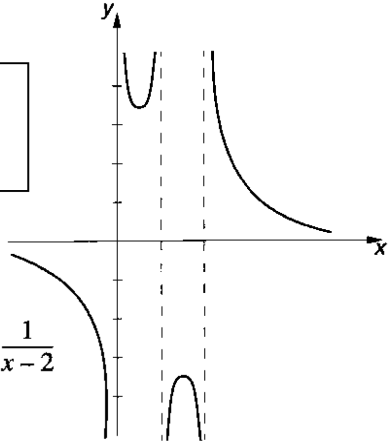
Rational functions

$$f(x)=\frac{P(x)}{Q(x)}=K(x)+\frac{R(x)}{Q(x)},$$



Asymptotic behaviour

$f(x)-K(x)\to 0$ as $x\to\pm\infty$.

$$y=\frac{x^2-2x+2}{x(x-1)(x-2)}=\frac{1}{x}-\frac{1}{x-1}+\frac{1}{x-2}$$

Partial fractions

Assume that

(*i*) $R(x)$, $Q(x)$ are real polynomials, deg $R<$deg Q,

(*ii*) $Q(x)$ is decomposed in real factors of degree ≤ 2, i.e.

$Q(x)=C(x-r)^m(x-s)^n\ \dots\ (x^2+2ax+b)^p(x^2+2cx+d)^q\ \dots,\ (a^2<b,\ c^2<d)$

Then

$$\frac{R(x)}{Q(x)}=\frac{R_1}{x-r}+\frac{R_2}{(x-r)^2}+\dots+\frac{R_m}{(x-r)^m}+$$
$$+\frac{S_1}{x-s}+\frac{S_2}{(x-s)^2}+\dots+\frac{S_n}{(x-s)^n}+\dots+$$
$$+\frac{A_1x+B_1}{(x^2+2ax+b)}+\dots+\frac{A_px+B_p}{(x^2+2ax+b)^p}+$$
$$+\frac{C_1x+D_1}{x^2+2cx+d}+\dots+\frac{C_qx+D_q}{(x^2+2cx+d)^q}+\dots,$$

The following example illustrates how to find the constants.

$$\frac{1}{(x^2+1)(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+1}=$$
$$=\frac{A(x+1)(x^2+1)+B(x^2+1)+(Cx+D)(x+1)^2}{(x+1)^2(x^2+1)}=$$
$$=\frac{(A+C)x^3+(A+B+2C+D)x^2+(A+C+2D)x+(A+B+D)}{(x+1)^2(x^2+1)}$$

Identification of coefficients:

$A+C=0,\ A+B+2C+D=0,\ A+C+2D=0,\ A+B+D=1$

$\Rightarrow A=B=-C=1/2,\ D=0.$ Thus

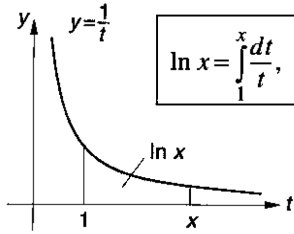
$$\frac{1}{(x^2+1)(x+1)^2}=\frac{1}{2(x+1)}+\frac{1}{2(x+1)^2}-\frac{x}{2(x^2+1)}$$

5.3 Logarithmic, Exponential, Power and Hyperbolic Functions

Logarithmic functions

$y=\ln x,\quad y'=\dfrac{1}{x}\quad (x>0)$

$y=\log_a x,\quad y'=\dfrac{1}{x\ln a}\quad (a>0,\ a\neq 1)$



$$\ln x=\int_1^x\frac{dt}{t},\quad x>0$$

$\ln 1=0,\ \ln e=1,\ \lim_{x\to 0^+}\ln x=-\infty,\ \lim_{x\to\infty}\ln x=\infty$

$\log_a x+\log_a y=\log_a xy$	$\log_a x-\log_a y=\log_a\frac{x}{y}$	$\log_a x^p=p\log_a x$
$\ln x+\ln y=\ln xy$	$\ln x-\ln y=\ln\frac{x}{y}$	$\ln x^p=p\ln x$
$\log_a\frac{1}{x}=-\log_a x$	$\ln\frac{1}{x}=-\ln x$	$\log_a x=\frac{\log_b x}{\log_b a}=\frac{\ln x}{\ln a}$

Complex case: $\log z=\ln|z|+i\arg z$

Inverses

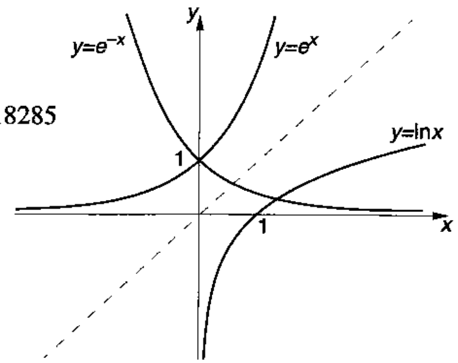
$y=\ln x\ \Leftrightarrow\ x=e^y\quad y=\log_a x\Leftrightarrow x=a^y=e^{y\ln a}$

Exponential functions

Natural base $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\approx 2.71828\ 18285$

$y=e^x=\exp x,\quad y'=e^x$

$y=a^x,\quad y'=a^x\ln a\ (a>0)$



$a^0=1,\ \lim_{x\to-\infty}e^x=0,\ \lim_{x\to\infty}e^x=\infty$

Construction of wavelets

Defining the *product filter* as $P(\omega) = \overline{H(\omega)}\tilde{H}(\omega)$ and inserting (8) into (6) gives $P(\omega) + P(\omega + \pi) = 1$ as a single condition for a biorthogonal MRA. In terms of the z-transform $P(z) = H(z^{-1})\tilde{H}(z)$ and the biorthogonality condition becomes

$$P(z) + P(-z) = 1. \tag{13}$$

The approximation properties of the scaling functions means that $P(z)$ should have zeros at $z = -1$. *Daubechies'* maxflat product filter, with $2N$ zeros at $z = -1$, is given by

$$P(z) = \left(\frac{1+z^{-1}}{2}\right)^N \left(\frac{1+z}{2}\right)^N Q_N(z), \tag{14}$$

where $Q_N(z) = a_{N-1}z^{N-1} + \ldots + a_{1-N}z^{1-N}$ is the unique polynomial of least degree such that (13) is satisfied.

