TD7, exercice II

$$f(z) = \frac{1}{z} - \operatorname{cotg} z$$

On commence par décomposer $\operatorname{cotg} z = \dfrac{\cos z}{\sin z} = \dfrac{\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!}}{\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}}$

$$= \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} - (\cdots) z^6 + \cdots}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - (\cdots) z^7 + \cdots} = \frac{\overbrace{1 - \frac{z^2}{2} + \frac{z^4}{4!} - (\cdots) z^6 + \cdots}^{b}}{z\Big(1 \underbrace{- \frac{z^2}{3!} + \frac{z^4}{5!} + (\cdots) z^6 + \cdots}_{a}\Big)} = \frac{b}{z} \cdot \frac{1}{1+a}$$

$$= \frac{1}{z}\, b\,(1 - a + a^2 - a^3 + \cdots)$$

(série géométrique valable pour $|a| < 1$)

$$= \frac{1}{z}\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} + (\cdots) z^6 + \cdots\right)\Big[1 - \underbrace{\left(-\frac{z^2}{3!} + \frac{z^4}{5!} + (\cdots) z^6 + \cdots\right)}_{a} + \underbrace{\left(\frac{z^4}{(3!)^2} + (\cdots) z^6 + \cdots\right)}_{a^2} + \underbrace{\big((\cdots) z^6 + \cdots\big)}_{a^3} + \cdots\Big]$$

$$= \frac{1}{z}\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} + (\cdots) z^6 + \cdots\right)\left[1 + \frac{1}{6} z^2 + \frac{7}{360} z^4 + (\cdots) z^6 + \cdots\right]$$

$$= \frac{1}{z}\Big(1 + z^2 \underbrace{\left(\frac{1}{6} - \frac{1}{2}\right)}_{-\frac{1}{3}} + z^4 \underbrace{\left(\frac{1}{4!} + \frac{7}{360}\right)}_{\frac{11}{180}} + (\cdots) z^6 + \cdots\Big) = \frac{1}{z} - \frac{z}{3} + \frac{11 z^3}{180} + (\cdots) z^5 + \cdots$$

→ Série de Laurent :
$A_{-1} = 1$, $A_0 = 0$, $A_1 = -\frac{1}{3}$, $A_2 = 0$, $A_3 = \frac{11}{180}$

Donc $\underbrace{\frac{1}{z} - \operatorname{cotg} z}_{f(z)} = \frac{z}{3} - \frac{11 z^3}{180} + (\cdots) z^5 + \cdots$ est prolongeable par continuité en l'origine : $f(0) = 0$

TD 8, ex III, question 4 (premiers termes de la série de Laurent de $\frac{1}{f-1}$)

$$g(z) = \frac{1}{f(z)-1} \quad , \quad f(z) = \frac{e^{\pi z}+e^{-\pi z}}{2} = \sum_{k=0}^{\infty} \frac{\pi^{2k}}{(2k)!} z^{2k}$$

$$= 1 + \frac{\pi^2}{2} z^2 + \frac{\pi^4}{4!} z^4 + \frac{\pi^6}{6!} z^6 + (\dots) z^8 + \dots$$

(On est intéressés aux termes A_n pour $n \leq 2$)

$$g = \frac{1}{\cancel{1} + \frac{\pi^2}{2} z^2 + \frac{\pi^4}{4!} z^4 + \frac{\pi^6}{6!} z^6 + (\dots) z^8 - \cancel{1}}$$

$$= \frac{1}{\frac{\pi^2 z^2}{2}} \times \frac{1}{\left(1 + \underbrace{\frac{2\pi^2}{4!} z^2 + \frac{2\pi^4}{6!} z^4 + (\dots) z^6 + \dots}_{a}\right)}$$

$$= \frac{2}{\pi^2 z^2}\left(\frac{1}{1+a}\right) = \frac{2}{\pi^2 z^2}\left(1 - a + a^2 - a^3 + \dots\right)$$

c'est la série géométrique, valable pour $|a| < 1$

$$= \frac{2}{\pi^2 z^2}\Big(1 - \underbrace{\left(\frac{2\pi^2}{4!} z^2 + \frac{2\pi^4}{6!} z^4 + (\dots) z^6 + \dots\right)}_{a}$$
$$+ \underbrace{\left(\frac{4\pi^4}{(4!)^2} z^4 + (\dots) z^6 + \dots\right)}_{a^2}$$
$$- \underbrace{\left((\dots) z^6 + \dots\right)}_{a^3}$$
$$+ \dots \Big)$$

