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Stochastic excitation of streaky boundary layers

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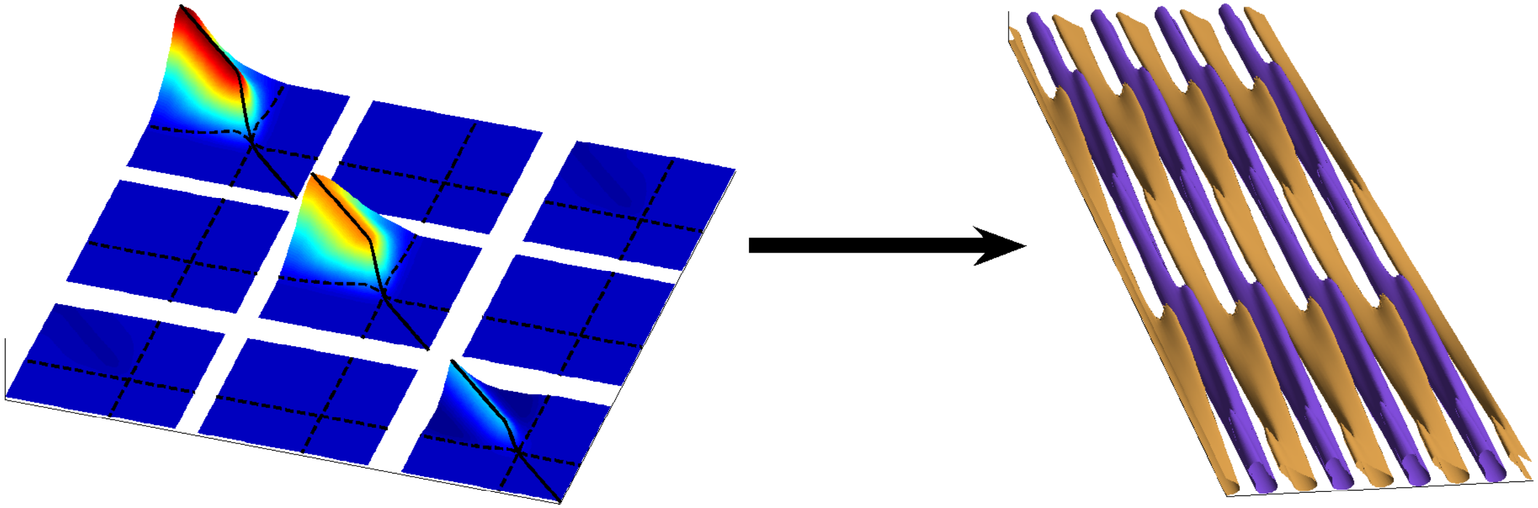
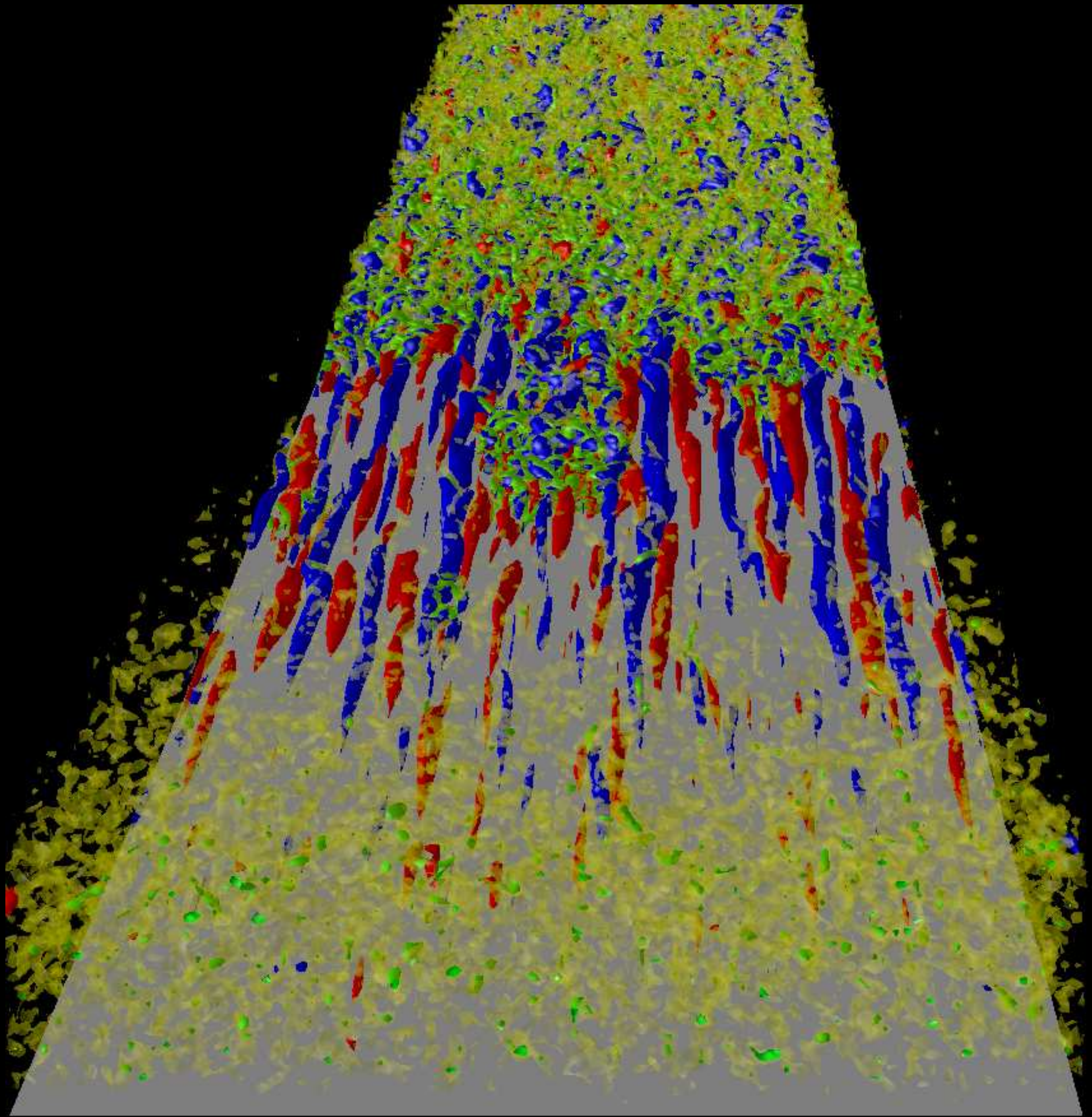


Boundary layer excited by free-stream turbulence



Fully turbulent inflow and flat plate

Turbulent free-stream
$\longrightarrow$ receptivity
$\longrightarrow$ streaks
$\longrightarrow$ streak instability
$\longrightarrow$ turbulent spots
$\longrightarrow$ turbulent boundary layer

(Image by Philipp Schlatter,

LES of bypass transition, Tuesday, D31)

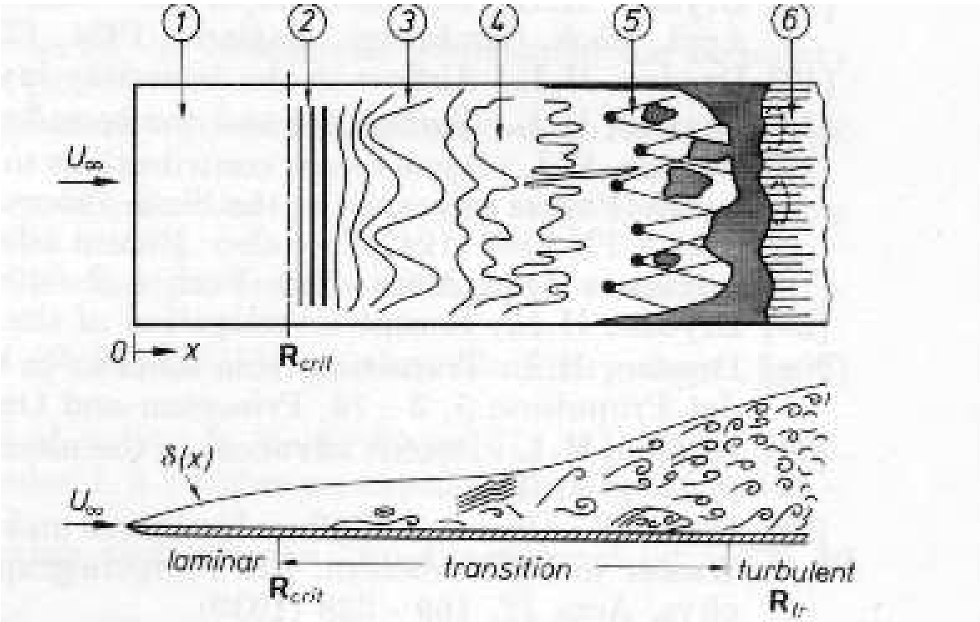
Boundary layer stability

TS waves:

Large Reynolds

2D waves

Exponential growth

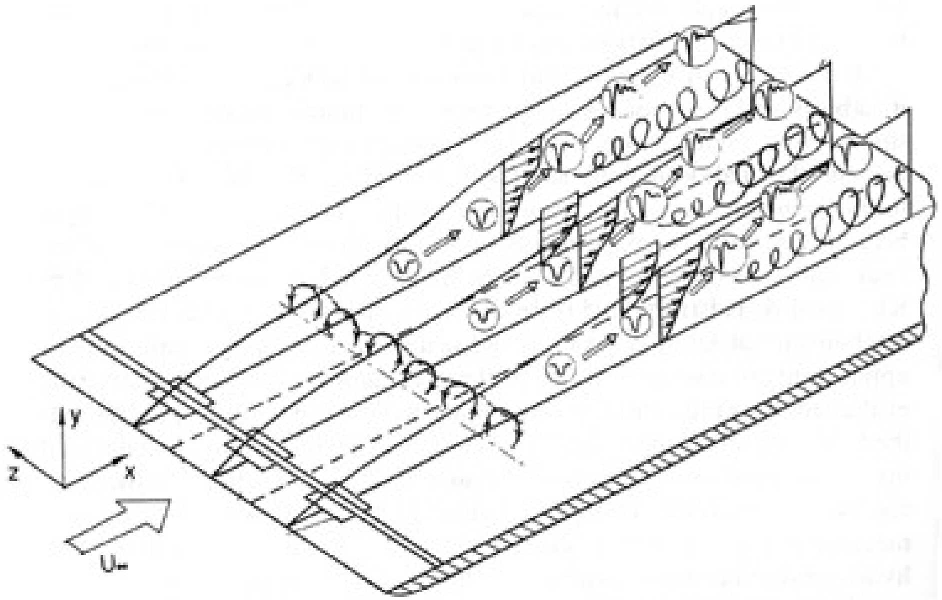


Streaks:

Subcritical Reynolds

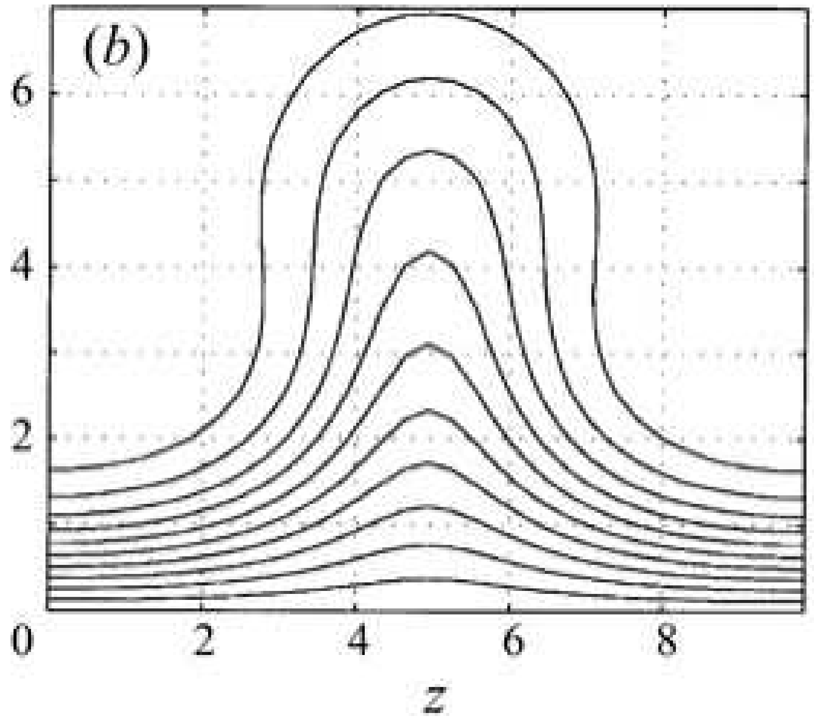
Large external disturbances

Transient growth



Secondary instability of streaks

Contours of streamwise velocity, yz plane:



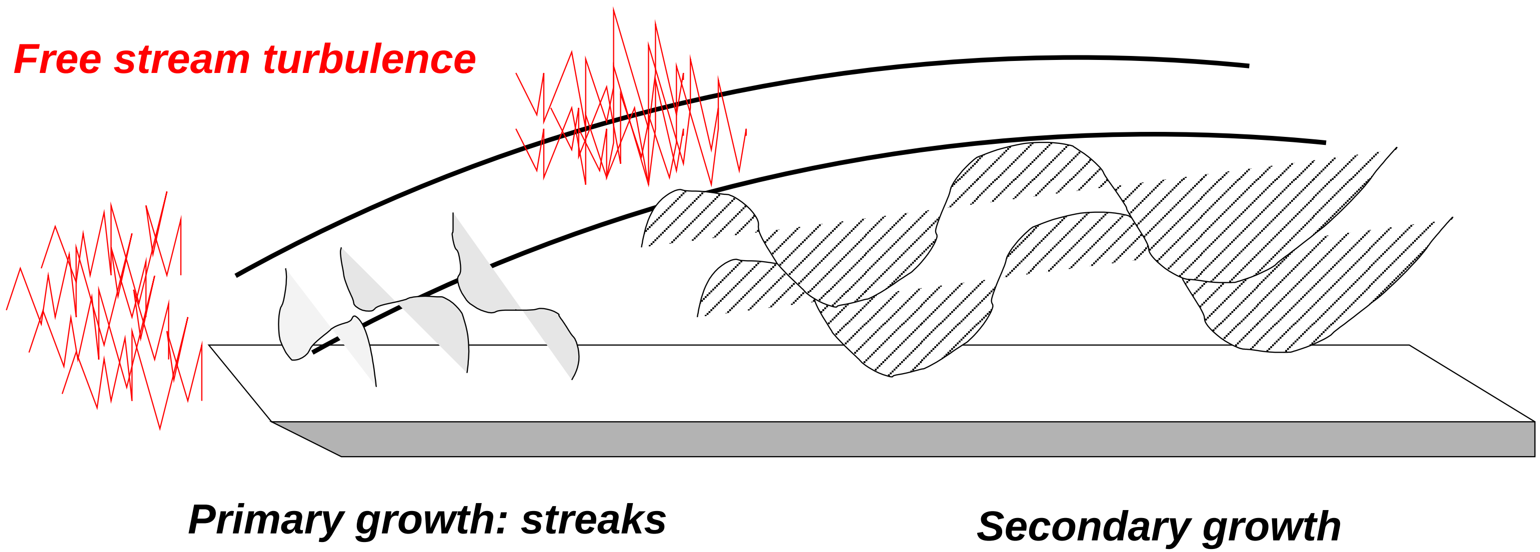
Streaks instability is at amplitude 26% of the free stream velocity

inflectional profile
$\rightarrow$ Inviscid instability

Maybe an other mechanism at lower amplitude?

(Luca Brandt, PhD thesis)

Secondary transient growth of the streaks

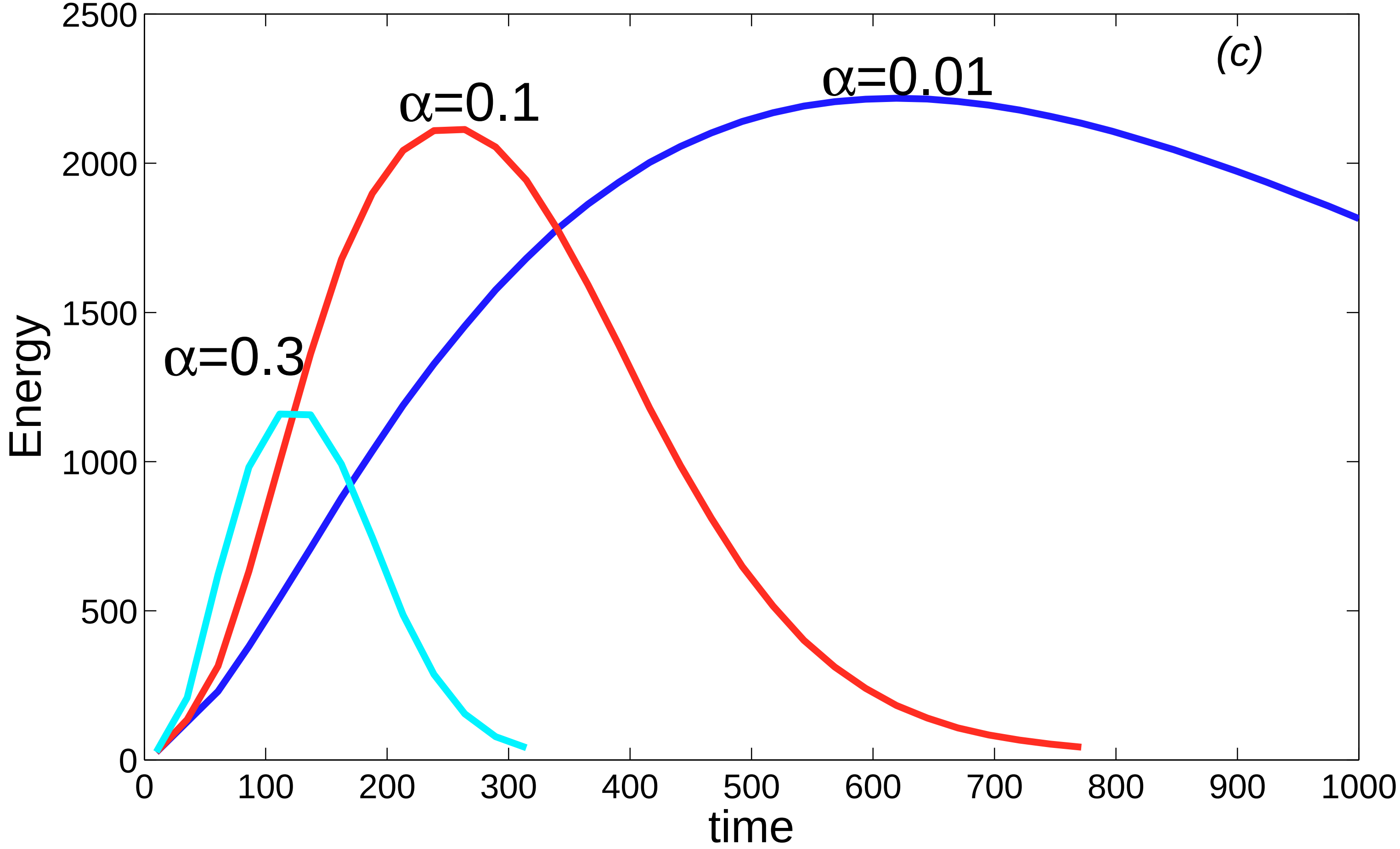


Assume the boundary layer is excited by the FST at all downstream position→ streaks is generated, then streaks is disturbed

Possibility of transient growth on top of the streaks: Schoppa & Hussain, JFM(2002).

Energy growth: sinuous perturbations

Transient growth: energy envelopes



Streak amplitude: 25% of free-stream velocity

Question

How likely are those initial excitation in a FST boundary layer?

Will this growth mechanism be active with realistic disturbances?

Statistical description

Farrell& Ioannou, POF(1993):Stochastic forcing of the linearized Navier–Stokes equations.

FST is erratic, instationnary, we describe it by its statistics$\longrightarrow$ two point correlation

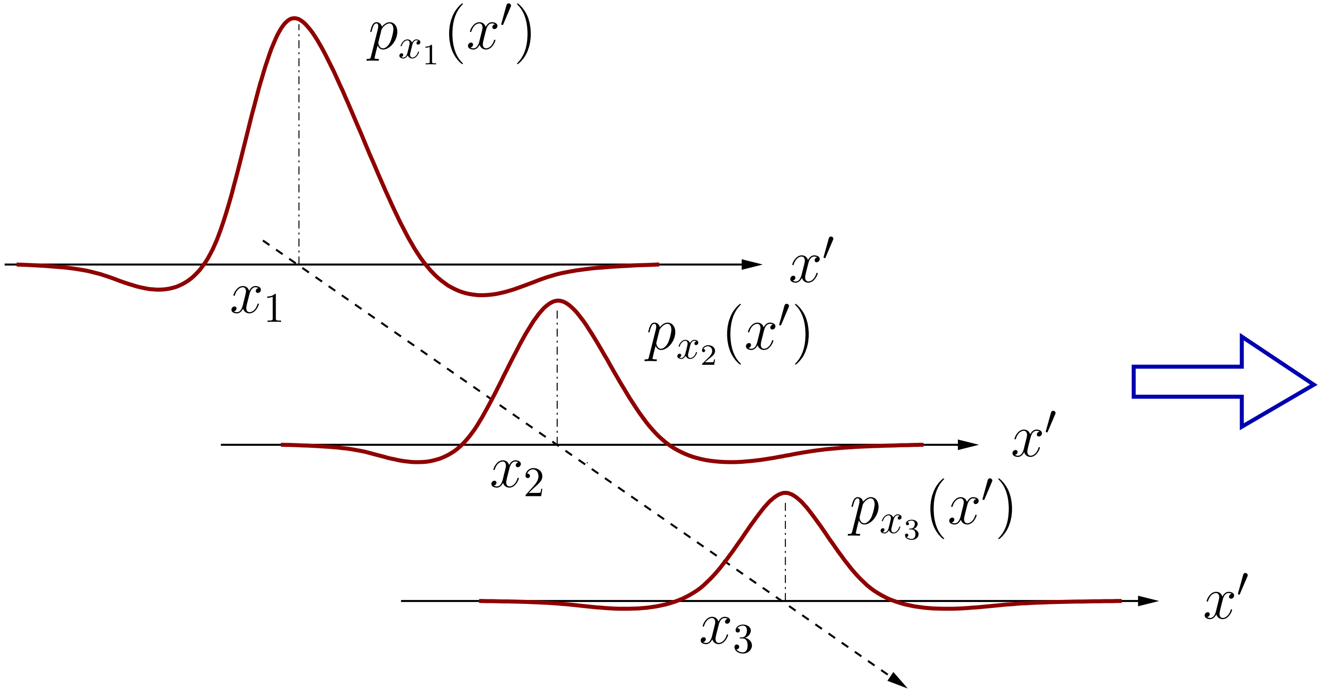
Average, or expectation operator E.

Two-point correlation: $P_u(x, x') = E[u(x)u(x')]$ of u at x and x'.

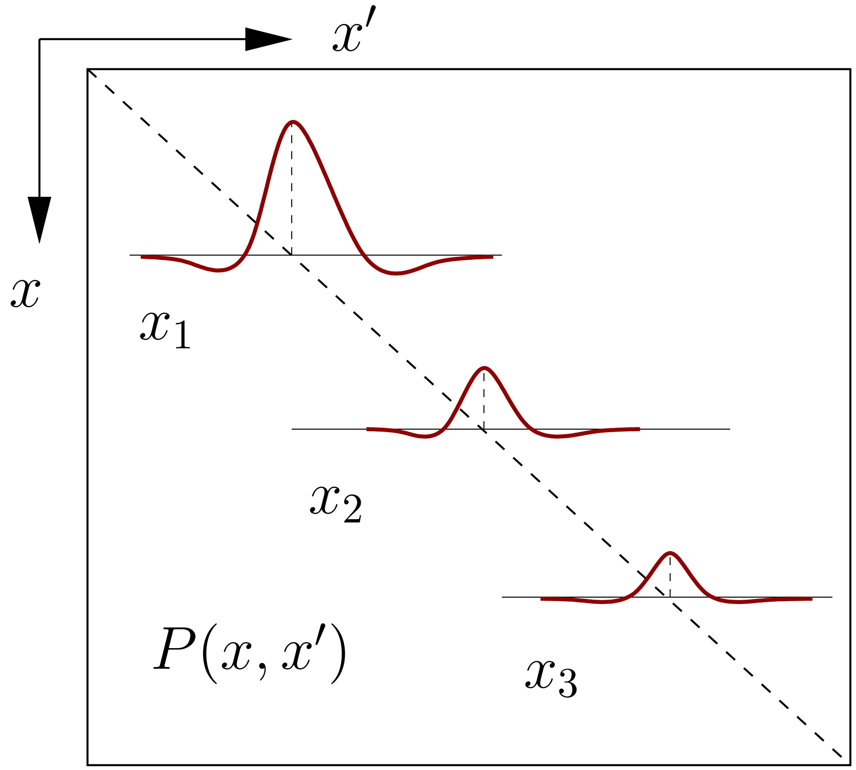
For 3D flow: $P(x, x', y, y', z, z', t, t')$.

Covariance matrix

Correlation is **covariance** normalized to unit variance.