The construction of biorthogonal wavelets proceeds in three steps:

1. Find a product filter with zeros at $z = -1$ satisfying (13).
2. Factor $P(z)$, in some way, into $H(z)$ and $\tilde{H}(z)$.
3. Define the wavelets by relation (8).

Remark. The scaling function and the wavelet are compactly supported if $H(z)$ and $G(z)$ are finite impulse response filters (FIR). The scaling function is symmetric whenever the zeros of $H(z)$ come in pairs as z_i and $1/z_i$. An orthogonal MRA is obtained when $H = \tilde{H}$ and then $P(z) = H(z^{-1})H(z)$. This means that the zeros of $P(z)$ come in pairs as z_i and $1/z_i$. So either z_i *or* $1/z_i$ is a zero of $H(z)$. Orthogonality thus prevents symmetry except for the simple Haar MRA, where all zeros of $P(z)$ are at $z = -1$. For non-compactly supported scaling functions and wavelets it is possible to combine orthogonality and symmetry though.

Example. For $N = 2$ Daubechies' product filter is

$$P(z) = \left(\frac{1+z^{-1}}{2}\right)^2 \left(\frac{1+z}{2}\right)^2 \left(\frac{-z+4-z^{-1}}{2}\right) = \frac{1}{32}(-z^3 + 9z + 16 + 9z^{-1} - z^{-3}).$$

Two possible factorizations of this product filter are:

1. Orthogonal and non-symmetric,

$$H(z) = \tilde{H}(z) = \frac{1}{8}((1+\sqrt{3}) + (3+\sqrt{3})z^{-1} + (3-\sqrt{3})z^{-2} + (1-\sqrt{3})z^{-3}).$$

2. Biorthogonal and symmetric,

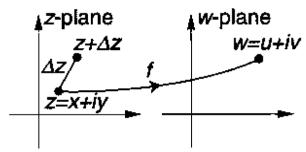
$$H(z) = \frac{1}{4}(z + 2 + z^{-1})$$

$$\tilde{H}(z) = \frac{1}{8}(-z^2 + 2z + 6 + 2z^{-1} - z^{-2}).$$

14 Complex Analysis

14.1 Functions of a Complex Variable

Complex numbers, see sec. 2.3.



Notation

$$w = f(z) = f(x+iy) = u(x, y) + iv(x, y)$$

Differentiation

$f(z)$ is *differentiable* at z if

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \text{exists.}$$

Remark. $f'(z) = u'_x + i v'_x = v'_y - i u'_y$

Analytic functions

Definition. The function $f(z)$ is analytic in a domain Ω if $f(z)$ is differentiable at every point of Ω. [$f(z)$ is analytic at ∞ if $f(1/z)$ is analytic at 0.]

Remark. $|z|$ and $\bar{z}$ are not analytic functions.

Some properties of analytic functions

Assume that $f(z)$ is analytic in Ω with boundary C. Then in Ω,

1. Any order derivative of $f(z)$ exists and is an analytic function.
2. (*Cauchy–Riemann's equations:*) In polar Coordinates:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \qquad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}$$

The converse is true if the partial derivatives are continuous in Ω.

Remark. $f(z) = u(z, 0) + iv(z, 0)$; $f'(z) = u'_x(z, 0) + i v'_x(z, 0) =$
$= u'_x(z, 0) - i u'_y(z, 0)$ etc.; $f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right) + C = 2iv\left(\frac{z}{2}, -\frac{iz}{2}\right) + C$
if $f(z)$ is analytic around zero.

3. $\Delta u = u''_{xx} + u''_{yy} = 0$, $\Delta v = 0$, i.e. u and v are (*conjugate*) *harmonic functions*.

4. $u(x, y) = C_1$, $v(x, y) = C_2$ represent two orthogonal families of curves.

5. l'Hospital's rule for limits is valid for a quotient of analytic functions.

6. (*Maximum – modulus principle.*)
   $|f(z)| \le M$ on C (C simple) $\Rightarrow$ $|f(z)| < M$ in Ω (if $f(z)$ is not constant).
   [$|f(z)|$ attains its maximum (and minimum if $f(z) \neq 0$) on the boundary].

7. $f'(a) \neq 0 \Rightarrow w = f(z)$ has an analytic inverse function $z = f^{-1}(w)$ in a neighborhood of a and
$$\frac{dz}{dw} = 1/\frac{dw}{dz}\,.$$

8. (*Liouville's theorem*). If $f(z)$ is analytic in the entire plane (i.e. an *entire function*) and bounded, then $f(z)$ is constant.

9. (*Schwarz' lemma*)
   (*i*) $f(z)$ analytic for $|z| < 1$ (*ii*) $|f(z)| \le 1, f(0) = 0 \Rightarrow$
   $|f(z)| \le |z|$ (equality only if $f(z) = cz$, $|c| = 1$)

Elementary functions

Single-valued functions

1. $z^n = (x + iy)^n$, n integer ($z \neq 0$ if $n < 0$)

2. $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$. Period $= 2\pi i$

3. $\cosh z = \frac{1}{2}(e^z + e^{-z})$, $\sinh z = \frac{1}{2}(e^z - e^{-z})$

   $\tanh z = \dfrac{\sinh z}{\cosh z}\left(z \neq \left(k + \frac{1}{2}\right)\pi i\right)$, $\coth z = \dfrac{\cosh z}{\sinh z}$ $(z \neq k\pi i)$

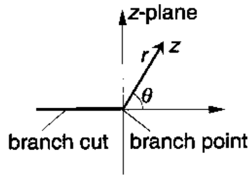
4. $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

   $\tan z = \dfrac{\sin z}{\cos z}\left(z \neq \left(k + \frac{1}{2}\right)\pi\right)$, $\cot z = \dfrac{\cos z}{\sin z}$ $(z \neq k\pi)$

$\sin iz = i \sinh z$	$\sinh iz = i \sin z$
$\cos iz = \cosh z$	$\cosh iz = \cos z$
$\tan iz = i \tanh z$	$\tanh iz = i \tan z$
$\cot iz = -i \coth z$	$\coth iz = -i \cot z$

(Formulas for real elementary functions (cf. chapt. 5) are valid also in the complex case.)

