Université Pierre et marie Curie
Master 2. Turbulence : dynamics and modelling (NSF02)

Exam - February 25st, 2013

All documents are forbidden
Duration : 2h

Exercice : Self-similar Free-shear flow

We consider the plane jet generated in a medium at rest by a plane nozzle of width H, in which the velocity $u = U_0$ is initially uniform. Pressure can be always assumed to be uniform. The Reynolds number $Re_0 = \frac{U_0 H}{\nu} \gg 1$, is high enough for the flow in the nozzle to be fully turbulent. **This allows us to neglect the effect of molecular viscosity**

The jet is directed along the longitudinal coordinate x, develops along the cross-stream y, and can be considered statistically independent of the span-wise direction z. Therefore, **the problem will be considered as perfectly 2-D**. The characteristic jet width is called δ, the characteristic longitudinal length is L.

Since we are interested in mean quantities a Reynolds decomposition will be performed, $u_i = \bar{u}_i + u'_i$ (the usual hydrodynamics symbols can also be used $\bar{\mathbf{u}} = (\bar{u}, \bar{v})$). The characteristic velocities are U, V for the x and y component respectively. U is the mean velocity of the jet in its centreline.

The flow is considered statistically stationnary.

We recall here that Navier-Stokes equations are

$$\begin{aligned} \frac{\partial u_i}{\partial t} &+ u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \Delta u_i \\ \frac{\partial u_j}{\partial x_j} &= 0 \end{aligned}$$

<u>Remark</u> : Much information is given inside questions, which can be used for the following points, even without giving the answer. The two parts are largely independent, except for the definitions.

Part 1 : setting of the problem

1. **Equations**
   Using the Reynolds decomposition, write the ensemble averaged continuity equation.
2. Write the momentum equations for the mean velocity field (RANS). The Reynolds stress tensor will be denoted $R_{ij} = \overline{u'_i u'_j}$

3. Given that the medium is initially at rest, give the boundary conditions at infinity ($y = \pm\infty$), for $\bar{u}, \bar{v}, R_{12}$.

   Réponse:

   $u = v = R_{xy} = 0\ ,\ y = \pm\infty$

4. **Dimensional analysis**

   Let's assume that the growth of the jet width is sufficiently slow to neglect longitudinal derivatives with respect to transversal ones :

$$\frac{\frac{\partial}{\partial x}}{\frac{\partial}{\partial y}} = O(\delta/x) \ll 1\ , \qquad (1)$$

   where the ratio $\delta/x = \alpha$ represents the spreading rate.

   Using the averaged continuity eq. (point 1), make a dimensional analysis and show that

$$V \sim \alpha U$$

   Réponse:

   $U/L \sim V/\delta$

5. In order to solve our problem, we have to model the Reynolds stress. We use an eddy viscosity approximation. The eddy viscosity will be of the form

$$\nu_t = CU^a\delta^b, \qquad (2)$$

   where C is a constant parameter without dimensions of the model. Give by dimensional analysis the exponent a, b.

   Estimate the parameter C of the model knowing that the turbulent Reynolds number defined as $Re_t = \frac{U\delta}{\nu_t}$ can be roughly said to be $Re_t \approx 20$. Justify the use of U, δ as length scales in (2).

   Réponse:

$$a = b = 1$$

$$Re_t = \frac{U\delta}{\nu_t} = 1/C\ \ \rightarrow C \approx 0.05$$

6. **Momentun Flux** Using the dimensional analysis of (1), and the fact that the flow is at rest at $|y| \gg \delta$, reduce the RANS longitudinal and transversal equations obtained in question (2) to a single equation for longitudinal momentum. It is recalled that both viscous and pressure terms can be neglected.

   Réponse:

By dimensional analysis, $x - eq \sim U^2/L \sim O(1)$; $y - eq \sim V^2/\delta \sim O(\alpha)$

$$u\partial_x u + v\partial_y v = -\partial_y R_{12}$$

7. With the help of the continuity equation, show that the longitudinal averaged momentum eq. can be written as

$$\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} = -\frac{\partial R_{12}}{\partial y} \tag{3}$$

Réponse:

Immédiat. Ici Je leur donne l'équation correcte pour pouvoir éventuellement même partir d'ici.

8. We define the total momentum flux per unit span as

$$J = \int_{-\infty}^{\infty} \bar{u}^2 dy \ , \tag{4}$$

Integrate the equation (3) over y and remembering that the flow is at rest at the infinity, using boundary conditions defined in question 3, show that the total momentum flux is independent of x :

$$\frac{dJ}{dx} = 0$$

Give the dimensions of J.

