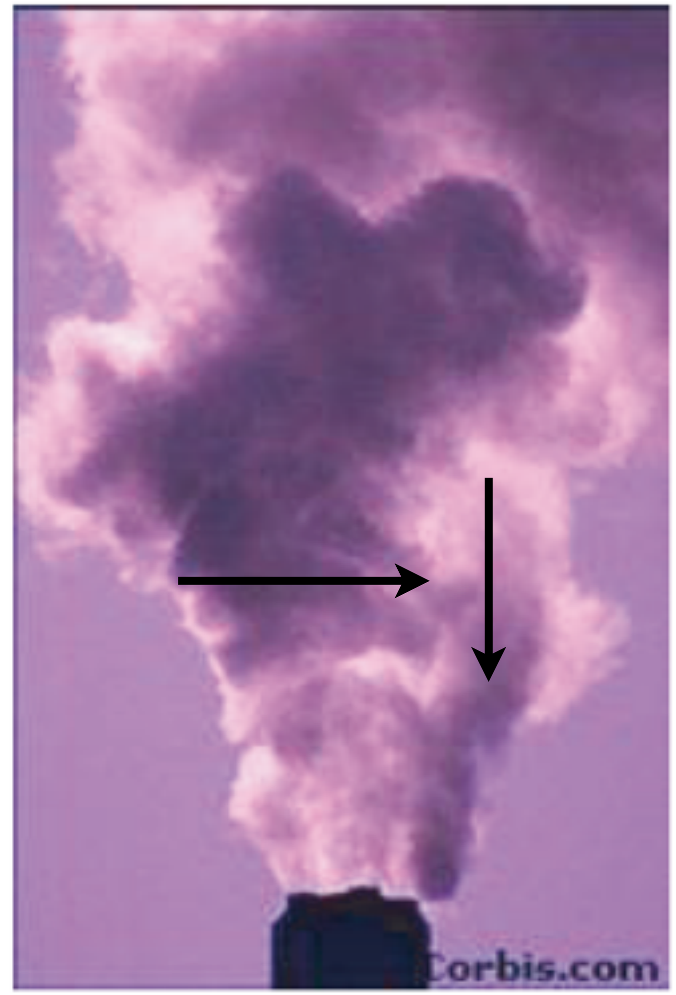
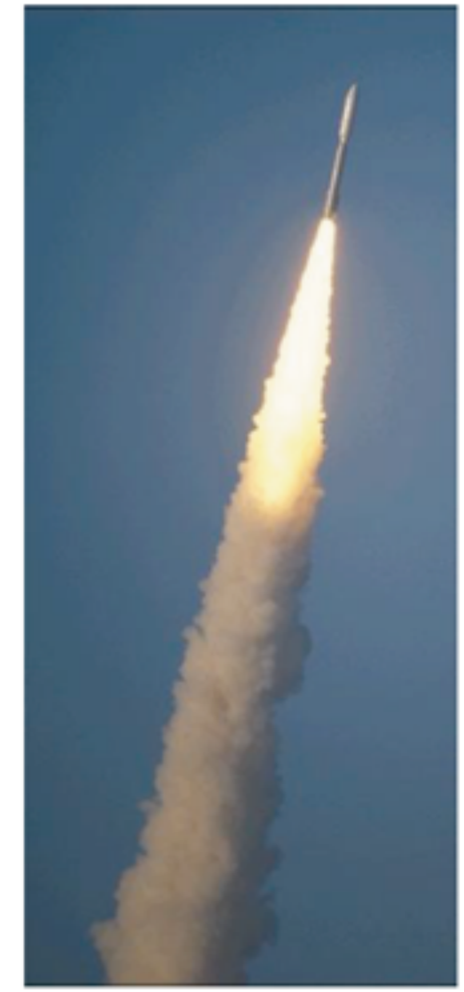
Free-shear flows

Pope: Chap. 5

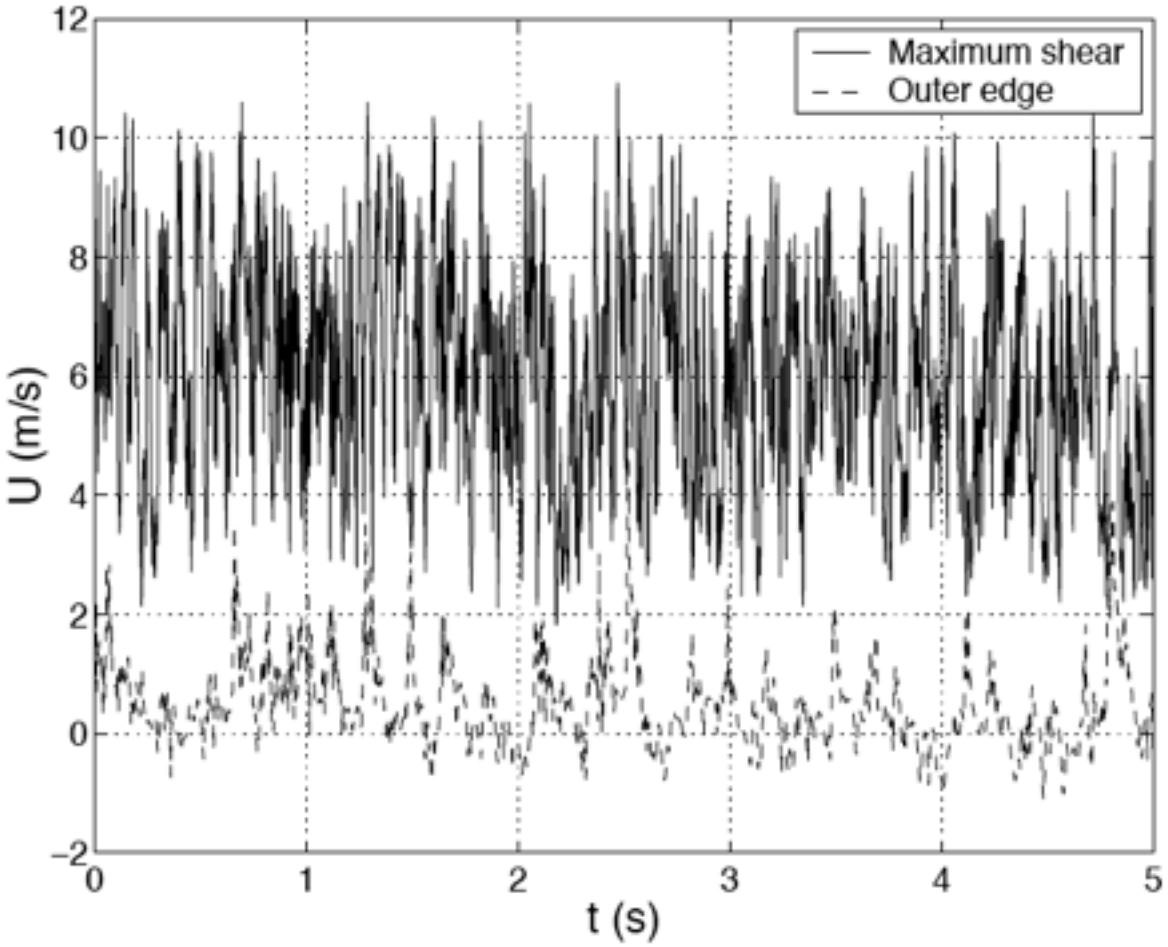
- No boundaries
- Velocity gradients





$$\left[\frac{\partial \omega_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\omega}_i}{\partial x_j}\right] = \tilde{\omega}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \epsilon_{ijk} \frac{\partial \tilde{\rho}}{\partial x_j} \frac{\partial \tilde{p}}{\partial x_k}$$

Possible effect of density gradients

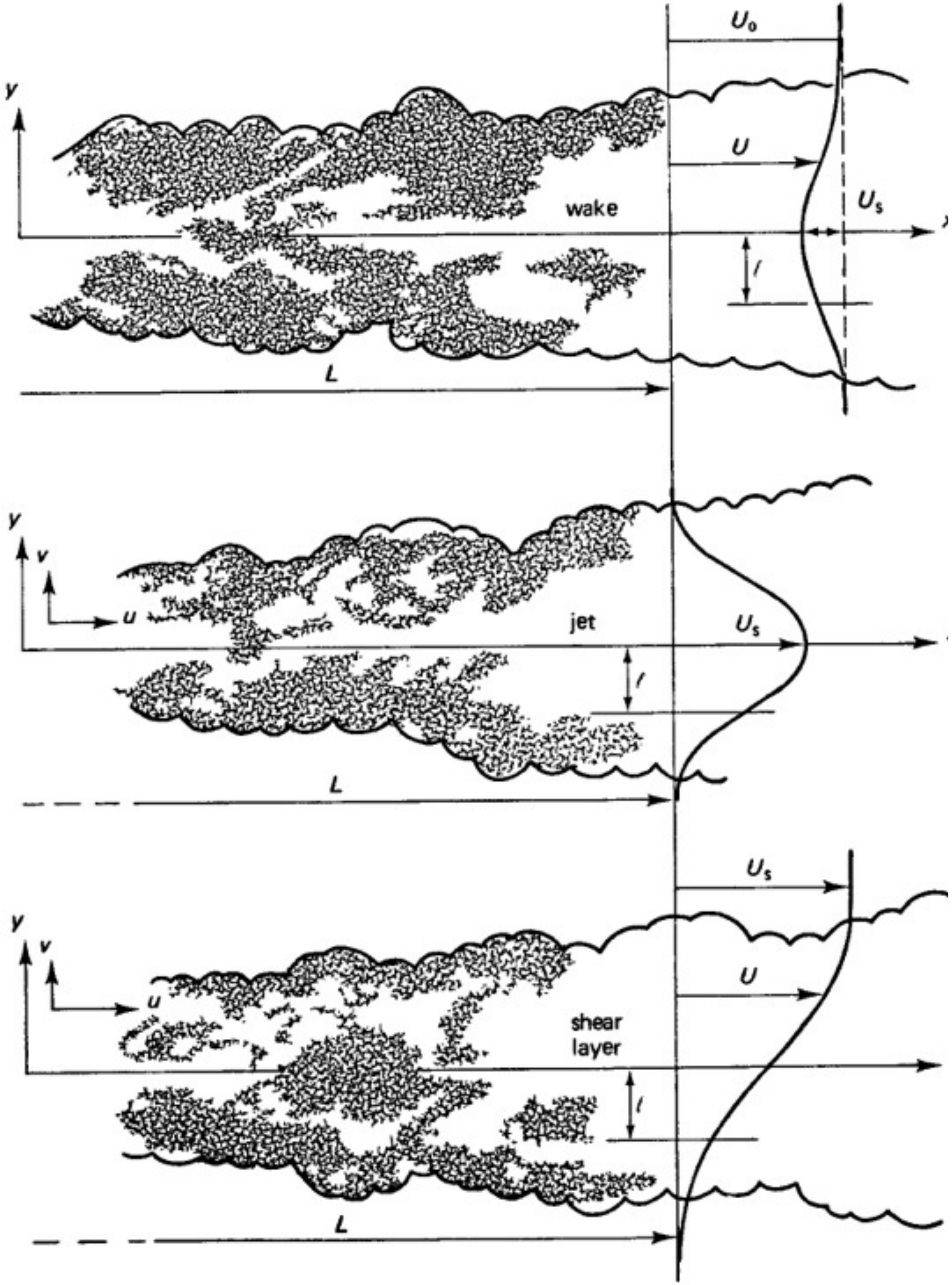


Turbulent-non-turbulent interface "Corrsin super-layer"

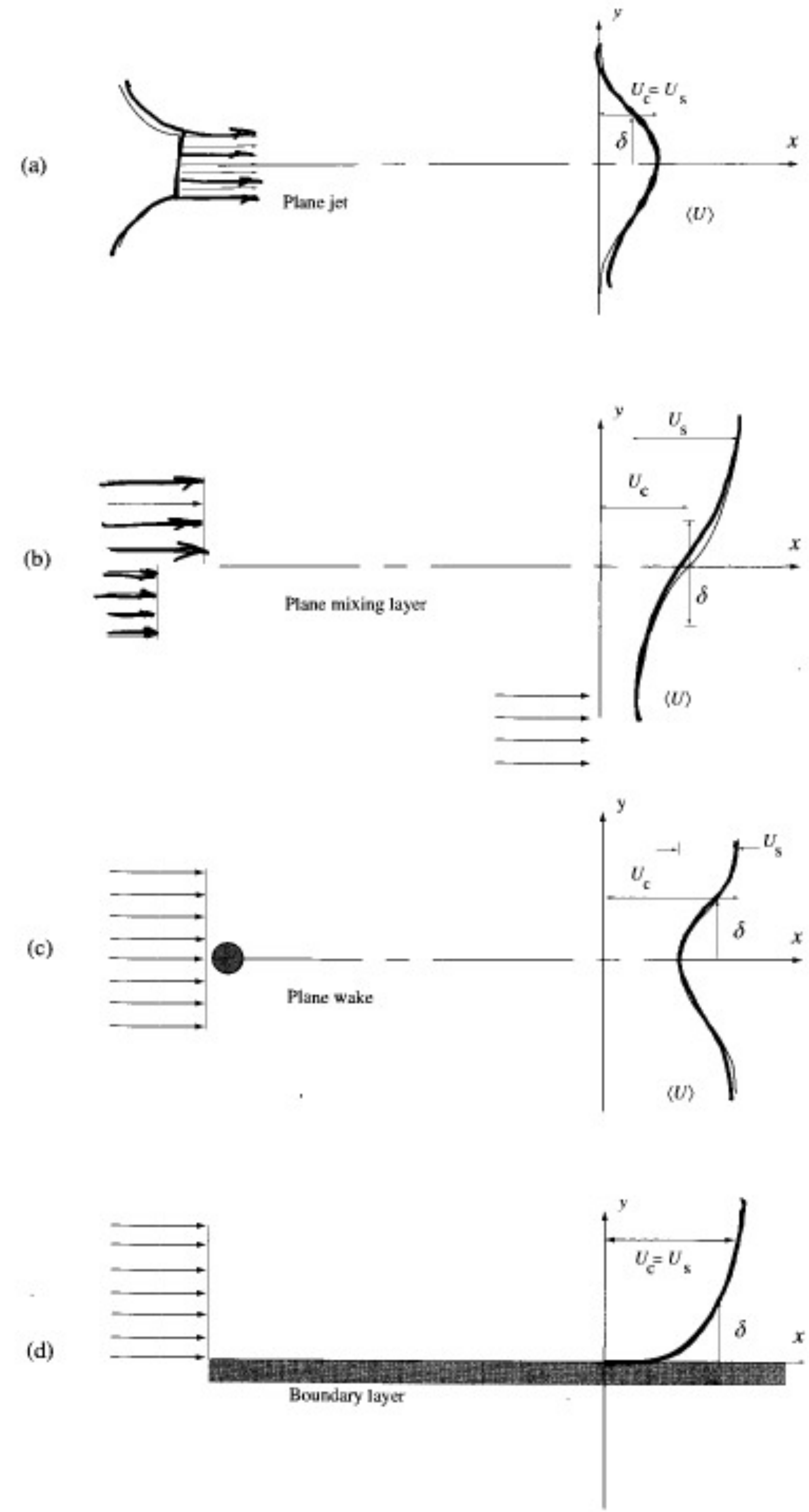
- Kolmogorov scale
- Intermittency
- Irrotational motion outside