Multiple – valued functions



5. $\log z = \ln|z| + i \arg z = \ln r + i(\theta + 2n\pi)$
   (infinitely-valued)
   Principal branch:
   $\operatorname{Log} z = \ln r + i\theta, -\pi < \theta \le \pi$

6. $z^a = e^{a \log z}$, a non-integer
   $\left(\text{if } a = \frac{p}{q} \in \mathbf{Q} \text{ then } z^a \text{ is } q\text{-valued, if } a \notin \mathbf{Q} \text{ then } z^a \text{ is } \infty\text{-valued}\right)$

Example

1. $\log 2i = \ln|2i| + i \arg 2i = \ln 2 + i\left(\frac{\pi}{2} + 2n\pi\right)$

2. $(2i)^i = e^{i \log 2i} = e^{-\left(\frac{\pi}{2} + 2n\pi\right) + i\ln 2} = e^{-\pi/2 - 2n\pi}[\cos(\ln 2) + i \sin(\ln 2)]$

A survey of elementary functions

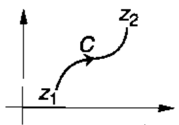
$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y),\ r = |z| = \sqrt{x^2 + y^2},\ \theta = \arg z$$

Function $w = f(z)$	Real part $u(x, y)$	Imaginary part $v(x, y)$	Zeros ($k = 0, \pm 1, \pm 2, \ldots$) m = order	Isolated singularities m = order	Inverse $z = f^{-1}(w)$
z	x	y	$0, m = 1$	$\infty, m = 1$ (pole)	w
z^2	$x^2 - y^2$	$2xy$	$0, m = 2$	$\infty, m = 2$ (pole)	$w^{1/2}$
$1/z$	$\frac{x}{r^2}$	$-\frac{y}{r^2}$	$\infty, m = 1$	$0, m = 1$ (pole)	$1/w$
$1/z^2$	$\frac{x^2 - y^2}{r^4}$	$-\frac{2xy}{r^4}$	$\infty, m = 2$	$0, m = 2$ (pole)	$w^{-1/2}$
$\sqrt{z}$	$\pm\left(\frac{x + r}{2}\right)^{\frac{1}{2}}$	$\pm\left(\frac{-x + r}{2}\right)^{\frac{1}{2}}$	0, branch point	$0, \infty$ branch points	w^2
e^z	$e^x \cos y$	$e^x \sin y$	–	∞ (ess. sing.)	$\log w$
$\cosh z$	$\cosh x \cos y$	$\sinh x \sin y$	$\left(k + \frac{1}{2}\right)\pi i, m = 1$	∞ (ess. sing.)	$\log(w + \sqrt{w^2 - 1})$
$\sinh z$	$\sinh x \cos y$	$\cosh x \sin y$	$k\pi i, m = 1$	∞ (ess. sing.)	$\log(w + \sqrt{w^2 + 1})$
$\tanh z$	$\frac{\sinh 2x}{\cosh 2x + \cos 2y}$	$\frac{\sin 2y}{\cosh 2x + \cos 2y}$	$k\pi i, m = 1$	$\left(k + \frac{1}{2}\right)\pi i, m = 1$ (poles) ∞, (ess. sing.)	$\frac{1}{2}\log\left(\frac{1 + w}{1 - w}\right)$
$\log z$	$\ln r$	$\theta + 2n\pi$	1 (princ. branch), $m = 1$	$0, \infty$ branch points	e^w
$\cos z$	$\cos x \cosh y$	$-\sin x \sinh y$	$\left(k + \frac{1}{2}\right)\pi, m = 1$	∞ (ess. sing.)	$-i \log(w + \sqrt{w^2 - 1})$
$\sin z$	$\sin x \cosh y$	$\cos x \sinh y$	$k\pi, m = 1$	∞ (ess. sing.)	$-i \log(iw + \sqrt{1 - w^2})$
$\tan z$	$\frac{\sin 2x}{\cos 2x + \cosh 2y}$	$\frac{\sinh 2y}{\cos 2x + \cosh 2y}$	$k\pi, m = 1$	$\left(k + \frac{1}{2}\right)\pi, m = 1$ (poles) ∞ (ess. sing.)	$-\frac{i}{2}\log\left(\frac{1 + iw}{1 - iw}\right)$

14.2 Complex Integration

Basic properties

Definition



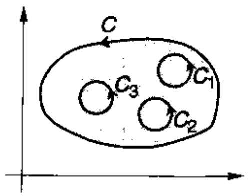
C: $z=z(t)$, $a \le t \le b$

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt =$$

$$= \int_C (u+iv)(dx+i\,dy)$$

Properties

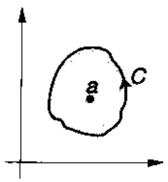
1. $\left|\int_C f(z)dz\right| \le \int_C |f(z)| \cdot |dz| \le M \cdot L$, if $|f(z)| \le M$ on C, L = length of C.
2. If $f(z)$ is analytic in a domain containing C and $F(z)$ is a primitive function of $f(z)$, then
$$\int_C f(z)dz = F(z_2) - F(z_1)$$
3. (*Cauchy's theorem*)
$f(z)$ analytic on and inside a closed curve $C \Rightarrow \oint_C f(z)dz = 0$
4. (*Morera's theorem*, converse of Cauchy's theorem)
(*i*) $f(z)$ continuous in a region Ω
(*ii*) $\oint_C f(z)dz = 0$, every simple closed curve C in Ω
$\Rightarrow f(z)$ is analytic in Ω.



5. If $f(z)$ is analytic in a region with a finite number of "holes" (where $f(z)$ is not necessarily analytic), then
$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \ldots$$
6. If $f(z)$ is analytic on and inside a simple closed curve C, and a is any point inside C, then
(*i*) (*Cauchy's integral formula*)

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$$



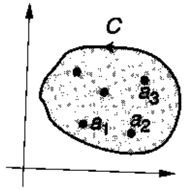
(*ii*) $|f^{(n)}(a)| \le \dfrac{M \cdot n!}{R^n}$ if C is a circle with centre at a and radius $= R$, $|f(z)| \le M$ on C.

