Notes on Triple Deck.

P.-Y. Lagrée

CNRS & UPMC Univ Paris 06, UMR 7190,
Institut Jean Le Rond ∂'Alembert, Boîte 162, F-75005 Paris, France
pyl@ccr.jussieu.fr ; www.lmm.jussieu.fr/$\sim$lagree

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Abstract

We present here briefly the famous "triple deck theory". In this framework, boundary layer separation is possible without singularity. Some numerical experiments describing the flow over bumps or wedges without or with flow reversal in various asymptotic régimes are presented.

1   Introduction.

Let us present a summing up of the preceding chapters. We presented what happens in a wall layer when a shear flow is disturbed, we next saw the perturbation of a Poiseuille flow. We observed in that case that perturbations can exist in the core flow (the Main Deck), the perturbations are expressed as a perturbation of the stream lines trough a function $-A$. We then presented the Blasius boundary layer, we emphasized the influence of the displacement thickness and the retroaction with the ideal fluid in the framework of Interacting Boundary Layer Theory.

2   Triple Deck

2.1   Overview

In the fifty's Lighthill [4] and Landau among a lot of others began to understand that boundary layer separation will be explained by new scales and a strong displacement of the boundary layer. This occurring at a small longitudinal scale, but larger than the boundary layer itself.

Then, simultaneously Neiland [6], Messiter [5] and Stewartson [13] in 69 proposed a new asymptotic structure in three decks (figure 1).

First, there is the basic boundary layer, which is now the "Main Deck". this layer is disturbed near the wall where the velocity is the smaller, the

length of this layer is small. Perturbations in this lower layer called "Lower Deck" are transmitted through the "Main Deck". In this layer the perturbation acts as a displacement of the stream lines (with a function called $-A(x)$). This deflexion of the stream lines is transmitted to the ideal fluid layer : the "Upper Deck". This deflexion creates a disturbance of pressure, and this disturbance of pressure will be transmitted back in the lower deck promoting the velocity disturbances. So that we will deal with a coupled system of equation: a disturbance of pressure creates a disturbance of stream lines which in turn creates a disturbance of pressure.
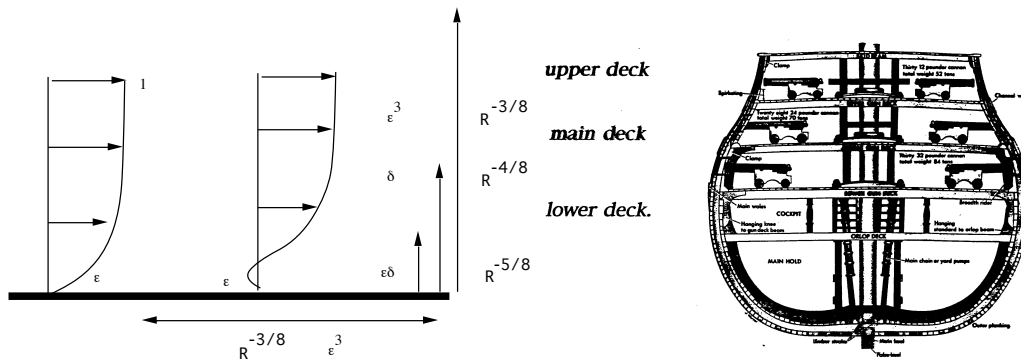

Figure 1: Left, the triple deck scales. Right, "triple decker ship of the line" from HMS victory brochure Porthmouth ("vaisseau de ligne à trois ponts"). In german "Dreierdeck-Theorie", a french translation of Triple Deck Theory may be "Triple Pont" instead of "Triple Couche".

2.2 Scales

2.2.1 Main Deck

The classical way to look at Triple Deck is to consider perturbations of the Boundary Layer. The first idea to introduce is the existence of a perturbation of small length compared to the boundary layer development itself.