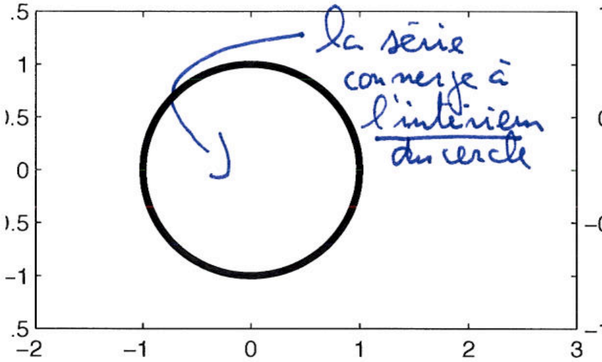
$$= \frac{2}{\pi^2 z^2}\left(1 - \frac{2\pi^2}{4!} z^2 + \left(\frac{4\pi^4}{(4!)^2} - \frac{2\pi^4}{6!}\right) z^4 + (\dots) z^6 + \dots\right)$$

$$= \frac{2}{\pi^2 z^2} - \frac{4}{4!} + \left(\frac{8\pi^2}{(4!)^2} - \frac{4\pi^2}{6!}\right) z^2 + (\dots) z^4 + \dots$$

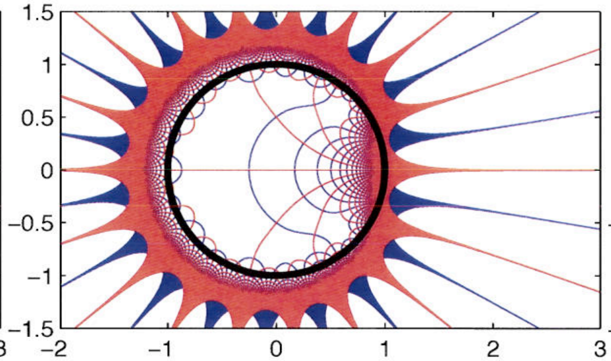
$$\boxed{= \underbrace{\frac{2}{\pi^2}}_{A_{-2}} z^{-2} \underbrace{- \frac{1}{6}}_{A_0} + \underbrace{\frac{\pi^2}{120}}_{A_2} z^2 + (\dots) z^4 + \dots}$$

c'est la série de Laurent pour $n \leq 2$

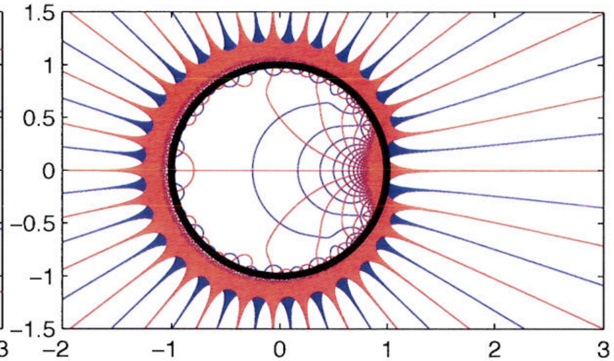
S_5



S_{10}



S_{15}



convergence de la série $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ pour $|z|<1$ (on rajoute cinq termes pour chaque graph)