Residues

$\operatorname{Res}_{z=a} f(z) = c_{-1}$, i.e. the coefficient of $(z-a)^{-1}$ in that Laurent series expansion of $f(z)$ [cf. sec. 14.3] which converges in $0<|z-a|<R$.

The residue theorem

Assume that $f(z)$ is analytic on and inside C except at finitely many points $a_1, a_2, \ldots, a_n$. Then



$$\frac{1}{2\pi i}\oint_C f(z)dz = \sum_{k=1}^{n} \operatorname{Res}_{z=a_k} f(z)$$

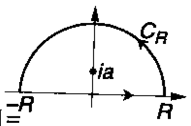
Calculation of residues

1. Determine c_{-1} in the Laurent series expansion.
2. Simple pole: $\operatorname{Res}_{z=a} f(z) = \lim_{z \to a}(z-a)f(z)$. [l'Hospital's rule may be used].
In particular, if $f(z)$, $g(z)$ analytic, $f(a) \ne 0$, $g(a)=0$, $g'(a) \ne 0$, then
$$\operatorname{Res}_{=a} \frac{f(z)}{g(z)} = \frac{f(a)}{g'(a)}$$
3. Pole of order m: $\operatorname{Res}_{z=a} f(z) = \lim_{z \to a} \dfrac{1}{(m-1)!}\left(\dfrac{d}{dz}\right)^{m-1}\{(z-a)^m f(z)\}$.

Calculation of definite integrals

1. $\int_0^{2\pi} R(\sin\theta, \cos\theta)d\theta = [z=e^{i\theta}] = \oint_{|z|=1} R\left(\dfrac{z-z^{-1}}{2i}, \dfrac{z+z^{-1}}{2}\right)\dfrac{dz}{iz}$.
2. If $f(z)$ is analytic in the upper half-plane $\operatorname{Im} z \ge 0$ except for a finite number of points $a_1, \ldots, a_n$ above the real axis, and if $|zf(z)| \to 0$ as $z \to \infty$, then
$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=a_k} f(z)$$
3. If C_R: $z=Re^{i\theta}$, $0 \le \theta \le \pi$ and if $|f(z)| \le M \cdot R^{-k}$, ($M, k > 0$ constants), then $\int_{C_R} f(z)e^{-\operatorname{Im} z}dz \to 0$ as $R \to \infty$.

Example. $I = \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx = \operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}dx$, $a>0$.



Set $f(z) = \dfrac{e^{iz}}{z^2+a^2}$: $\operatorname{Res}_{z=ia} f(z) = e^{-a}\lim_{z \to ia}\dfrac{z-ia}{z^2+a^2}$ = [l'Hospital's rule] =

$= e^{-a}\lim_{z \to ia}\dfrac{1}{2z} = \dfrac{e^{-a}}{2ia}$. Furthermore, $|zf(z)| = \dfrac{|z|e^{-y}}{|z^2+a^2|} \le \dfrac{|z|}{|z^2+a^2|} \to 0$ as $z \to \infty$.

Thus, $I = \operatorname{Re}\left(2\pi i \cdot \dfrac{e^{-a}}{2ia}\right) = \dfrac{\pi e^{-a}}{a}$

Calculation of sum of infinite series

Assume that $|f(z)| \le \text{constant}\, |z|^{-a}$, $a>1$ as $z \to \infty$.

1. $\sum_{-\infty}^{\infty} f(n) = -$[sum of residues of $\pi f(z)\cot \pi z$ at all poles of $f(z)$].

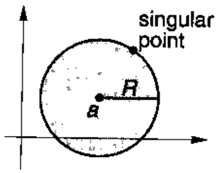
2. $\sum_{-\infty}^{\infty} (-1)^n f(n) = -$[sum of residues of $\dfrac{\pi f(z)}{\sin \pi z}$ at all poles of $f(z)$].

Example. $\sum_{-\infty}^{\infty} \dfrac{1}{n^2+a^2} = \left(\operatorname*{Res}_{z=ia} \dfrac{\pi\cot \pi z}{z^2+a^2} + \operatorname*{Res}_{z=-ia} \dfrac{\pi \cot \pi z}{z^2+a^2}\right) = \dfrac{\pi}{a}\coth \pi a$

14.3 Power Series Expansions

Taylor series

If $f(z)$ is analytic in a neighborhood of $z=a$, then



$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad a_n = \frac{f^{(n)}(a)}{n!}$$

Radius of convergence $R =$ distance to the nearest singular point, or

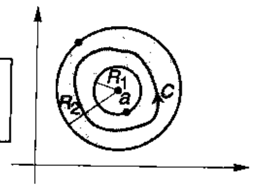
$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \left[\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \text{ if they exist}\right].$$

Example. Sought: Taylor series of $\operatorname{Log}(2z-i)$ about $z=0$; $\operatorname{Log}(2z-i) = \operatorname{Log}[-i(1+2iz)] = \operatorname{Log}(-i) + \operatorname{Log}(1+2iz) = -i\pi/2 + 2iz - 1/2(2iz)^2 + \dots$

Table of series expansions, see sec. 8.6.

Laurent series

If $f(z)$ is analytic in an annulus about $z=a$, then



$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \quad c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

R_1 and R_2 radii of convergence:

$$\frac{1}{R_2} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}; \quad R_1 = \limsup_{n\to\infty} \sqrt[n]{|c_{-n}|}$$

$f(z)$ has singular points on the circles $|z-a| = R_i$, $i = 1, 2$.

Example.
Sought: Laurent series expansion of $f(z) = \dfrac{2}{z^2-1}$ in the annulus $1 < |z-2| < 3$.

Solution. $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z+1} = [z-2 = w] = \dfrac{1}{w+1} - \dfrac{1}{w+3} = \dfrac{1}{w\left(1+\frac{1}{w}\right)} - \dfrac{1}{3\left(1+\frac{w}{3}\right)} =$

$= \dfrac{1}{w}\left(1 - \dfrac{1}{w} + \dfrac{1}{w^2} - \dots\right) - \dfrac{1}{3}\left(1 - \dfrac{w}{3} + \dfrac{w^2}{9} - \dots\right) = \sum_{n=0}^{\infty} (-1)^n (z-2)^{-n-1} - \dfrac{1}{3}\sum_{n=0}^{\infty}\left(-\dfrac{1}{3}\right)^n (z-2)^n$

14.4 Zeros and Singularities

Zeros

Assume that $f(z)$ is analytic (and $\not\equiv 0$) in a neighbourhood of $z=a$. The point a is a *zero* of order n if $f(z) = (z-a)^n g(z)$, where $g(z)$ is analytic and $g(a) \neq 0$.

