curve with two arcs

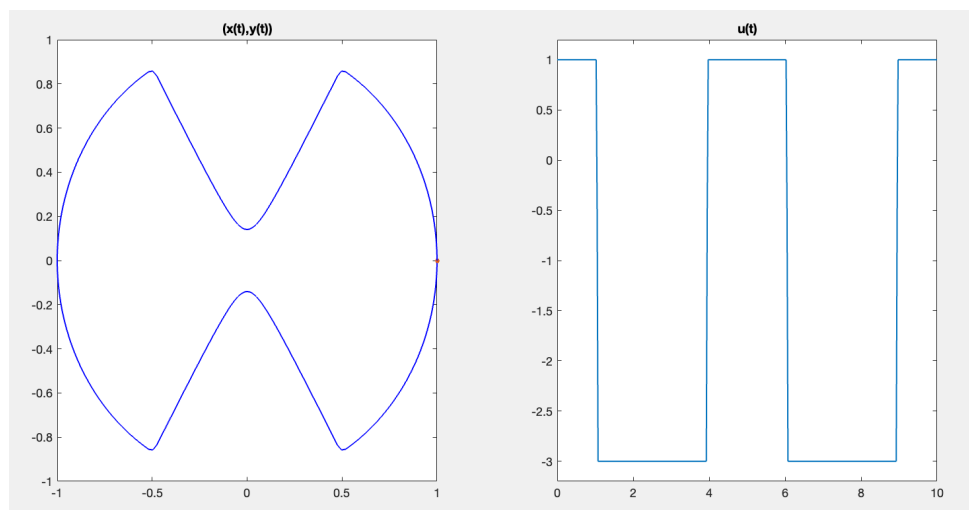


FIG. 6. Numerical simulation of the butterfly solution, with $u_{\min} = -3$, $u_{\max} = 1$, $T = 10$.

of hyperbola, as it can be seen on Figure 6. When $T \rightarrow +\infty$, these arcs of hyperbola tend to a “V”-shape, with the edge of the “V” tending to $(0,0)$: this is the turnpike phenomenon, similar to what has been discussed in section 3.3.

4. Applications in quantum and classical physics.

4.1. Ground state of 1D Schrödinger operators with a finite potential well. The physical motivation of the optimal control problem (1.1) and (1.2) lies in the study of one-dimensional Schrödinger operators with a finite potential well, given by

$$P_{t_1,M} = -\frac{d^2}{dt^2} + V_{t_1,M} \text{id}$$

generally considered on the whole real line $\mathbb{R}$ (i.e., $T = +\infty$) or on an interval $(-\frac{T}{2}, \frac{T}{2})$, with suitable boundary conditions, where the potential $V_{t_1,M}$ is defined by

$$V_{t_1,M}(t) = \begin{cases} 0 & \text{if } |t| \leq t_1, \\ M & \text{if } |t| > t_1 \end{cases}$$

depending on two parameters $M > 0$ and $t_1 > 0$ that are respectively the height and the width of the potential well. More precisely, when $T < +\infty$ we consider $P_{t_1,M}$ on the domain of periodic functions

$$\begin{aligned} D(P_{t_1,M}) &= H_{\text{per}}^2\left(-\frac{T}{2}, \frac{T}{2}\right) \\ &= \left\{ \psi \in H^2\left(-\frac{T}{2}, \frac{T}{2}\right) \mid \psi\left(-\frac{T}{2}\right) = \psi\left(\frac{T}{2}\right), \quad \dot{\psi}\left(-\frac{T}{2}\right) = \dot{\psi}\left(\frac{T}{2}\right) \right\} \end{aligned}$$

when $T = +\infty$ we consider the operator on $D(P_{t_1,M}) = H^2(-\frac{T}{2}, \frac{T}{2})$.