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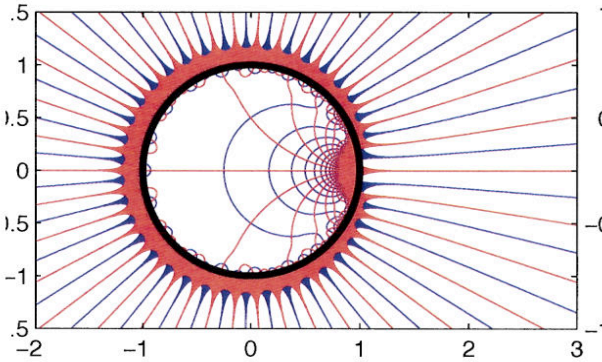
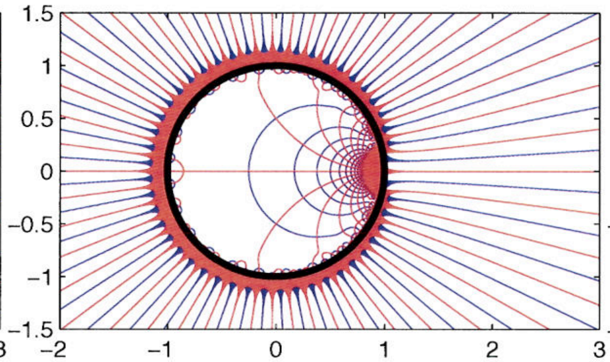
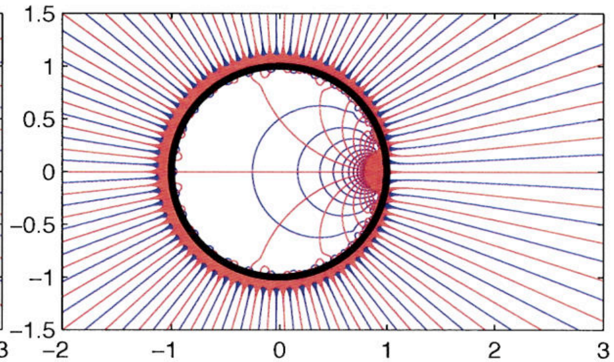
S_{20}



S_{25}



S_{30}



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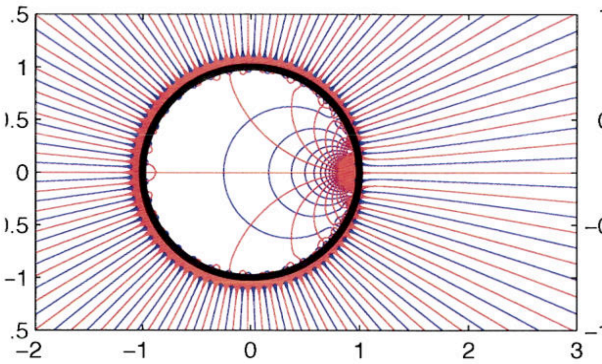
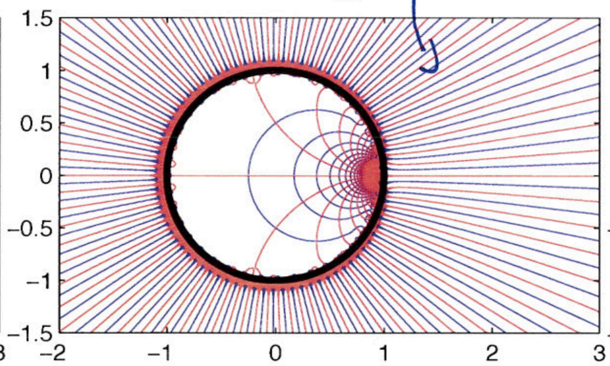
ça converge pas pour $|z|>1$

