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# Surface tension force on a partly submerged body

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The vertical component of the surface tension force on a body partly submerged in a liquid is shown to equal the weight of liquid displaced by the meniscus. It is upward if the meniscus is depressed and downward if the meniscus is elevated. Previously this was known for vertical axially symmetric bodies and for two-dimensional vertical plates. The vertical component of the pressure force on the body is shown to equal the weight of liquid which would fill the volume bounded by the wetted surface of the body, a vertical cylinder through the waterline, and the original horizontal free surface. © *1998 American Institute of Physics.* [S1070-6631(98)02111-4]

A surface tension force $\mathbf{F}_T$ and a pressure force $\mathbf{F}_P$ act on a body partly submerged in a liquid at rest. The vertical component of $\mathbf{F}_T$ will be shown to equal the weight of liquid displaced by the meniscus, directed upward if the meniscus is depressed and downward if it is elevated. This extends to bodies of any shape the two-dimensional result for the force on a vertical flat plate (Rusanov and Prokhorov[1]) and the axially symmetric results of Laplace[2] for the volume of fluid raised in a vertical capillary tube and of Schulze[3] for the force on any vertical axially symmetric body. The vertical component of $\mathbf{F}_P$ will be shown to equal the weight of liquid which would fill a volume bounded by the wetted surface of the body, a vertical cylinder through the waterline, and the original horizontal free surface (see Fig. 1). The horizontal components of $\mathbf{F}_T$ and $\mathbf{F}_P$ will be considered also.

The interface $z=\eta(x,y)$ between a liquid and gas at rest satisfies the Young–Laplace equation

$$-T(\partial_x n_1+\partial_y n_2)=\rho g\,\eta(x,y). \tag{1}$$

Here $T$ is the coefficient of surface tension, $\mathbf{n}(x,y)\equiv(n_1,n_2,n_3)=[-\eta_x,-\eta_y,1](1+\eta_x^2+\eta_y^2)^{-1/2}$ is the unit normal to the interface pointing out of the liquid, $\rho$ is the density of the liquid, and $g$ is the acceleration of gravity which acts in the $-z$ direction. We assume that the interface extends from a curve $C$ on the surface $S$ of a partly submerged body to infinity, where $\eta$=0. $C$ is given by $\mathbf{r}(s)$ where $s$ is arclength on $C$.

The projection of $C$ on the $(x,y)$ plane is a curve $\Gamma$ with arclength $t$ and inward normal $\boldsymbol{\nu}(t)$. We integrate (1) over the $(x,y)$ plane outside $\Gamma$, apply the divergence theorem to the left side, and assume that $n_1$ and $n_2$ vanish sufficiently rapidly at infinity. Then we get

$$T\int_\Gamma \nu\cdot n\,dt=W_M. \tag{2}$$

$W_M$ is $\rho g$ times the volume between the interface $z=\eta(x,y)$ and the surface $z=0$, counted negative if $\eta$>0. Thus $W_M$ is the weight of the liquid displaced by the meniscus.

To interpret the left side of (2) we note that

$$\mathbf{F}_T=T\int_C \dot{\mathbf{r}}(s)\times\mathbf{n}[x(s),y(s)]ds. \tag{3}$$

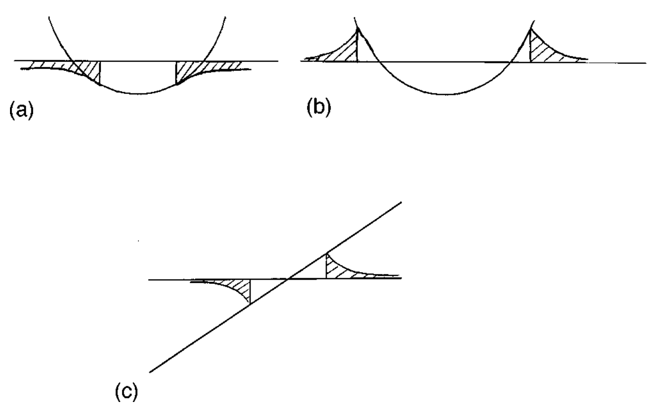

FIG. 1. The vertical component of the force $\mathbf{F}_T$ due to surface tension is positive in (a) and negative in (b). Its magnitude is the weight of liquid which would fill the cross-hatched regions. The vertical component of the pressure force $\mathbf{F}_P$ is positive in both cases. In (a) it is equal to the weight of the liquid which would fill the unhatched region between the body surface and the $x$ axis. In (b) the area of the two small triangular regions above the axis must be subtracted from the area below the axis. Thus in both cases, the force is less than the weight of the liquid displaced by the body. In (c) both vertical forces are differences between volumes below and above the axis.

