How to build a
quantitatively accurate
simple model

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How to build a quantitatively accurate simple model

when you are given a

quantitatively accurate complicated model

➡ Model reduction…

… and how to build the complicated model

Motivations

1) control:
Perturbations to objective function
Perturbation to measurement
Actuators to objective function
Actuators to measurements

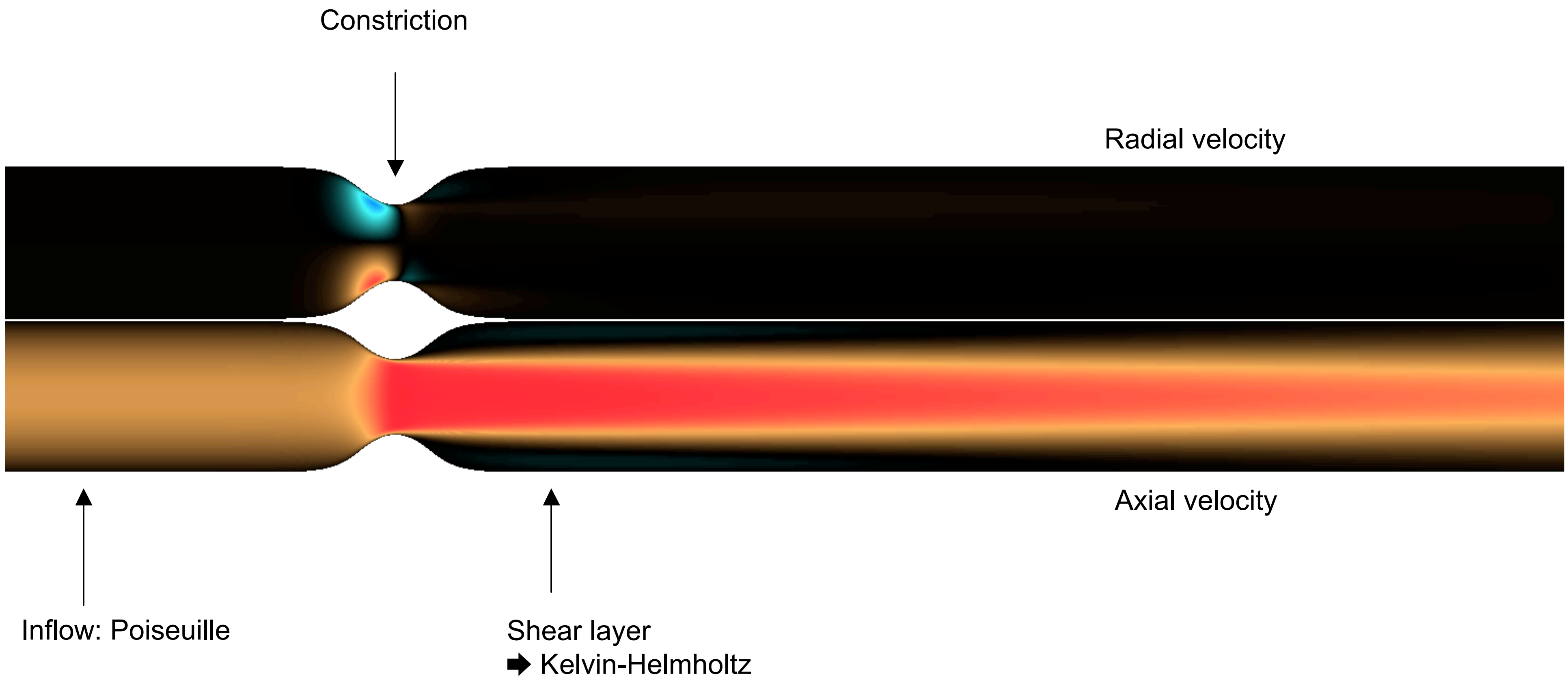
2) Flow part of a larger system: « blockwise » modeling.