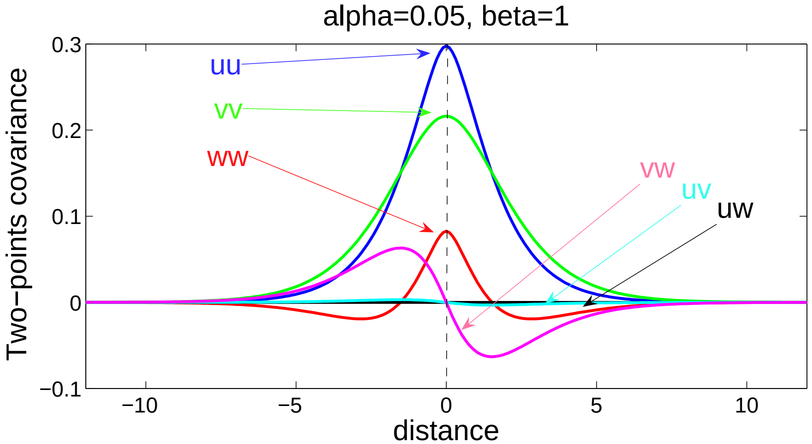
Two point correlation, varying in x

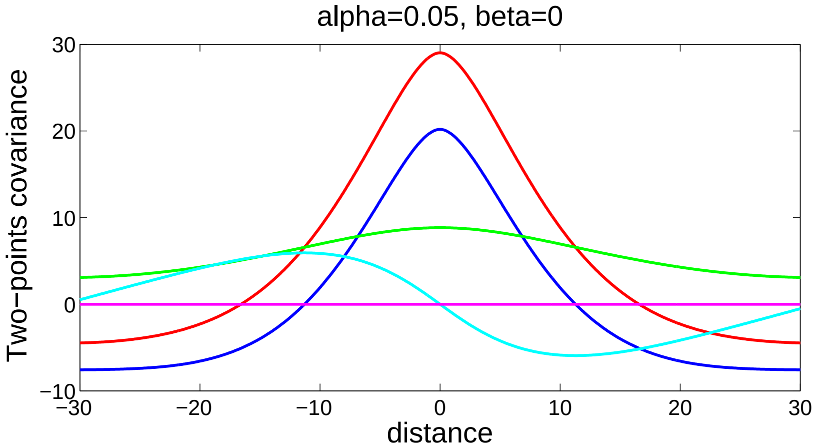


Correlation matrix

Two-point correlation of FST

Covariance in y:

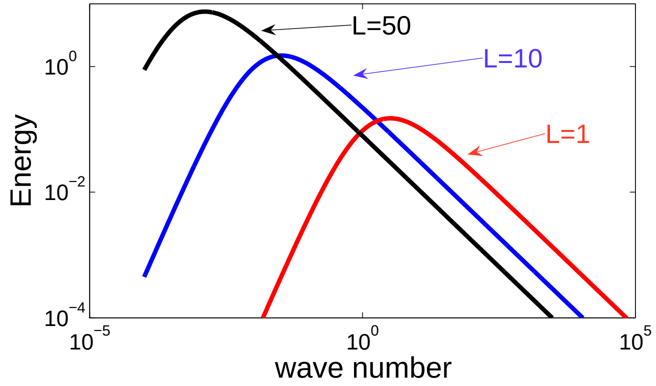




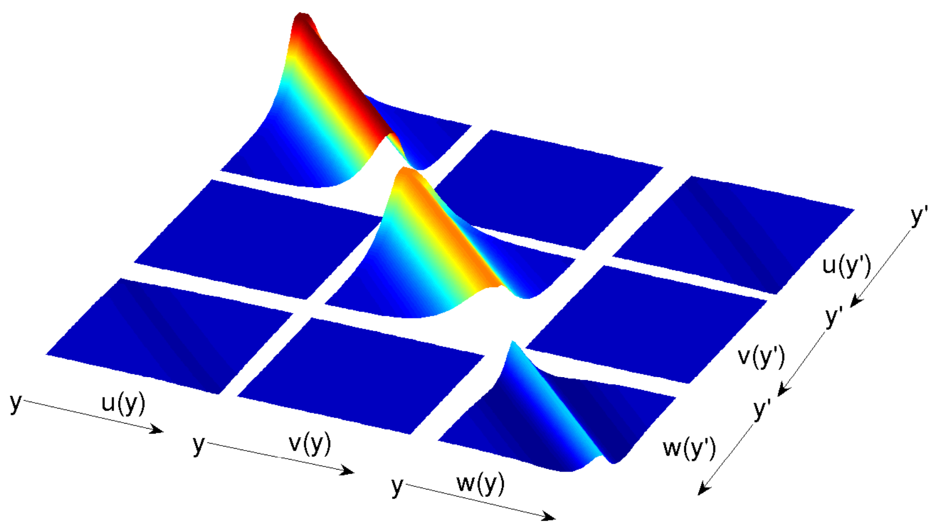
Isotropic turbulence: Von-Karman spectrum

Energy spectra:

$$E(k) = \frac{2}{3} kL \frac{a(kL)^4}{(b+(kL)^2)^{(17/6)}}$$



Covariance matrix:



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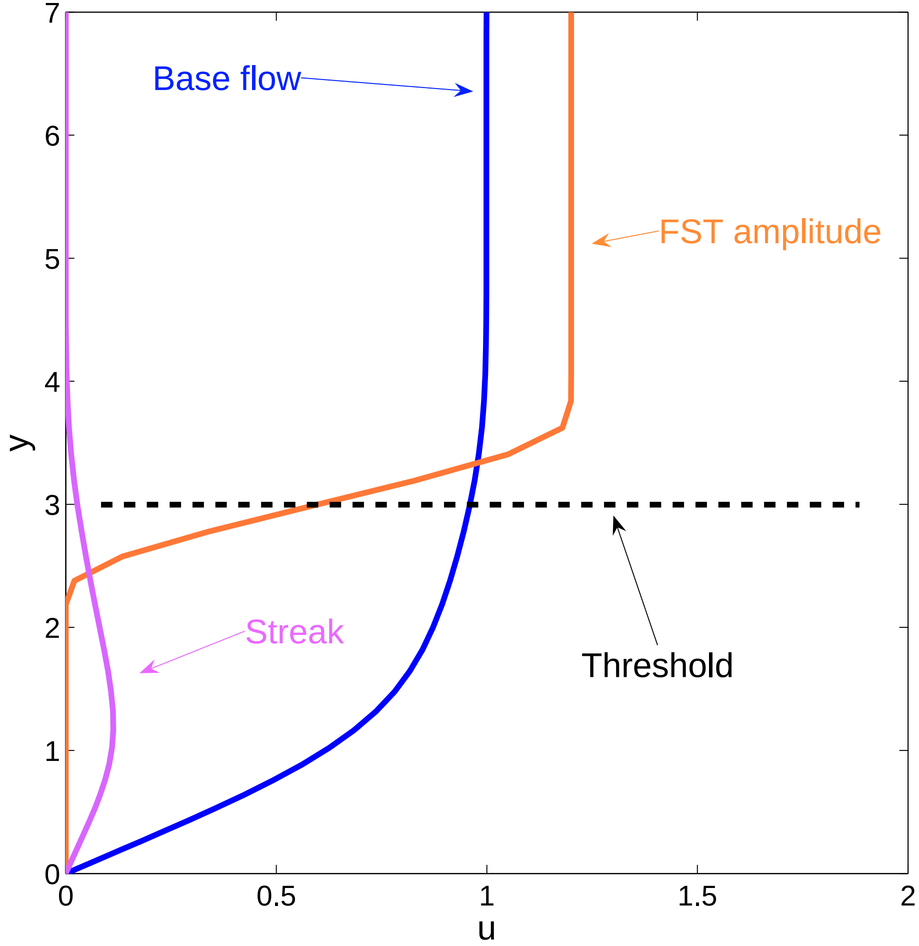
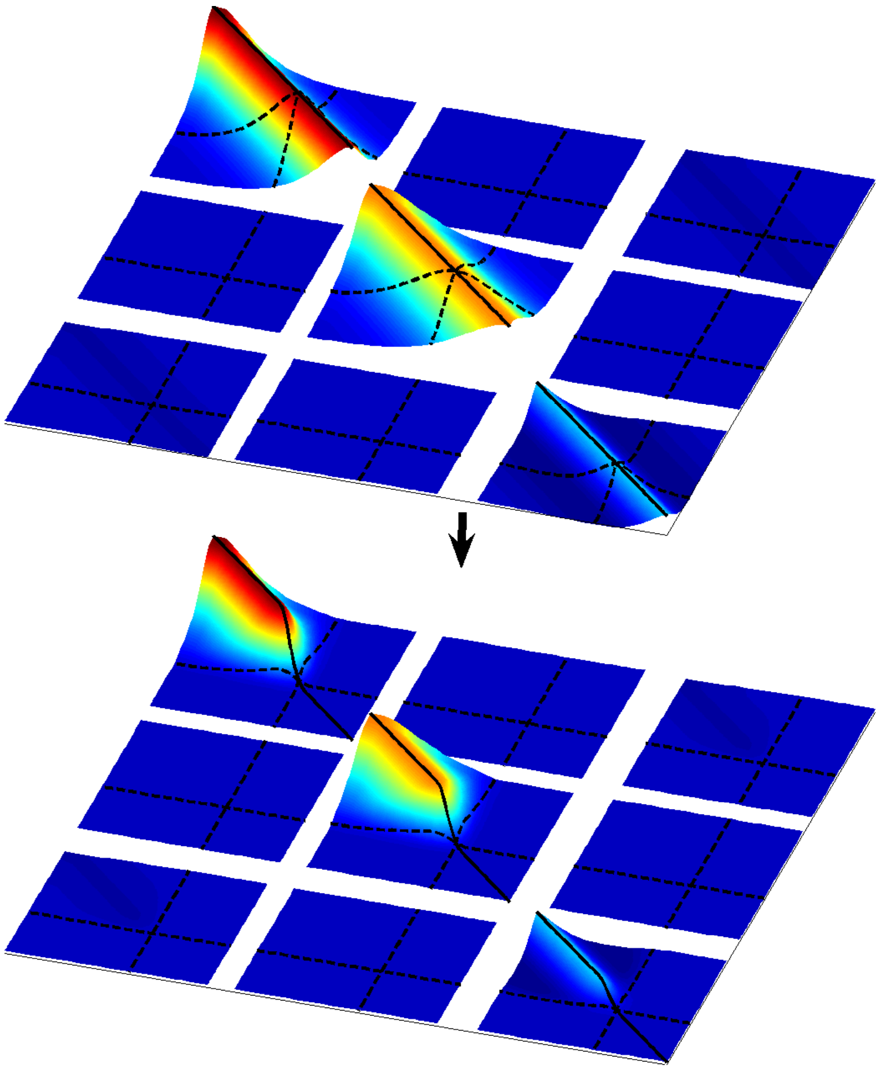
FST is only in the free-stream

Zero disturbances inside the boundary layer, Threshold at $y = 3\delta^\star$





Covariance matrix with ramping

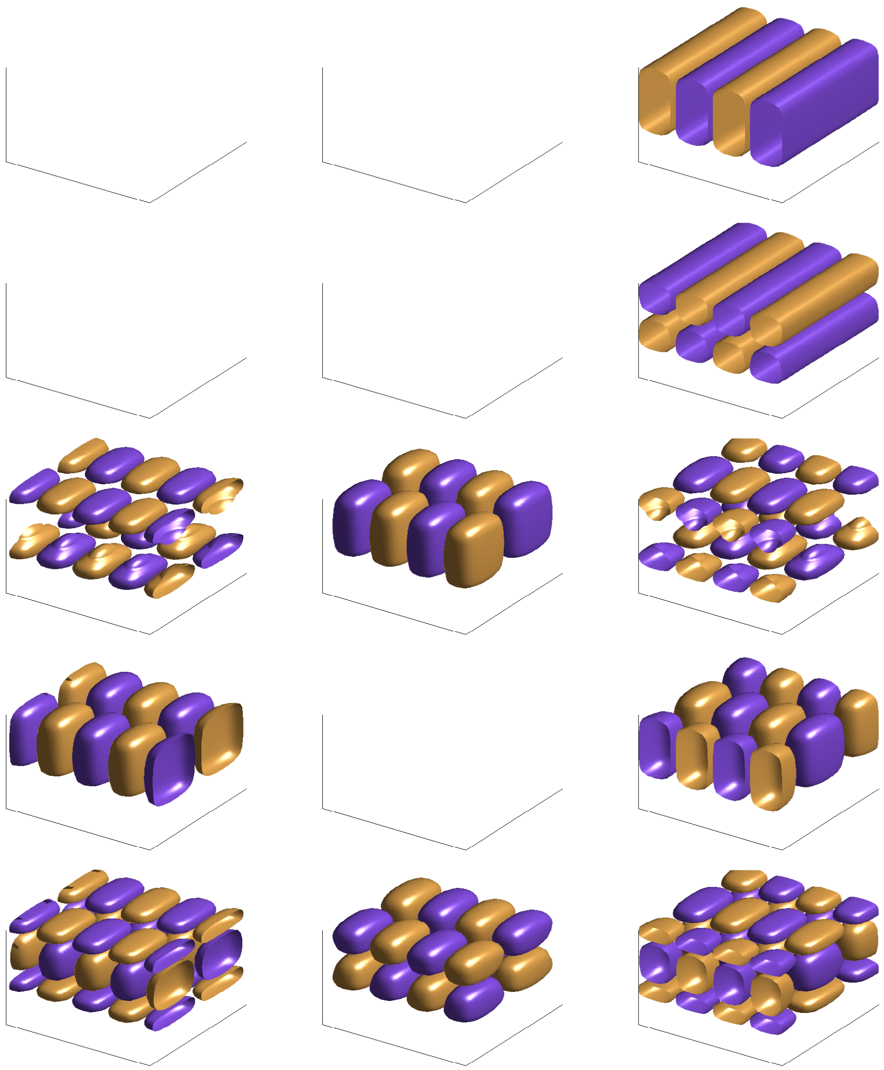
POD modes from the covariance matrix

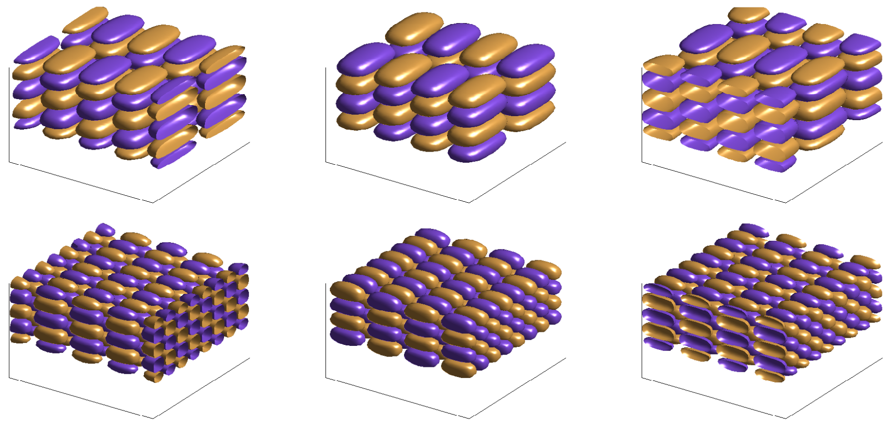
Two-point correlation data has many dimensions $\longrightarrow$ show coherent structures

