(3)

$$\underline{\underline{A}} = \begin{pmatrix} -D & -\Omega & 0 \\ \Omega & D & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\underline{A}} = \nabla \underline{u}$$

$$A_{ij} = \begin{pmatrix} \partial_x u_x & \partial_y u_x & 0 \\ \partial_x u_y & \partial_y u_y & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a)

(Strain)

$D = -\partial_x u_x = +\partial_y u_y \Rightarrow D =$ contraction suivant $\underline{e}_x$

dilatation " $\underline{e}_y$

(stress symmetric, rotation antisymmetric)

$\Omega \Rightarrow$ rotation ! dans un référentiel inertiel !

$\underline{u}^{(0)} = \Omega \, \underline{n} \times \underline{x} \qquad \nabla \times \underline{u}^0 = 2 \Omega \, \underline{n}$

$$\Rightarrow A = \left(\frac{\partial u_i^{(0)}}{\partial x_j} \right) = \begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ antisymmetric

(b) Nota sui sistemi riferimento:

$x_i = x_i^L(\underline{X}, t, t_0) \Rightarrow$

$dx_i = u_i dt + F_{ij} X_j \qquad \dot{x} = u_i = \frac{\partial x_i^L}{\partial t}$ e $F_{ij} = \frac{\partial x_i^L}{\partial X_j}$

(Cauchy matrix)

$\Rightarrow d^3x = J \, d^3X \qquad J(X, t, t_0) = \det \underline{\underline{F}}$

nel caso di pura rotazione $D = 0$

$\boxed{F^{-1} = F^{+}}$ (ortogonale) $\qquad x_i^{(0)} = F_{ij}(t - t_0) X_j$

• $F_{ij}(X, t_0, t_0) = \delta_{ij}$ (déf)

$$\dot F_{ij} = A_{ik} F_{kj} \qquad \begin{pmatrix} -D & -\Omega & 0 \\ \Omega & D & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -D & -\Omega & 0 \\ \Omega & D & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} D^2-\Omega^2 & 0 & 0 \\ 0 & D^2-\Omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\ddot F_{ij} = A_{ik} \dot F_{kj} \Rightarrow$$

$$\Uparrow$$

$$\Rightarrow \ddot F_{ij} = A_{ik} A_{ke} F_{ej} \qquad B_{ij} = A_{ik} A_{ke} = \begin{pmatrix} D^2-\Omega^2 & \cancel{D\Omega-\Omega D} & 0 \\ 0 & -\Omega^2+D^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$j = 1, 2$

$$\ddot F_{ij} = \underbrace{(D^2-\Omega^2)}_{\sigma^2} F_{ij} \qquad \boxed{j = 1, 2} \qquad \sigma = \sqrt{D^2-\Omega^2}$$

$$\Downarrow$$

$$\ddot F_{11} = \sigma^2 F_{11} \quad \ddot F_{12} = \sigma^2 F_{12} \quad \ddot F_{22} = \sigma^2 F_{22} \quad \ddot F_{21} = \sigma^2 F_{21}$$

$$\Rightarrow \boxed{F_{ij} = c_1 e^{-\sigma t} + c_2 e^{\sigma t}} \qquad \text{per semplicità } \boxed{t - T_0 = t} \ (t_0 = 0)$$

$$\begin{cases} F_{ij}(0) = \delta_{ij} \Rightarrow c_1 + c_2 = \delta_{ij} \\ \dot F_{ij}(0) = A_{ik} F_{kj}(0) = A_{ik}\delta_{kj} = -c_1\sigma + c_2\sigma \end{cases}$$

$$\Rightarrow c_2 - c_1 = \frac{A_{ik}\delta_{kj}}{\sigma} \qquad c_1 = \delta_{ij} - c_2$$

$$c_2 - \delta_{ij} + c_2 = \frac{A_{ik}\delta_{kj}}{\sigma} \Rightarrow \boxed{c_2 = \frac{1}{2}\frac{A_{ik}\delta_{kj}}{\sigma} + \frac{\delta_{ij}}{2}}$$

$$\boxed{c_1 = \frac{\delta_{ij}}{2} - \frac{A_{ik}\delta_{kj}}{2\sigma}}$$

$$\Rightarrow F_{ij} = \frac{\delta_{ij}}{2} e^{-\sigma t} - \frac{A_{ij}}{2\sigma} e^{-\sigma t} + \frac{\delta_{ij}}{2} e^{\sigma t} + \frac{A_{ij}}{2\sigma} e^{\sigma t}$$

$$\boxed{F_{ij} = \delta_{ij} \underbrace{\frac{e^{-\sigma t} + e^{\sigma t}}{2}}_{\cosh \sigma t} + \frac{A_{ij}}{\sigma} \sinh \sigma t}$$

pour $\boxed{ij=3}$

$\dot{F}_{33} = 0 \Rightarrow \boxed{F_{33} = \text{cost.}}$

$\ddot{F}_{\alpha 3} = \sigma^2 F_{\alpha 3}$

$\ddot{F}_{3\alpha} = \sigma^2 F_{3\alpha} \Rightarrow F_{3\alpha} = c_1 e^{-\sigma t} + c_2 e^{\sigma t}$