➡ When you need a « working » model

Study case

Constricted pipe

Inner flow, convectively unstable

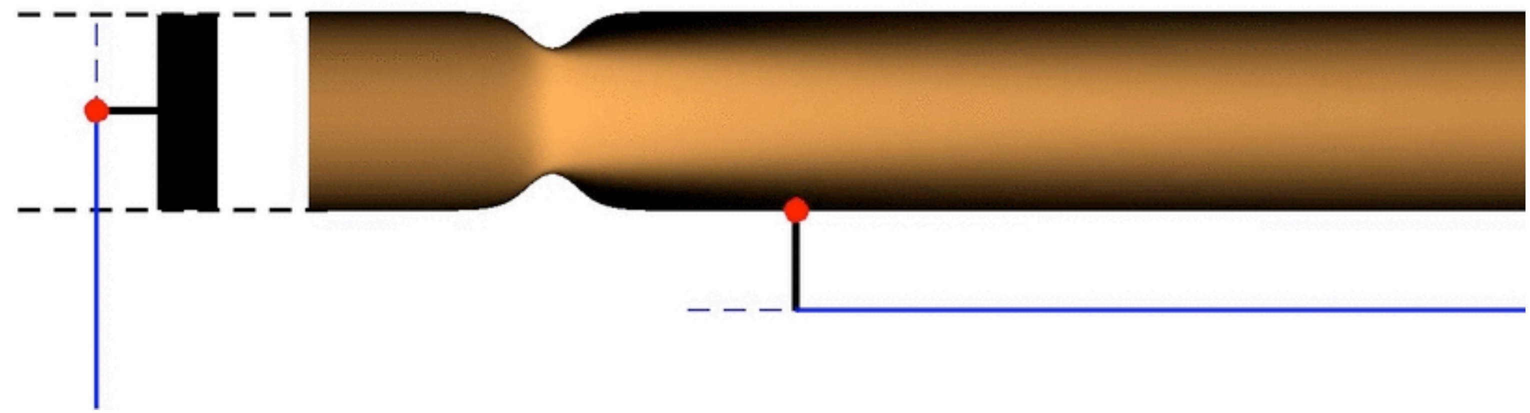


(Comparaison with experiment: Martin Grifftih, Thomas Leweke, IRPHÉ)

How *this* affects *that*

- in the pipe -

This: (the input), piston upstream
That: (the output), Shear stress downstream



The « complicated » model

Dynamic of perturbations to base flow U

$$\underbrace{\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} \dot{u} \\ \dot{p} \end{pmatrix}}_{\dot{q}} = \underbrace{\begin{pmatrix} U\cdot\nabla + \nabla U \cdot + \Delta/Re & -\nabla \\ \nabla\cdot & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ p \end{pmatrix}}_{q}$$

(See Gallaire & Ehrenstein, JFM (2006))

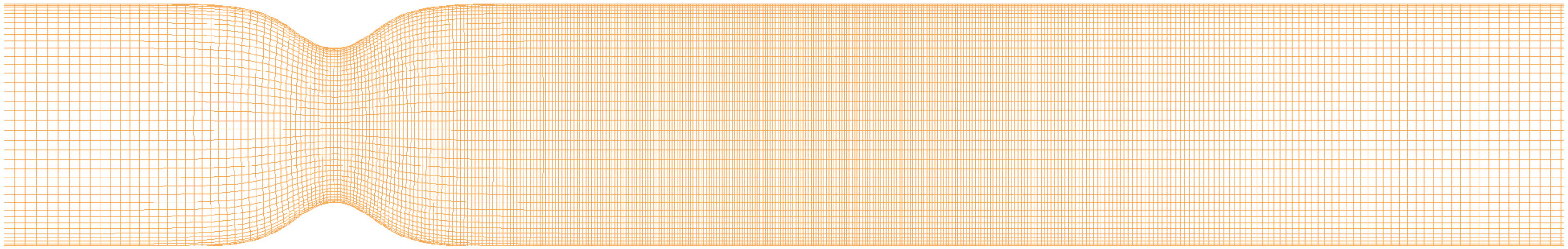
Use of the model:

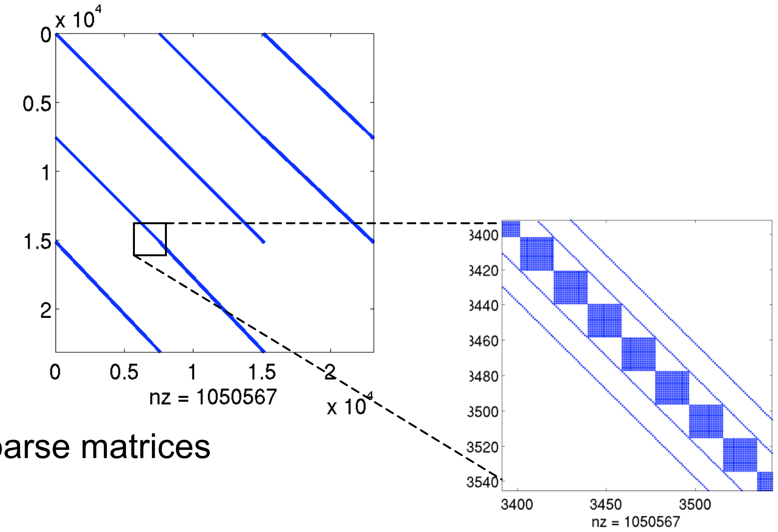
- Eigenmodes: $\frac{1}{\lambda} v = A^{-1} E v$
- Response to harmonic forcing: $\hat{q} = (i\omega E - A)^{-1} \hat{f}$
- Response to initial conditions $q^{n+1} = (E - Ah/2)^{-1} (E + Ah/2) q^n$

Décomposition LU of A
Décomposition LU of (E-Ah/2)
Décomposition LU of (iωE-A)

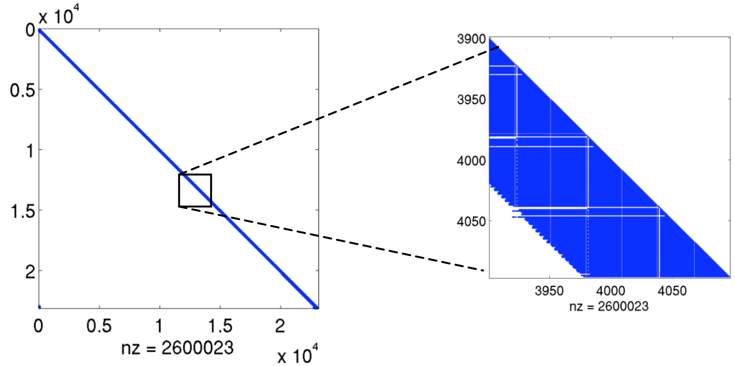
➡ No « treatment » of pressure

Spatial discretization





Sparse matrices



LU decomposition
(upper/Lower triangular)