We have the basic non dimensional Blasius profile $U_B(\tilde{y})$ in the boundary layer, where $\tilde{y}$ is the transverse variable scaled by $L/\sqrt{Re}$. Now suppose that at longitudinal scale say $x_3 L$ there is a perturbation of this basic profile. Of course $x_3 \ll 1$. We will call "Main Deck" the region considered which is of relative scale x_3 but which is of boundary layer scale in the transverse direction. As this scale is small, the boundary layer as not evolved, and at first order $V_B = 0$. So, suppose that at longitudinal scale say x_3 there is a perturbation of this basic profile of magnitude ε, then:

$$\tilde{u} = U_B(\tilde{y}) + \varepsilon \tilde{u}_1$$

In order to retain all the terms in the incompressibility and in the total derivative equation,

$$\tilde{u} = U_B(\tilde{y}) + \varepsilon\tilde{u}_1, \quad \tilde{v} = \frac{\varepsilon}{x_3\sqrt{Re}}\tilde{v}_1$$

longitudinal equation of momentum $(U_B\frac{\partial\tilde{u}_1}{\partial x} + \tilde{v}_1 U'_B)$, is of order ε/x_3. The previous analysis show that the relevant pressure term is in ε^2/x_3 which is negligible as are the viscous terms. This small value of pressure may be considered here as a first hypothesis that we will verify after. The system to solve is then

$$\frac{\partial\tilde{u}_1}{\partial x} + \frac{\partial\tilde{v}_1}{\partial\tilde{y}} = 0, \; (U_B\frac{\partial\tilde{u}_1}{\partial x} + \tilde{v}_1 U'_B) = 0, \quad \frac{\partial\tilde{p}_1}{\partial\tilde{y}} = 0.$$

By elimination we find: $U_B^2\frac{\partial}{\partial\tilde{y}}(\frac{\tilde{v}_1}{U_B}) = 0$, the classical notation is then to introduce a function of x say $A(x)$ introduced as a constant of integration, such as

$$\tilde{u}_1 = A(x)U'_B(\tilde{y}) \text{ and then } \tilde{v}_1 = -A'(x)U_B(\tilde{y})$$

is solution of the system.

With this description, the velocity is not zero but $\varepsilon A(x)U'_B(0)$ on the wall, so we have to introduce a new layer to full fit the no slip condition.
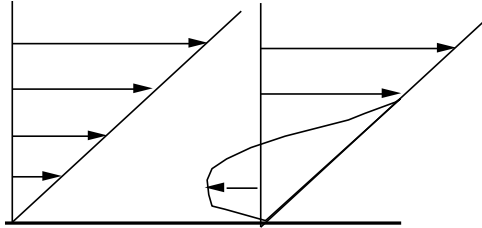
2.2.2 Lower Deck

Figure 2:   Near the wall the velocity profile is linear, the order of magnitude of the variation of velocity must be the same than the basic flow in order to obtain separation.

The purpose of the lower deck is to introduce a layer in which this perturbation of velocity will be annihilated. So the scale of velocity is ε, then as the velocity of the boundary layer is linear near the wall it is natural to guess that the lower deck will by of size $\varepsilon L/\sqrt{Re}$.

The behavior of the velocity in the Main Deck is

$$\tilde{u} = U_B(\tilde{y}) + \varepsilon A(x)U'_B(\tilde{y}).$$

We look at it near the wall. For $\tilde{y} \to 0$ the Blasius profile is linear near the wall $U_B(\tilde{y}) \to \tilde{U}'_B(0)\tilde{y}$ and then the velocity is $U'_B(0)\tilde{y} + \varepsilon A(x)U'_B(0)$, written in the inner variables of the lower deck this is (as $\tilde{y} = \varepsilon y$)

$$\varepsilon(y + A(x))U'_B(0).$$

So we deduce that in the lower deck the velocity should match to this quantity:

$$\lim_{y \to +\infty} u = (y + A(x))U'_B(0)$$

The convective diffusive equilibrium of the Navier Stokes equations

$$u\frac{\partial}{\partial x} \simeq Re^{-1}\frac{\partial^2}{\partial y^2}$$

written with the longitudinal x_3 and transversal $\varepsilon Re^{-1/2}$ scales reads:

$$\frac{\varepsilon}{x_3} \sim Re^{-1}\frac{1}{(\varepsilon Re^{-1/2})^2}$$

so that the longitudinal scale is :

$$x_3 = \varepsilon^3.$$

The pressure comes from the non linear balance

$$u\frac{\partial u}{\partial x} \simeq -\frac{\partial p}{\partial x},$$

it is of order ε^2, and the transverse equations of momentum gives as in the classical boundary layer:

$$\frac{\partial p}{\partial y} = 0$$

so, the pressure does not depend on y and is constant across the lower deck.