Entrainment:
- Spreading
- Dilution
- Transport

Structures



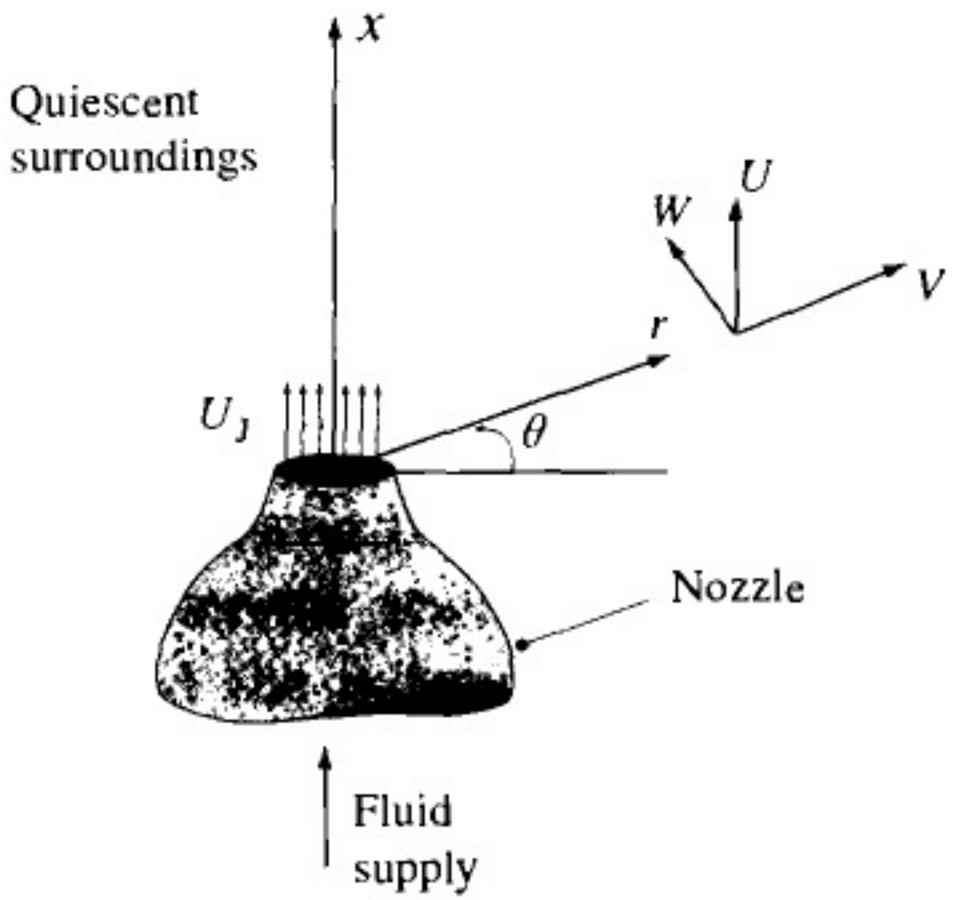
U₀
U
U_s
wake
l
L
y
v
u
jet
shear
layer

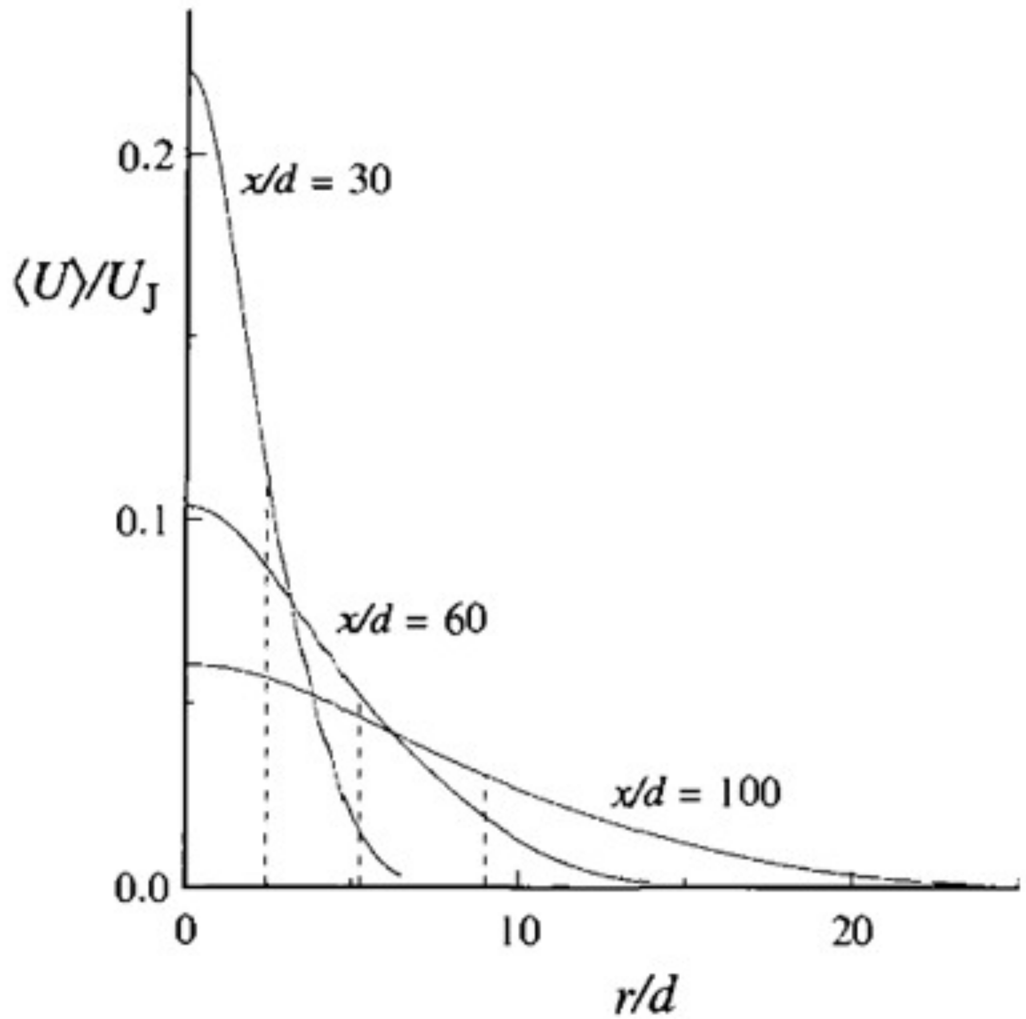


(a)
Plane jet
U_c = U_s
δ
x
y
⟨U⟩
(b)
Plane mixing layer
U_s
U_c
(c)
Plane wake
(d)
Boundary layer

In ideal experiment U_J, d, ν

Only dimensionless parameter $Re = \frac{U_J d}{\nu}$





mean velocity predominant in the axial direction

$$< U > \gg < V >$$

Fig. 5.2. Radial profiles of mean axial velocity in a turbulent round jet, Re = 95,500. The dashed lines indicate the half-width, $r_{1/2}(x)$, of the profiles. (Adapted from the data of Hussein *et al.* (1994).)

Centerline velocity

$$U_0(x) \equiv \langle U(x,0,0) \rangle$$

jet half-width

$$\langle U(x, r_{1/2}(x), 0) \rangle = \frac{1}{2} U_0(x).$$

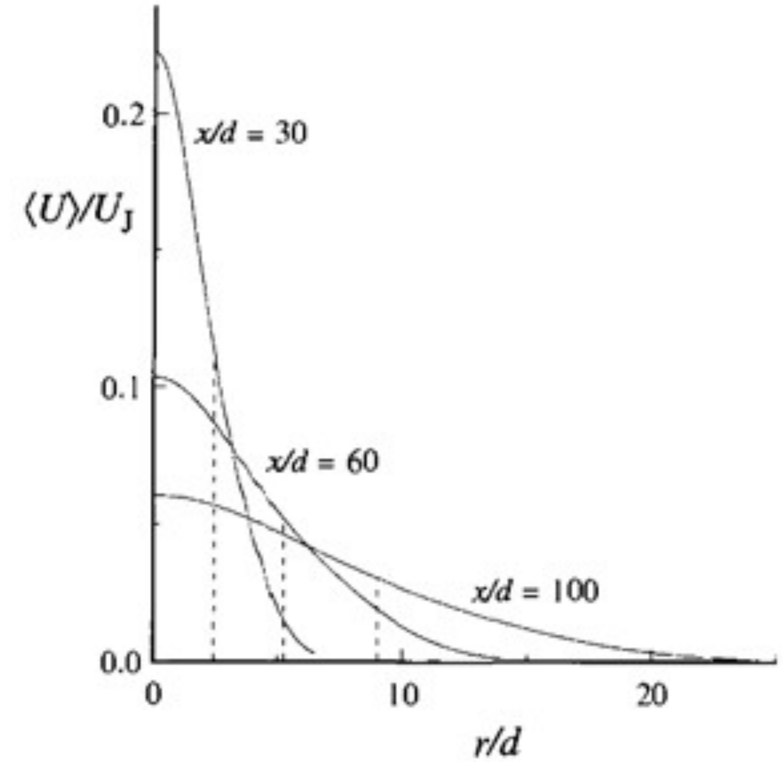


Fig. 5.2. Radial profiles of mean axial velocity in a turbulent round jet, Re = 95,500. The dashed lines indicate the half-width, $r_{1/2}(x)$, of the profiles. (Adapted from the data of Hussein *et al.* (1994).)

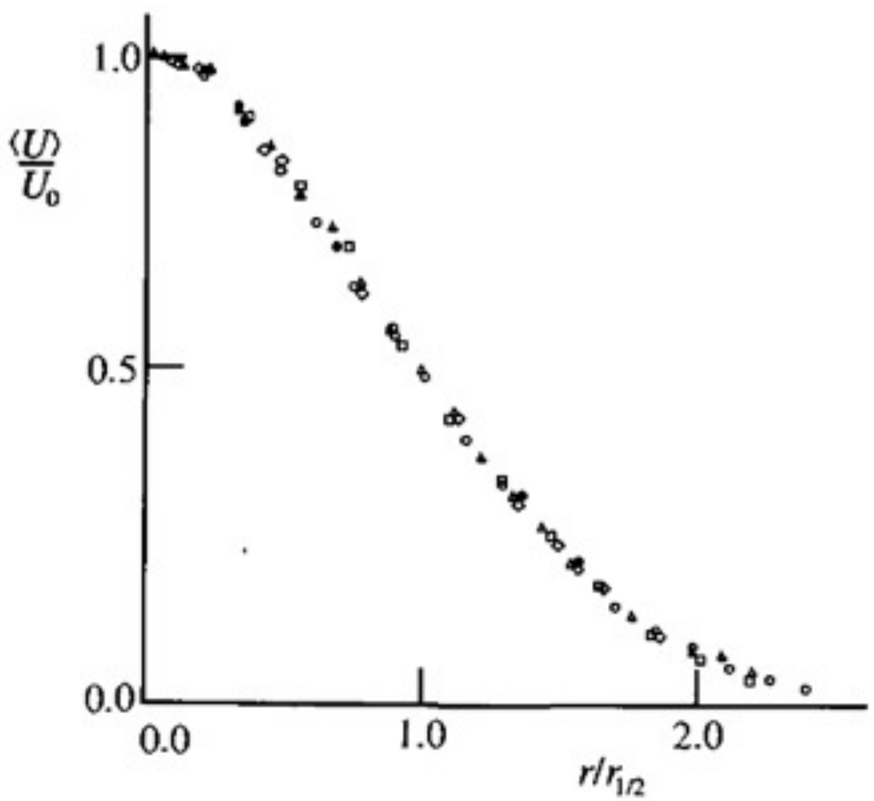


Fig. 5.3: Mean axial velocity against radial distance in a turbulent round jet, Re ≈ 10^5; measurements of Wygnanski and Fiedler (1969). Symbols: ○, $x/d = 40$; △, $x/d = 50$; □, $x/d = 60$; ◇, $x/d = 75$; •, $x/d = 97.5$.

<u>Self-similarity</u>

Self-similarity

$$\xi \equiv \frac{y}{\delta(x)},$$

$$\tilde{Q}(\xi, x) \equiv \frac{Q(x, y)}{Q_0(x)}.$$

$$\tilde{Q}(\xi, x) = \hat{Q}(\xi)$$

$Q(x, y)$ is self-similar

$U_0, r_{1/2} = ?$

$$\boxed{\frac{U_0(x)}{U_J} = \frac{B}{(x - x_0)/d},}$$

$$S \equiv \frac{dr_{1/2}(x)}{dx}$$

$$\boxed{r_{1/2}(x) = S(x - x_0)}$$

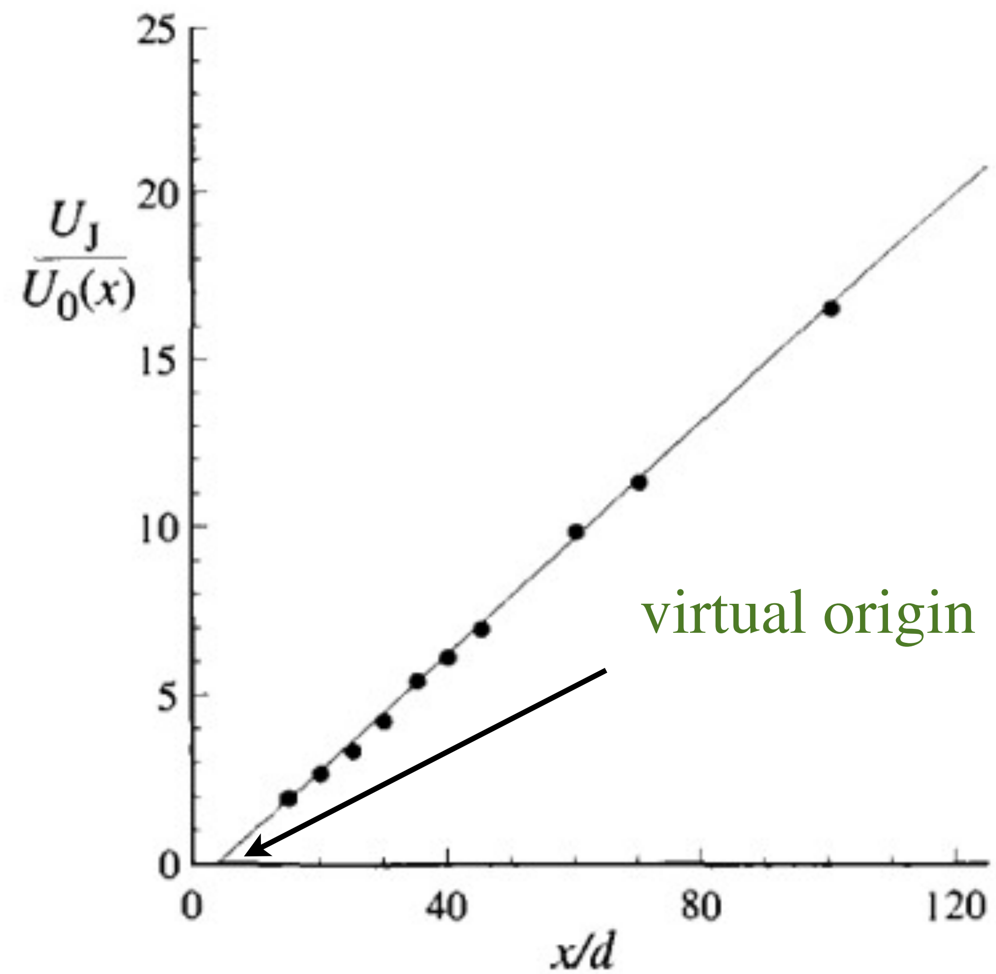
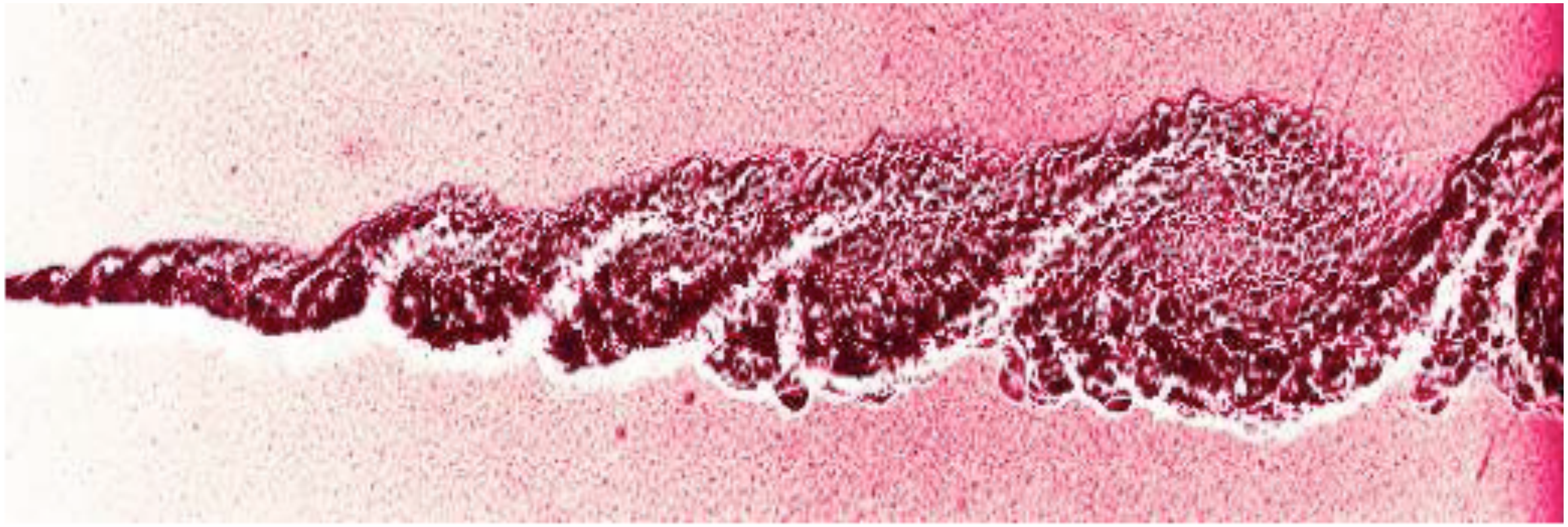


Fig. 5.4. The variation with axial distance of the mean velocity along the centerlin in a turbulent round jet, Re = 95,500: symbols, experimental data of Hussein *et a* (1994); and line, Eq. (5.6) with $x_0/d = 4$ and $B = 5.8$.