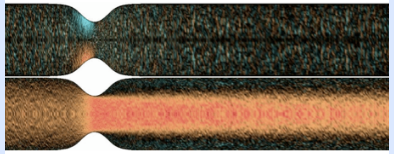
Chebychev/finite difference
Conformal mapping

➡ **Stationary state**: 6 Newton/Raphson iterations :
2 LU decompositions, 4 GMRES solves. (less than a minute)

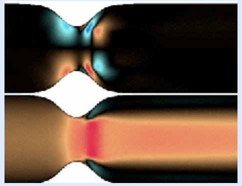
(Optimisation des conditions initiales)

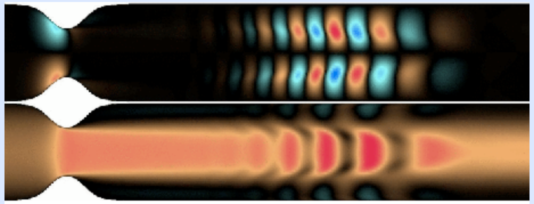
Itération puissance: $q^k = (\mathcal{H}_T^+ \mathcal{H}_T)^n q^0$

Random initial guess:

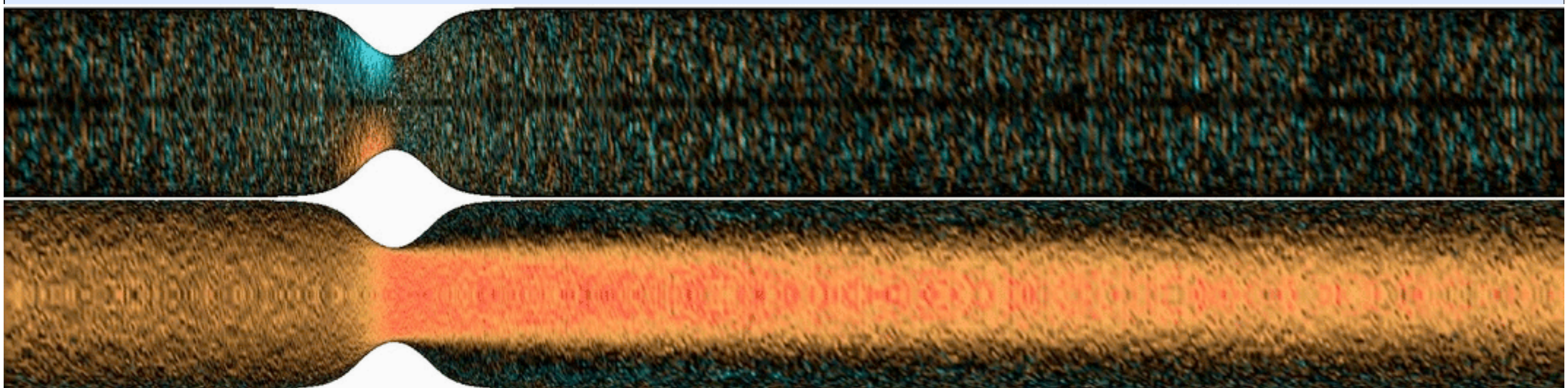


Optimal initial condition