Remark. a is a zero of order $n \Leftrightarrow$

$$f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0, f^{(n)}(a) \neq 0$$

Singularities

$z=a$ is a *singular point* of $f(z)$ if $f(z)$ fails to be analytic at a. It is isolated if there is a neighbourhood of a in which there are no more singular points.

Classification of isolated singularities.

The point $z=a$ is

(*i*) a *removable singularity* if $\lim_{z\to a} f(z)$ exists.

(*ii*) a *pole* of order n if $f(z) = (z-a)^{-n} g(z)$, where $g(z)$ is analytic, $g(a) \neq 0$. [The Laurent series expansion about a contains finitely many negative power terms.]

(*iii*) an *essential singularity* otherwise, in which case there are infinitely many negative power terms in the Laurent series expansion about a.

Furthermore, branch points of multiple-valued function are examples of non-isolated singular points.

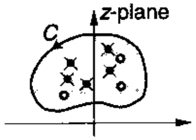
1. (*Picard's theorem*)

The point $z=a$ is an essential singularity of $f(z) \Rightarrow$ Every neighborhood of a contains an infinite set of points z such that $f(z) = w$ for every complex number w (with the possible exception of a single value of w).

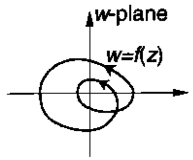
[*Example.* $f(z) = e^{1/z}$. Essential singularity at $z=0$, exceptional value $w=0$].

2. An isolated singular point $z=a$ is a pole $\Leftrightarrow \lim_{z\to a} |f(z)| = \infty$.

The argument principle



Assume that $f(z)$ is analytic inside and on a simple curve C except for a finite number of poles inside C, $f(z) \neq 0$ on C. Let $N =$ number of zeros, $P =$ number of poles inside C (including multiplicity). Then



$$N - P = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\Delta_C \arg f(z)$$

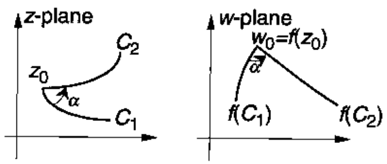
$\Delta_C \arg f(z) = 4\pi$
$N-P=2$

Rouché's theorem

Assume (*i*) $f(z), g(z)$ analytic on and inside a simple closed curve C (*ii*) $|g(z)| < |f(z)|$ on C. Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.

14.5 Conformal Mappings

Assume that $f(z)$ is analytic. The mapping $w=f(z)$ is *conformal* (i.e. preserves angles both in magnitude and sense) at z_0 if $f'(z_0)\neq 0$.



Remark. The Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}=|f'(z)|^2$.

Riemann's mapping theorem

Assume that Ω is a simply connected region with boundary C. Then there exists a mapping $w=f(z)$, analytic in Ω, which maps Ω one-to-one and conformally onto the unit disc and C onto the unit circle.

The bilinear (Möbius) transformation

The mapping $w=\dfrac{az+b}{cz+d}$ $(ad-bc\neq 0)$ maps

(*i*) circle $\to$ circle or straight line

(*ii*) straight line $\to$ circle or straight line

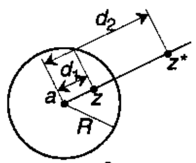
Invariance of cross ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad [w_k=w(z_k)]$$

Inverse points

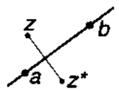
z and z^* are inverse points

(*i*) with respect to a circle if
$(z^*-a)(\bar{z}-\bar{a})=R^2$



$d_1d_2=R^2$

(*ii*) with respect to a line if
$(\bar{b}-\bar{a})(z^*-a)=(b-a)(\bar{z}-\bar{a})$



Invariance of inverse points

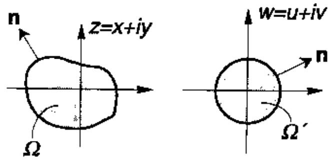
Pairs of inverse points are mapped to pairs of inverse points (with respect to corresponding circles or lines).

Preservation of harmonicity by conformal mappings

Assume that

(*i*) $h(u, v)$ is harmonic in w-plane.

(*ii*) $f(z)=u(x, y)+iv(x, y)$ is an analytic function mapping Ω conformally into Ω'.



Then

$H(x, y)=h(u(x, y), v(x, y))$ is harmonic in Ω.

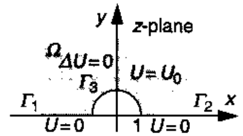
Remark. $\dfrac{\partial h}{\partial n}=0$ on $\partial\Omega' \Rightarrow \dfrac{\partial H}{\partial n}=0$ on $\partial\Omega$.

Cf. Poisson's integral formulas, sec. 10.9.

Example. (*Solving a Dirichlet's problem by conformal mapping.*)

Problem: Determine the electric potential $U(x, y)$ in the unbounded shadowed region Ω if the potential is given on the boundary as indicated in the figure, i.e. solve the following Dirichlet problem:

$$(*)\quad \begin{cases} \Delta U=0 & \text{in } \Omega \text{ (i.e. } U \text{ is harmonic in } \Omega) \\ U=0 & \text{on } \Gamma_1 \text{ and } \Gamma_2 \\ U=U_0 & \text{on } \Gamma_3 \end{cases}$$



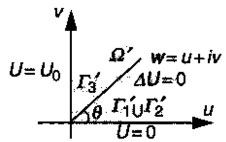
Solution. Set $z=x+iy$ and $w=u+iv$. By the bilinear transformation

$$w=\frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}=\frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}=u+iv,$$

Ω is conformally mapped onto Ω' = the first quadrant of the w-plane. Furthermore,
$\Gamma_1\mapsto\Gamma_1': 0<u<1,\ v=0 \quad \Gamma_2\mapsto\Gamma_2': u>1,\ v=0 \quad \Gamma_3\mapsto\Gamma_3': u<0,\ v>0.$