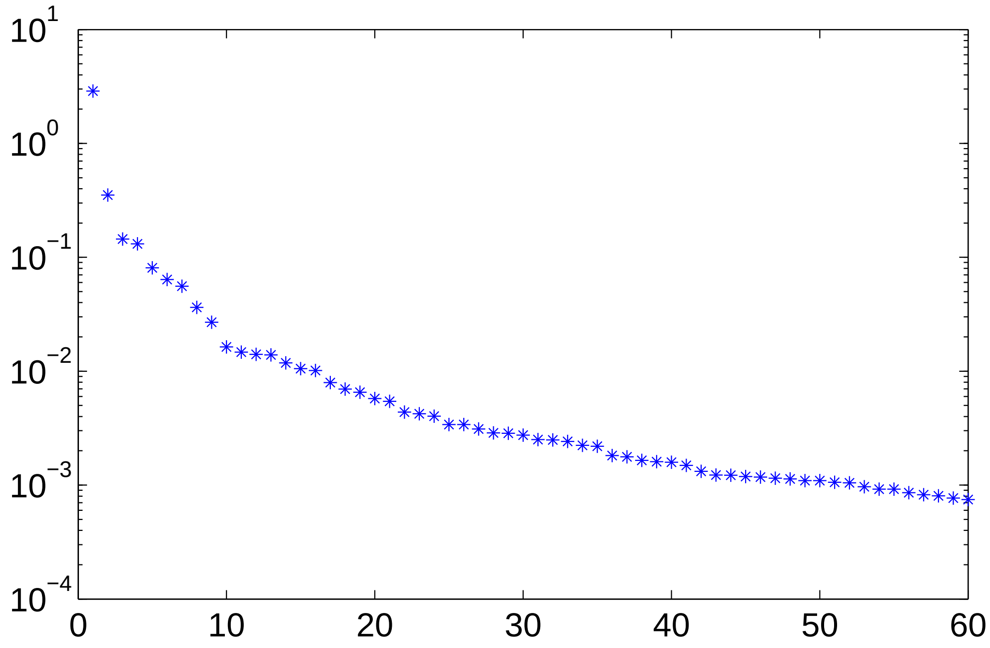
Energy content of the flow structures tells about flow coherence

POD modes are the eigenmodes of the covariance matrix

POD modes of the FST







Faster oscillation$\rightarrow$ more damped, many spatial frequencies are present

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Lyapunov equation

Explicit state solution:

$$\dot{q} = Aq + w \quad , q(0) = q_0, \quad \Rightarrow \quad q(t) = e^{At} q_0 + \int_{\tau=0}^{\infty} e^{A(t-\tau)} w(\tau) \mathrm{d}\tau$$

State covariance:

$$\underbrace{E[q(t)q(t)^H]}_{P(t,t)} = e^{At} \overbrace{E[q_0 q_0^H]}^{P_0} e^{A^H t} + \int_0^{\infty} \int_0^{\infty} e^{A(t-\tau)} \overbrace{E[w(\tau)w(\tau')^H]}^{W\delta(\tau-\tau')} e^{A^H(t-\tau')} \mathrm{d}\tau \mathrm{d}\tau'$$

$$= e^{At} P_0 e^{A^H t} + \int_0^{\infty} e^{A(t-\tau)} W e^{A^H(t-\tau)} \mathrm{d}\tau$$

Differentiating this convolution integral:

$$\frac{\partial P(t,t)}{\partial t} = A \left(\underbrace{e^{At} P_0 e^{A^H t} + \int_0^{\infty} e^{A(t-\tau)} W e^{A^H(t-\tau)} \mathrm{d}\tau}_{P(t,t)} \right) + \left(\underbrace{e^{At} P_0 e^{A^H t} + \int_0^{\infty} e^{A(t-\tau)} W e^{A^H(t-\tau)} \mathrm{d}\tau}_{P(t,t)} \right) A^H + \underbrace{e^{A0} W e^{A^H 0}}_{W}$$

Lyapunov equation

$$\frac{\partial P}{\partial t} = AP + PA^H + W, \quad P(0) = P_0$$

Initial condition problem

No stochastic forcing: $W = 0$

Stochastic initial condition

$\frac{\partial P}{\partial t} = AP + PA^H, \quad P(0,0) = P_0$

$\rightarrow$ **Covariance varies in time**

Forced problem

With stochastic forcing: $W \neq 0$

Long time after initial condition

$0 = AP + PA^H + W$

$\rightarrow$ **Reached statistical steady state**

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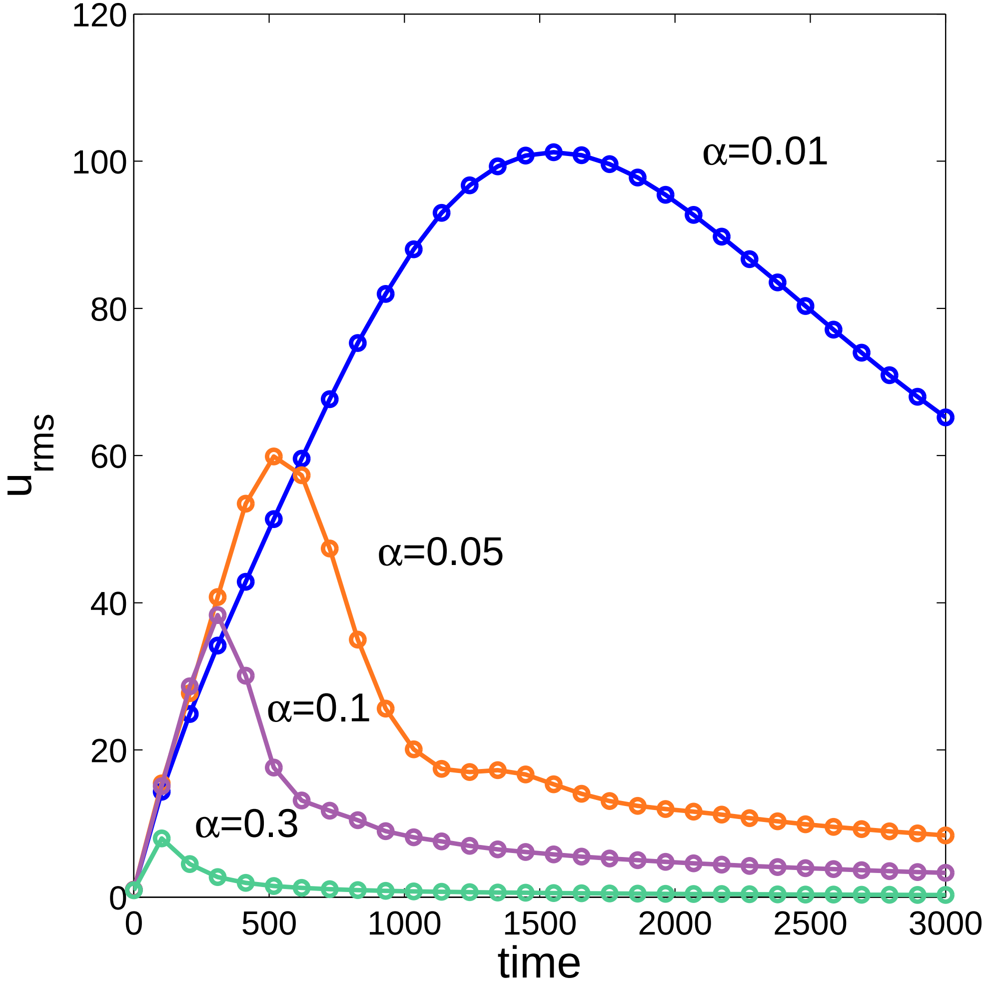
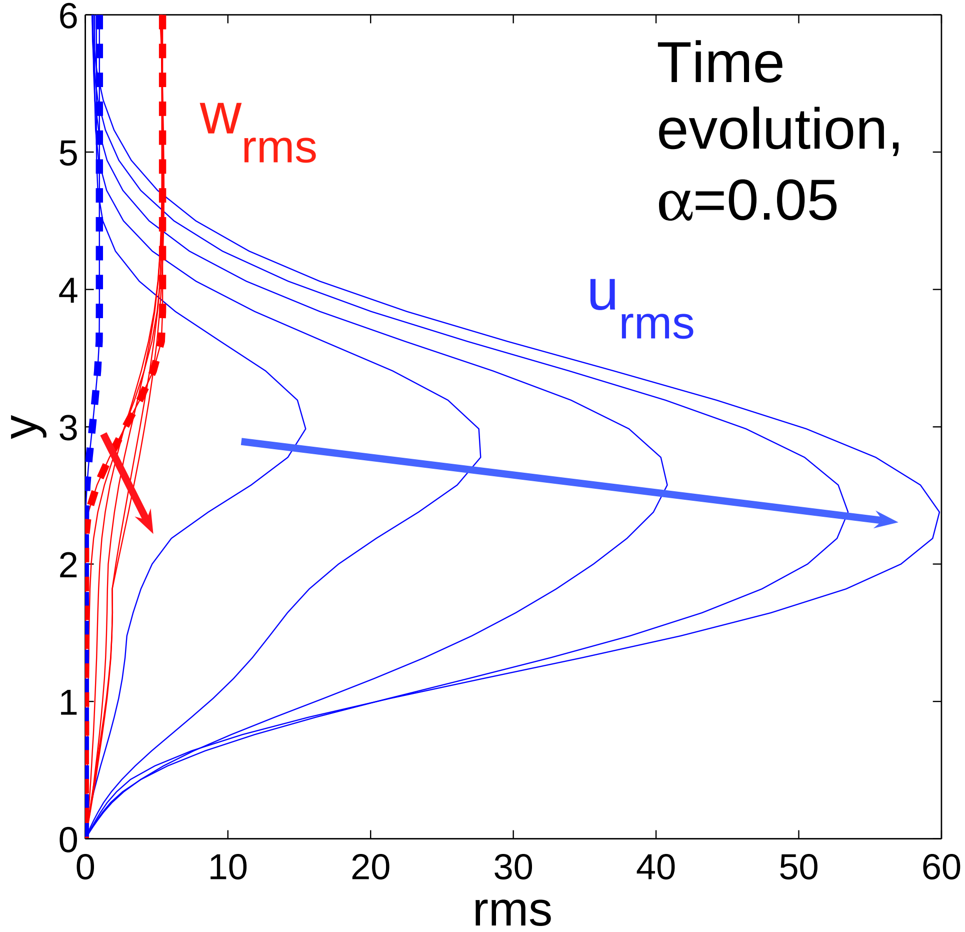
Streaky flow excited by FST: Stochastic initial value problem