The relationship with the optimal control problem (1.1) and (1.2) is the following. Let us consider any nontrivial optimal solution $(x(\cdot), y(\cdot), u(\cdot))$ of (1.1) and (1.3). As stated in Theorem 1.1, they are all given by the same control (1.5), maybe shifted in time. In Theorem 1.1, as well as in its proof, for convenience we assumed that the trajectories were defined on the time interval $[0, T]$. Shifting in time (see **(A₂)** in Lemma 2.1, let us now assume that they are defined on the time interval $(-\frac{T}{2}, \frac{T}{2})$. Then, now, the optimal control is given by

$$u(t) = \begin{cases} u_{\max} & \text{if } |t| < t_1, \\ u_{\min} & \text{if } |t| > t_1 \end{cases}$$

for almost every $t \in (-\frac{T}{2}, \frac{T}{2})$. Using (1.1a) and (1.1b), we infer that, for $t \in (-\frac{T}{2}, \frac{T}{2})$,

$$\begin{aligned} -\frac{d^2}{dt^2}x(t) &= u_{\max}x(t) & \text{if } |t| < t_1, \\ \left(-\frac{d^2}{dt^2} + (u_{\max} - u_{\min})\text{id}\right)x(t) &= u_{\max}x(t) & \text{if } |t| > t_1. \end{aligned}$$

Therefore

$$P_{t_1,M}x(\cdot) = u_{\max}x(\cdot) \quad \text{with} \quad M = u_{\max} - u_{\min} = \omega_{\max}^2 + \omega_{\min}^2.$$

Since $x(\cdot)$ is nontrivial and belongs to $D(P_{t_1,M})$, this means that $x(\cdot)$ is an eigenfunction of $P_{t_1,M}$ associated with the eigenvalue $u_{\max}$. Actually it corresponds to the ground state of $P_{t_1,M}$, i.e., $u_{\max}$ is the smallest eigenvalue of $P_{t_1,M}$.

When $T \rightarrow +\infty$, we know, by Remark 1.2, that

$$x\left(-\frac{T}{2}\right) = x\left(\frac{T}{2}\right) \sim \frac{\omega_{\max}}{\omega_{\min}^2 + \omega_{\max}^2} \exp\left(\frac{\omega_{\min}}{\omega_{\max}} \operatorname{Arctan}\left(\frac{\omega_{\min}}{\omega_{\max}}\right) - \omega_{\min} \frac{T}{2}\right) \rightarrow 0,$$

$$t_1 \rightarrow t_1^\infty = \frac{1}{\omega_{\max}} \operatorname{Arctan}\left(\frac{\omega_{\min}}{\omega_{\max}}\right).$$

This limit value of t_1^∞ satisfies $\omega_{\max} \tan(\omega_{\max} t_1^\infty) = \omega_{\min}$. It corresponds to well known computations in quantum physics, as explained next.

Computation of the ground state of $P_{t_1, M}$. Given a height $M > 0$ and a width $t_1 > 0$, a well known problem in quantum physics consists of determining the ground state of $P_{t_1, M}$ – and more generally, the discrete eigenvalues of $P_{t_1, M}$ and their corresponding eigenfunctions. Recall that the discrete eigenvalues of $P_{t_1, M}$ belong to the interval $(0, M)$ (beyond M , we have continuous spectrum).

We first explain this fact for $T = +\infty$, thus recovering classical computations in quantum physics (see, e.g., [8]).

Let $u_{\max} \in (0, M)$ be the lowest eigenvalue of $P_{t_1^\infty, M}$. We set $\omega_{\min} = \sqrt{M - \omega_{\max}^2}$, so that $M = u_{\max} - u_{\min}$. Now, we must have

$$t_1^\infty = \frac{1}{\omega_{\max}} \operatorname{Arctan}\left(\frac{\sqrt{M - \omega_{\max}^2}}{\omega_{\max}}\right).$$

Since the function $\omega_{\max} \mapsto t_1^\infty(\omega_{\max})$ is bijective on $(0, M)$, there exists a unique solution $\omega_{\max} \in (0, M)$ of the above equation. In quantum physics, this solution is seen as the first solution of a quantification relation. The corresponding eigenfunction (or ground state) is the function $x(\cdot)$ given by Theorem 1.1.