Reynolds Number

	Panchapakesan and Lumley (1993a)	Hussein *et al.* (1994), hot-wire data	Hussein *et al.* (1994), laser-Doppler data
Re	11,000	95,500	95,500
S	0.096	0.102	0.094
B	6.06	5.9	5.8

Only small scale structures



Self-similar region

$$x/d > 30$$

$$\frac{U_0(x)}{U_J} = \frac{B}{(x - x_0)/d},$$

$$B = 5.8$$

$$r_{1/2}(x) = S(x - x_0)$$

$$S = 0.094$$

similarity variable

$$\xi \equiv r/r_{1/2}$$

$$\eta \equiv r/(x - x_0).$$

$$f(\eta) = \bar{f}(\xi) = \langle U(x, r, 0)\rangle / U_0(x)$$

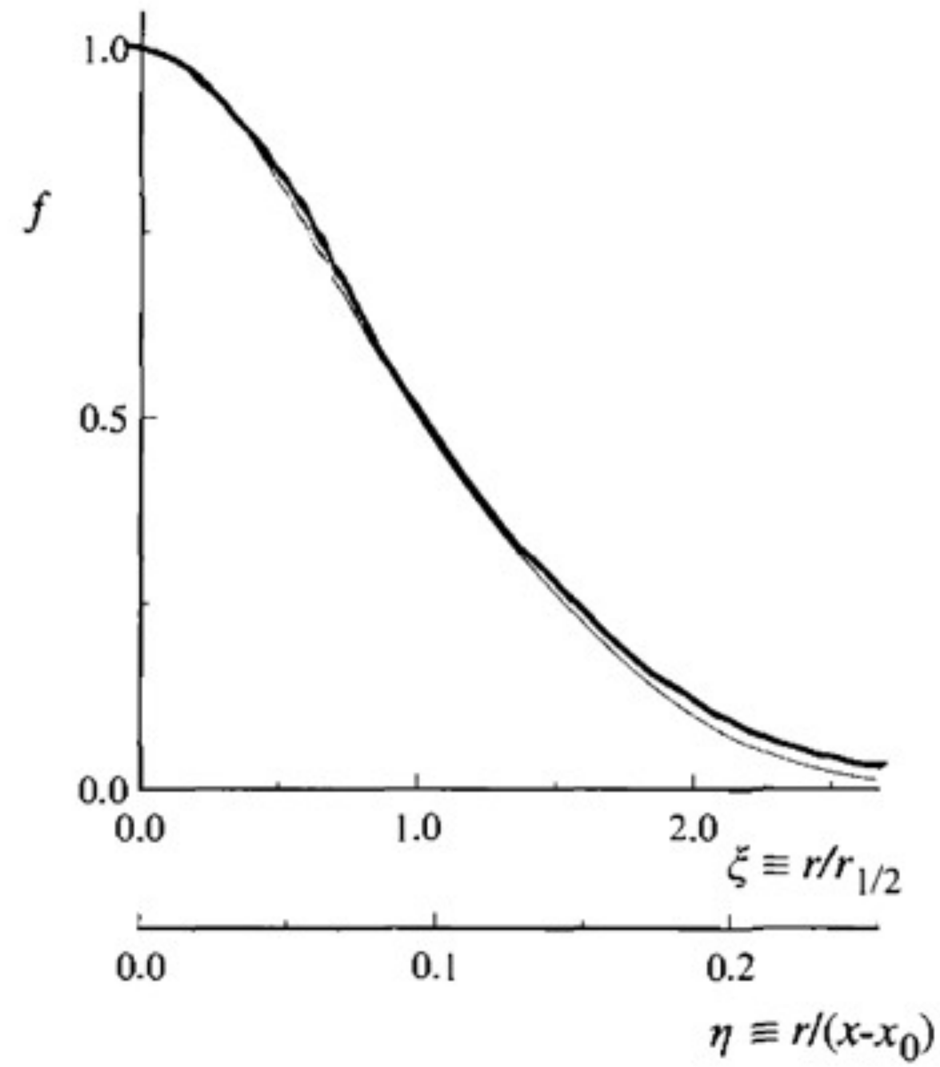


Fig. 5.5. The self-similar profile of the mean axial velocity in the self-similar round jet: curve fit to the LDA data of Hussein *et al.* (1994).

Lateral Velocity

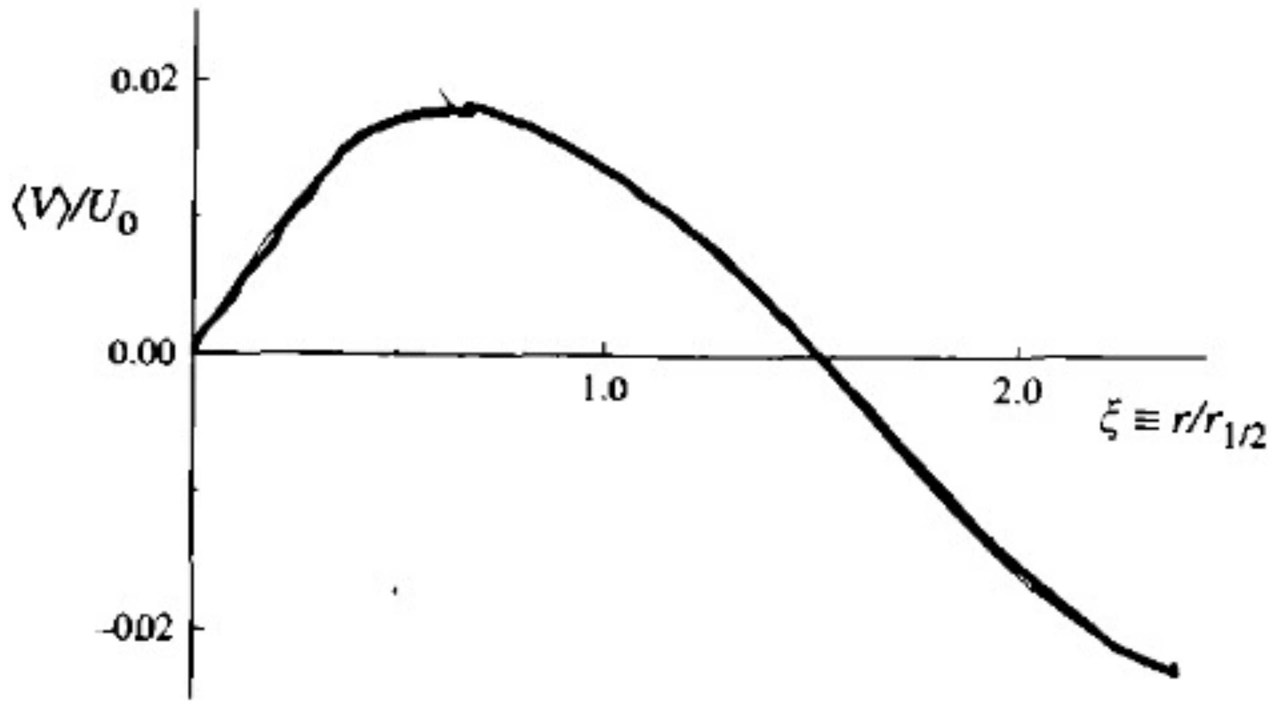


Fig. 5.6. The mean lateral velocity in the self-similar round jet. From the LDA data of Hussein *et al.* (1994).

Entrainment

Reynolds stresses

Symmetry
$$\begin{bmatrix} \langle u^2\rangle & \langle uv\rangle & 0 \\ \langle uv\rangle & \langle v^2\rangle & 0 \\ 0 & 0 & \langle w^2\rangle \end{bmatrix}$$

$$u_0'(x) \equiv \langle u^2\rangle_{r=0}^{1/2}$$

$$\boxed{u'/U_0(x) \approx const}$$

Reynolds stresses
self-similar

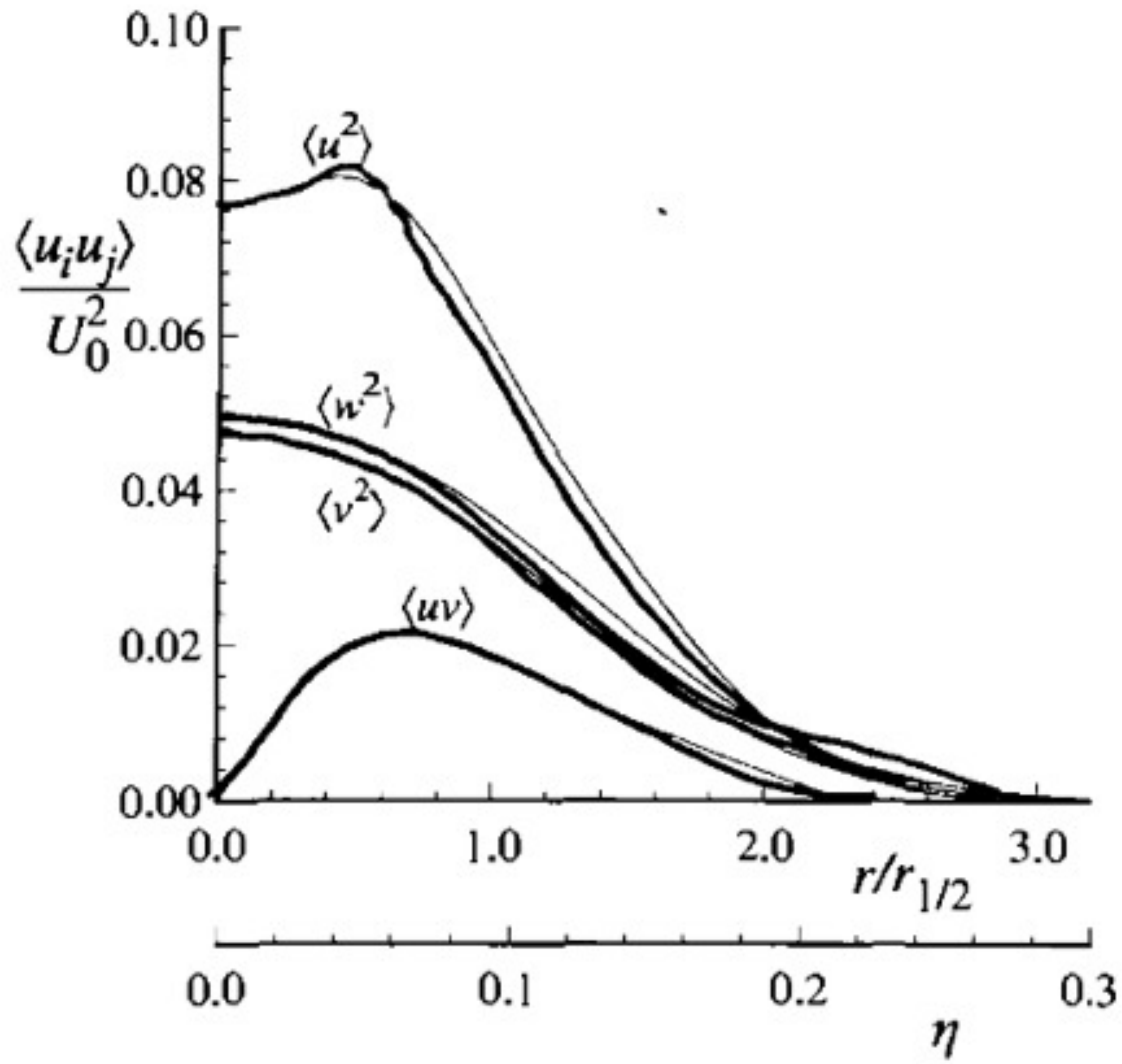


Fig. 5.7. Profiles of Reynolds stresses in the self-similar round jet: curve fit t
LDA data of Hussein *et al.* (1994).

(i) center r.ms. 25% mean
(ii)ratio rms mean increases
(iii)anisotropy
(iv)shear-stress 25%
(v) gradient hypothesis
(vi)eddy viscosity self-similar
(vii)Turbulent scale self-similar

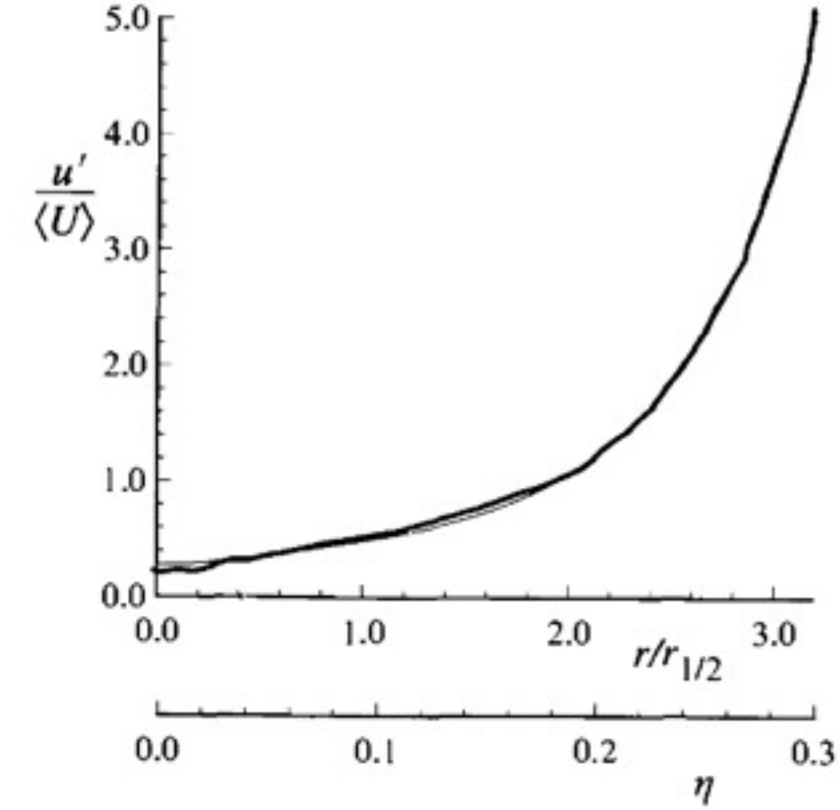


Fig. 5.8. The profile of the local turbulence intensity – $\langle u^2 \rangle^{1/2}/\langle U \rangle$ – in the self-similar round jet. From the curve fit to the experimental data of Hussein *et al.* (1994).