Time evolution of u_{rms}

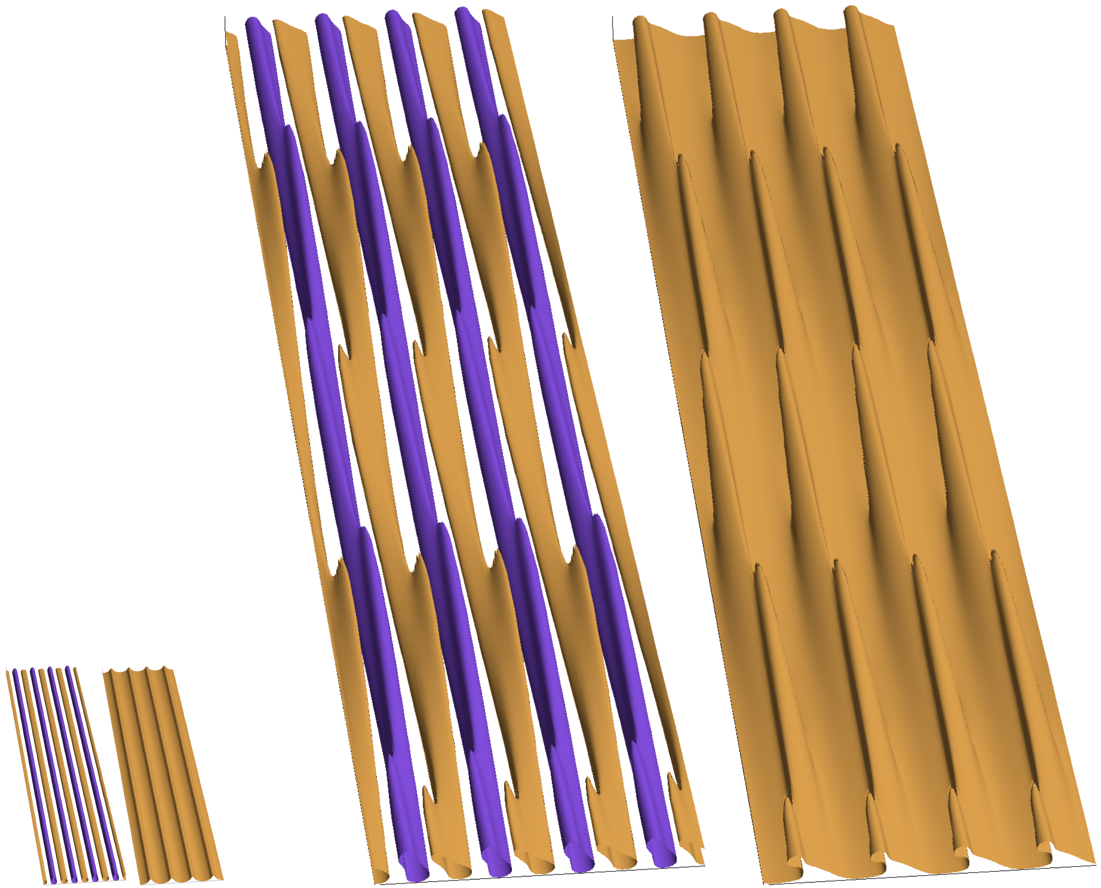


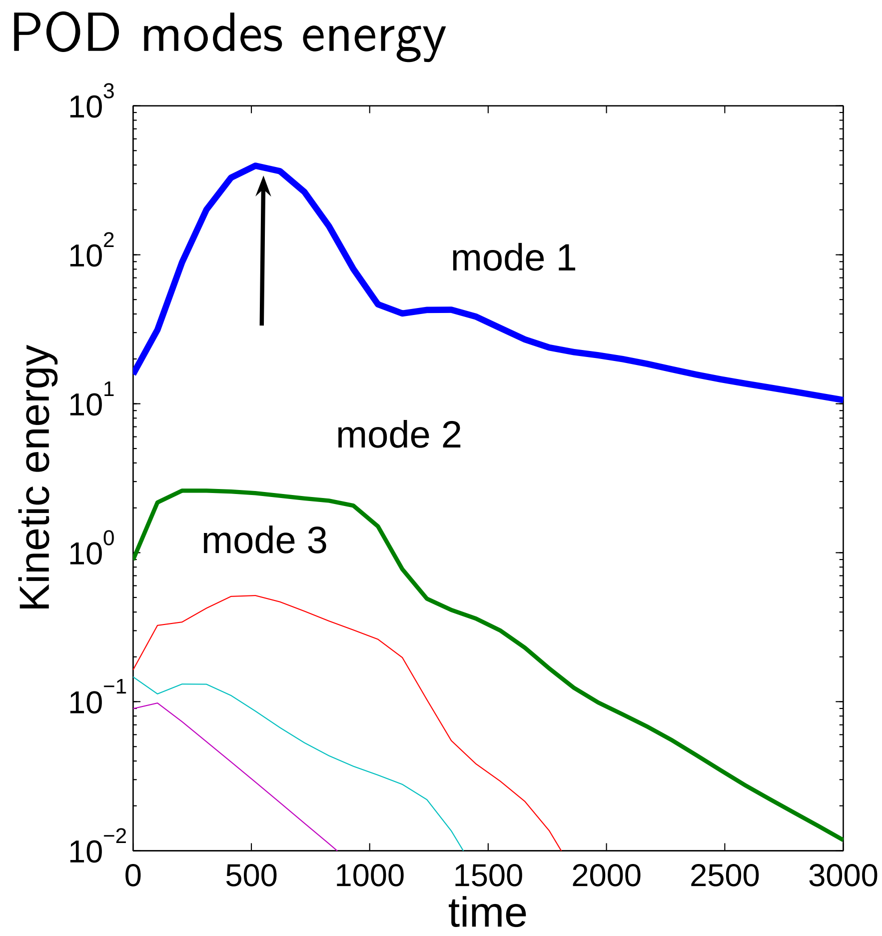
rms profile:



The FST is the flow initial condition

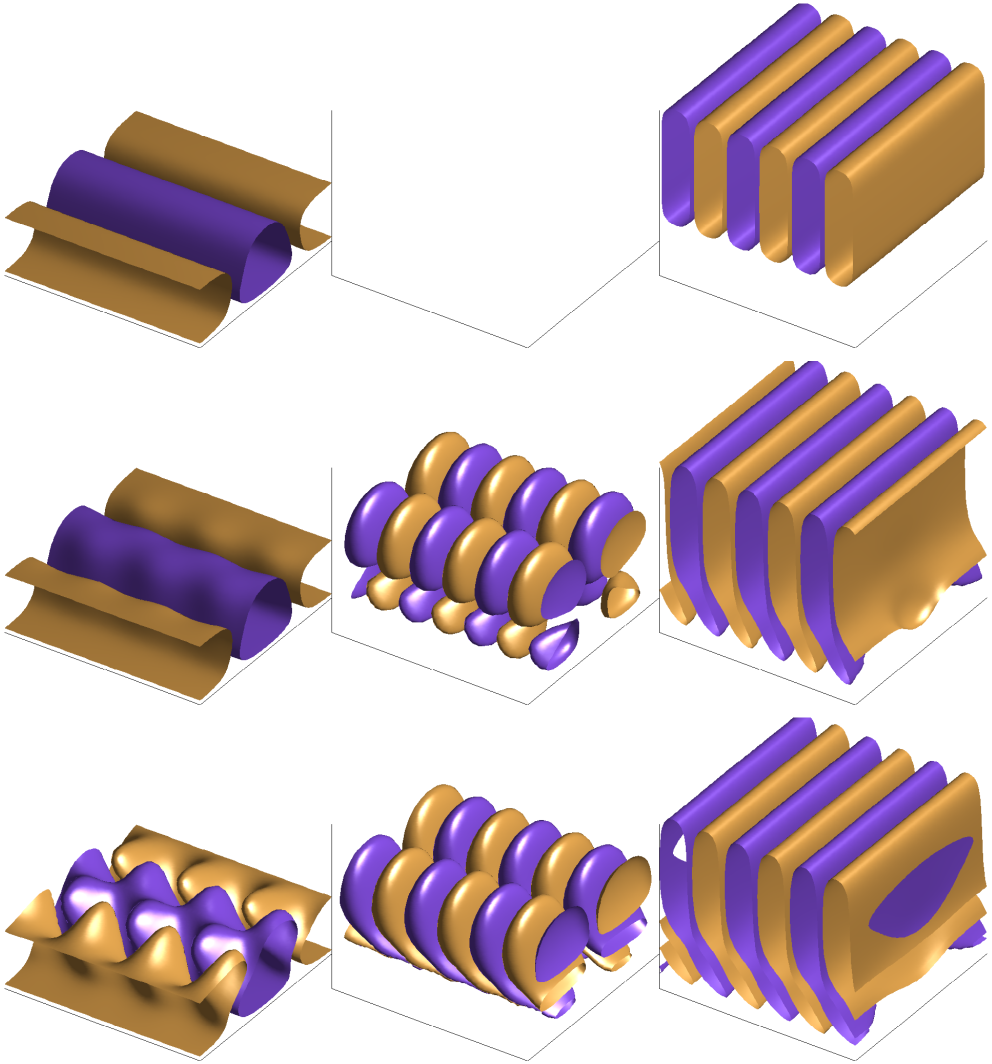
POD modes, $\alpha = 0.05$



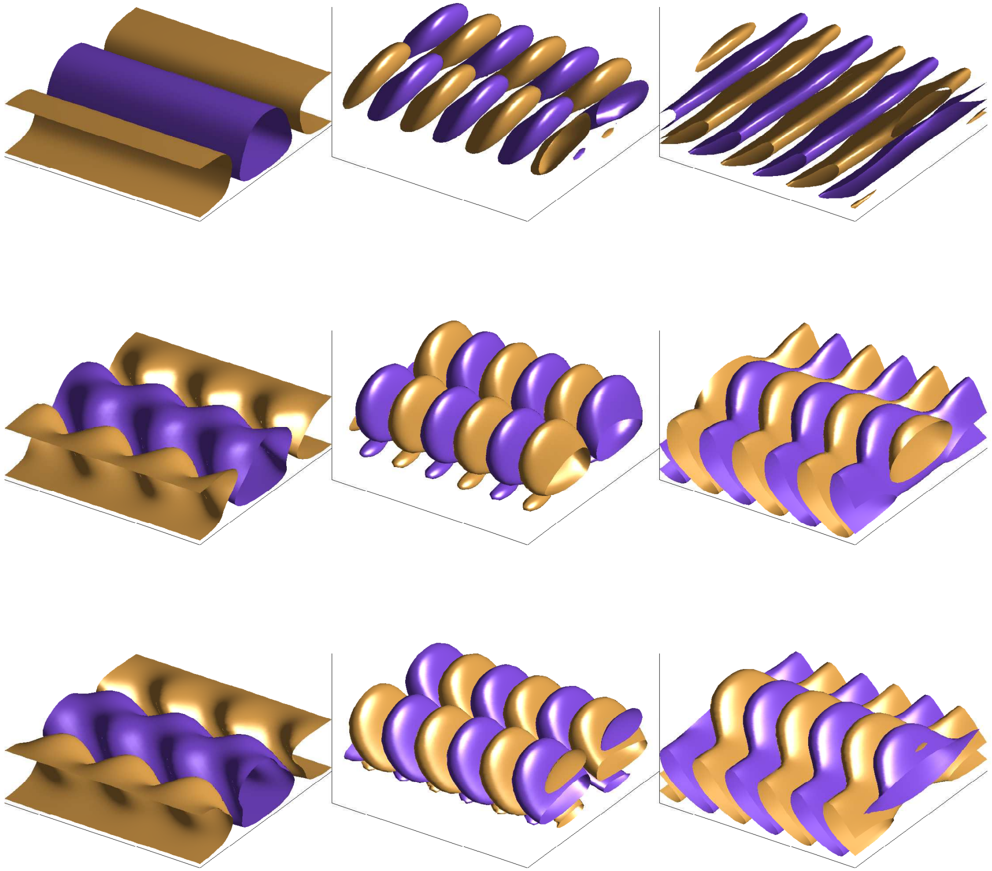


Flow structure with maximum energy

Comparison FST/optimal

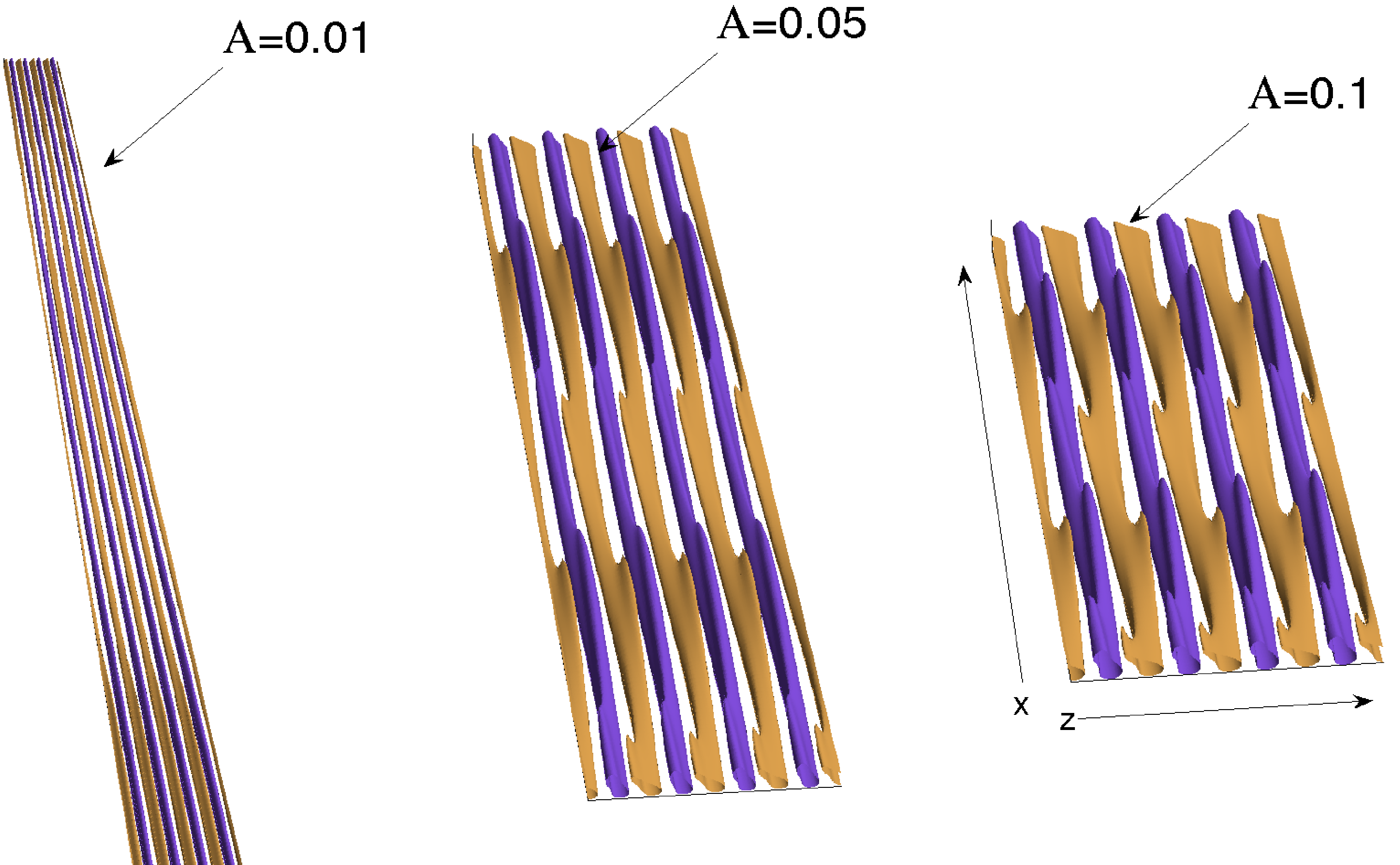


FST initial condition: u,v,w

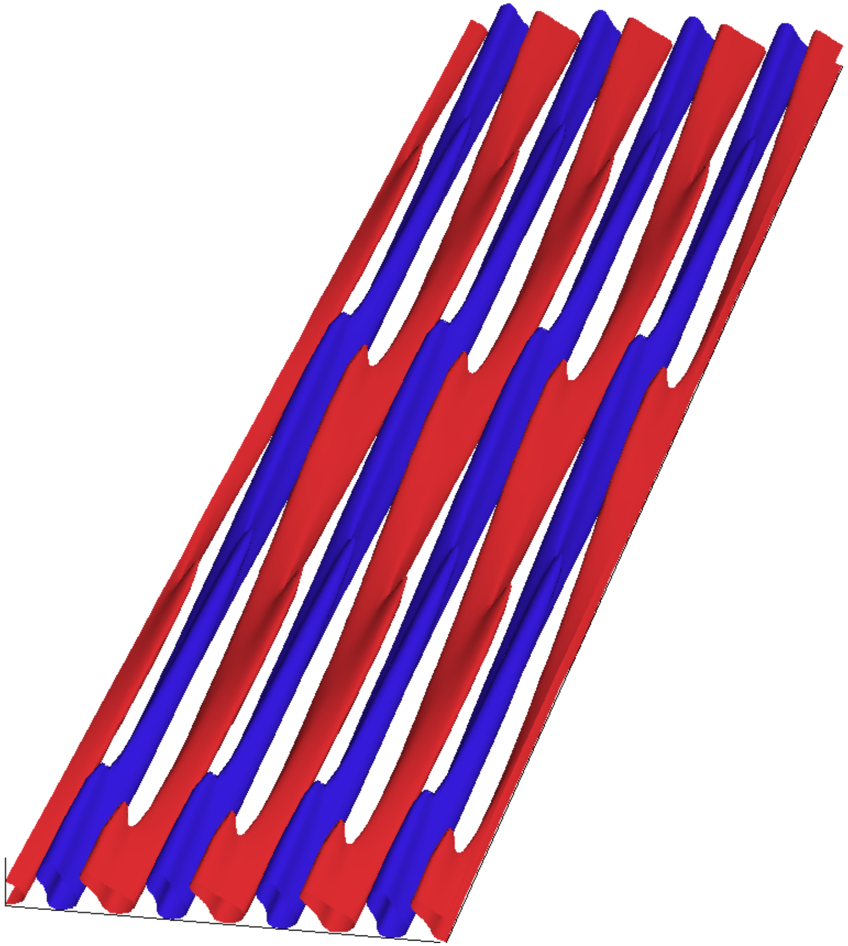


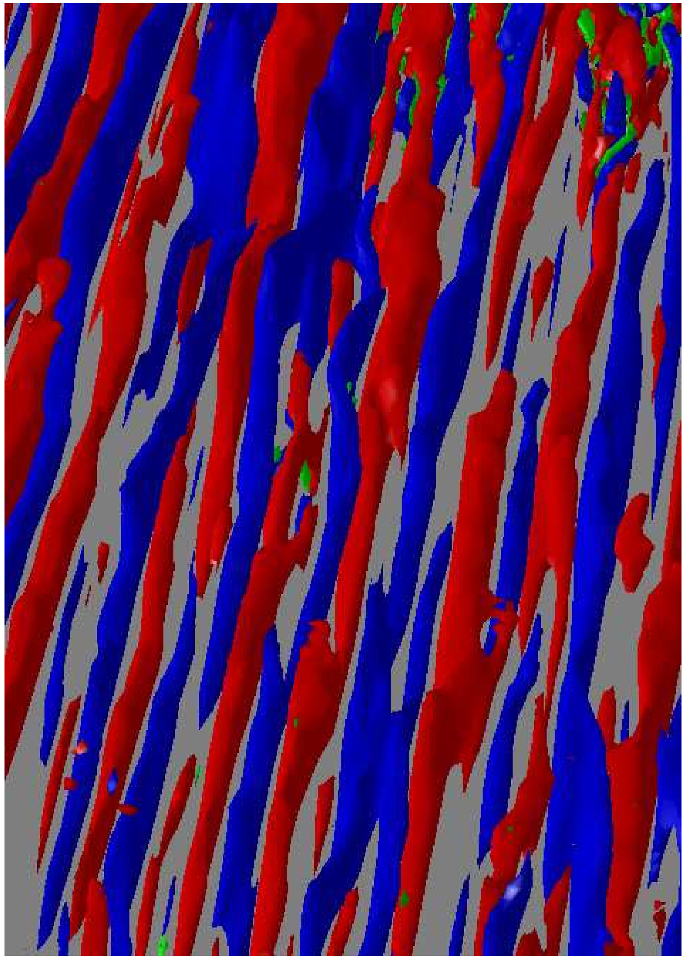
Worst case initial condition: u,v,w

Flow structures at time of max energy



Comparison of flow structures: Streamwise velocity





(LES of bypass transition : Philipp Schlatter)

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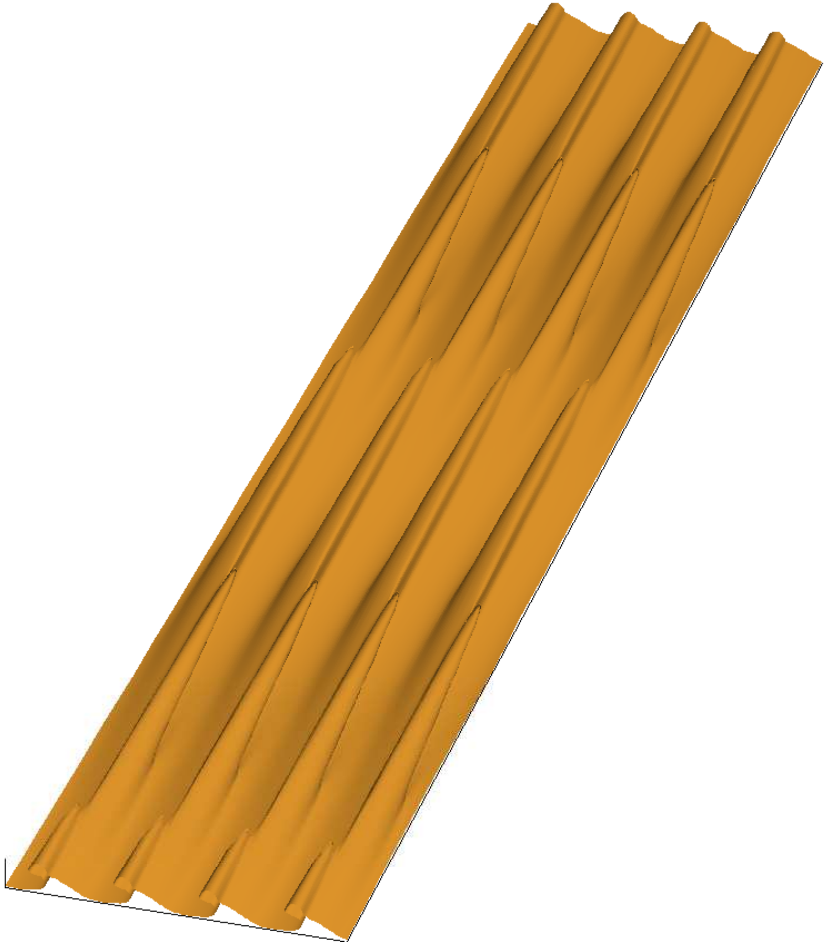
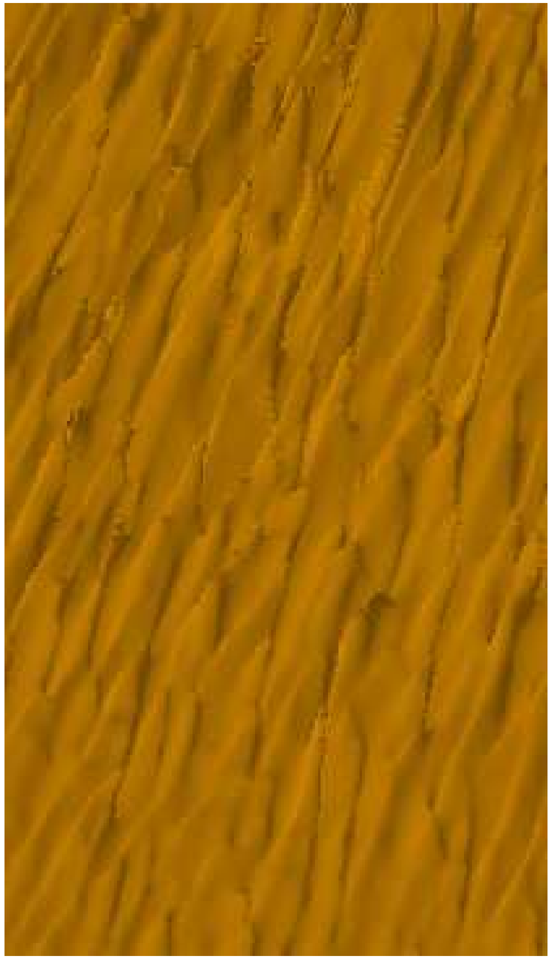
Comparison of flow structures: Streamwise shear

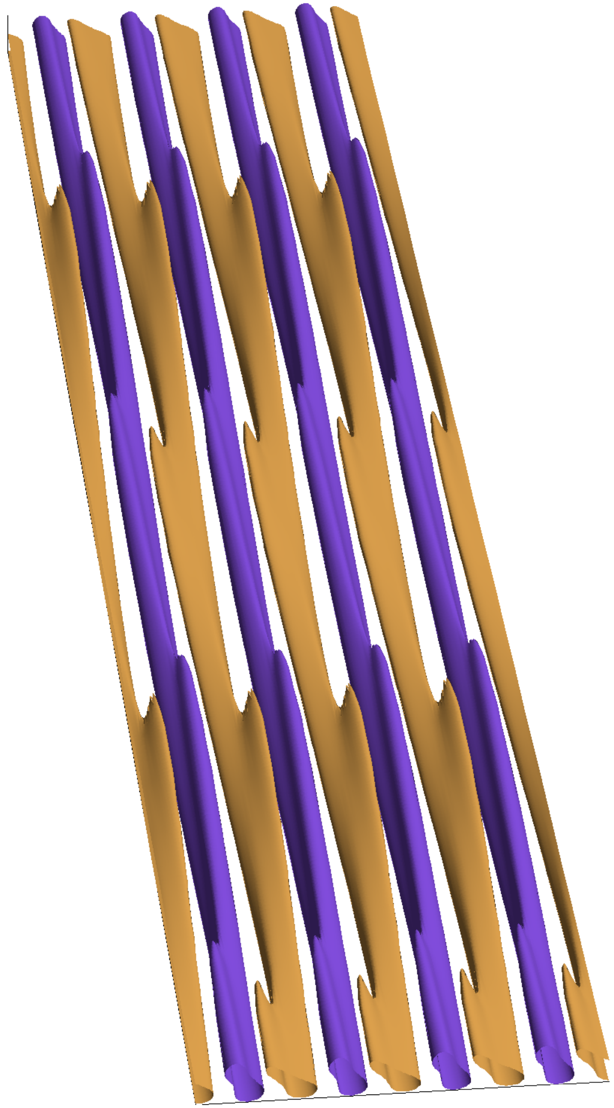




(LES of bypass transition : Philipp Schlatter)

Conclusions

1. Possibility of energy growth of $\mathcal{O}(1000)$ for subcritical streak
2. FST description using two-point correlation
3. Computation of state two-points correlation using Lyapunov equation
4. Response to FST involves transient growth mechanism
5. Secondary transient growth explains observed streak structure



→ **Bypass transition involves TG mechanism twice**

Does this explain streak breakdown?

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Computation of the transient growth

Power Iteration:

- Consider initial guess $q^0(0)$
- March forward in time with dynamic equation : $q^0(\tau) = \mathcal{H}_\tau q^0(0)$
- March backward in time with adjoint equation: $q^1(0) = \mathcal{H}_\tau^+ q^0(\tau)$
- Renormalize energy

Each of these power iteration magnifies
the component of the initial guess on the optimal initial condition.