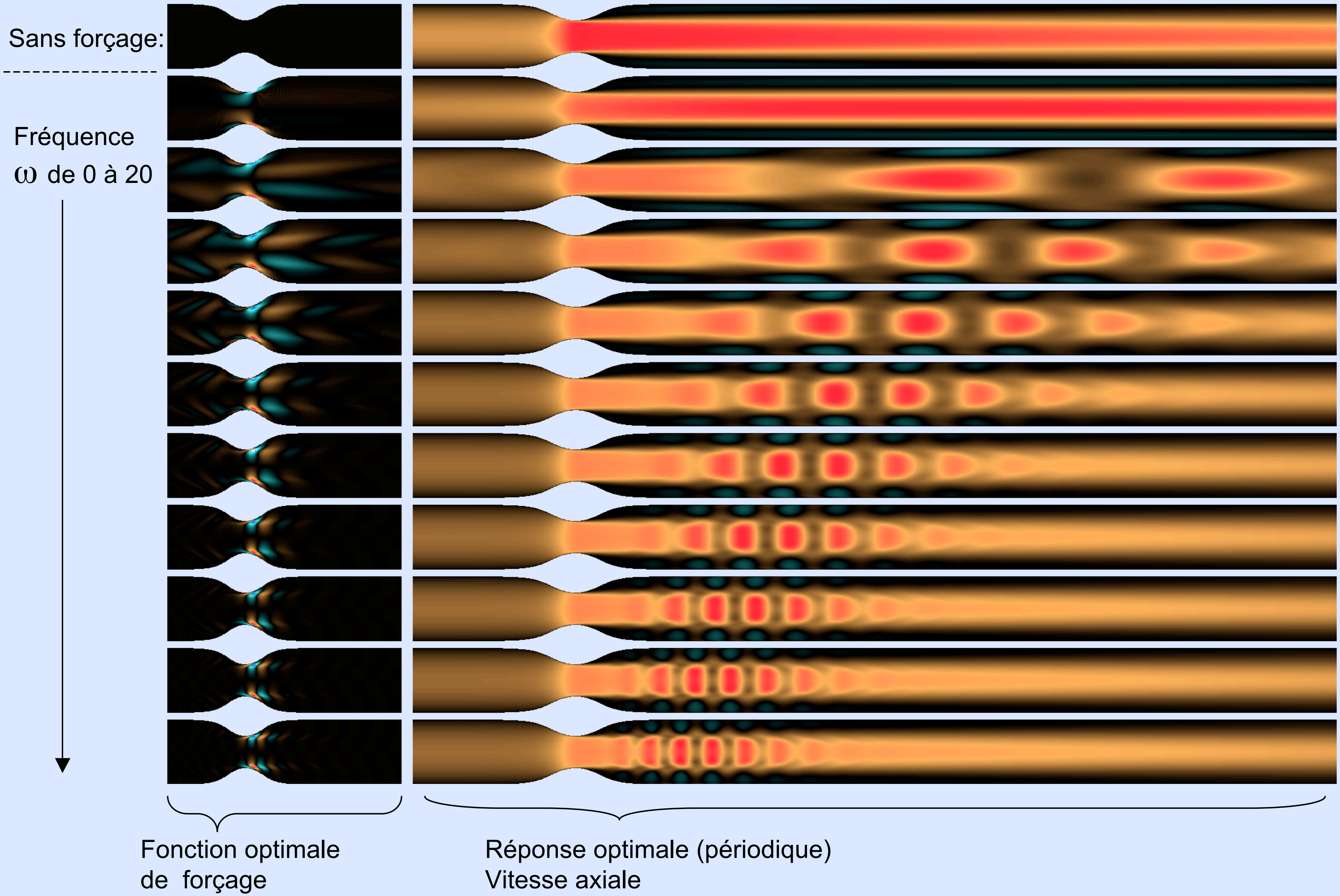




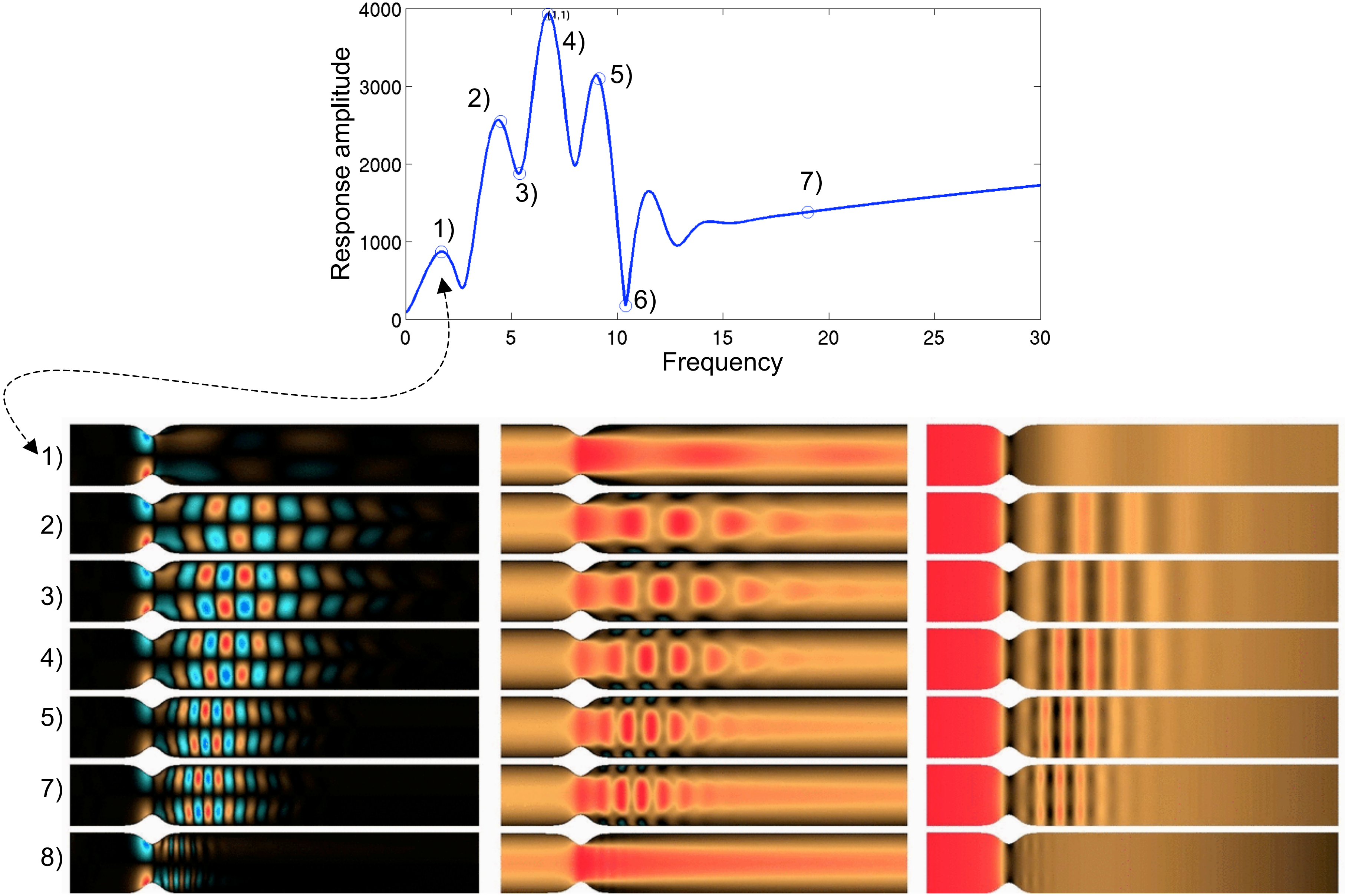
Optimal response at t=4



(Forçage optimal)

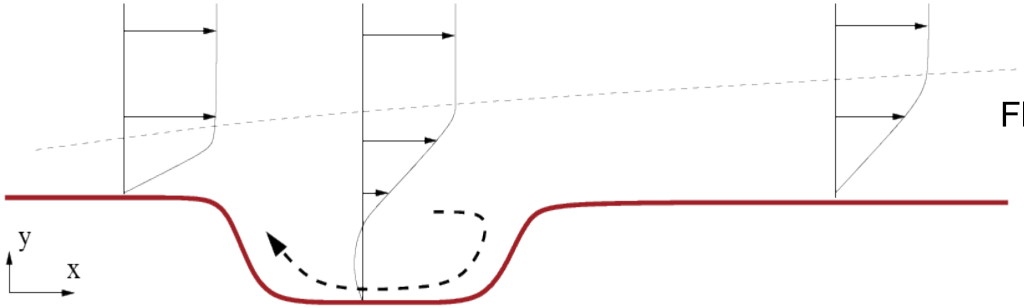


Frequency response



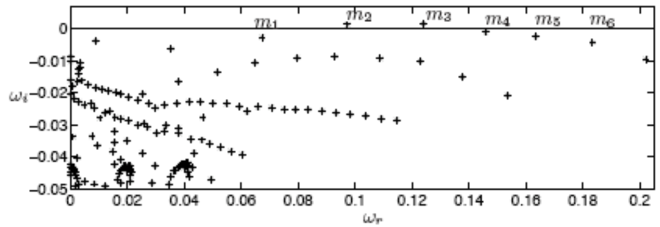
First « simple » model:
Projection on eigenmodes

Åkervik, Hoepffner, Ehrenstein, Henningson, JFM (2007)



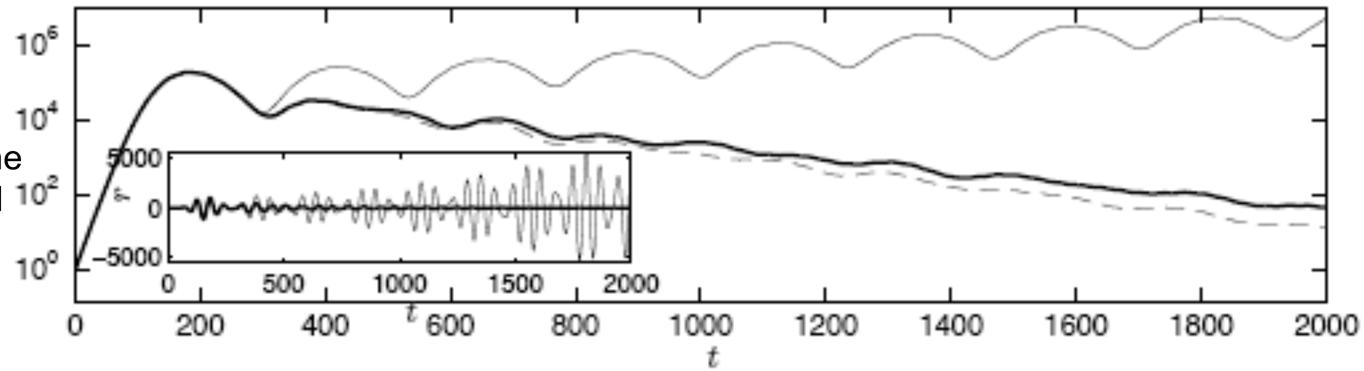
Flow over a smooth cavity

Spectra



2D eigenmodes

Control using the
Reduced model



Choice of basis based on input/output

1) Flow structures with maximium energy
2) Flow structures with maximum observability

➡ **Combine** to get flow structures with maximum transfer energy

$$AP + PA^{+} + M = 0$$

N-S

State correlation

Adjoint N-S

Input correlation

Lyapunov equation

(see Schmid, Ann. Rev. Fluid Mech (2007))

(continuous time) Lyapunov

(discrete time) Lyapunov, ou équation de Stein

$$AP + PA^H + R^H R = 0 \quad \Leftrightarrow \quad \tilde{A} P \tilde{A}^H + \tilde{R}^H \tilde{R} = 0$$