The final system is then:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2}.$$

With no slip condition at the wall ($u = v = 0$), the entrance velocity profile $u(x \to -\infty, y) = U'_B(0)y$, and the matching condition with the Main Deck: $u(x, y \to \infty) = (y + A)U'_B(0)$. Note, that the system is parabolic, there is no output condition needed to solve it.

2.3 Upper Deck

The disturbed velocity in the Main Deck is :

$$\tilde{u} = U_B(\tilde{y}) + \varepsilon A(x) U'_B(\tilde{y}); \ \tilde{v} = \frac{1}{\varepsilon^2 \sqrt{Re}} A'(x) U_B(\tilde{y})$$

and for the pressure

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0$$

Now let us see what happens at the top of the Main Deck, for $\tilde{y} \to \infty$:

$$\tilde{u} = 1; \ \tilde{v} = \frac{1}{\varepsilon^2 \sqrt{Re}} A'(x),$$

there is no more longitudinal perturbation of the velocity at order ε, but there is a transverse velocity, a kind of "blowing velocity" at the edge of the Main Deck. Note that the pressure remains the same order ε^2.

   Therefore we look at a layer of longitudinal size $x_3 = \varepsilon^3$ and of same thickness in which we have a blowing velocity at the wall of order $\frac{1}{\sqrt{Re}\varepsilon^2}$ and a pressure of order ε^2. This rings us a bell: the problem of perturbation of an dial fluid by a bump, which is equivalent of a flat plate with a blowing velocity. To have a consistent problem both orders of magnitude of blowing velocity and pressure perturbation should be equal $\varepsilon^2 = \frac{1}{\sqrt{Re}\varepsilon^2}$, so that we obtain the final magic parameter:

$$\varepsilon = Re^{-1/8}$$

3   The various régimes

3.1   Upper Deck, coupling relation incompressible

The velocity at the top of the Main Deck is then the velocity at the bottom of the upper deck: $-A'$. Depending on the ideal fluid régime, one may compute the pressure. For a incompressible flow on has the Hilbert relation:

$$p = \frac{-1}{\pi} \int -\frac{\frac{dA}{dx}}{x - \xi} d\xi$$

One see that there in the equations one can remove U'_0 in the equations in changing the scales, say u multiplied by U, p is multiplied by P, x by X and y by Y, so by invariance ($u\partial_x u$ versus $\partial_x p$) $P = U^2$, and ($u\partial_x u$ versus $\partial_y^2 u$) gives $U = X/Y^2$. At infinity $u \sim U'_0 y + ...$ so that $X = U'_0 Y^3$. The pressure displacement relation tells that the pressure is proportional to A', so $U^2 = Y/X$ which is $(X/Y^2)^2 = Y/X$ or $X^3 = Y^5$. But remember that $X = U'_0 Y^3$, then $X = U_0'^{-5/4}$ so, finally

$$x_3 = (U'_0)^{-5/4} Re^{-3/8}, \ \delta_3 = (U'_0)^{-3/4} Re^{-5/8}$$

$$u_3 = (U_0')^{1/4} Re^{-1/8}, \quad v_3 = (U_0')^{3/4} Re^{-1/4}, \quad \pi = (U_0')^{1/2} Re^{-1/4}$$

so that the final system is independent of the base flow.

3.2 Upper Deck, coupling relation supersonic

For a compressible supersonic flow, one has the Ackeret formula:

$$p = -\gamma \frac{M}{\sqrt{M^2 - 1}} \frac{dA}{dx}$$

again, changing the scales, one can remove the U_0' :

$$x_3 = C^{3/8} (U_0')^{-5/4} (M^2 - 1)^{-3/8} Re^{-3/8}, ...$$

then the relation is $p = -A'$.

3.3 Upper Deck, coupling relation transcritical

In transcritical flows a new parameter $K = (M^2 - 1)(U_0')^{-2/5} C^{-1/5} Re^{-1/5}$
and

$$p = -(\frac{3}{2\sqrt{\gamma + 1}} A')^{2/3}$$

see Bodoniy, Bartels & Rothmayer, Bodonyi and Kluwick [2].