Convergence in less than 20 iterations $\rightarrow$ well separated eigenvalues

Numerical solution of Lyapunov equation

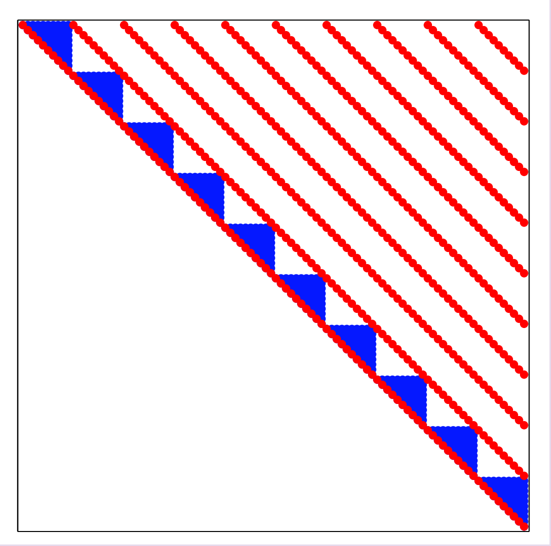
Solve: $AX + XA^H + W = 0$

1. Schur decomposition $A = UA'U^H$, $\rightarrow$ A' upper diagonal, U orthogonal.
2. Resulting equation $A' \overbrace{U^H X U}^{X'} + \overbrace{U^H X U}^{X'} A'^H + \overbrace{U^H W U}^{W'} = 0$
3. Use Kronecker product $\otimes$

$$A \otimes B \triangleq \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{pmatrix}$$

$$\vec{(}A'X' + X'A'^H + W') = 0$$

$$= \underbrace{(I \otimes A' + \overline{A}' \otimes I)}_{\mathcal{F}} \vec{(}X') + \vec{(}W')$$



$\mathcal{F}$ has upper diagonal structure

4. Solve by backward substitution

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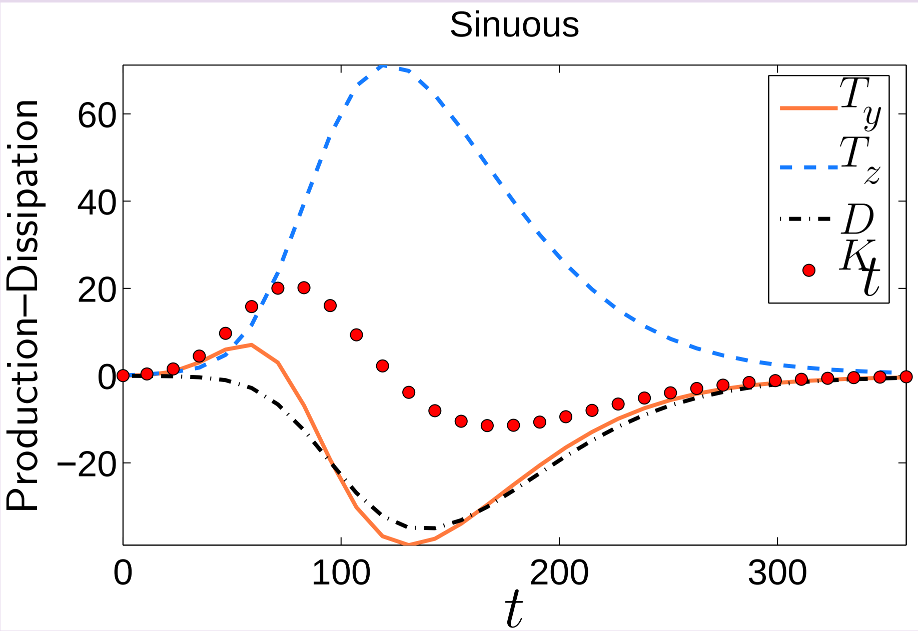
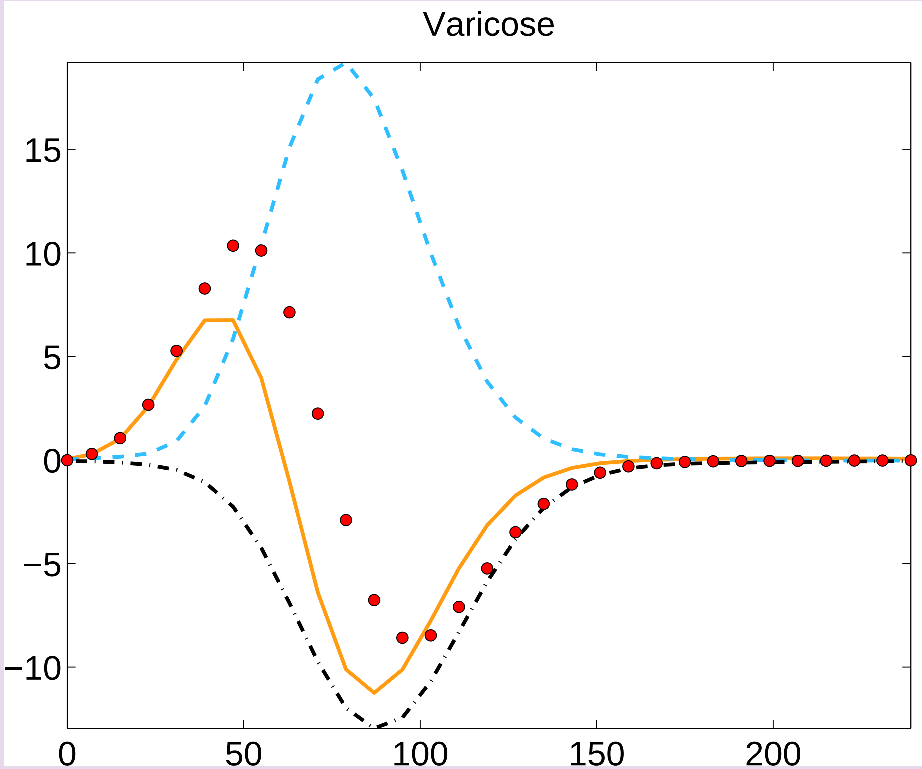
Energy balance
in streak secondary transient growth

$$K_t = \int (\underbrace{-uv\,U_y}_{T_y} \underbrace{-uw\,U_z}_{T_z} \underbrace{-\boldsymbol{\omega}\cdot\boldsymbol{\omega}/\mathrm{Re}}_{D})\, dy\, dz\, dx,$$

- K_t: time variation of kinetic energy
- T_y: production due to interaction with wall normal mean shear
- T_z: production due to interaction with spanwise mean shear
- D: dissipation due to viscosity

Production and dissipation
in streak secondary transient growth





- Spanwise shear always contributes to energy growth
- Wall-normal shear gives then takes: Orr mechanism related to structure tilting

Disturbance can gain energy

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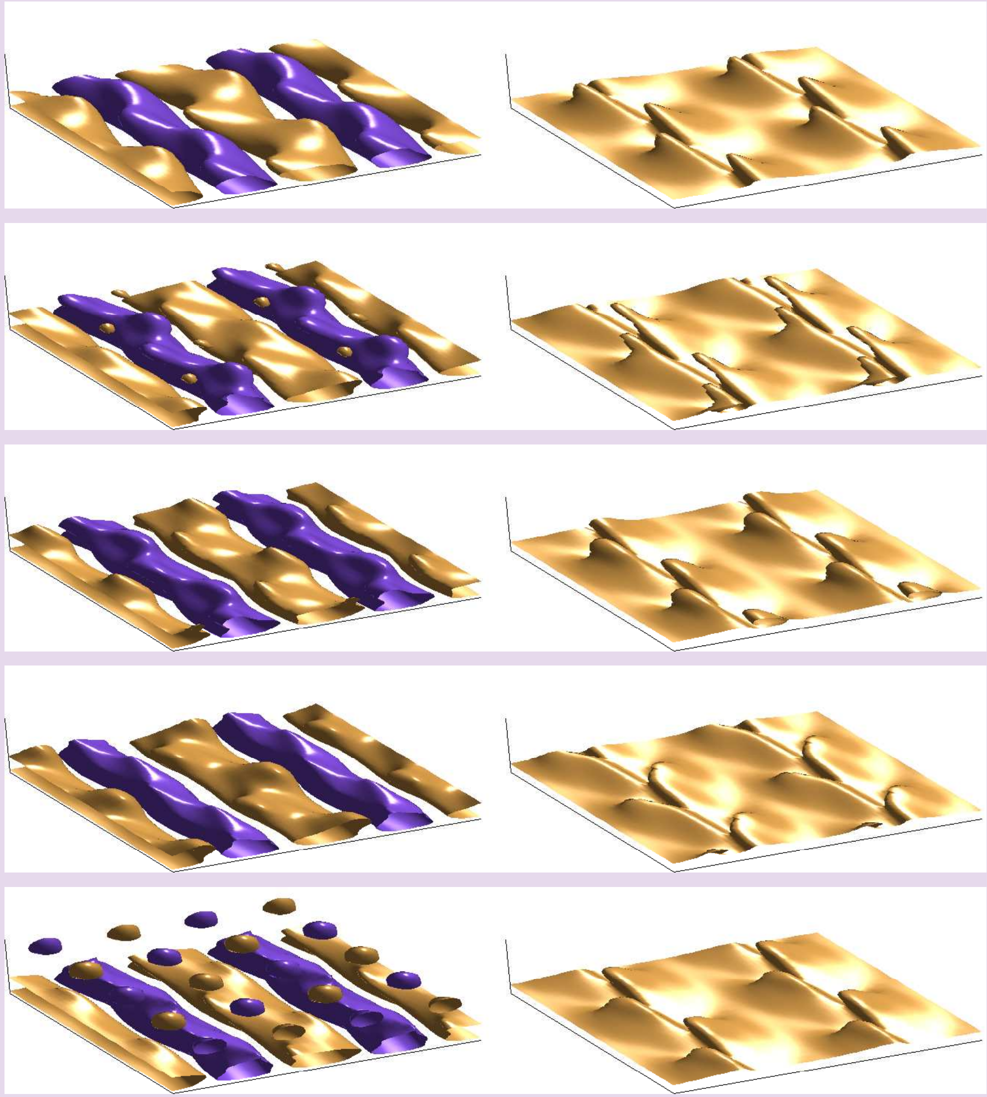
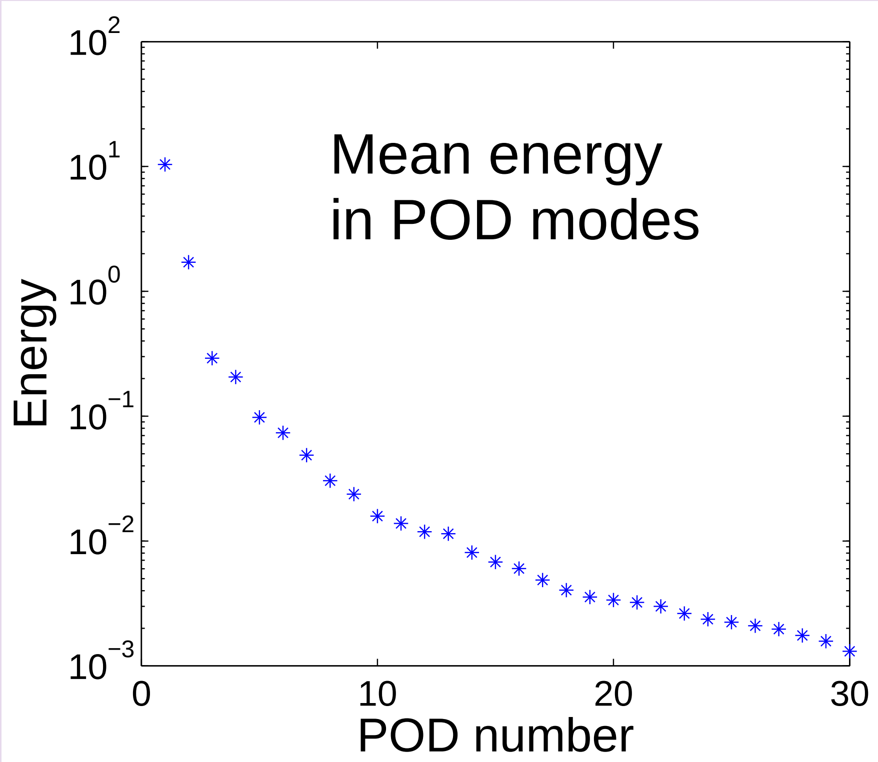
Streaky flow excited by FST: Forced problem



The flow is constantly excited by FST



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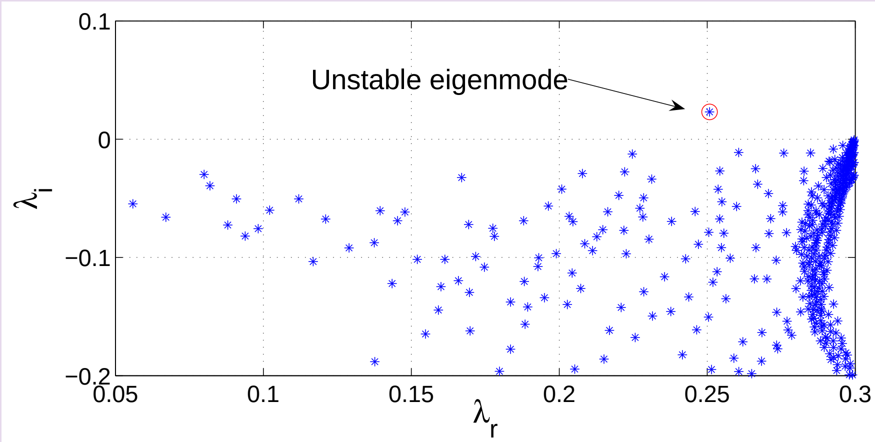
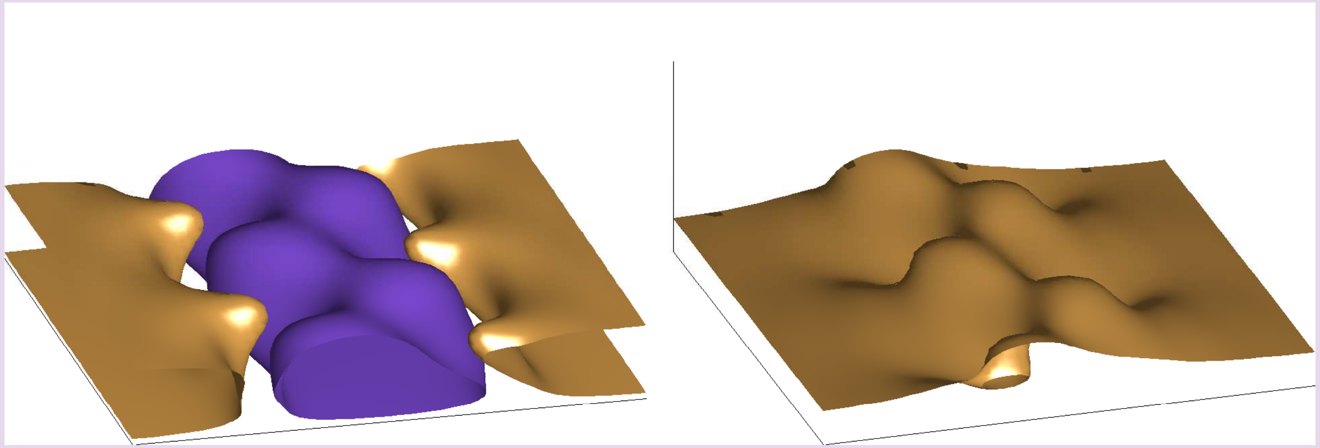
Unstable sinuous eigenmode