Shifting

$$\begin{cases} \tilde{A} = (A - \sigma I)^{-1}(A + \sigma I) \\ \tilde{R} = \sqrt{2\sigma}(A - \sigma I)R \end{cases}$$

$$P = \tilde{R}^H \tilde{R} + \tilde{A}\tilde{R}^H \tilde{R}\tilde{A}^H + \tilde{A}^2 \tilde{R}^H \tilde{R}\tilde{A}^{2H} + \ldots$$

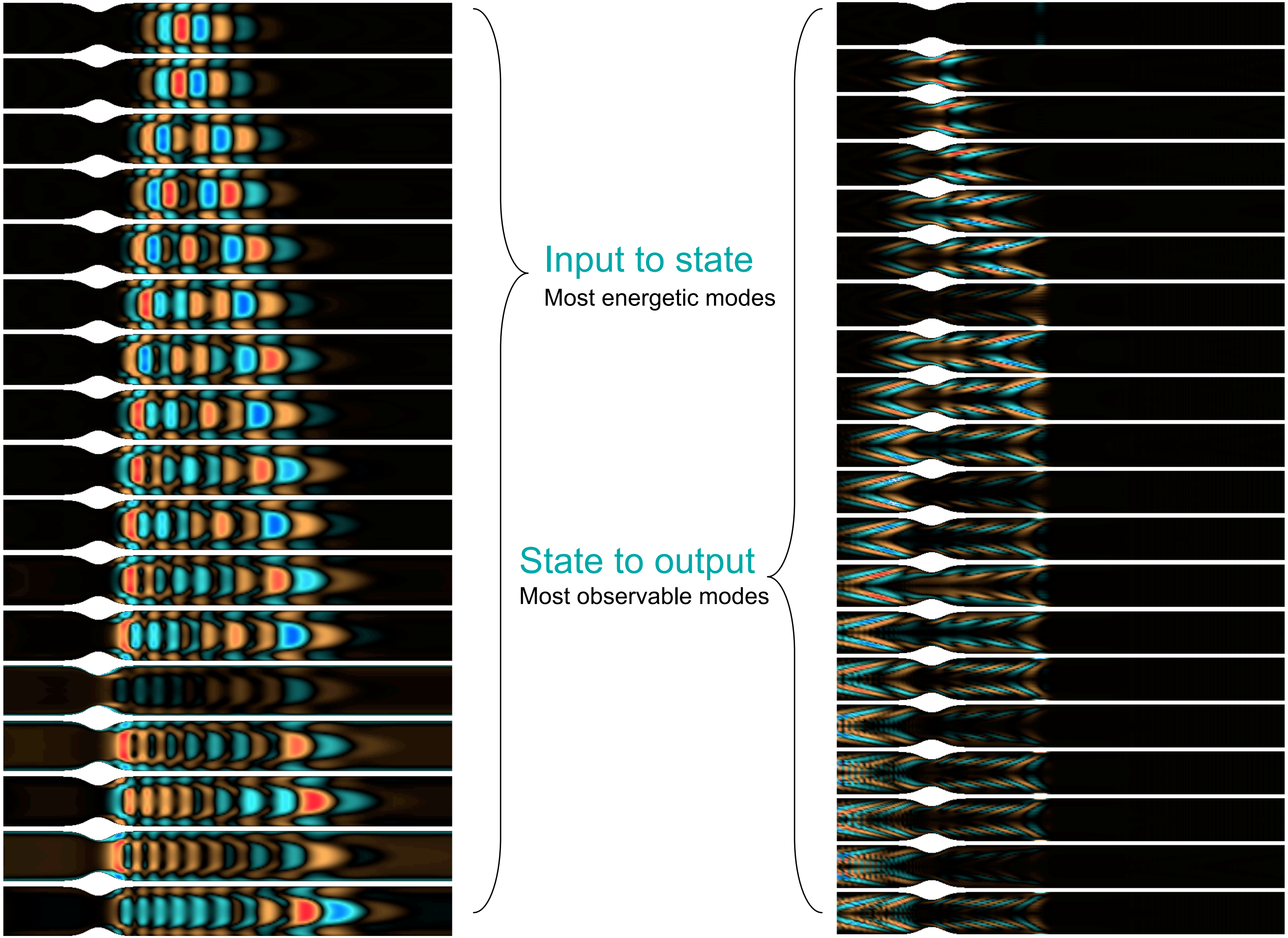
Solution itérative (de Smith)