Therefore, the problem (*) is transformed to the corresponding Dirichlet problem in the (u, v)-plane:

$$\begin{cases} \Delta U=0 & \text{in } \Omega' \\ U=0 & \text{on } \Gamma_1'\cup\Gamma_2' \\ U=U_0 & \text{on } \Gamma_3' \end{cases}$$

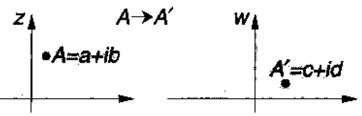
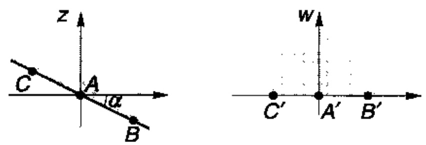


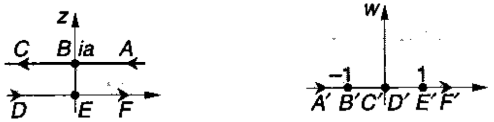
Because $\theta=\arg w=\arctan\dfrac{v}{u}$ is harmonic in the first quadrant (it is the imaginary part of the analytic function $\log w=\ln|w|+i\arg w$), the solution of problem (*) is

$$U=\frac{2U_0}{\pi}\theta=\frac{2U_0}{\pi}\arg w=\frac{2U_0}{\pi}\arctan\frac{v}{u}=\frac{2U_0}{\pi}\arctan\frac{2y}{x^2+y^2-1}$$

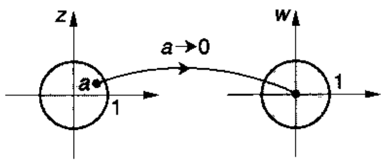
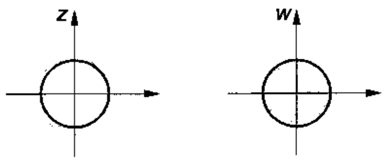
Special conformal mappings

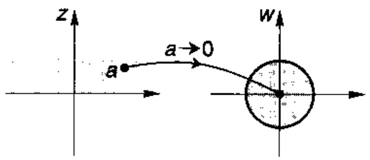
Mappings onto the upper half plane

		Mapping
1.		$w=\dfrac{d}{b}(z-a)+c$
2.		$w=e^{i\alpha}z$

		Mapping
3.		$w = z^{\pi/\alpha}$
4.		$w = e^{\pi z/a}$
5.		$w = \cosh \frac{\pi z}{a}$
6.		$w = -\cos \frac{\pi z}{a}$
7.		$w = \frac{1 - iz}{z - i}$
8.		$w = \left(\frac{1 + z^{\pi/\alpha}}{1 - z^{\pi/\alpha}}\right)^2$

Mappings onto the unit circle

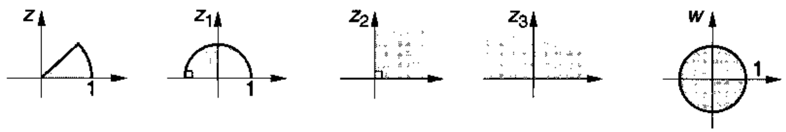
		Mapping
9.		$w = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$ (θ arbitrary)
10.		$w = \frac{1}{z}$

		Mapping
11.		$w = \frac{z - a}{z - \bar{a}}$

Composite mappings

Example. Find a conformal mapping of the circle sector $0 < \arg z < \pi/4$, $|z| < 1$ onto the unit disc $|z| < 1$.

Solution.



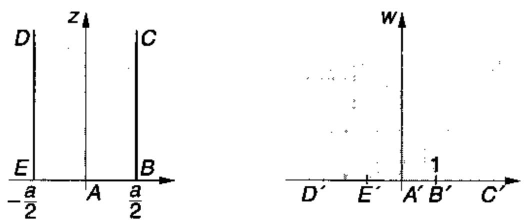
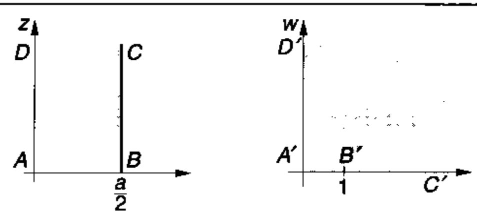
(*i*) $z_1 = z^4$

(*ii*) $z_2 = \frac{1 + z_1}{1 - z_1}$

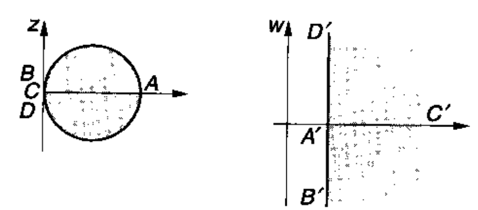
(*iii*) $z_3 = z_2{}^2$ or directly by 8: $z_3 = \left(\frac{1 + z^4}{1 - z^4}\right)^2$

(*iv*) By 11: $w = \frac{z_3 - i}{z_3 + i} = \frac{(1 + z^4)^2 - i(1 - z^4)^2}{(1 + z^4)^2 + i(1 - z^4)^2}$

Miscellaneous mappings

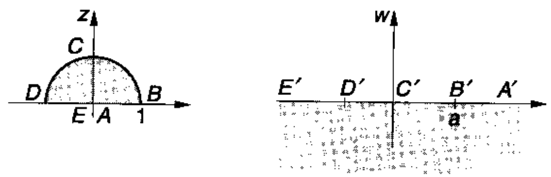
12.		$w = \sin \frac{\pi z}{a}$
13.		$w = \sin \frac{\pi z}{a}$

14.



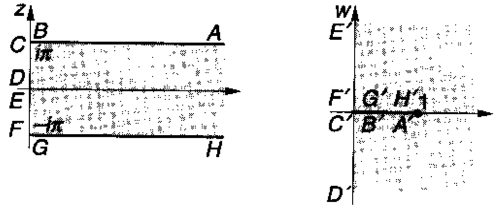
$w = \dfrac{1}{z}$

15.