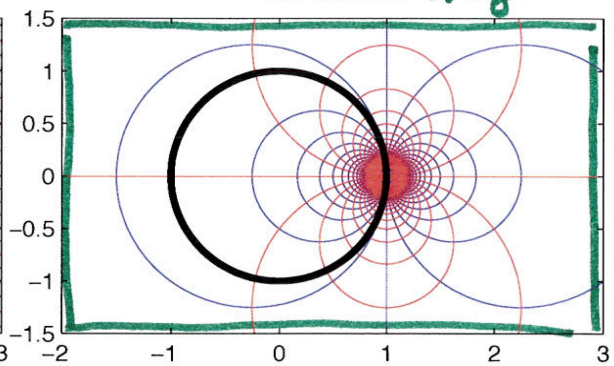
S_{35}

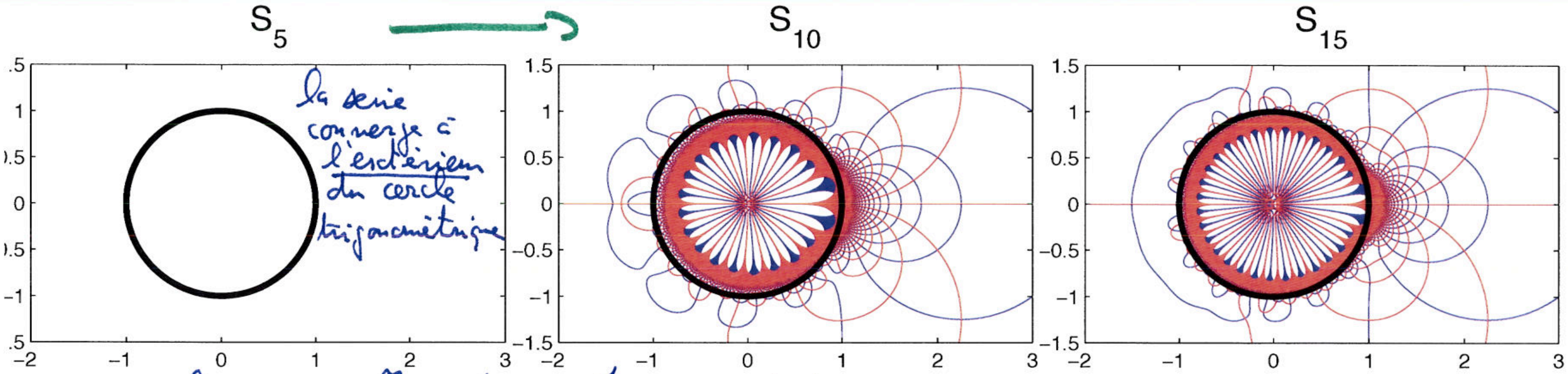


S_{40}

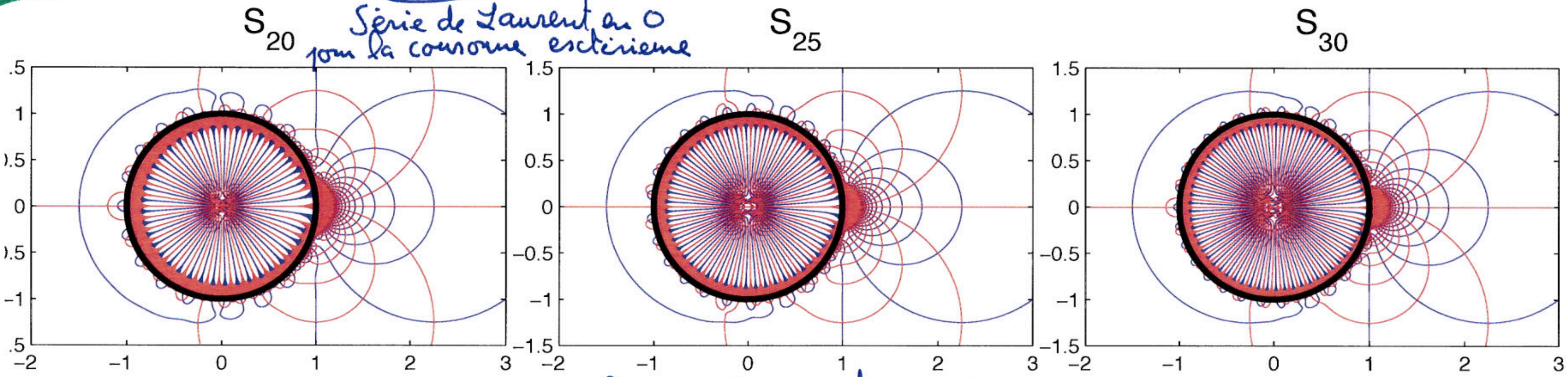


fonction $\frac{1}{1-z}$





Convergence de la serie $\underbrace{\sum_{k=0}^{\infty} -\frac{1}{z^{k+1}}}_{\text{Série de Laurent en 0 pour la couronne extérieure}} = \frac{1}{1-z}$ pour $|z|>1$



ça converge pas pour $|z|<1$

On rajoute cinq termes de la série à chaque graph.