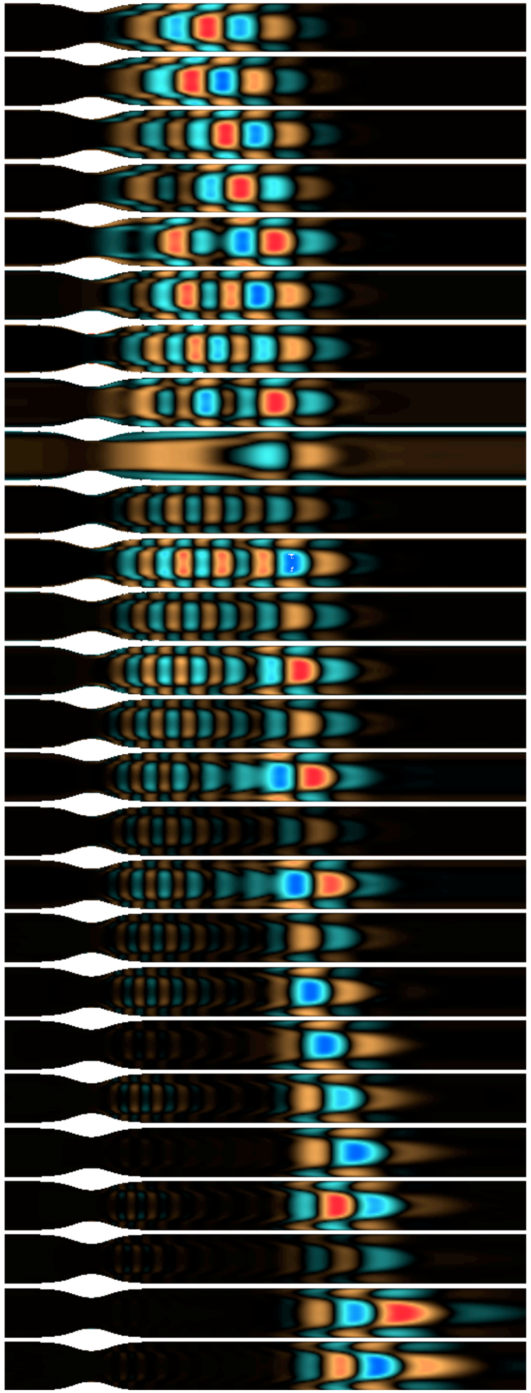
$$P = Z^H Z, \quad Z = \begin{bmatrix} | & | & | & \vdots \\ \tilde{R} & \tilde{A}\tilde{R} & \tilde{A}^2\tilde{R} & \ldots \\ | & | & | & \vdots \end{bmatrix}$$