$w = \dfrac{a}{2}\left(z + \dfrac{1}{z}\right)$

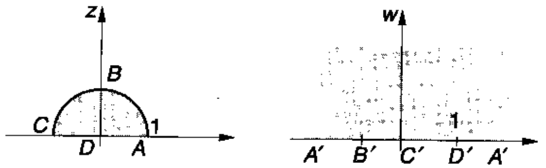
16.



$w = \coth \dfrac{z}{2}$

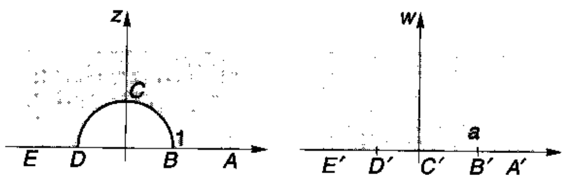
$= \dfrac{e^z + 1}{e^z - 1}$

17.



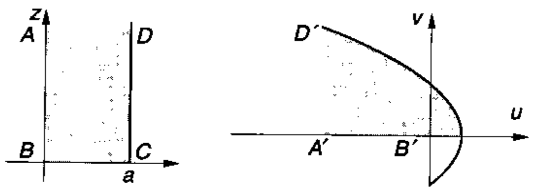
$w = \left(\dfrac{1+z}{1-z}\right)^2$

18.



$w = \dfrac{a}{2}\left(z + \dfrac{1}{z}\right)$

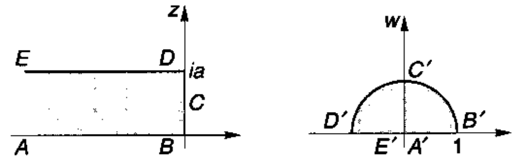
19.



Parabola $v^2 = -4a^2(u - a^2)$

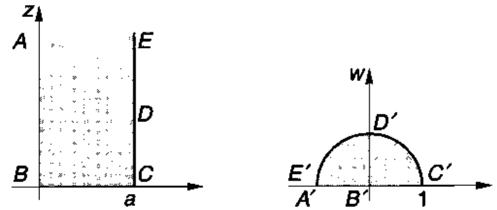
$w = z^2$

20.



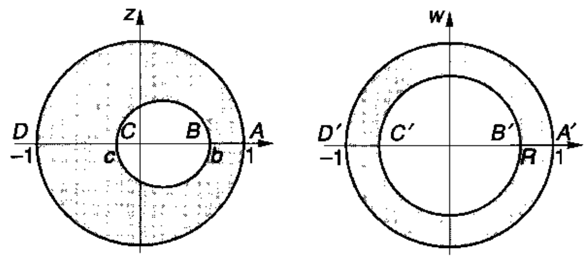
$w = e^{\pi z/a}$

21.



$w = \tan^2 \dfrac{\pi z}{4a}$

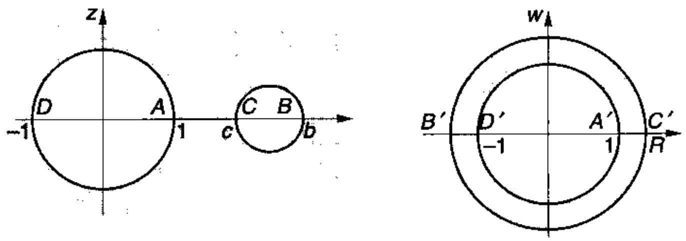
22.



$w = \dfrac{z-a}{1-az}$

$a = \dfrac{1 + bc - \sqrt{(1-b^2)(1-c^2)}}{b+c}$ $R = \dfrac{1 - bc - \sqrt{(1-b^2)(1-c^2)}}{b-c}$

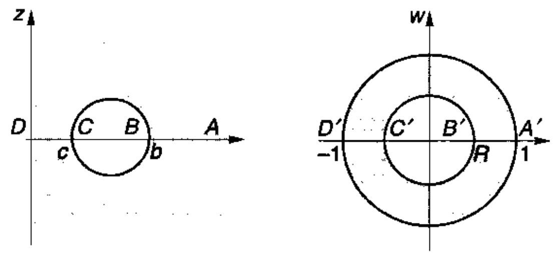
23.



$$\boxed{w = \dfrac{z-a}{1-az}}$$

$a = \dfrac{1 + bc - \sqrt{(b^2-1)(c^2-1)}}{b+c}$ $R = \dfrac{bc - 1 + \sqrt{(b^2-1)(c^2-1)}}{b-c}$

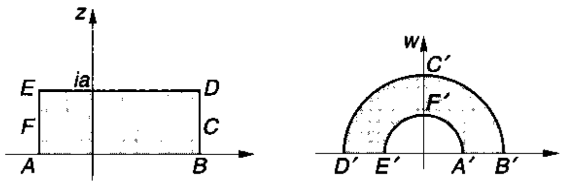
24.



$$w = \frac{z-a}{z+a}$$

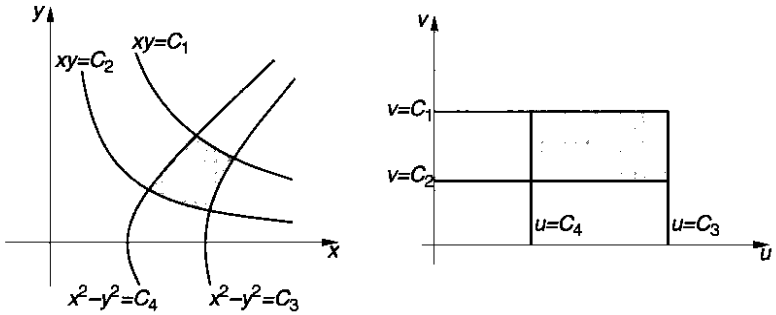
$a = \sqrt{bc}$ $\qquad R = \dfrac{\sqrt{b}-\sqrt{c}}{\sqrt{b}+\sqrt{c}}$

25.



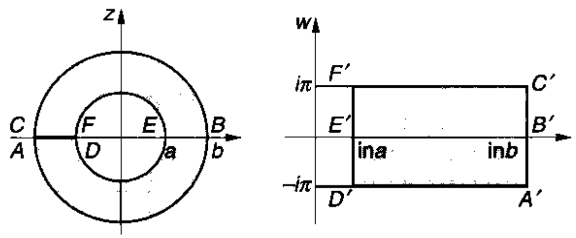
$$w = e^{\pi z/a}$$

26.