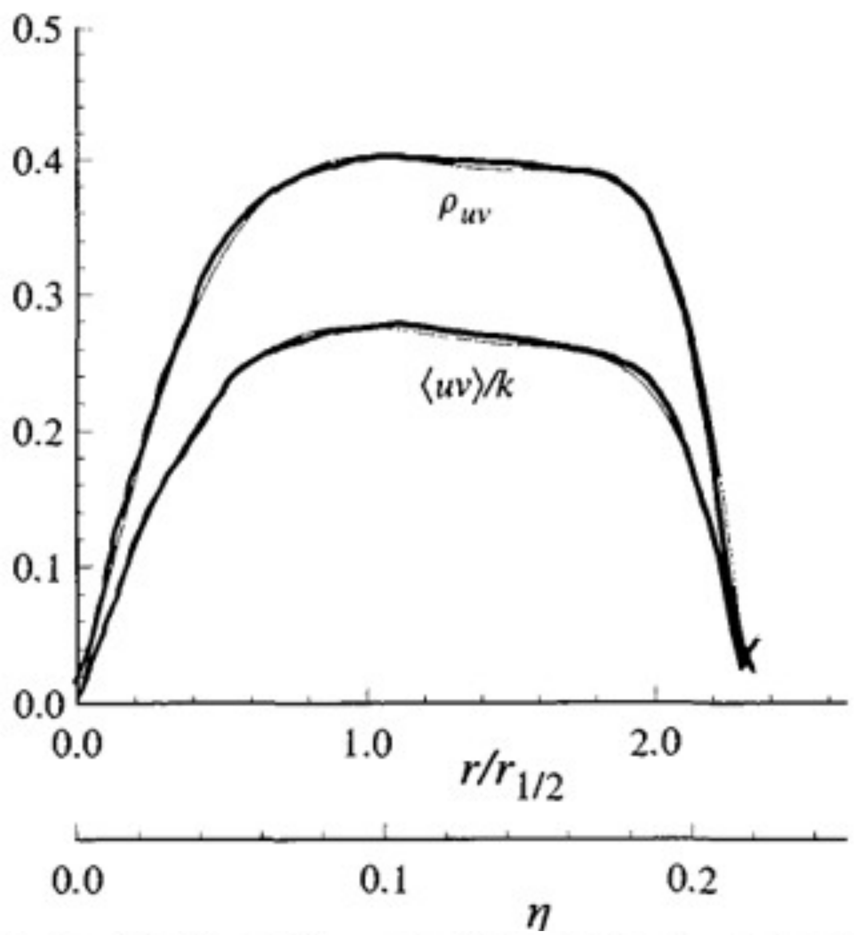
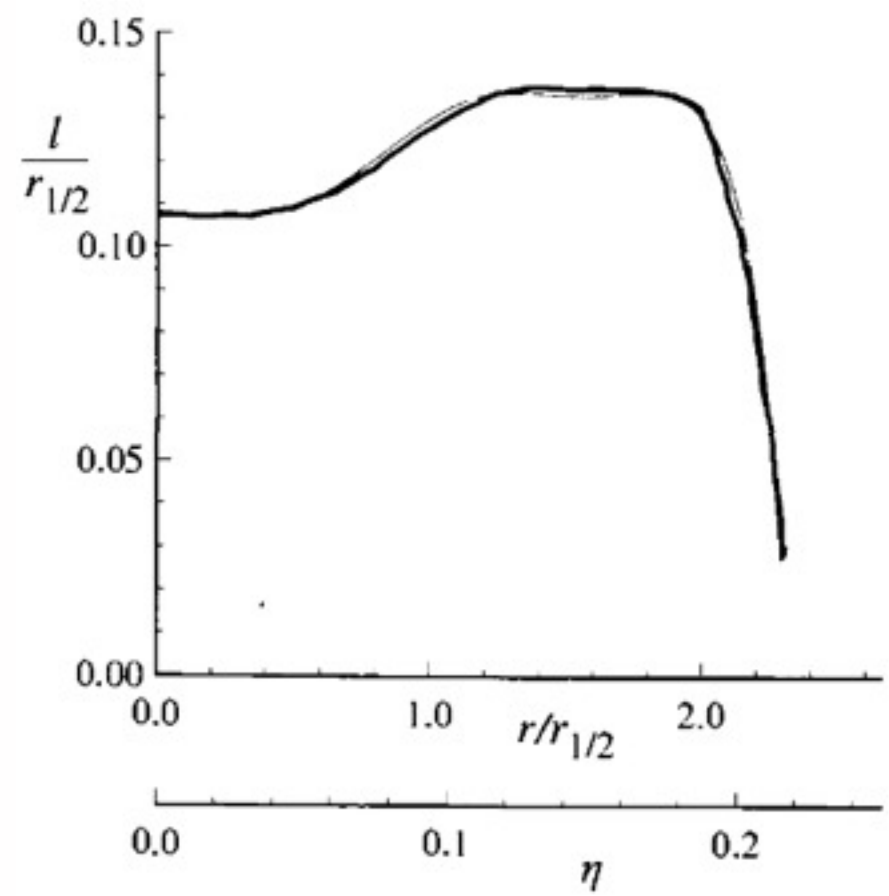
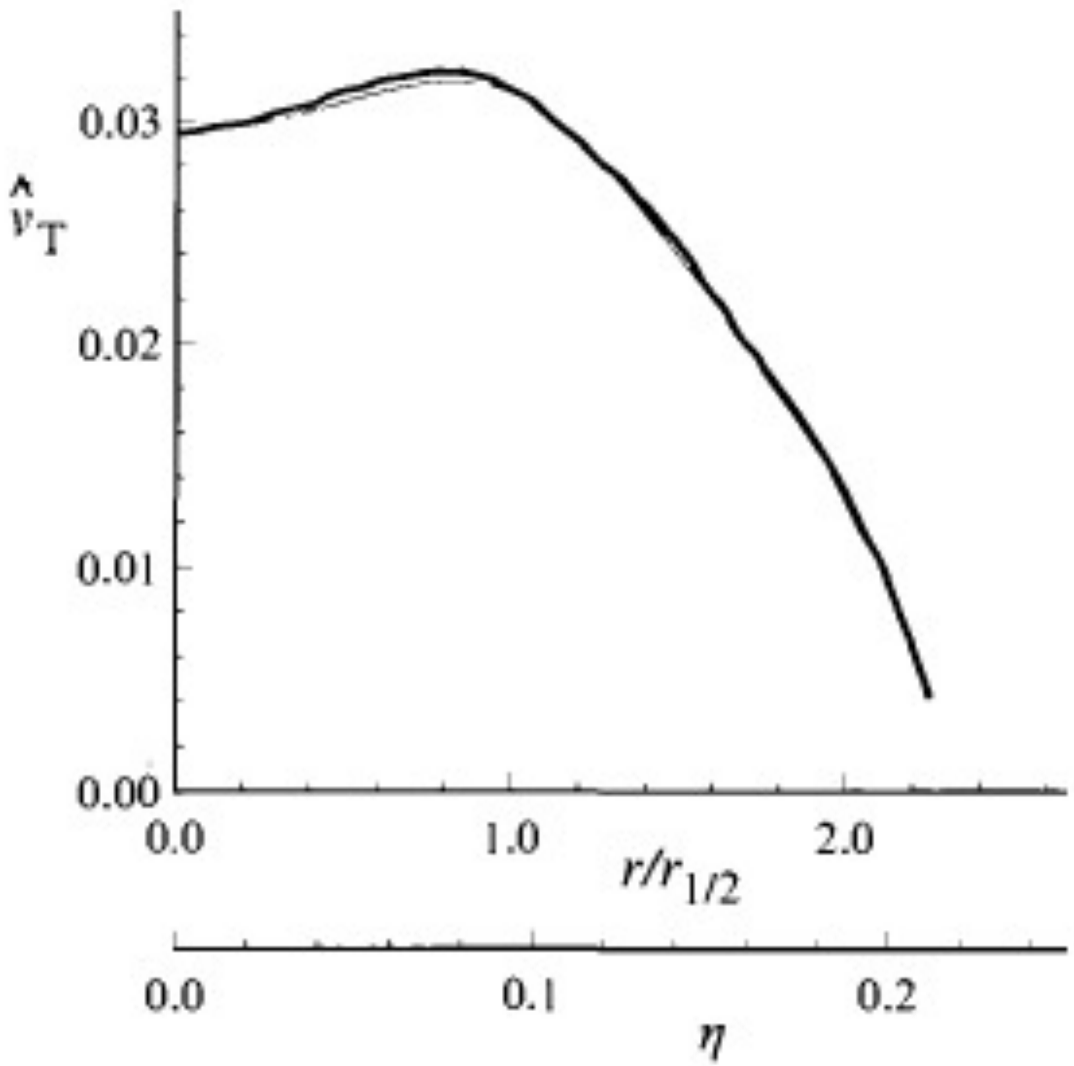
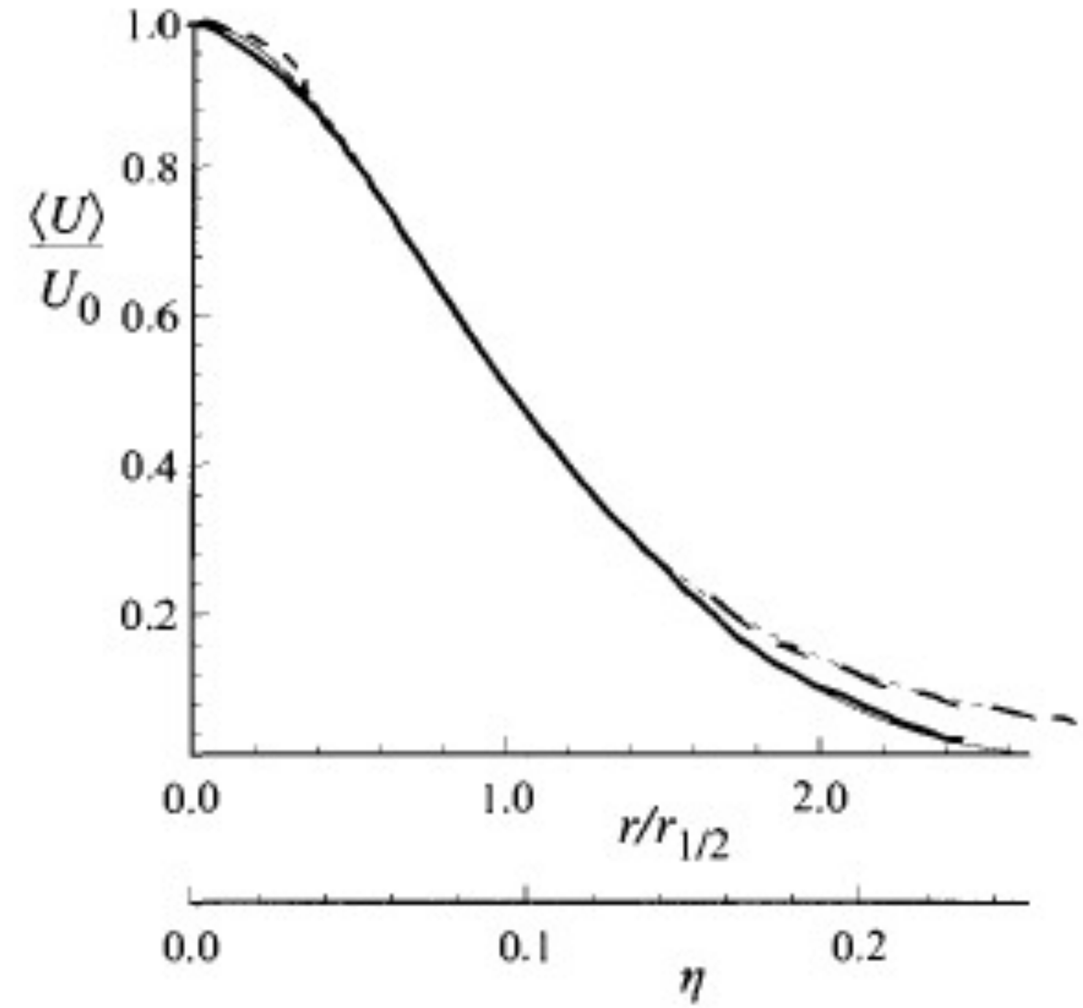


Fig. 5.9. Profiles of $\langle uv \rangle/k$ and the u–v correlation coefficient ρ_{uv} in the self-similar round jet. From the curve fit to the experimental data of Hussein *et al.* (1994).



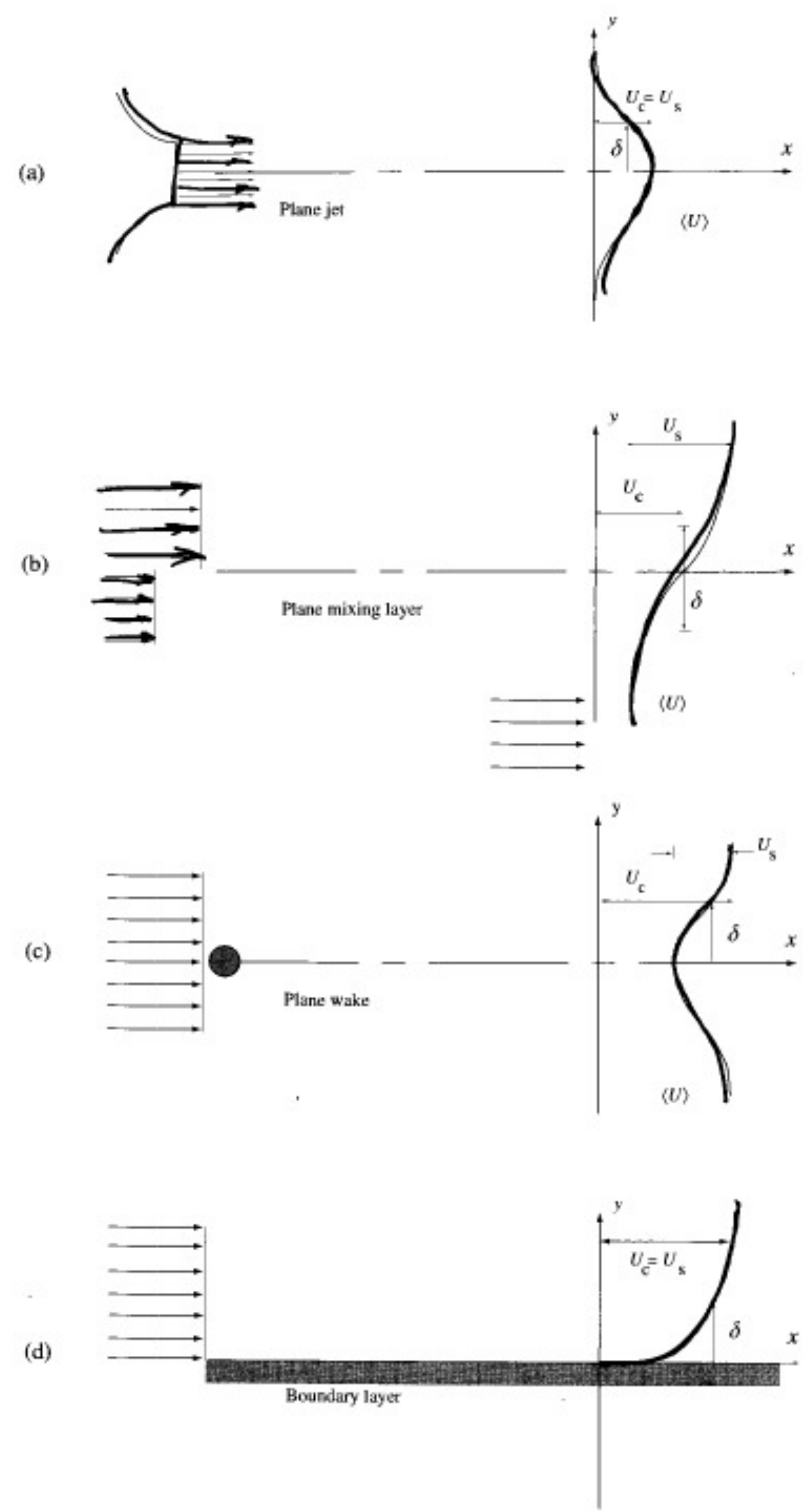

0.03
$\hat{\nu}_T$
0.02
0.01
0.00
0.0
1.0
2.0
$r/r_{1/2}$
0.0
0.1
0.2
η


1.0
0.8
$\frac{\langle U \rangle}{U_0}$
0.6
0.4
0.2
0.0
1.0
2.0
$r/r_{1/2}$
0.0
0.1
0.2
η

Free-shear flows

Part II: round jet

- Pope: Chap. 5


(a)
Plane jet
y
x
$U_c = U_s$
δ
$\langle U \rangle$
(b)
Plane mixing layer
y
x
U_s
U_c
δ
$\langle U \rangle$
(c)
Plane wake
y
x
U_s
U_c
δ
$\langle U \rangle$
(d)
Boundary layer
y
x
$U_c = U_s$
δ

Round Jet

For statistically axisymmetric, stationary non-swirling flows - such as the round jet or the wake behind a sphere - the corresponding turbulent boundary-layer equations are

$$\frac{\partial \langle U \rangle}{\partial x} + \frac{1}{r}\frac{\partial (r\langle V \rangle)}{\partial r} = 0,$$

$$\langle U \rangle \frac{\partial \langle U \rangle}{\partial x} + \langle V \rangle \frac{\partial \langle U \rangle}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \langle U \rangle}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial r}(r\langle uv \rangle)$$

$$\langle p \rangle/\rho = p_0/\rho - \langle v^2 \rangle + \int_r^{\infty} \frac{\langle v^2 \rangle - \langle w^2 \rangle}{r'}\, \mathrm{d}r'$$

We neglect

$$-\frac{\partial}{\partial x}\left(\langle u^2 \rangle - \langle v^2 \rangle + \int_r^{\infty} \frac{\langle v^2 \rangle - \langle w^2 \rangle}{r'}\, \mathrm{d}r' \right)$$

Flow rate momentum, mass, energy

$$\frac{\partial}{\partial x}(r\langle U\rangle^2) + \frac{\partial}{\partial r}(r\langle U\rangle\langle V\rangle + r\langle uv\rangle) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty r\langle U\rangle^2\,\mathrm{d}r = -\left[r\langle U\rangle\langle V\rangle + r\langle uv\rangle\right]_0^\infty$$
$$= 0,$$

$$\dot{M}(x) \equiv \int_0^\infty 2\pi r\rho\langle U\rangle^2\,\mathrm{d}r$$

Momentum flow rate conserved true for all jets and wakes!

$$\langle U(x,r,0)\rangle = U_0(x)\bar{f}(\xi),$$

$$\xi \equiv r/r_{1/2}(x)$$

$$\dot{M} = 2\pi\rho\left(r_{1/2}U_0\right)^2\int_0^\infty \xi\bar{f}(\xi)^2\,\mathrm{d}\xi$$

Mass & energy flow rate

$$\dot{m}(x) \equiv \int_0^\infty 2\pi r\rho\langle U\rangle\,\mathrm{d}r$$
$$= 2\pi\rho r_{1/2}\left(r_{1/2}U_0\right)\int_0^\infty \xi\bar{f}(\xi)\,\mathrm{d}\xi,$$

$$\dot{E}(x) \equiv \int_0^\infty 2\pi r\rho\tfrac{1}{2}\langle U\rangle^3\,\mathrm{d}r$$
$$= \frac{\pi\rho}{r_{1/2}}\left(r_{1/2}U_0\right)^3\int_0^\infty \xi\bar{f}(\xi)^3\,\mathrm{d}\xi.$$

Self-similarity

Empirical observation: $\langle U \rangle / U_0(x)$, $\langle u_i u_j \rangle / U_0(x)^2$ self-similar as functions of : $\xi \equiv r / r_{1/2}(x)$

Self-similarity implies linear spreading of jet

Defining $\bar{g}(\xi) \equiv \langle uv \rangle / U_0(x)^2$

Momentum Equation

$$[\xi \bar{f}^2] \left\{ \frac{r_{1/2}}{U_0} \frac{\mathrm{d}U_0}{\mathrm{d}x} \right\} - \left[\bar{f}' \int_0^\xi \xi \bar{f} \, \mathrm{d}\xi \right] \left\{ \frac{r_{1/2}}{U_0} \frac{\mathrm{d}U_0}{\mathrm{d}x} + 2 \frac{\mathrm{d}r_{1/2}}{\mathrm{d}x} \right\} = - [(\xi \bar{g})']$$