Décomposition racine de la matrice de covariance

Solution itérative sous forme décomposée

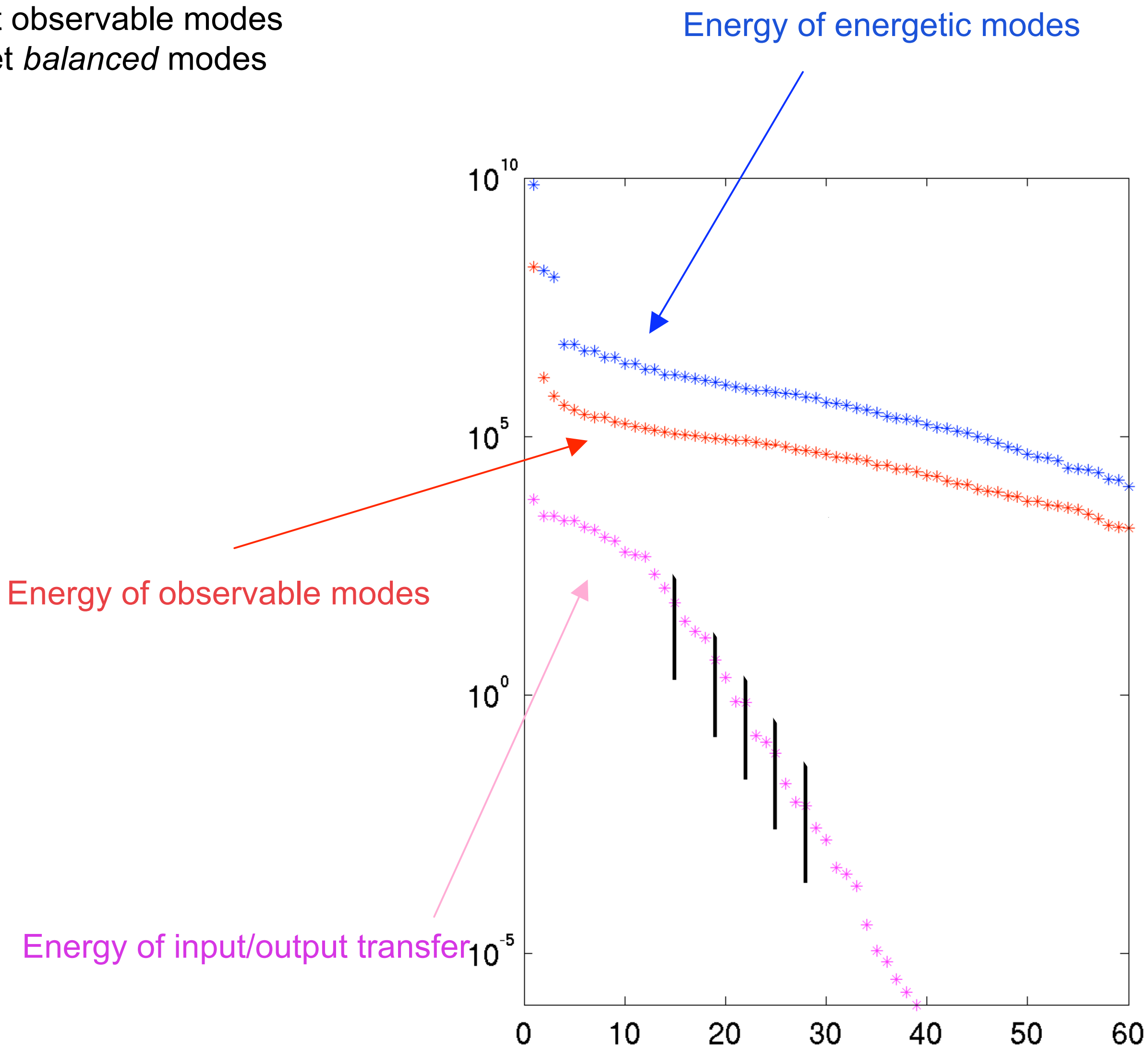
➡Low-Rank cyclic Smith


Input to state
Most energetic modes
State to output
Most observable modes

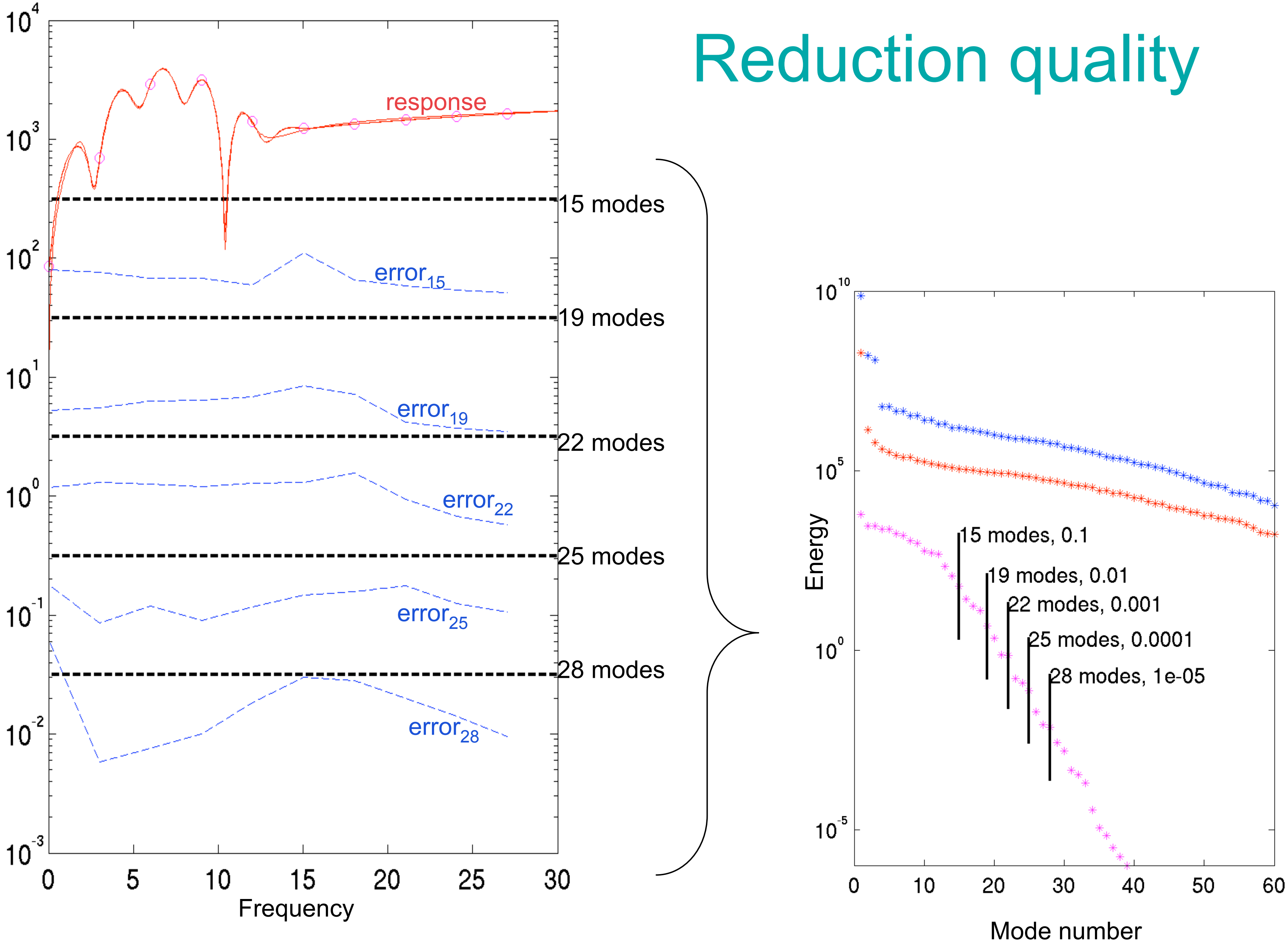


Balanced modes

Combine most energetic and
Most observable modes
to get *balanced* modes



Reduction quality



Conclusions

How to build a simple model

Define the input and the output, find a set of spatial structures to project the dynamic equations, while preserving the transfer function

Eigenmodes do not account for input/input
Energetic coherent structures: eigenmodes of covariance matrix
Observable coherent structures: eigenmodes of adjoint covariance matrix

Basis for reduction obtained by balancing dominant *input-to-state* and *state-to-output subspaces*

A priori bound on reduction error based on « transfer energy » of neglected structures