Figure 7 shows an example for $u_{\max} = 1$ and $u_{\min} = -3$, i.e., $M = 4$. Figure 7a) shows the evolution of T as a function of t_1 as given by (1.6). When $T = +\infty$, this function tends to t_1^∞ . Figure 7b) shows the operator $V_{t_1, M}$ for $T = +\infty$ ($T = 10$ for numerical purposes), $t_1 = t_1^\infty$ and $M = 4$, as well as the two eigenvalues belonging

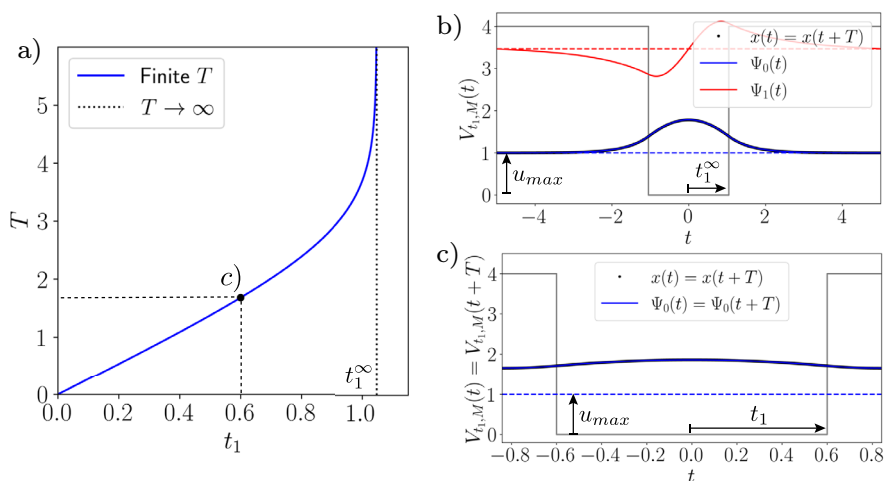


FIG. 7. Example of the spectrum computation of a 1D Schrödinger operator with a finite potential well for $u_{\min} = -3$ and $u_{\max} = 1$. a) Evolution of T as a function of t_1 for the optimal solution of (1.1) and (1.2) given in (1.6). b) “Bound states” of a 1D Schrödinger operator with a finite potential with $M = u_{\max} - u_{\min} = 4$ and $t_1 = t_1^\infty$. c) Case with periodic boundary conditions and $M = u_{\max} - u_{\min} = 4$, $t_1 = 0.6$ and T given in (1.6) illustrated in a).

to the interval $(0, M)$ in dashed lines and the two associated eigenfunctions $\Psi_0(t)$ and $\Psi_1(t)$ (the local x -axis of those eigenfunctions is located at the level of their eigenvalues). The eigenfunctions are normalized so that $\int_{-T/2}^{T/2} \Psi_n(t)^2 dt = 1$. As expected, the smallest eigenvalue is $u_{\max} = 1$ and the associated eigenfunction Ψ_0 corresponds to the solution $x(\cdot)$ of the optimal problem (1.1) and (1.2) with $u_{\max} = 1$, $u_{\min} = -3$ and $T = +\infty$.

When $T < +\infty$, the situation is a bit more complicated. Since t_1 is given in the quantum problem, the value of T is determined by (1.6) as shown in Figure 7a). For $u_{\max} = 1$ and $u_{\min} = -3$, the $(T$ -periodic) solution $x(\cdot)$ of the optimal problem (1.1) and (1.2), with $t_1 = 0.6$ for example, corresponds to a period $T \approx 1.682$ as shown on Figure 7a). Looking for the eigenvalues of the operator $P_{t_1, M}$ with $M = u_{\max} - u_{\min} = 4$, $t_1 = 0.6$, $T \approx 1.682$, we find only one eigenvalue $u_{\max} \in (0, M)$ as shown on Figure 7c). The T -periodic eigenfunction Ψ_0 , normalized so that $\int_{-T/2}^{T/2} \Psi_n(t)^2 dt = 1$, is drawn on Figure 7c) on the interval $[-T/2, T/2]$. This eigenfunction corresponds to the optimal solution $x(\cdot)$ of (1.1) and (1.2) computed with the same parameters.

4.2. Optimal dynamical stabilization. Another important application where the optimal control problem (1.1) and (1.2) matters is the so-called dynamical stabilization phenomenon, known in classical physics as the process by which charged particles can be trapped in periodically varying electromagnetic fields like in mass spectrometers or in trapped ion quantum computers (see [13]). Dynamical stabilization consists of periodically modulating in time the properties of a system to dynamically sustain one of its naturally unstable configurational states. Recent works in this field (see [7]) have shown a new modulation parameter regime that exhibits some mathematical analogy with the computation of ground states of 1D Schrödinger operators $P_{t_1, M}$ with potential, as described in the previous subsection. In this new framework, optimal control theory can therefore be used to determine the minimal modulation needed to stabilize a system, hence the term “optimal dynamical stabilization.”