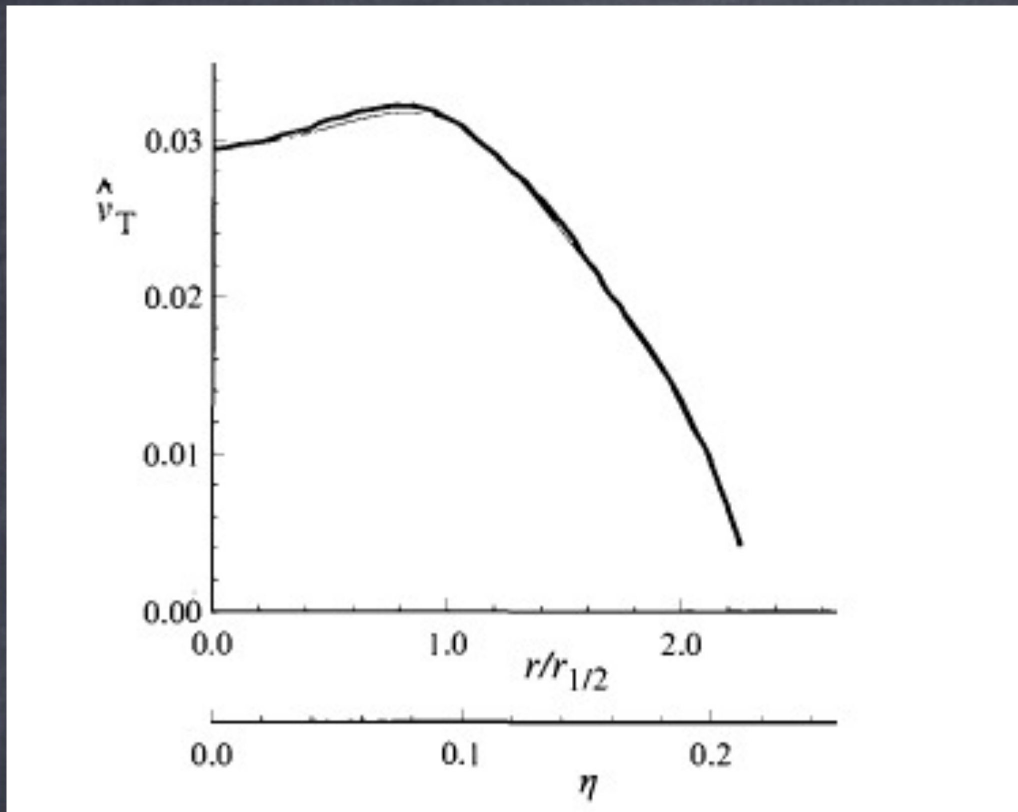
$$\frac{r_{1/2}}{U_0} \frac{\mathrm{d}U_0}{\mathrm{d}x} = C,$$

$$\frac{r_{1/2}}{U_0} \frac{\mathrm{d}U_0}{\mathrm{d}x} + 2 \frac{\mathrm{d}r_{1/2}}{\mathrm{d}x} = C + 2S$$

$$\frac{\mathrm{d}r_{1/2}}{\mathrm{d}x} = S$$

$$C \equiv \frac{r_{1/2}}{U_0} \frac{\mathrm{d}U_0}{\mathrm{d}x} = -S$$

Constant-viscosity solution



Schlichting (1933)

$$f(\eta) = \frac{1}{(1 + a\eta^2)^2}$$

$$a = \left(\sqrt{2} - 1\right)/S^2$$

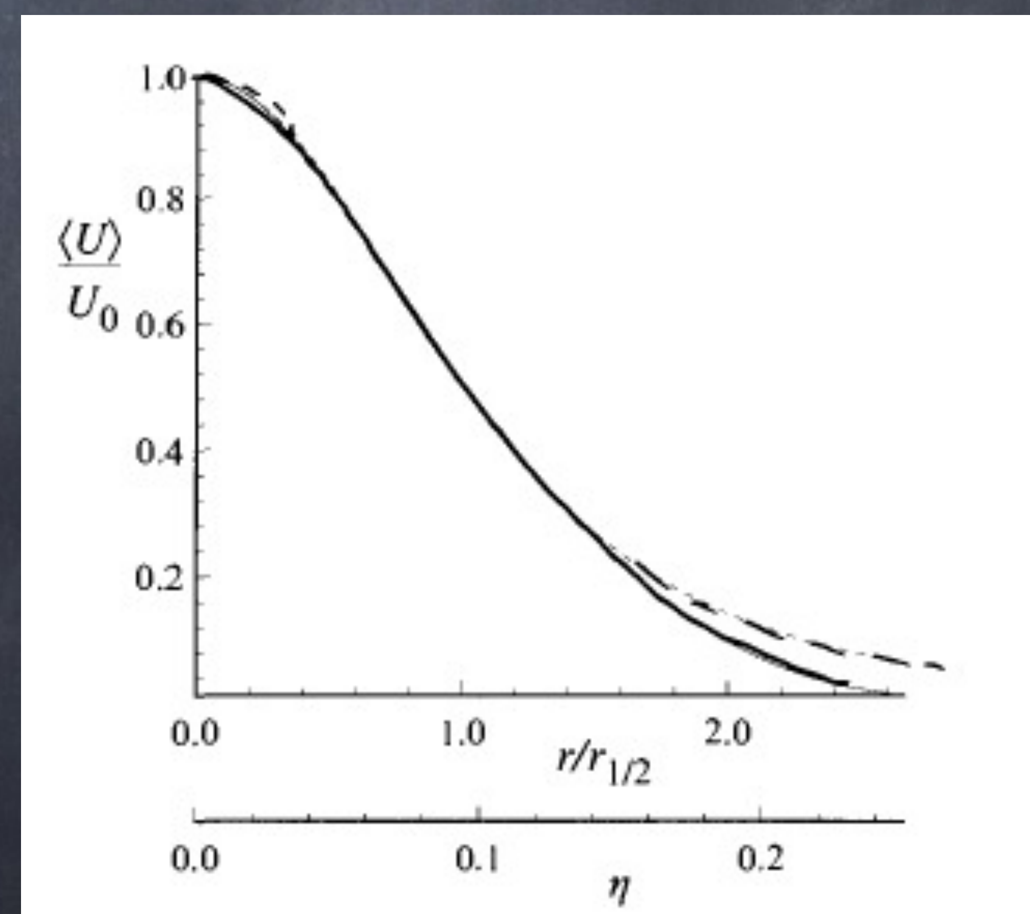
$$S = 8\left(\sqrt{2} - 1\right)\hat{\nu}_T$$

$$R_T \equiv \frac{U_0(x) r_{1/2}(x)}{\nu_T} = \frac{1}{\hat{\nu}_T} \approx 35$$

$$\langle uv \rangle = -\nu_T \frac{\partial \langle U \rangle}{\partial r}$$

$$\nu_T(x, r) = r_{1/2}(x) U_0(x) \hat{\nu}_T(\eta),$$

$$\langle U \rangle \frac{\partial \langle U \rangle}{\partial x} + \langle V \rangle \frac{\partial \langle U \rangle}{\partial r} = \frac{\nu_T}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \langle U \rangle}{\partial r}\right)$$



Kinetic Energy

Instantaneous Energy

$$E(\boldsymbol{x}, t) \equiv \tfrac{1}{2}\boldsymbol{U}(\boldsymbol{x}, t) \cdot \boldsymbol{U}(\boldsymbol{x}, t)$$

$$\langle E(\boldsymbol{x}, t)\rangle = \bar{E}(\boldsymbol{x}, t) + k(\boldsymbol{x}, t)$$

$$\bar{E} \equiv \tfrac{1}{2}\langle \boldsymbol{U}\rangle \cdot \langle \boldsymbol{U}\rangle$$

$$k \equiv \tfrac{1}{2}\langle \boldsymbol{u} \cdot \boldsymbol{u}\rangle = \tfrac{1}{2}\langle u_i u_i\rangle$$

$$\frac{\mathrm{D}E}{\mathrm{D}t} + \nabla \cdot \boldsymbol{T} = -2\nu S_{ij} S_{ij}$$

$$T_i \equiv U_i p/\rho - 2\nu U_j S_{ij}$$

Mean Kinetic Energy

$$\frac{\bar{\mathrm{D}}\langle E\rangle}{\bar{\mathrm{D}}t} + \nabla \cdot (\langle \boldsymbol{u}E\rangle + \langle \boldsymbol{T}\rangle) = -\bar{\varepsilon} - \varepsilon$$

$$\bar{\varepsilon} \equiv 2\nu \bar{S}_{ij} \bar{S}_{ij},$$

$$\varepsilon \equiv 2\nu \langle s_{ij} s_{ij}\rangle,$$

$$\bar{S}_{ij} = \langle S_{ij}\rangle = \frac{1}{2}\left(\frac{\partial \langle U_i\rangle}{\partial x_j} + \frac{\partial \langle U_j\rangle}{\partial x_i}\right)$$

$$s_{ij} = S_{ij} - \langle S_{ij}\rangle = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\frac{\bar{D}\langle E\rangle}{\bar{D}t} + \nabla\cdot(\langle \boldsymbol{u}E\rangle + \langle \boldsymbol{T}\rangle) = -\bar{\varepsilon} - \varepsilon.$$

$$\frac{\bar{D}\bar{E}}{\bar{D}t} + \nabla\cdot\bar{\boldsymbol{T}} = -\mathcal{P} - \bar{\varepsilon},$$

$$\frac{\bar{D}k}{\bar{D}t} + \nabla\cdot\boldsymbol{T}' = \mathcal{P} - \varepsilon,$$

$$\bar{T}_i \equiv \langle U_j\rangle\langle u_i u_j\rangle + \langle U_i\rangle\langle p\rangle/\rho - 2\nu\langle U_j\rangle\bar{S}_{ij}$$

$$T'_i \equiv \tfrac{1}{2}\langle u_i u_j u_j\rangle + \langle u_i p'\rangle/\rho - 2\nu\langle u_j s_{ij}\rangle$$

(i) Only the symmetric part of the velocity-gradient $\quad \mathcal{P} = -\langle u_i u_j\rangle\bar{S}_{ij}.$

(ii) Only the anisotropic part of Reynolds stress counts $\quad \mathcal{P} = -a_{ij}\bar{S}_{ij}.$

$$\mathcal{P} \equiv -\langle u_i u_j\rangle \frac{\partial\langle U_i\rangle}{\partial x_j}$$

(iii) With turbulent viscosity hypothesis $\quad \mathcal{P} = 2\nu_T\bar{S}_{ij}\bar{S}_{ij} \geq 0.$

<u>**Production**</u>

(iv) Boundary-layer approximation $\quad \mathcal{P} = -\langle uv\rangle \frac{\partial\langle U\rangle}{\partial y}$

(v) Boundary-layer + turbulent viscosity $\quad \mathcal{P} = \nu_T\left(\frac{\partial\langle U\rangle}{\partial y}\right)^2$

Dissipation

$$\frac{\bar{\mathrm{D}}\bar{E}}{\bar{\mathrm{D}}t} + \nabla \cdot \bar{\boldsymbol{T}} = -\mathcal{P} - \bar{\varepsilon},$$

$$\frac{\bar{\mathrm{D}}k}{\bar{\mathrm{D}}t} + \nabla \cdot \boldsymbol{T}' = \mathcal{P} - \varepsilon,$$

$$\bar{\varepsilon} \equiv 2\nu \bar{S}_{ij}\bar{S}_{ij},$$

Mean dissipation negligible

$$\varepsilon \equiv 2\nu \langle s_{ij}s_{ij} \rangle,$$

Dissipation non-negative

Pseudo-dissipation

$$\tilde{\varepsilon} \equiv \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle$$

$$\tilde{\varepsilon} = \varepsilon - \nu \frac{\partial^2 \langle u_i u_j \rangle}{\partial x_i \, \partial x_j}$$

k/U_0^2

$$\widehat{\mathcal{P}} \equiv \mathcal{P}/(U_0^3/r_{1/2}) \approx -\frac{\langle uv \rangle}{U_0^2} \frac{r_{1/2}}{U_0} \frac{\partial \langle U \rangle}{\partial r}$$

Self-similar

→ $\hat{\varepsilon} \equiv \varepsilon/(U_0^3/r_{1/2})$

Independent of Re!

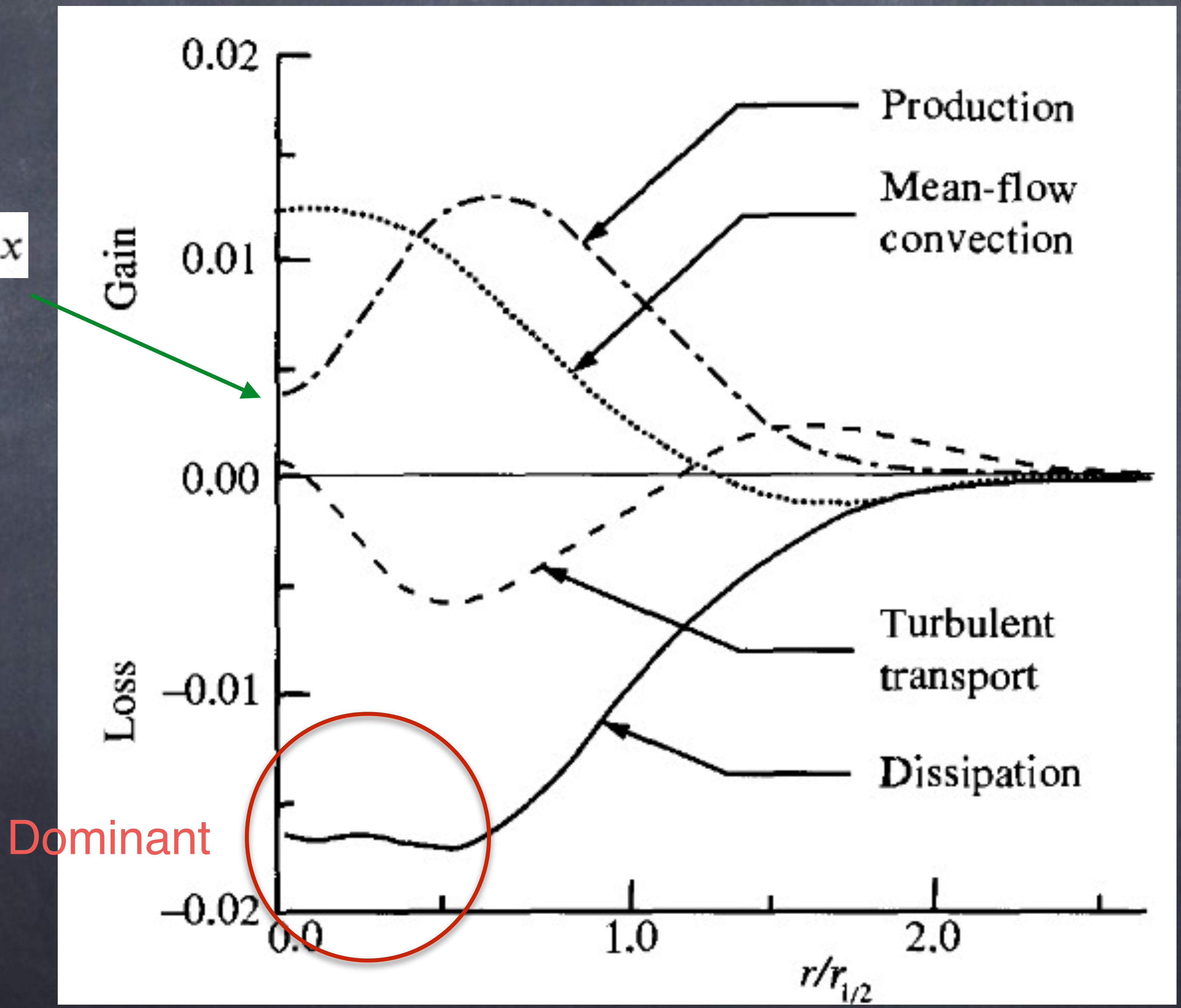
Different viscosities

$$\varepsilon_{\mathrm{a}}(x, r) = \varepsilon_{\mathrm{b}}(x, r) = \hat{\varepsilon}(r/r_{1/2}(x)) \, \frac{U_0^3(x)}{r_{1/2}(x)}$$

Kinetic Energy

Budgets

$-(\langle u^2\rangle - \langle v^2\rangle)\,\partial\langle U\rangle/\partial x$