We multiply (3) by $\hat{z}\cdot$ and observe that $\hat{\mathbf{z}}\cdot\dot{\mathbf{r}}\times\mathbf{n}=\mathbf{n}\cdot\hat{\mathbf{z}}\times\dot{\mathbf{r}}=\mathbf{n}\cdot\boldsymbol{\nu}\sin\varphi$ where $\varphi$ is the angle between $\hat{\mathbf{z}}$ and $\dot{\mathbf{r}}$. Since $\sin\varphi\, ds=dt$, we get from (3)

$$\hat{\mathbf{z}}\cdot\mathbf{F}_T=T\int_C \mathbf{n}\cdot\boldsymbol{\nu}\sin\varphi\; ds=T\int_\Gamma \mathbf{n}\cdot\boldsymbol{\nu}dt. \tag{4}$$

Now (2) and (4) yield the result

$$\hat{\mathbf{z}}\cdot\mathbf{F}_T=W_M. \tag{5}$$

The force $\mathbf{F}_P$ due to hydrostatic pressure is given by

$$\mathbf{F}_P=\int \rho g z\mathbf{N}(\mathbf{x})dA. \tag{6}$$

Here $dA$ is the area element of the wetted surface $S$ and $\mathbf{N}(\mathbf{x})$ is the unit normal to $S$ directed out of the body. To compute the vertical component of $\mathbf{F}_P$ we note that $\hat{\mathbf{z}}\cdot\mathbf{N}(\mathbf{x})dA=\pm dx\; dy$. This is the signed area element on the $x,y$ plane into which $dA$ projects, where the sign is that of $\hat{\mathbf{z}}\cdot\mathbf{N}$. When the sign is negative at all points of $S$, $S$ projects onto the interior $A_\Gamma$ of $\Gamma$ and we obtain

$$\hat{\mathbf{z}}\cdot\mathbf{F}_P=-\rho g\int_{A_\Gamma} z(x,y)dx\; dy. \tag{7}$$

The integral in (7) is just minus the volume $V$ of the domain bounded below by $S$, above by the plane $z=0$, and laterally by the vertical cylinder through $C$, which intersects the horizontal plane in the curve $\Gamma$. Thus the right side of (7) is just the weight $W_V$ of the fluid contained in this domain, so we have

$$\hat{\mathbf{z}}\cdot\mathbf{F}_P=W_V. \tag{8}$$

This result remains true when $\hat{\mathbf{z}}\cdot\mathbf{N}$ changes sign, as one can show by considering separately the regions where it is positive and where it is negative.

To obtain the total vertical force on the body, we add (8) and (5):

$$\hat{\mathbf{z}}\cdot(\mathbf{F}_T+\mathbf{F}_P)=W_M+W_V. \tag{9}$$

The right side of (9) is the weight of liquid that would fill the region bounded below by the interface and the wetted surface of the body, and above by the undisturbed interface $z=0$, when $\eta(x,y)$ is single valued. If this region were filled with liquid, it would obviously be held at rest by the liquid below it. This provides a direct proof of (9). Subtracting (8) from (9) gives a direct proof of (5). Weights or volumes above $z=0$ must be counted as negative. Similarly, the total horizontal force on the body is equal to the horizontal force on a vertical cylinder of large radius surrounding the body. That force is zero, so we have

$$\hat{\mathbf{x}}\cdot(\mathbf{F}_T+\mathbf{F}_P)=0,\quad \hat{\mathbf{y}}\cdot(\mathbf{F}_T+\mathbf{F}_P)=0. \tag{10}$$

$\mathbf{F}_P^h$, the horizontal component of $\mathbf{F}_P$, is the same as the horizontal component of the force exerted by the external fluid on the vertical cylinder which extends downward from $C$. We assume that the wetted surface of the body lies inside this cylinder. The forces are equal because there is no net horizontal force on the fluid bounded by $S$ and this cylinder. Thus

$$\mathbf{F}_P^h=\rho g\int_\Gamma\int_{z_0}^{z(t)}\boldsymbol{\nu}(t)z\; dz\; dt=\frac{\rho g}{2}\int_\Gamma z^2(t)\boldsymbol{\nu}(t)dt. \tag{11}$$

Here $z(t)$ is the $z$ coordinate of $C$ at arclength position $t$ on $\Gamma$. The lower limit $z_0$, which is chosen below all points of $S$, drops out of the calculation because the integral of $\boldsymbol{\nu}(t)$ around the closed curve $\Gamma$ vanishes. Thus $\mathbf{F}_P^h$ is determined by the depression of the waterline due to surface tension.

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