For large streak amplitude $\alpha = 0.3$ eigenvalues:



Corresponding eigenfunction:



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Transient growth analysis

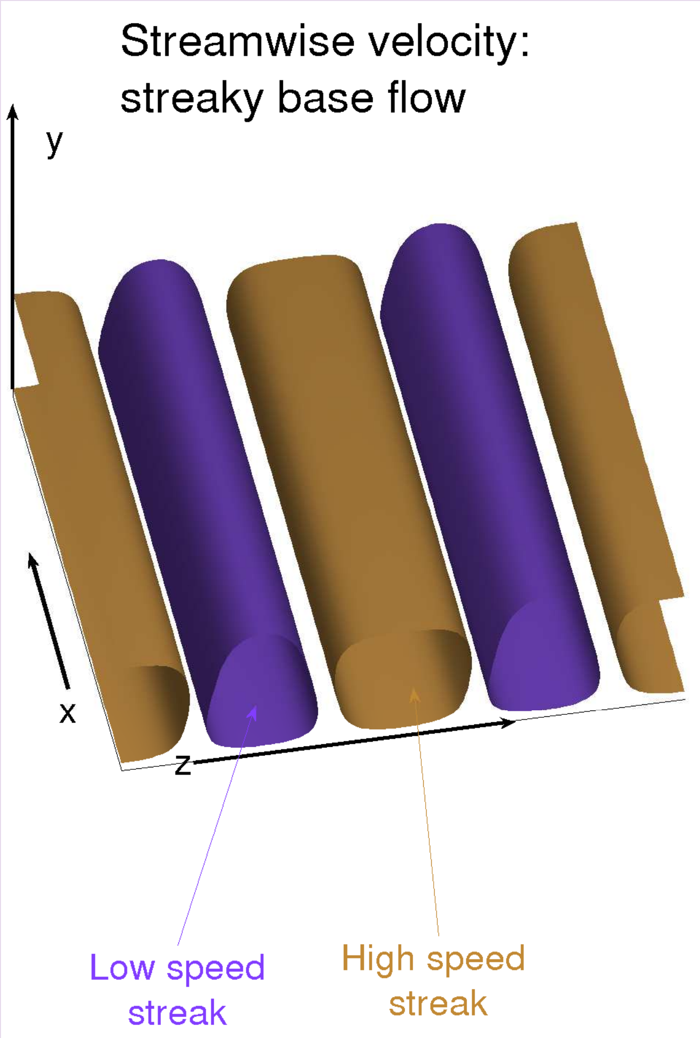
To build the dynamical operator:

- Use PSE to optimize disturbance
- Input forcing in DNS
- Extract fully saturated streamwise velocity profile $U(y, z)$
- Apply linear stability analysis, using Floquet

Build matrix $A \rightarrow$

eigenvalues for asymptotic stability

singular values for transient growth



Stability equations

Perturbation (v, η) on the base flow $U(y, z)$

Wavelike behaviour in the streamwise direction:

$$[v, \eta] = [\widehat{v}(y, z, t), \widehat{\eta}(y, z, t)] \; e^{i\alpha x} + c.c.$$

Derivation similar to the Orr–Sommerfeld/Squire equation:

$$\begin{cases} \Delta v_t + U \Delta v_x + U_{zz} v_x + 2U_z v_{xz} - U_{yy} v_x - 2U_z w_{xy} - 2U_{yz} w_x = \frac{1}{Re} \Delta\Delta v, \\ \eta_t + U \eta_x - U_z v_y + U_{yz} v + U_y v_z + U_{zz} w = \frac{1}{Re} \Delta \eta. \\ (\text{with} \quad w_{xx} + w_{zz} = -\eta_x - v_{yz}) \end{cases}$$

+ Floquet analysis: base flow and disturbance are periodic in spanwise direction. Look only at fundamental modes

Chebyshev discretization in wall-normal direction

Computation of the transient growth

- Dynamic system with initial condition:

$$\dot{q} = Aq, \quad q(0) = q_0$$

- Input-output operator $\mathcal{H}_\tau$:

$$q(t) = \mathcal{H}_\tau(q_0)$$

- Maximum possible growth:

$$G(\tau) = \max_q \frac{||\mathcal{H}_\tau q||_E}{||q||_E} = \max_q \frac{(\mathcal{H}_\tau q, \mathcal{H}_\tau q)}{(q, q)} \triangleq \max_q \frac{(q, \mathcal{H}_\tau^+ \mathcal{H}_\tau q)}{(q, q)},$$

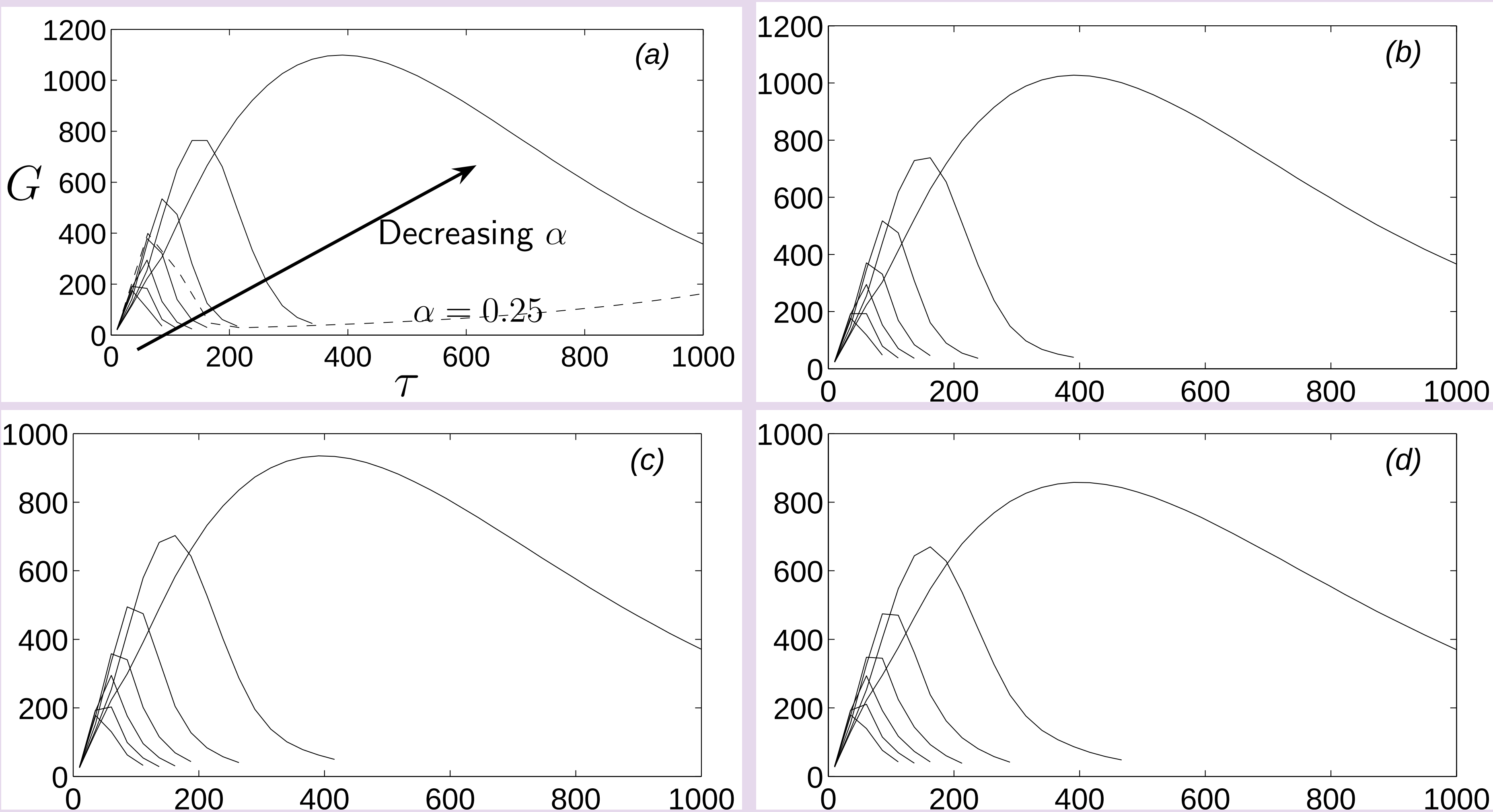
- with adjoint operator:

$$(\mathcal{H} q_1, q_2) = (q_1, \mathcal{H}^+ q_2), \quad \forall q_1, q_2$$

Max $G(\tau)$ is the largest eigenvalue of operator $\mathcal{H}_\tau^+ \mathcal{H}_\tau$

Varicose energy evolution

For 4 different streak amplitudes: energy envelope for several α.



α =0.01, 0.1, 0.2, ..., 0.6

(a),(b),(c),(d): Amplitudes: A=0.14, 0.2, 0.25, 0.29.

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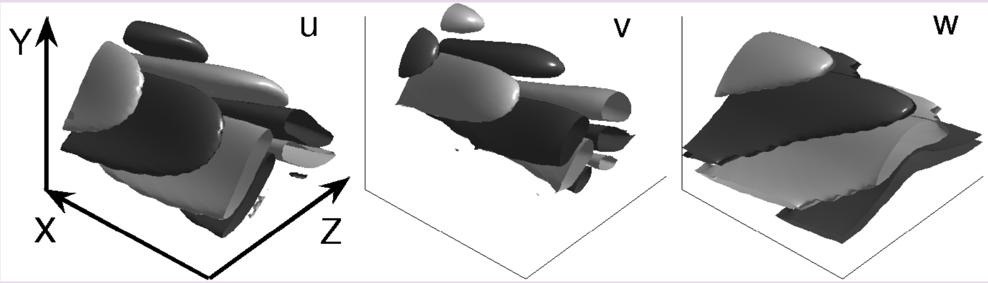
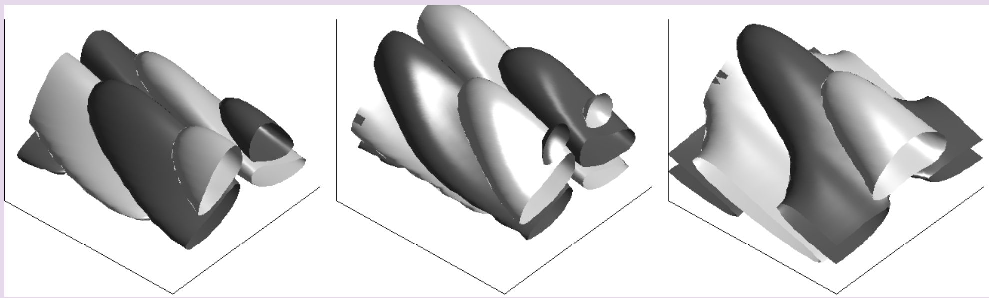
Flow structures for streak transient growth

Sinuous:

Optimal disturbance:

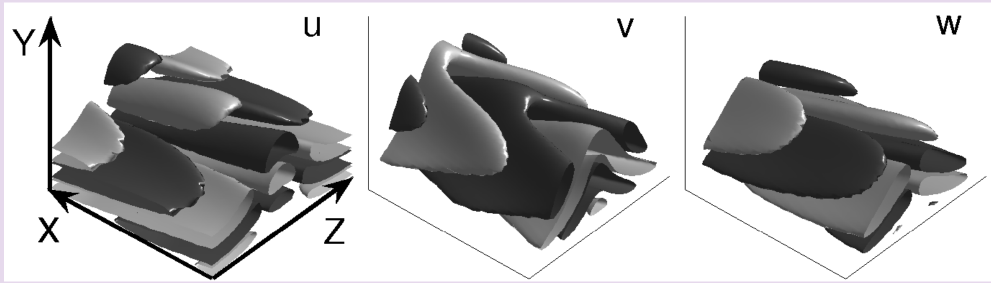


Optimal response

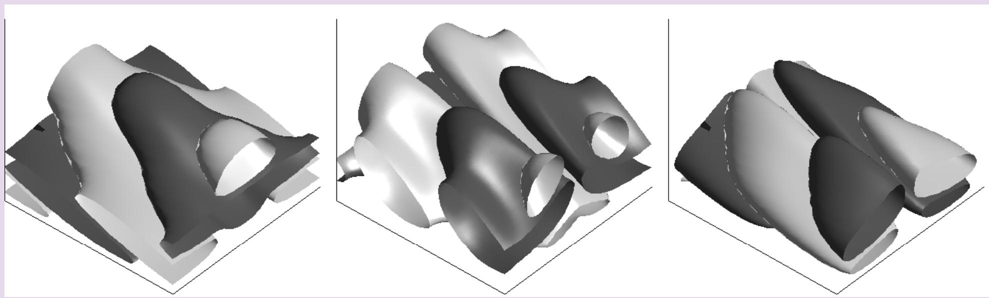


Varicose:

Optimal disturbance:



Optimal response



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