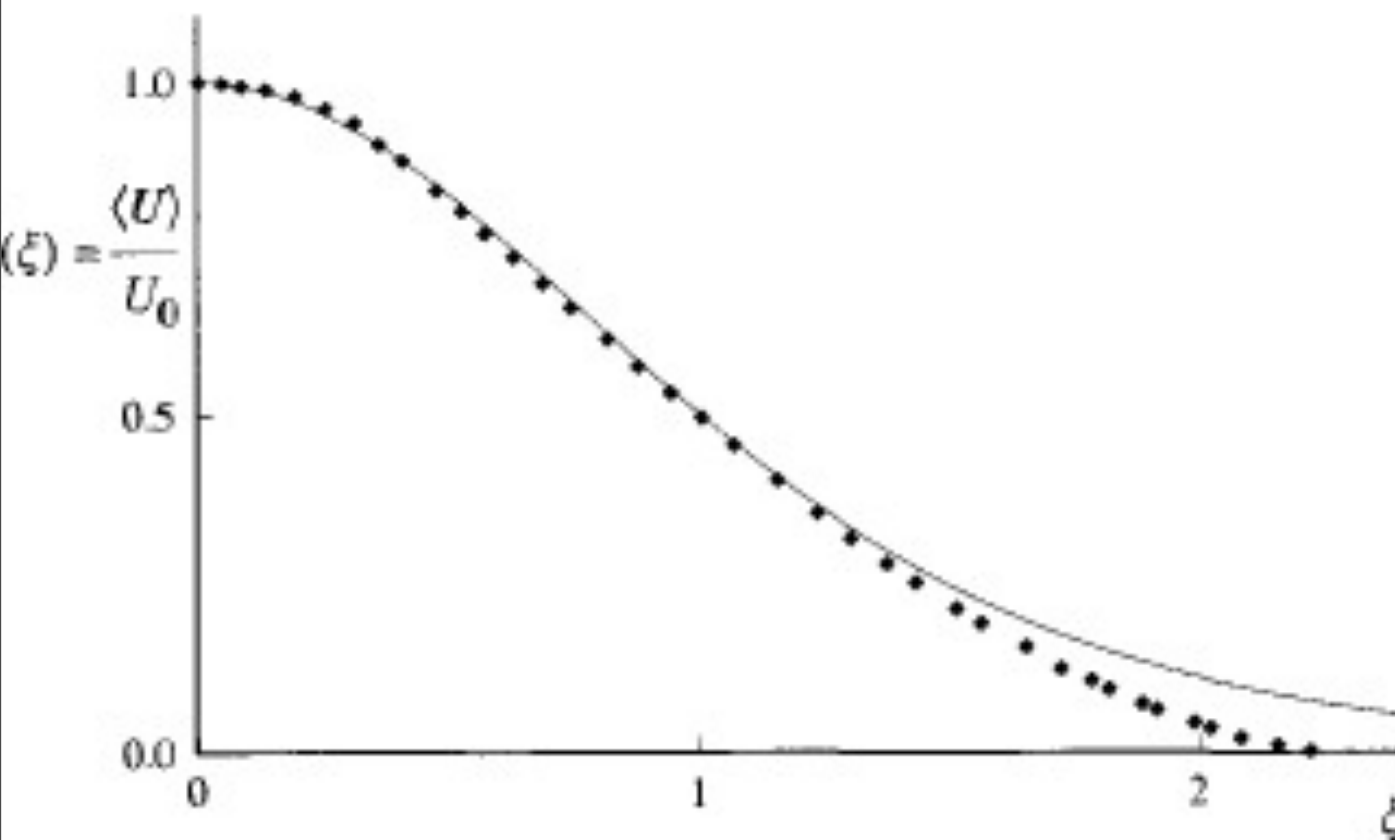


Free-shear flows

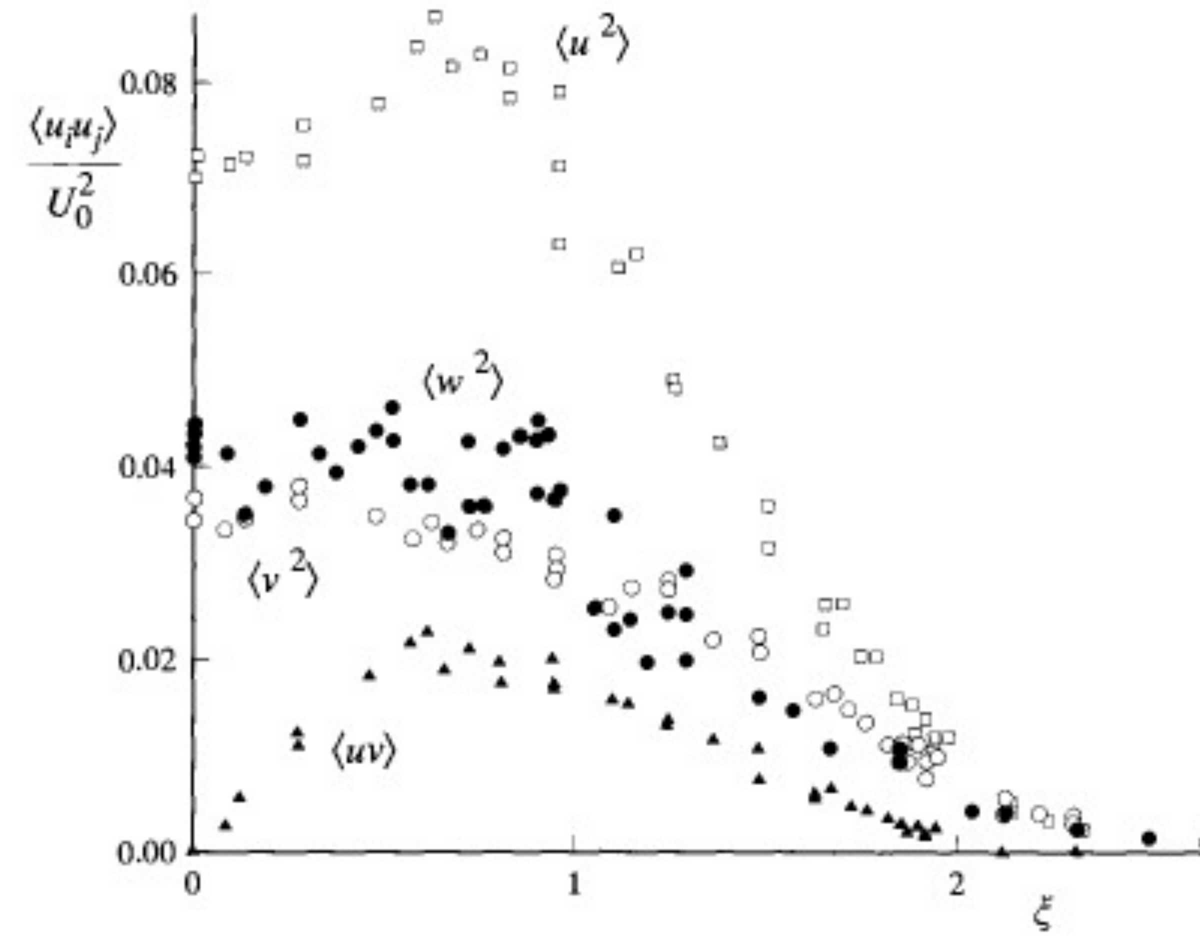
Part III: Other self-similar flows

- Pope: Chap. 5

Jet plan

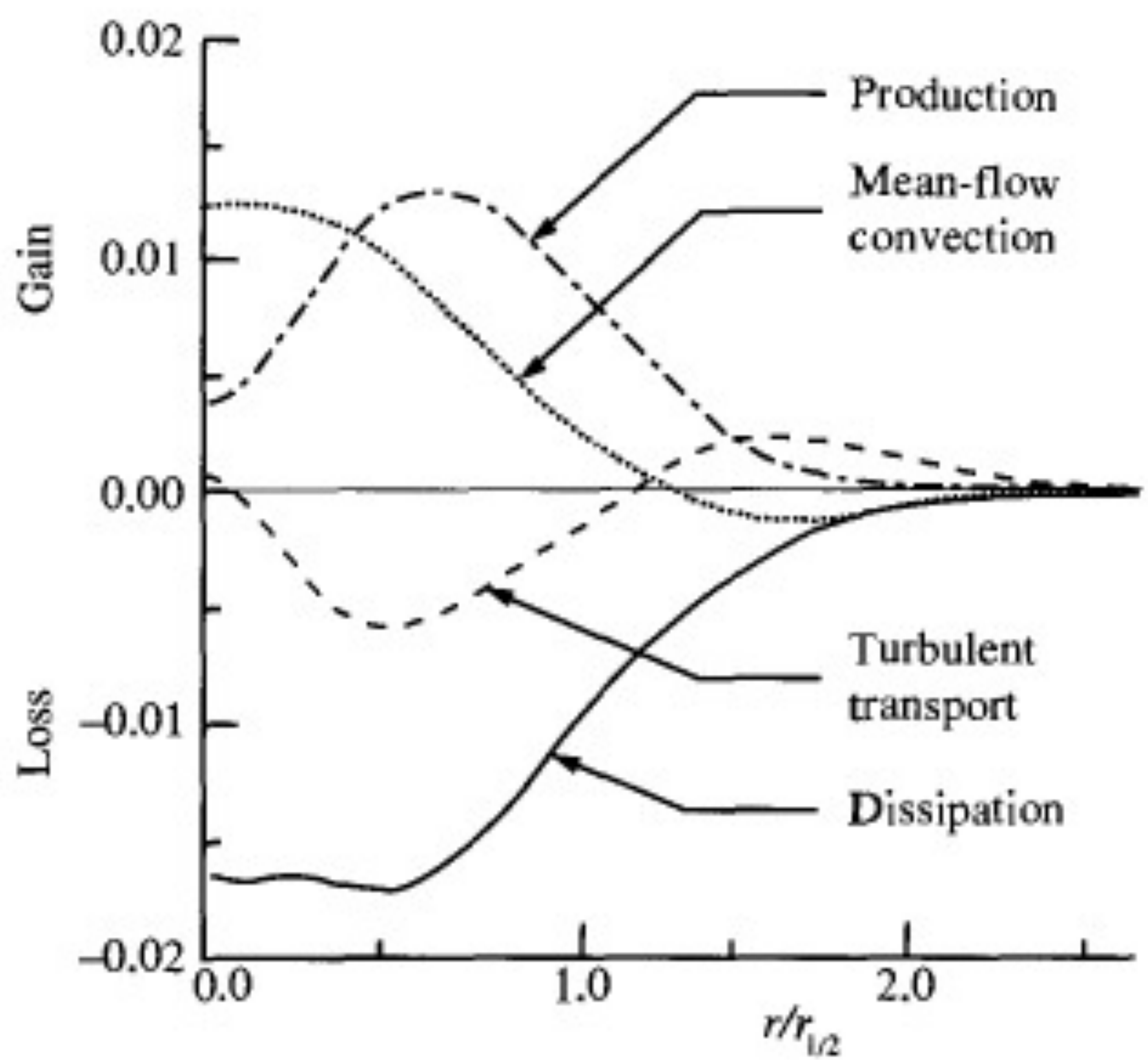


$(\xi) = \frac{\langle U \rangle}{U_0}$
1.0
0.5
0.0
0
1
2



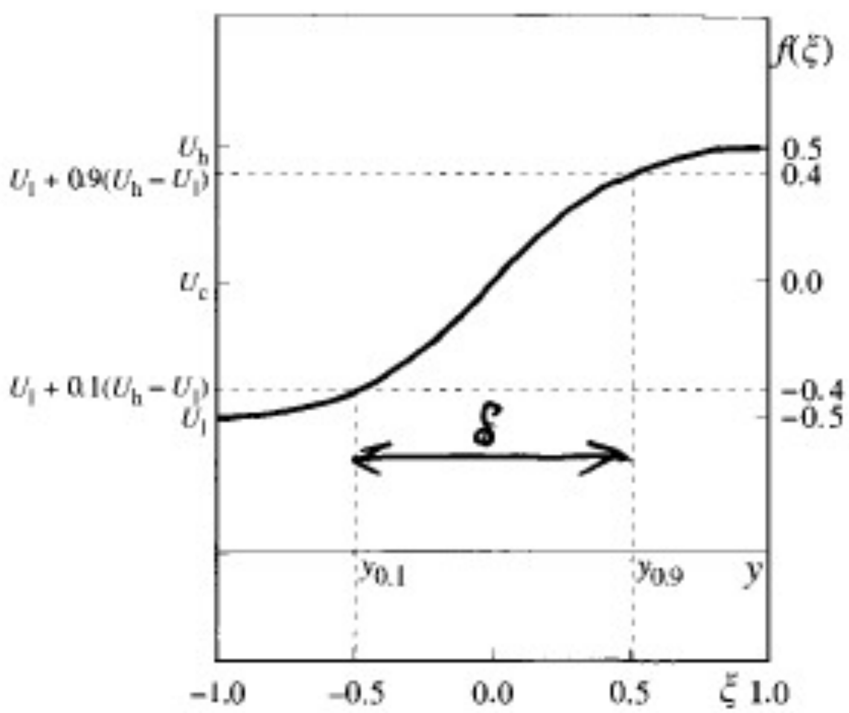
$\frac{\langle u_i u_j \rangle}{U_0^2}$
$\langle u^2 \rangle$
$\langle w^2 \rangle$
$\langle v^2 \rangle$
$\langle uv \rangle$
0.08
0.06
0.04
0.02
0.00
0
1
2
ξ

Round Jet



0.02
0.01
0.00
−0.01
−0.02
Gain
Loss
Production
Mean-flow
convection
Turbulent
transport
Dissipation
0.0
1.0
2.0
$r/r_{1/2}$

Couche de mélange

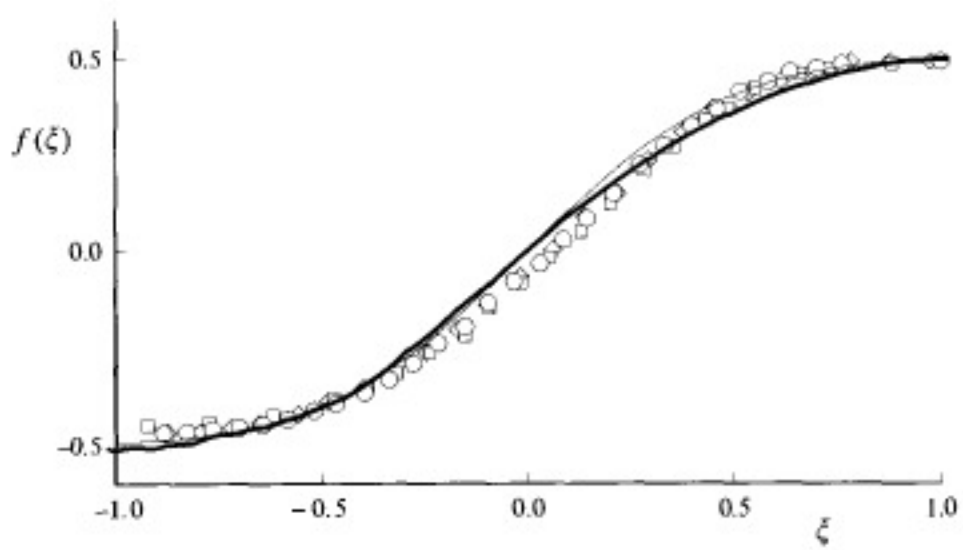


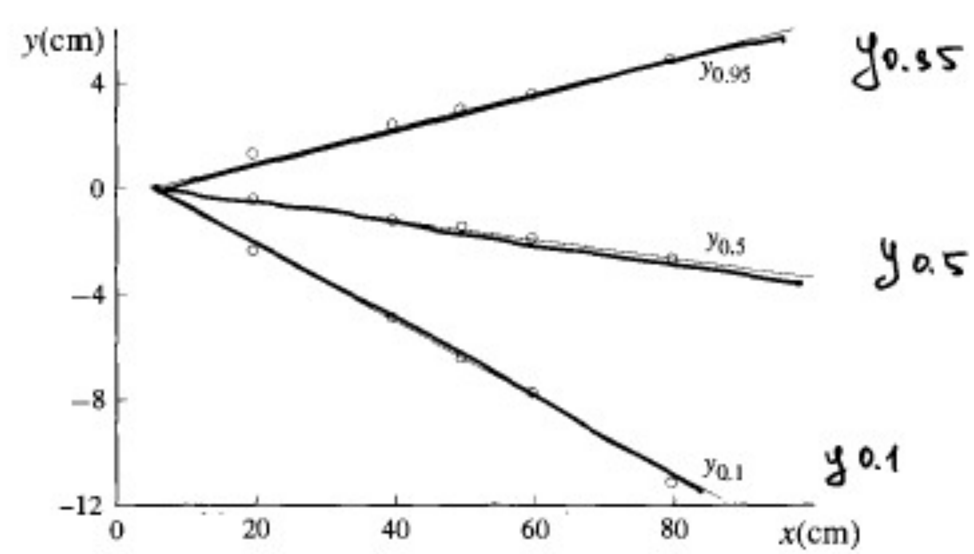
$$\langle U(x, y_\alpha(x), 0)\rangle = U_l + \alpha(U_h - U_l), \qquad \delta(x) = y_{0.9}(x) - y_{0.1}(x),$$