Réponse:

Banal. $[J] = [L]^3/[T]^2$

9. Far from the nozzle exit, $x \gg H$, and as long the Reynolds number of the jet stays large enough fro the effect of viscosity on the mean flow to be negligible, the only parameters which can control the behaviour of the jet are : the conserved momentum J, and x itself. By dimensional analysis, find the two dimensionless groups that can be formed with the only relevant quantities : J, x, δ, U.

Réponse:

$$\alpha = \delta/x \ , \ B = U^2 x/J$$

10. Using this information, obtain the evolution laws for $U(x)$, $\delta(x)$ and the local Reynolds number of the flow $Re = \frac{U\delta}{\nu}$. (Hint : The dimensionless groups can depend on nothing and thus they are constant)

Réponse:

The dimensionless groups have to be constant, since they cannot depend on anything. Therefore, $\delta = \alpha x$, and $U = (BJ/x)^{1/2}$.
The local Re is then $Re = \alpha(BJx)^{1/2}/\nu$

11. If the jet is initially turbulent, would it stay turbulent forever, or relaminarize downstream ?
Réponse:

Since the Reynolds number increases downstream, it will stay turbulent.

Part 2 : Self-similar solution

We seek for a similarity solution for the mean velocity profile.

Let us define precisely the thickness of the jet δ as the point where the mean velocity is $\bar{u}(\delta) = U/2$. The evolutionary laws previously obtained suggest that the flow should be expressible in terms of a single variable $\xi = y/\delta$.

1. Since the flow is incompressible, it is useful to use the averaged stream function ψ, instead of the velocity, to enforce continuity equation. Let us recall its definition

$$\bar{u} = \frac{\partial \psi}{\partial y}$$
$$\bar{v} = -\frac{\partial \psi}{\partial x}$$

The similarity solution has the form

$$\psi = Af(\xi) \ ,$$

where A is given by the relevant length scales. By dimensional analysis give A in terms of J, δ. Justify the choice of these two scales.
Réponse:

$$A = (J\delta)^{1/2}$$

2. Remembering that $\delta = \alpha x$, eq. (1), calculate the velocities in terms of f. These are the similarity solutions in symbolic form.
Réponse:

$$\bar{u} = (\frac{J}{\delta})^{1/2} f', \quad \bar{v} = \alpha(\frac{J}{\delta})^{1/2}(\xi f' - f/2)$$

3. Using the definition (4), show that

$$\int_{-\infty}^{\infty} (f')^2 d\xi = 1 \qquad (5)$$

Give the boundary conditions for f'.
Moreover, since by symmetry $\xi = 0$ is a streamline, we can define $f(0) = 0$.
Réponse:

$f'(\pm\infty) = 0$

4. We now look for us write the differential equation for f.
Using eddy viscosity approximation $R_{xy} = \nu_t \frac{\partial \bar{u}}{\partial y}$, with ν_t given by (2), write the momentum differential equation in terms of f.
Réponse:

$$2(ff')' + Kf''' = 0, \quad K = 4Cf'(0)/\alpha$$

5. Integrate twice the previous equation, using the boundary conditions found in question 4, and show that we obtain the following equation

$$Kf' = f_\infty^2 - f^2,$$

where K is a constant whose expression can be deduced from the previous point.
Réponse:

$$2(ff') + Kf'' = A1 \ ;$$
$$f^2 + Kf' = A\xi + B$$
$$\xi = \infty \ \Rightarrow \ A = 0, \ B = f_\infty^2$$

6. The solution of the equation is given by

$$f(\xi) = f_\infty tanh(f_\infty \xi/K) \tag{6}$$

From this equation, compute $f'(0)$, and using the definition of K give the value for f_∞/K.
Réponse:

$$f_\infty/K = (\alpha/4C)^{1/2}$$

7. We can use the experimental value for the growth rate, $d\delta/dx \approx 0.1$, to estimate the empirical constant C used in (2). It turns out to be $C \approx 0.32$. Does it agree roughly with the answer to point 5 of first part ? What is the implied value of the turbulent Reynolds number $Re_t = U\delta/\nu_t$?
Réponse:

$$Re_\epsilon \approx 30$$

8. The mean mass flux per unit spam is defined as

$$M = \int_{-\infty}^{\infty} \bar{u} dy \ .$$

By dimensional analysis compute the behaviour with x of the mean mass flux. What can you deduce about entrainment of the fluid by (from) the jet ? Is there ingestion (or expulsion) ?

Réponse:

$$M \sim U\delta \sim x^{1/2}$$

As it increases downstream, the jet must ingest fluid from infinity.

9. Compute the evolution of the corresponding cross-stream velocity at infinity, v_I.

Réponse:

$$v_I = -\frac{\alpha f_\infty}{2}(J/\delta)^{1/2} = -\frac{f_\infty}{2}(J\alpha/x)^{1/2}$$