The relationship between optimal dynamical stabilization and the optimal control problem (1.1) and (1.2) was discovered and exploited in [10]. Figure 8a) is a sketch of the model experiment reported in [10] that is an academic platform to explore the optimal dynamical stabilization of a one degree-of-freedom system in a periodically time-varying potential energy landscape. The experiment consists of a compass centered and aligned between two Helmholtz coils. The dipole's configurational state is fully parametrized in time by the angle $\theta(t)$ between the axis of the coils $\mathbf{e}_x$, coincident with the North-South magnetic axis of Earth, and the $N - S$ axis of the magnetized needle. This classical one degree-of-freedom nonlinear oscillator can be modeled by the nonlinear evolution equation

$$(4.1) \quad \ddot{\theta}(t) - 2\xi \sqrt{\frac{|B(i(t))|\mu}{I}} \dot{\theta}(t) + \frac{B(i(t))\mu}{I} \sin(\theta(t)) = 0,$$

where $\xi = 0.3$ is the experimental damping ratio, $\mu/I = 6.4 \times 10^4 \text{ A.kg}^{-1}$ is the ratio between the magnetic moment μ and moment of inertia I , and $\sqrt{|B(i(t))|\mu/I}$ is the natural frequency of the dipole around its stable equilibrium position that can be either $\theta(t) = 0$ or $\theta(t) = \pi$ depending on the current $i(t)$ in the coils. Here, $B(i(t)) = -(B_T + A i(t))$ is the magnitude of the uniform magnetic field felt by the dipole with $B_T = 47 \text{ } \mu\text{T}$ (Earth magnetic field), $A = 4496 \text{ } \mu\text{T/A}$ (a property related to the Helmholtz coils configuration) and $i(t)$ is the current in the coils.

When $i(t) = 0$, meaning that no power is supplied to the coils, the magnetic field points in the direction of $-\mathbf{e}_x$ and so does the North pole of the compass. In this