$$\bar{y}(x) = \tfrac{1}{2}[y_{0.9}(x) + y_{0.1}(x)].$$

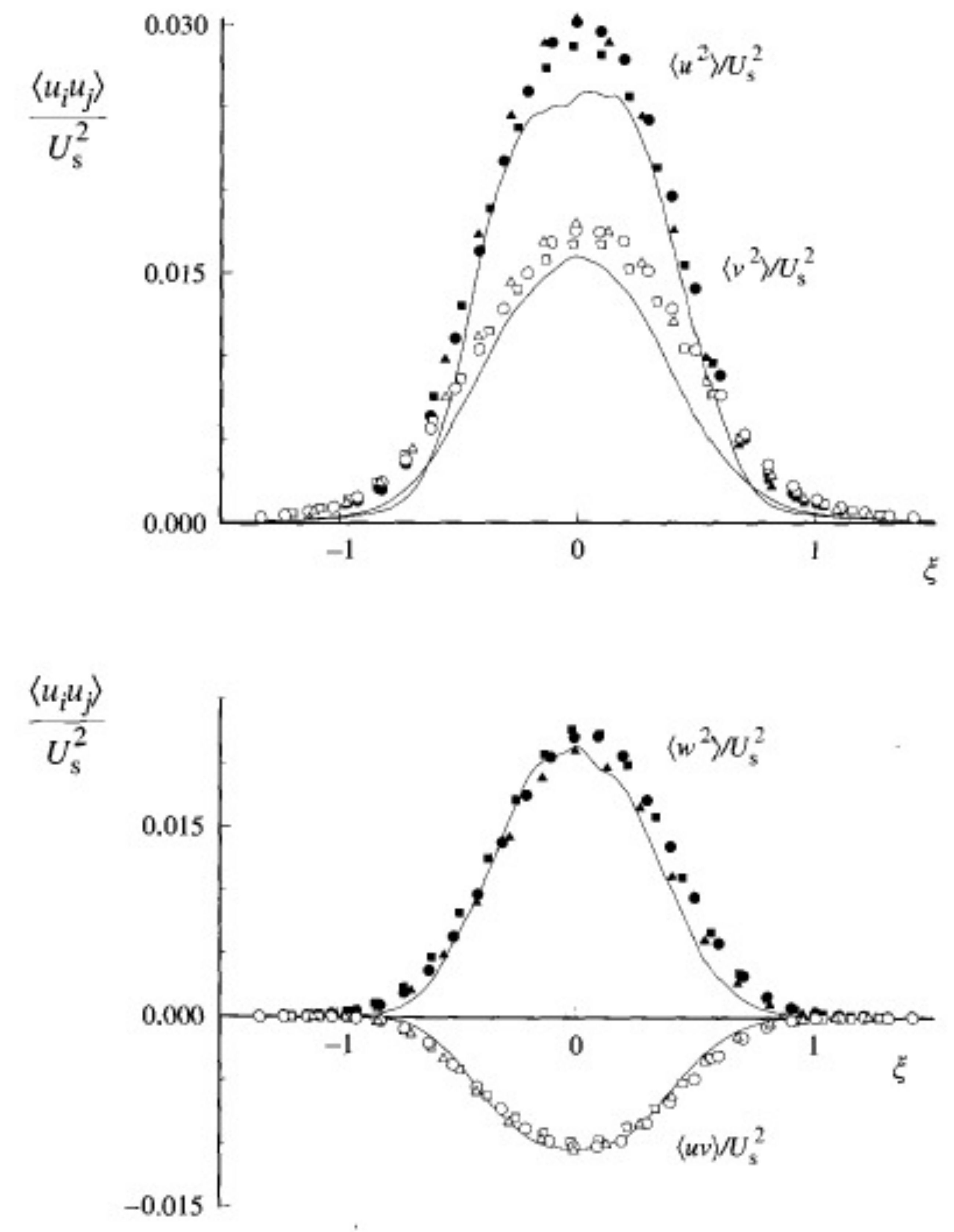
$$\xi = [y - \bar{y}(x)]/\delta(x),$$

$$f(\xi) = (\langle U\rangle - U_c)/U_s.$$





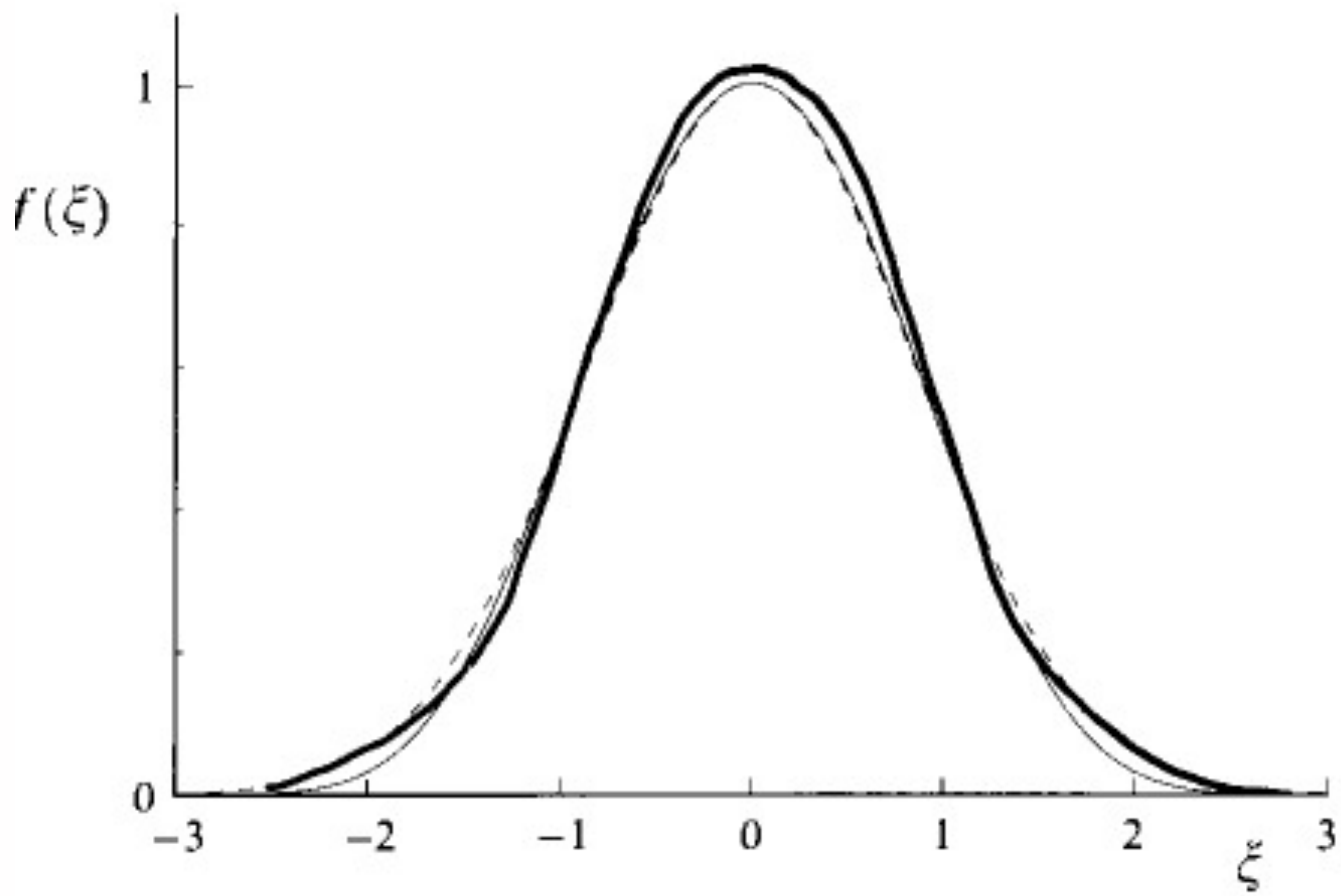
Couche de mélange



Sillage plan

$U_s(x) \equiv U_c - \langle U(x,0,0)\rangle.$ $\langle U(x, \pm y_{1/2}, 0)\rangle = U_c - \frac{1}{2}U_s(x).$ $\langle U\rangle = U_c - U_s(x) f(\xi).$

$f(\xi) = [U_c - \langle U(x,y,0)\rangle]/U_s(x),$ $S \equiv \frac{U_c}{U_s}\frac{dy_{1/2}}{dx},$ $\nu_T = \hat{\nu}_T U_s(x) y_{1/2}(x),$

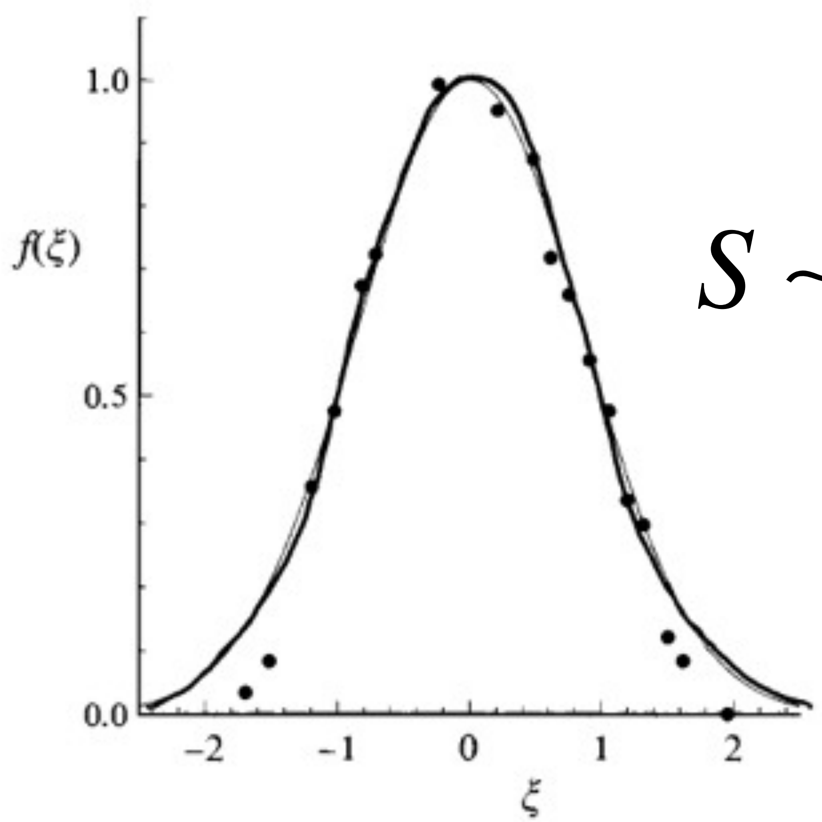


$$f(\xi) = \exp(-\alpha\xi^2),$$

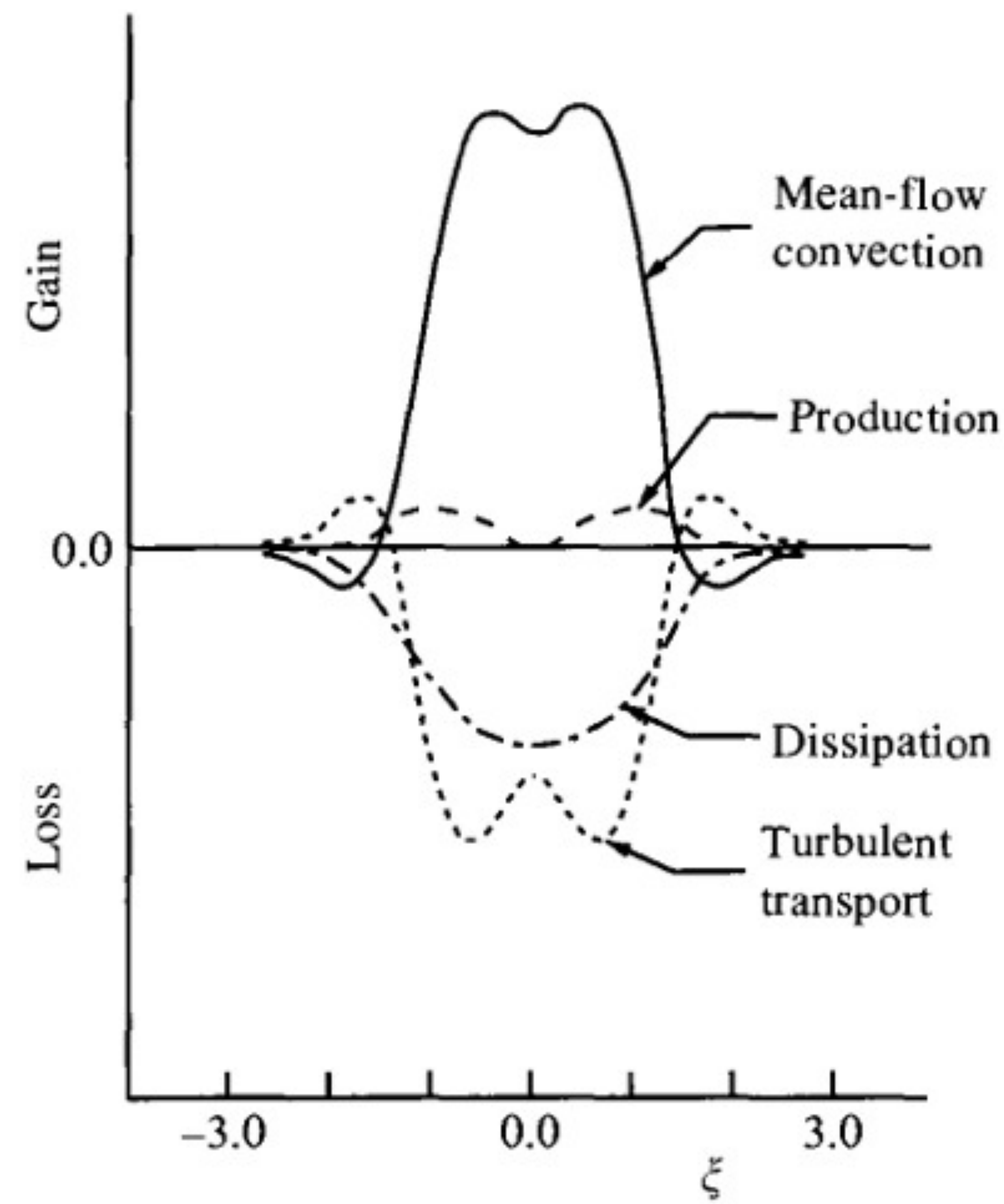
Self-similarity
Geometry dependence

Sillage axialsymétrique

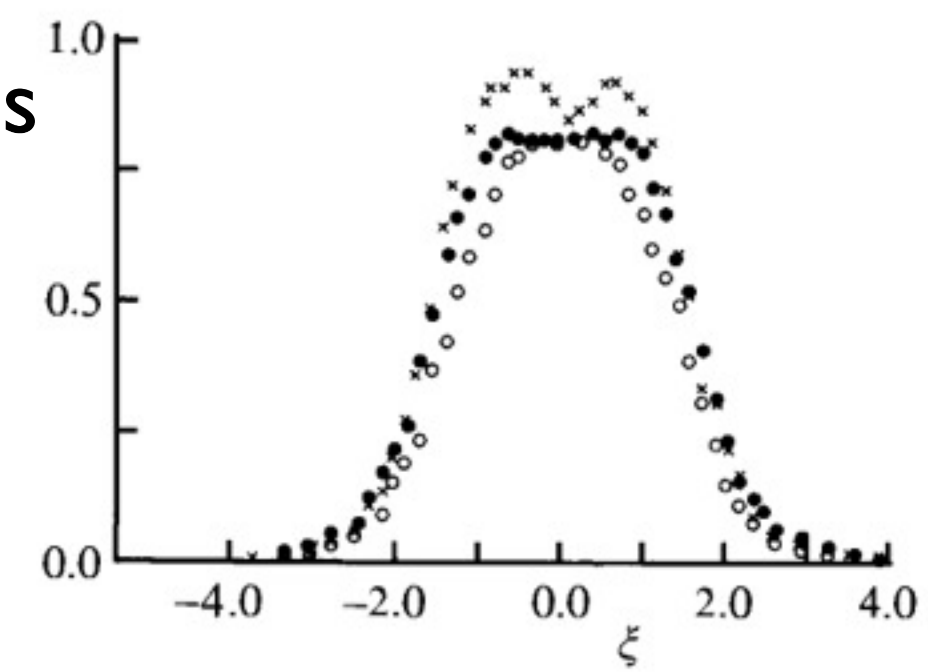
$$U_s \sim x^{-2/3} \quad r_{1/2} \sim x^{1/3}$$