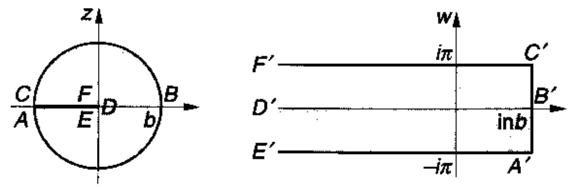
$$w = z^2$$

27.



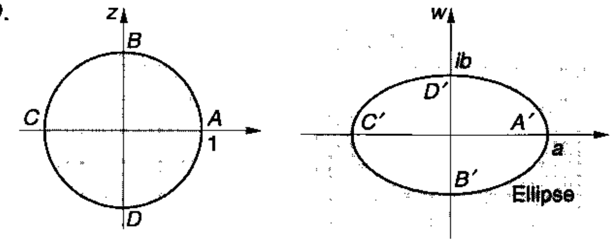
$$w = \operatorname{Log} z$$

28.



$$w = \operatorname{Log} z$$

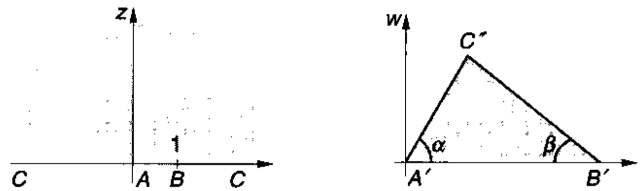
29.



$$w = pz + \frac{q}{z}$$

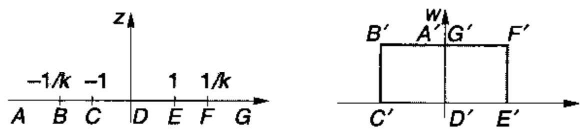
$p = \dfrac{a-b}{2}$ $\qquad q = \dfrac{a+b}{2}$

30.



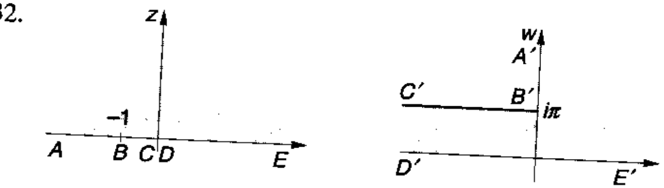
$$w = \int_0^z t^{\alpha/\pi - 1}(1-t)^{\beta/\pi - 1}\,dt$$

31.



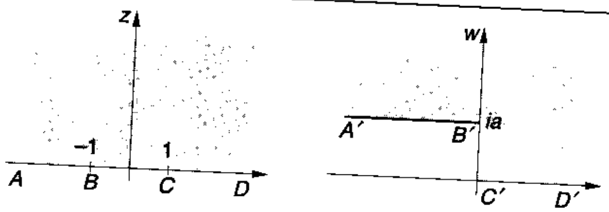
$$w = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad 0 < k < 1$$

32.



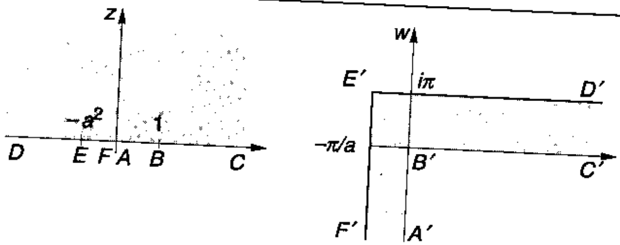
$$w=2\sqrt{z+1}+\mathrm{Log}\,\frac{\sqrt{z+1}-1}{\sqrt{z+1}+1}$$

33.



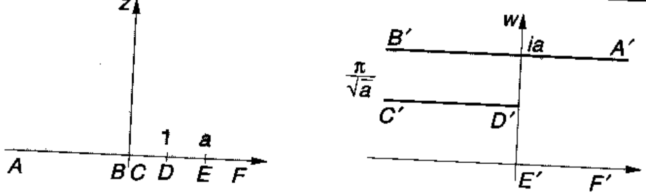
$$w=\frac{a}{\pi}\left(\sqrt{z^2-1}+\cosh^{-1}z\right)$$

34.



$$w=\frac{i}{a}\,\mathrm{Log}\,\frac{1+iat}{1-iat}+\mathrm{Log}\,\frac{1+t}{1-t}\,,\quad t=\sqrt{\frac{z-1}{z+a^2}}$$

35.



$$w=\cosh^{-1}\left(\frac{2z-a-1}{a-1}\right)-\frac{1}{\sqrt{a}}\cosh^{-1}\left[\frac{(a+1)z-2a}{(a-1)z}\right]$$

15 Optimization

(In this chapter all functions are assumed to be "smooth enough".)

15.1 Calculus of Variations

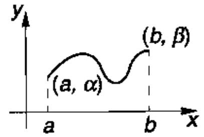
The calculus of variations treats the problem of finding extrema of *functionals*, i.e. real valued functions having *functions* as "independent variables". Below, *necessary* conditions (the *Euler-Lagrange equation* (15.1), the solutions of which are called *extremals*) are stated for some different kinds of variational problems. *Sufficient* conditions can be formulated (e.g. Weierstrass' theory on strong extrema). However, "common sense" may often be used to establish the sufficiency.

Problem 1 (fixed end points)

Find a function $y=y(x)$ that *minimizes*

$$I(y)=\int_a^b F(x,\,y,\,y')\,dx$$

$$y(a)=\alpha,\quad y(b)=\beta$$



for a given function $F(x,\,y,\,y')$.

Necessary condition for solution:

(15.1) $$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \iff$$

$$F'_y-F''_{xy'}-y'F''_{yy'}-y''F''_{y'y'}=0$$

In particular, if $F=F(y,\,y')$ then (15.1) implies

(15.2) $F-y'F'_{y'}=C$ (C constant)

Remark. The equation (15.1) is an ordinary differential equation of 2nd order. Combined with the boundary conditions $y(a)=\alpha$ and $y(b)=\beta$ the problem to be solved is a boundary-value problem.