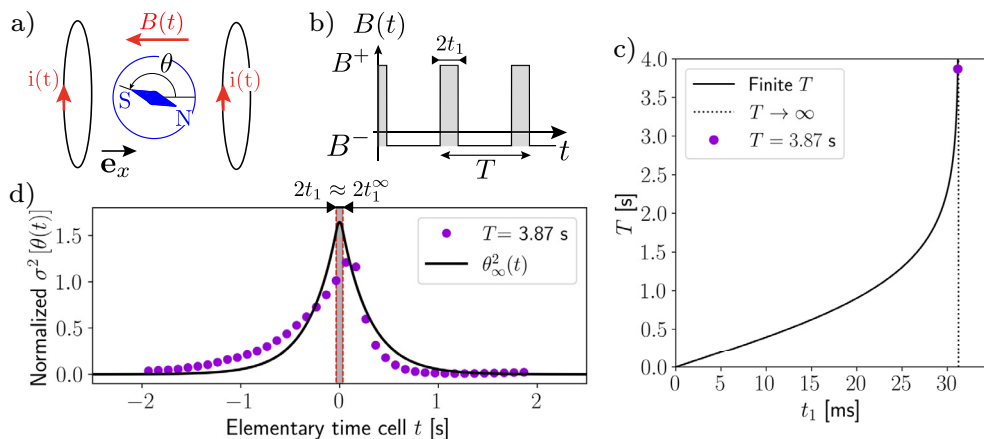


FIG. 8. An example of experimental results in optimal dynamical stabilization (see [10]). a) A compass, fully parameterized by the angle $\theta(t)$ between $\mathbf{e}_x$ and its $N-S$ local axis is placed at the center of two Helmholtz coils oriented along $\mathbf{e}_x$. b) We impose a T -periodic magnetic field $B(t) = B(t+T)$ with $B(t) = B^+ = 852 \mu\text{T}$ ($i = -200 \text{ mA}$) during a time $2t_1$ and $B(t) = B^- = -47 \mu\text{T}$ ($i = 0 \text{ A}$) during $T - 2t_1$. c) Evolution of T as a function of t_1 as given in (1.6) for $u_{\max} = B^+ \mu/I = 54.5 \text{ (rad/s)}^2$ and $u_{\min} = B^- \mu/I = -3 \text{ (rad/s)}^2$. d) Variance of the dynamically stable oscillatory motion $\theta(t)$ about $\theta = \pi$ for two experiments with $T = 3.87 \text{ s}$ and $2t_1 = 70 \approx t_1^\infty = 62 \text{ ms}$. The square of the optimal solution of (1.1) and (1.2) for $T = \infty$, $\theta_\infty(t)$, is shown in black line.

case, $B(t) = B^- = -47 < 0$, making $\theta = 0$ a stable equilibrium and $\theta = \pi$ unstable. A compass starting from any initial condition will eventually converge to $\theta = 0$ with damped oscillations, moving away from $\theta = \pi$. A natural physical question is

What is the minimal current over time, or more precisely the minimal value of $\int |i(t)| dt$, required to stabilize $\theta = \pi$?

If we ignore time and only consider constant currents, the answer is straightforward: in the experiment, for any constant current $i(t) = i < -10 \text{ mA}$, the stability of the equilibrium configurations is reversed, and $\theta = \pi$ becomes asymptotically stable. The minimal value of $\int |i(t)| dt$ is at least 10 mA multiplied by the total time. However, when time modulations are allowed, for example with a periodic current $i(t) = i(t+T)$, it becomes possible to reduce the value of $\int |i(t)| dt$, even more by switching off the coils ($i(t) = 0$) during part of the cycle, while still maintaining dynamic stability. The problem of minimizing $\int |i(t)| dt$ by increasing the duration of $i(t) = 0$ while keeping the periodicity condition $i(t) = i(t+T)$ can be framed as the optimal control problem (1.1) and (1.2).

Figure 8b) shows an example of a typical periodic current applied in the coils, leading to a positive magnetic field $B(t) = B^+ = +852 \mu\text{T}$ when $i(t) = -0.2 \text{ A}$ ($\theta = \pi$ is an attractor) and a negative magnetic field $B(t) = B^- = -47 \mu\text{T}$ when $i(t) = 0 \text{ A}$ ($\theta = \pi$ is a repeller). The experiment reported in [10] consists of seeking, in the (t_1, T) modulation parameter space, dynamically stable responses $\theta(t)$, remaining in the vicinity of $\theta = \pi$ for initial conditions $\theta(0) \approx \pi$ and $\dot{\theta}(0) = 0$. This stability can be rationalized by linearizing (4.1) about $\theta = \pi$, leading to

$$(4.2) \quad \ddot{\theta}(t) + u(t)\theta(t) = 0,$$

where $u(t) = u(t+T) = u_{\max} = B^+ \mu/I = 54.5 \text{ (rad/s)}^2$ during $2t_1$ and $u(t) = u_{\min} = B^- \mu/I = -3 \text{ (rad/s)}^2$ during $T - 2t_1$ (since the compass is only doing about half an oscillation for the considered t_1 , the damping factor can be neglected in a first approximation). By Floquet theory, the solution of (4.2) can be expressed as

$\theta(t) = \Psi(t)e^{st} + \bar{\Psi}(t)e^{-st}$, where Ψ is a complex T -periodic function and $s \in \mathbb{C}$. In the (t_1, T) modulation parameter space, one observe an alternance of unstable ($\Re(s) > 0$) and dynamically stable ($\Re(s) = 0$) tongues (see [10]). The dynamically stable tongues are bounded by T and $2T$ periodic solutions θ , that become narrower as T increases (see [7, 10]).

From the viewpoint of Floquet theory for the Hill equation (4.2), the boundaries of the (dynamically) stable tongues correspond to Floquet multipliers ± 1 , i.e., to the existence of nontrivial T -periodic or $2T$ -periodic solutions (see, e.g., [1, 12, 18]). In particular, in the conservative linear model (4.2), the optimal modulation provided by Theorem 1.1 yields a periodic solution and thus lies on such a boundary (the monodromy matrix has eigenvalues equal to 1). In practice, the small dissipation of the experimental setup shifts these boundaries and creates a genuine stability region around the theoretical optimum, which explains why the optimal control law remains effective under perturbations.