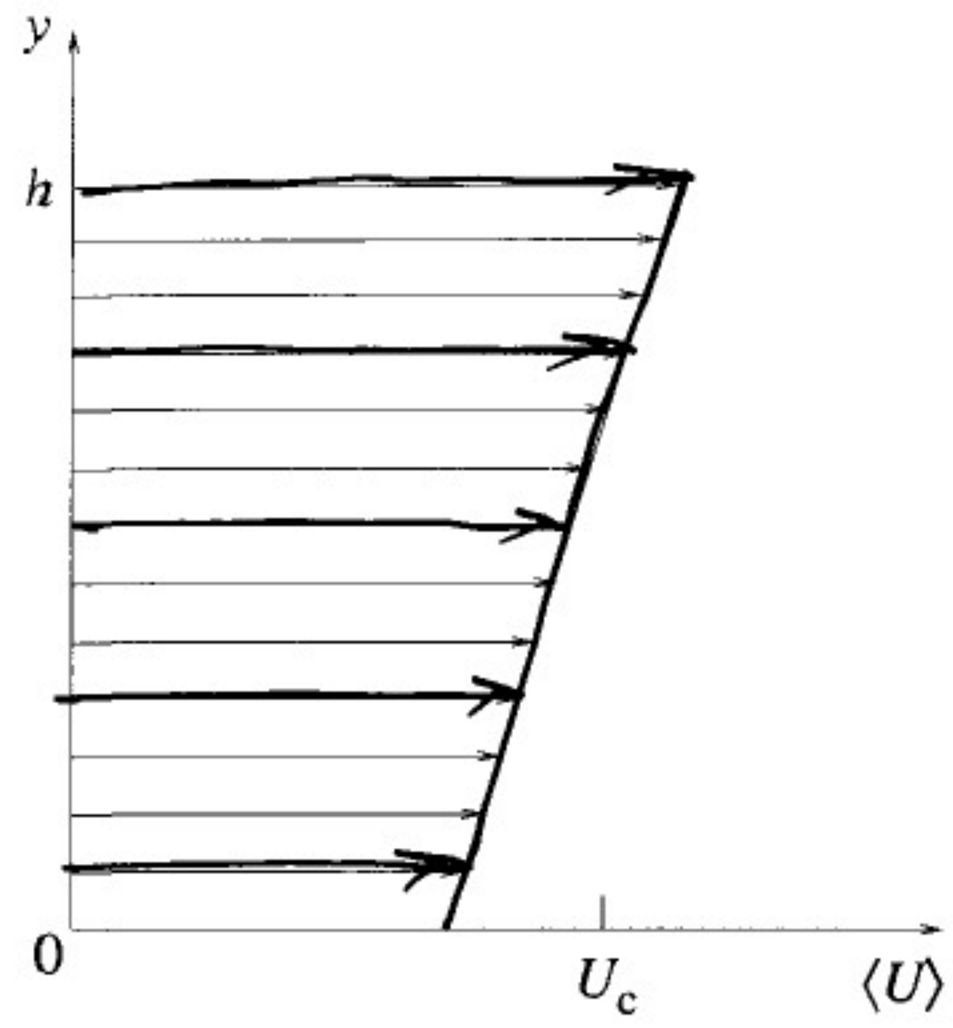
$$S \sim 5 \div 10$$

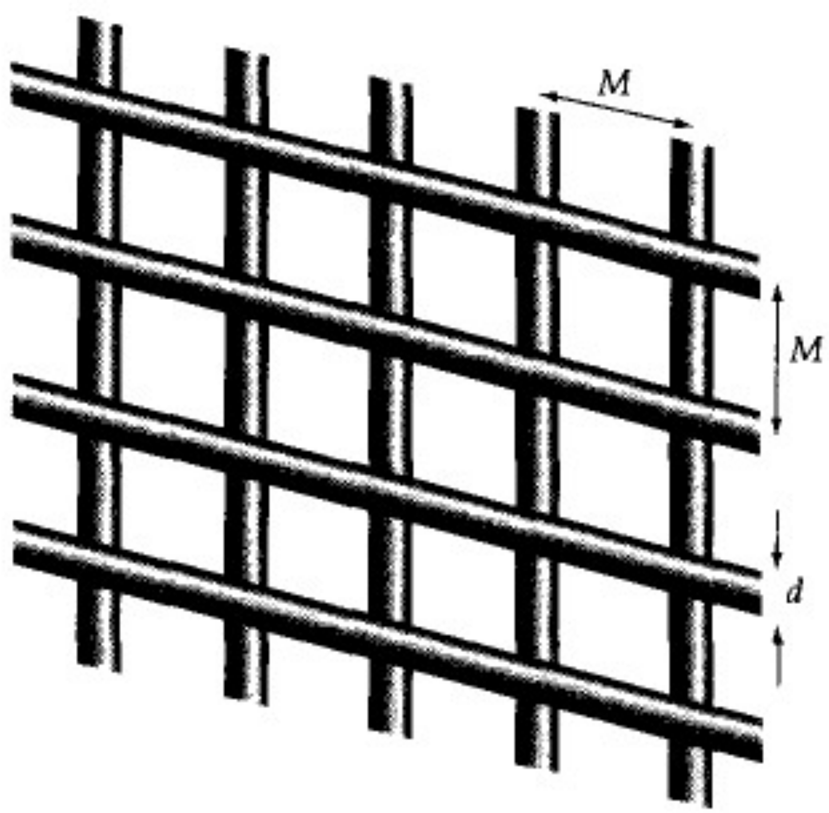


rms



Autre écoulements





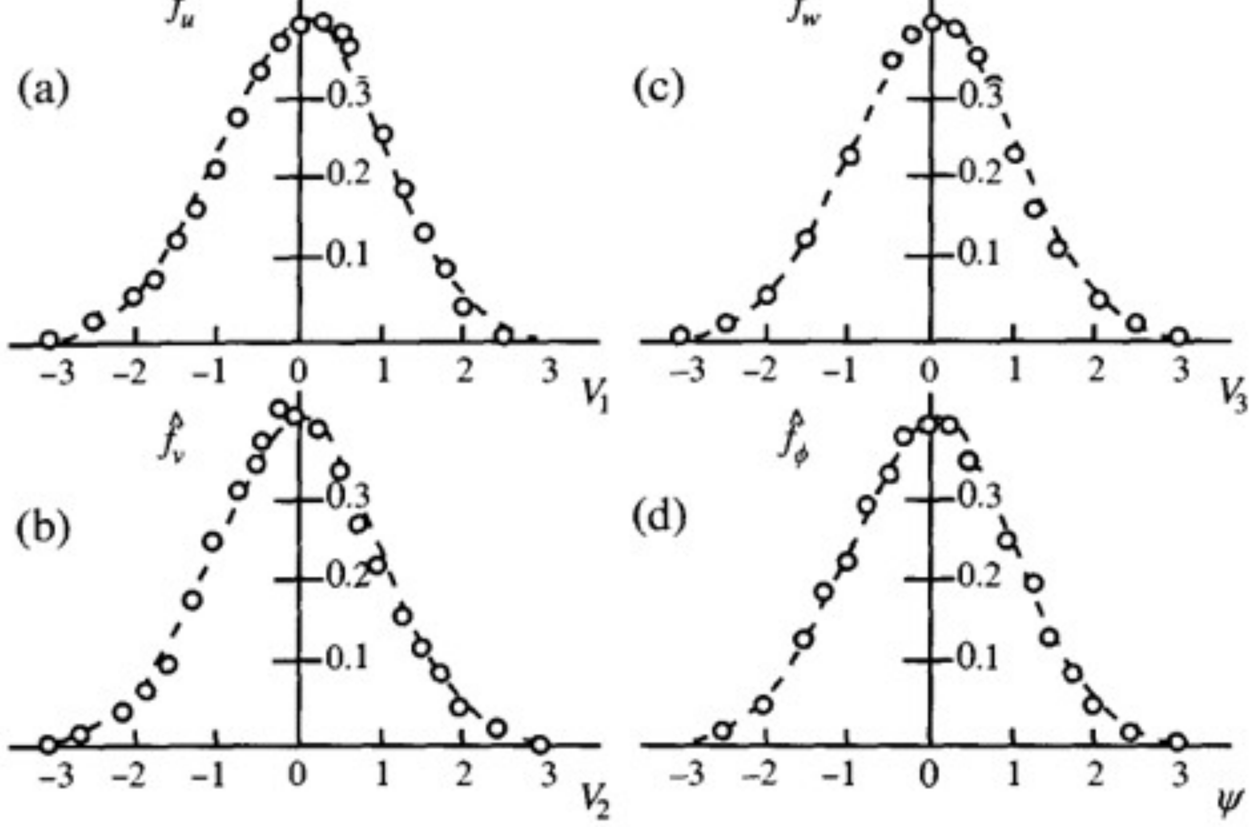
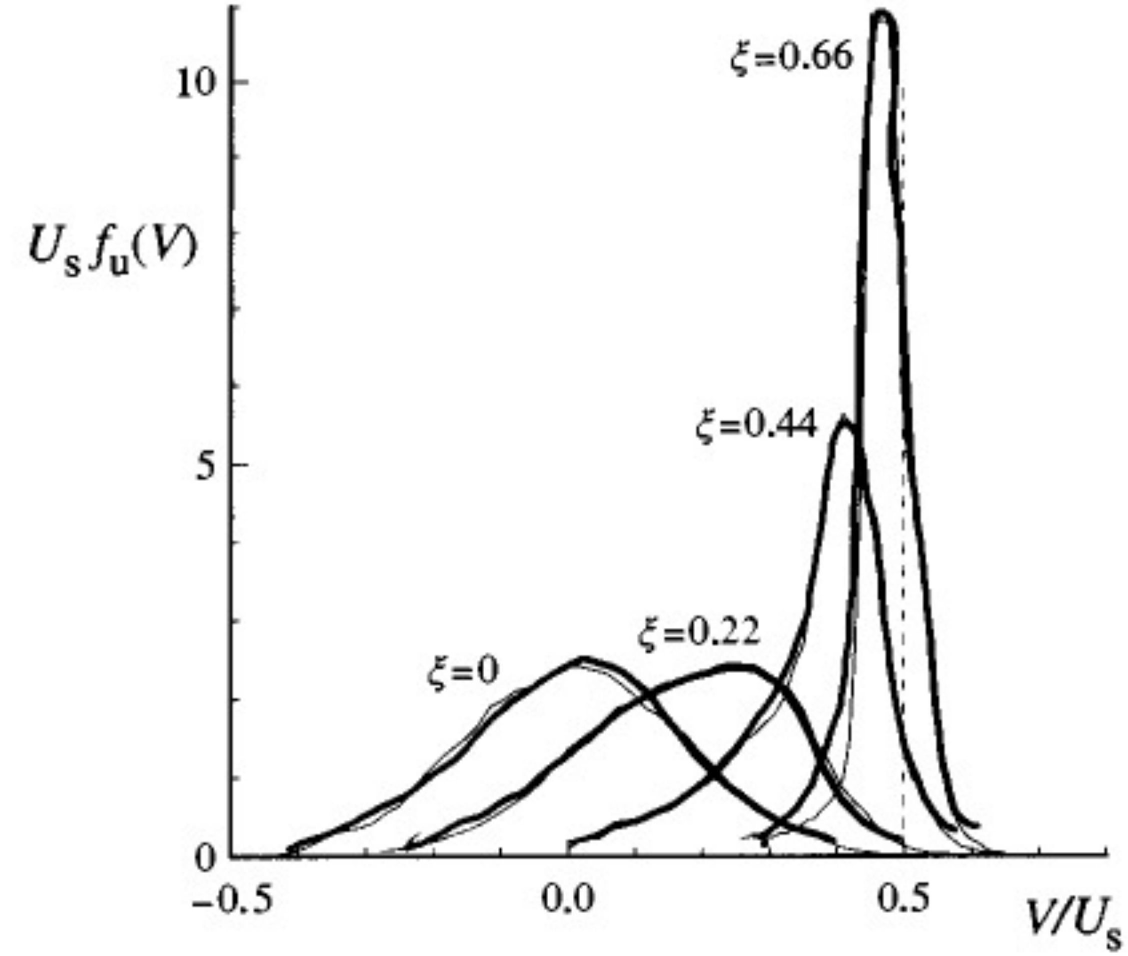
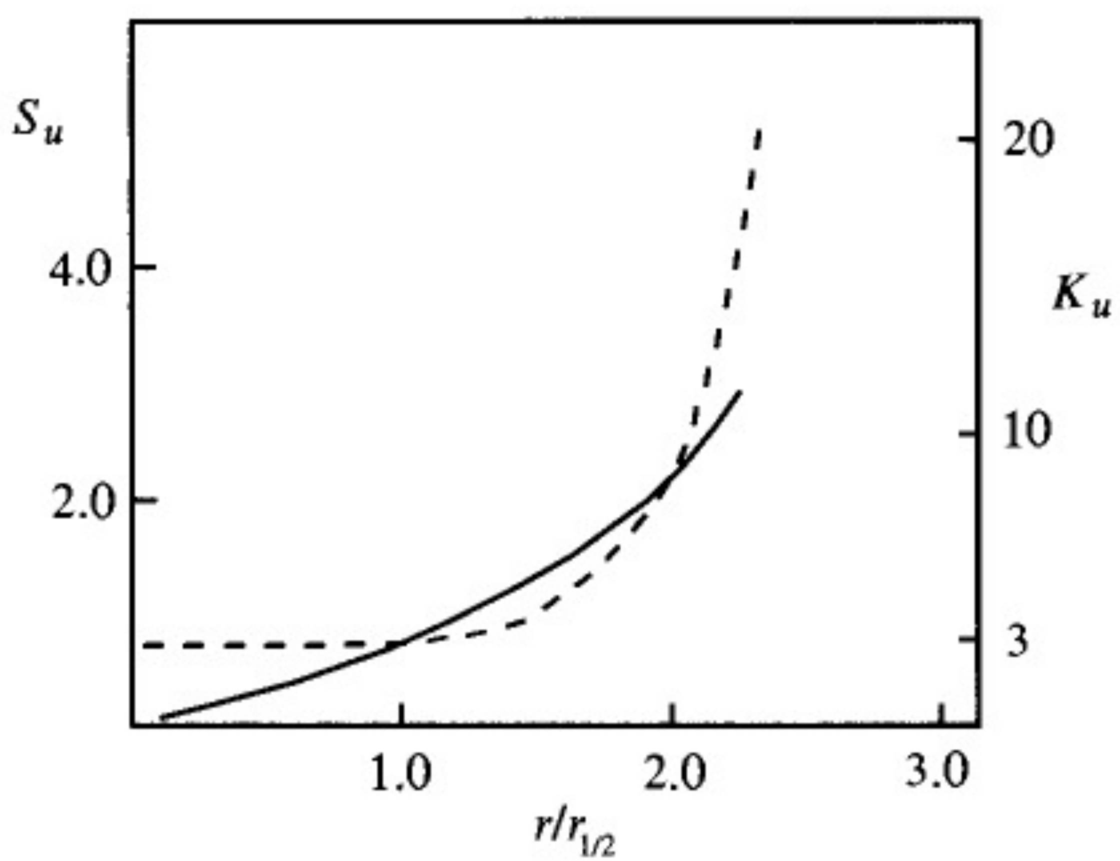


Fig. 5.45. Standardized PDFs of (a) u, (b) v, (c) w, and (d) ϕ in homogeneous shear flow. Dashed lines are standardized Gaussians. (From Tavoularis and Corrsin (1981).)





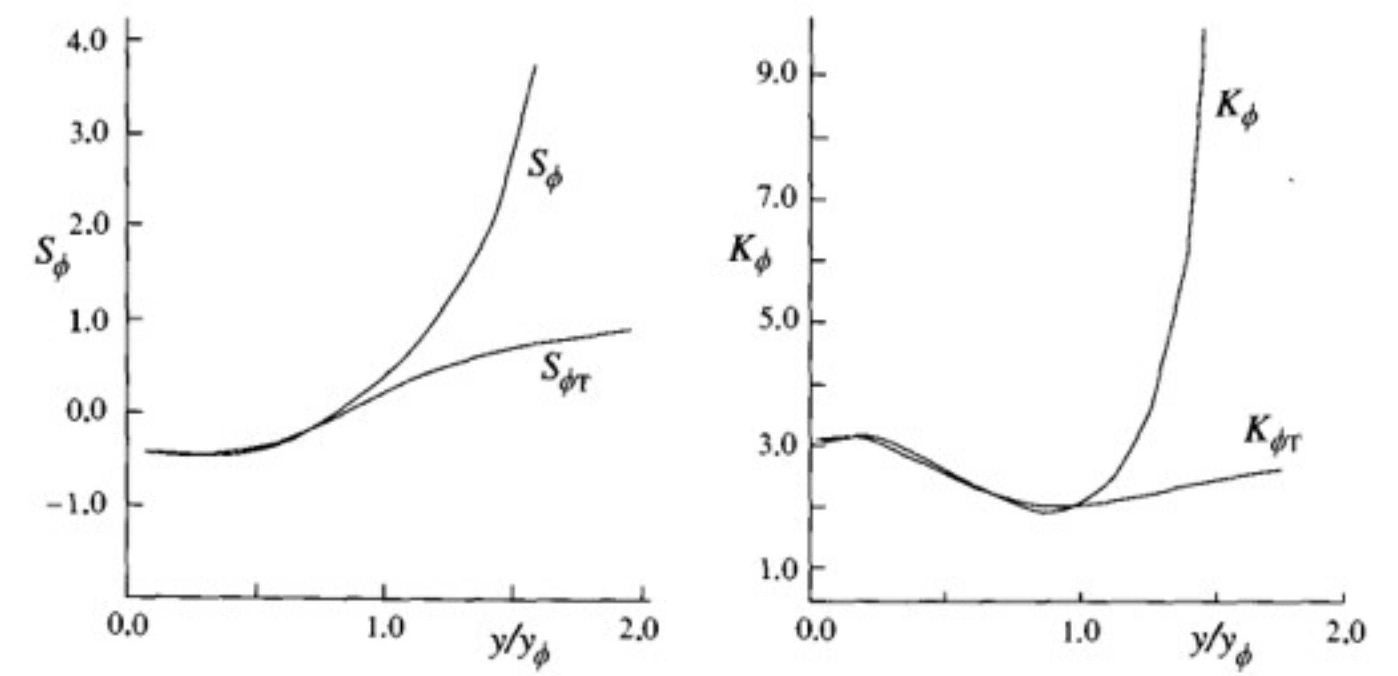


Fig. 5.48. Profiles of unconditional (S_ϕ and K_ϕ) and conditional turbulent ($S_{\phi T}$ and $K_{\phi T}$) skewness and kurtosis of a conserved passive scalar in the self-similar plane wake. From the experimental data of LaRue and Libby (1974).

177. Coherent structure at higher Reynolds number. This flow is as above but at twice the pressure. Doubling the Reynolds number has produced more small-scale structure without significantly altering the large-scale structure. *M. R. Rebollo, Ph.D. thesis, Calif. Inst. of Tech., 1976; Brown & Roshko 1974*