$F_{23}(0) = F_{13}(0) = F_{31}(0) = F_{32}(0) = 0 \Rightarrow \boxed{F_{3\alpha} = F_{\alpha 3} = 0}$

$F_{33}(0) = 1 \Rightarrow \boxed{F_{33} = 1}$

$\Longrightarrow$ dietro

G_α F ortogonal $\Rightarrow F^{-1} = F^{t} = G \Rightarrow$

$$G_{\alpha\beta} = \begin{cases} \delta_{\alpha\beta} \operatorname{ch}\sigma(t) - \dfrac{A_{\alpha\beta} \operatorname{sh}(\sigma t)}{\sigma} & \alpha,\beta = 1,2 \\ G_{33} = 1 \qquad G_{\alpha 3} = G_{3\alpha} = 0 \end{cases}$$

$\gamma^2 > \Omega^2 \Rightarrow \sigma \in \mathbb{R}^+ \Rightarrow$ évolution exponentielle

$\gamma^2 < \Omega^2 \Rightarrow \sigma \in \mathbb{C}\ (i\mathbb{R}) \Rightarrow$ év. sinusoidale

$\gamma^2 = \Omega^2 \Rightarrow \sigma = 0$ " algébrique.

(c) $R_{ij} = G_{ik} R_{kl}(X, t_0) G_{lj}$

initialement isotrope $\Rightarrow \boxed{R_{ij}(0) = \frac{2}{3} K \delta_{ij}}$

$\Rightarrow R_{ij} = \frac{2}{3} K G_{ik} \delta_{kl} G_{lj} = \frac{2}{3} K(0) G_{ik} G_{kj}$

$$GRG = \begin{pmatrix} g_{11}^2 + g_{12}g_{21} & g_{11}g_{12} + g_{12}g_{22} & 0 \\ g_{21}g_{11} + g_{22}g_{12} & g_{21}g_{12} + g_{22}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow R_{11} = \frac{2}{3}K\left[\left(ch\,\sigma(t-t_0) + \frac{D\,sh\,\sigma(t-t_0)}{\sigma}\right)^2 - \left(\frac{\Omega\,sh\,\sigma(t-t_0)}{\sigma}\right)^2\right]$$

$$R_{12} = \frac{2}{3}K(0)\left\{\left(\frac{ch\,\sigma t + D\,sh\,\sigma t}{\sigma}\right)\frac{\Omega}{\sigma}sh\,\sigma t + \frac{\Omega}{\sigma}sh\,\sigma t \times \right.$$
$$\left. \times \left(ch\,\sigma t - \frac{D}{\sigma}sh\,\sigma t\right)\right.$$

$$= \frac{2}{3}K(0)\left\{\frac{2\Omega}{\sigma}ch\,\sigma(t-t_0)\,sh\,\sigma(t-t_0)\right\}$$

$R_{21} = R_{12}$

$$R_{22}(t) = \frac{2}{3}K(0)\left\{ch^2\sigma t + sh^2\sigma - \frac{2D}{\sigma}sh\,\sigma(t)\,ch\,\sigma(t)\right\}$$

$$R_{33}(t) = R_{33}(0) = \frac{2}{3}K(0)$$

$$K = \frac{1}{2}R_{ii} = \frac{\cancel{2}K(0)}{\cancel{2}\,3}\left[1 + 2\left(ch^2\sigma(t) + sh^2\sigma t\right)\right]$$

$$\boxed{b_{ij} = \frac{R_{ij}}{2K} - \frac{\delta_{ij}}{3}}$$ tenseur d'anisotropie

$$b_{11} = -\frac{1}{3} + 2\left\{\frac{ch^2\sigma t + sh^2\sigma t + \frac{2D}{\sigma}sh\,\sigma t\,ch\,\sigma t}{1 + 2(ch^2\sigma t + sh^2\sigma t)}\right\}$$

$t \to \infty$ $ch\,t \sim \frac{e^t}{2} \sim sh\,t$ $\Rightarrow$ $t \to \infty$ $b_{11} \to \frac{2\sigma + D}{1 + 2\sigma}$