The T -periodic solutions θ of (4.2) as well as their associated T -periodic modulation function u , correspond to the solutions of the optimal control problem (1.1) and (1.2) (and the lower boundary of the first stability tongue). Notably, they minimize $\int_0^T u(t) dt$, i.e., $\int |i(t)| dt$, given that $u_{\min} < 0$ corresponds to $i(t) = 0$ in the experimental setup. When $T \rightarrow +\infty$, one should theoretically be able to dynamically stabilize the compass in $\theta = \pi$ with almost no current! Figure 8c) shows the evolution of T as a function of t_1 according to (1.6) for the optimal $u(\cdot)$ with $u_{\max} = 54.5$ and $u_{\min} = -3$. When $T = +\infty$, the optimal control $u(\cdot)$ has a constant duration $2t_1^\infty \approx 62$ ms, and a periodic solution, denoted $\theta_\infty(\cdot)$, should exist (teardrop optimal trajectory in the state space). In practice, by experimentally imposing $2t_1 = 70$ ms of $B(t) = B^+$ (the dissipation slightly switches the stability regions in the modulation parameter space), it has been possible to turn off the coils during $T - 2t_1 = 3.8$ s, i.e., more than 98% of the time as shown on Figure 8d). Beyond this period of $T = 3.87$ s, the compass is no more stable in practice because of the inherent imperfections of the setup and of the basin of attraction of initial conditions that shrinks about $\theta(0) = \dot{\theta}(0) = 0$ (see [7]).

The theoretical optimal solution $\theta_\infty(\cdot)$, that is almost a teardrop in the state space for $T = 3.87$ s, should predict the periodic oscillations of the compass about $\theta = \pi$. However, due to the nature of optimal dynamical stabilization for large T that is a symmetry breaking that repeats periodically, the observed experimental oscillations are actually quasi-periodic, consisting of a succession of scaled functions $\theta_\infty(\cdot)$ periods after periods (see [10]). Moreover, the experiments being sensitive to initial conditions, various experiments with the seemingly same parameters do not lead to the same quasi-periodic responses $\theta(t)$. *In fine*, it is not the oscillations $\theta(\cdot)$ that are well predicted over time by the optimal control problem (1.1) and (1.2), but rather their variance $\sigma^2[\theta(t)]$ over one period, that are consistently predicted by $\theta_\infty^2(t)$ as shown on Figure 8d).

5. Conclusion and perspectives. We have solved explicitly a periodic optimal control problem for the bilinear oscillator $\ddot{x}(t) + u(t)x(t) = 0$ with bounded control $u(t) \in [u_{\min}, u_{\max}]$ and periodic boundary conditions. The optimal control is bang-bang with at most two switchings, and the corresponding optimal trajectory admits a simple geometric description as the concatenation of arcs of ellipses (when $u = u_{\max}$) and hyperbolas (when $u = u_{\min}$). Beyond the explicit formulas, the proof combines Pontryagin's maximum principle with global arguments (surgery of extremals and clock-form techniques) that may be of independent interest and yields in particular a complete optimal synthesis.

Several extensions and open directions seem natural. On the modeling side, one may incorporate dissipation and/or additional state constraints, which are relevant in the dynamical stabilization experiment, and study how the present explicit structure persists or bifurcates (in particular, how robust stability properties evolve when moving away from the boundary of Floquet stability regions). On the mathematical side, it would be interesting to treat analogous periodic problems for higher-dimensional linear Hamiltonian systems, or for other cost functionals (e.g., quadratic costs, or costs involving $|u|$) where singular arcs may appear.

A natural extension is to impose a nodal constraint (or an index/rotation constraint) and study the corresponding optimal synthesis; this would connect the locally optimal “butterfly” extremals to higher eigenmodes in the Schrödinger interpretation.

Finally, the present problem can be viewed as a fully explicit example within the general theory of optimal periodic control [4, 6], and we hope it will serve as a benchmark for numerical methods and for further theoretical developments at the interface of periodic control and Floquet/Hill theory.

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