3.4 Upper Deck, coupling relation sub/supercritical

In the case of water flow, the ideal fluid response was for the pressure the disturbance divided by $F^2 - 1$. So by rescaling:
$x = (x^*/L - 1) U_B'(0)^5 \left| Fr^2 - 1 \right|^3 / (Re^{-3/8})$, $y = (y^*/L) U_B'(0)^2 \left| Fr^2 - 1 \right|^{-1} / (Re^{-5/8})$,
$p = (p^*/(\rho U_0^2)) U_B'(0)^2 \left| Fr^2 - 1 \right|^{-2} / (Re^{-2/8})$,
$p = A$ for subcritical flows $(F < 1)$ and $p = -A$ for supercritical flows
$(F > 1)$.

3.5 Jet Flow

Nearly the same configuration may exist for a wall jet of thickness $\delta = Re^{-1/2}$ near the wall l:

$$\frac{1}{x_3} \varepsilon^2 \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} \sim \frac{\varepsilon}{\varepsilon^2 \delta^2 Re} (\frac{\partial^2 \bar{u}}{\partial \bar{y}^2})$$

so that $x_3 = \varepsilon^3$ and $u = U_0 + \varepsilon A(x) U_0'$ and $v = -\frac{\varepsilon \delta}{x_3} A'(x) U_0$

$$U_0 \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}, \quad \text{so the scale is} \quad \frac{\varepsilon \delta}{x_3^2}(-A''(\bar{x})) U_0^2 \sim -\frac{\varepsilon^2}{\delta} \frac{\partial p_1}{\partial y},$$

so that $\varepsilon = Re^{-1/7}$ which gives $x_3 = Re^{-3/7}$ and $\varepsilon \delta = Re^{-9/14}$

$$p(\bar{x}, 0) = -A''(\bar{x}) \int_0^\infty U_0^2(y) dy$$

3.6   The various régimes, canonical system

The canonical system is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2}.$$

With at the wall ($u = v = 0$), at the entrance $u(x \to -\infty, y) = y$, and at the infinity $u(x, y \to \infty) = (y + A)$.

- $p = \frac{-1}{\pi} \int -\frac{\frac{dA}{dx}}{x-\xi} d\xi$ incompressible case

- $p = -A'$ supersonic

- $p = -A$ hypersonic case

- $p = A$ fluvial.

- $p = -A$ torrential.

- $p = -A$ mixed convection.

- $-A = 0$ pipes, Couette.

- $p = -A''$ pipes, wall jets.

- $p = -\partial_x \phi$, $A' = \partial_y \phi$, with $\partial_y^2 \phi = (1 + \partial_x \phi)\partial_x^2 \phi$, transsonic.

3.7   Linearised solution: self induced solution

We can look at a linearised solution of the canonical system. The linearised system is:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad y\frac{\partial u_1}{\partial x} + v_1 = -\frac{dp_1}{dx} + \frac{\partial^2 u_1}{\partial y^2}.$$

With at the wall ($u_1 = v_1 = 0$), at the entrance $u_1(x \to -\infty, y) = 0$, and at the infinity $u_1(x, y \to \infty) = A_1$.

We test e^{Kx} solutions on the linearized system, with $K > 0$.

$$u_1 = e^{Kx}\phi'(y), \quad v_1 = -Ke^{Kx}\phi(y), \quad p_1 = e^{Kx}P$$

with $\phi(0) = \phi'(0) = 0$ and say $\phi'(\infty) = 1$ so that $A_1 = e^{Kx}$; as the incompressibility is fulfilled, the momentum is

$$Ky\phi'(y) - \phi(y) = -KP + \frac{\partial^2 \phi'(y)}{\partial y^2}, \tag{1}$$

so by differentiation wez have to solve:

$$\frac{\partial^2 \phi''(y)}{\partial y^2} = Ky\phi''(y),$$

with $\phi'''(0) = KP$. If we write $z = K^{-1/3}y$ and $\phi'' = f$, then the equation is

$$\frac{d^2f(z))}{dz^2} = zf(z),$$

whose form is Airy function. $f(z) = Ai(z)$. Hence, from the boundary condition in 0 (ϕ'''), we have the derivative of the Airy function Ai'. Then ϕ'' is $K^{2/3}Ai(K^{1/3}y)P/Ai'(0)$ and $\phi' = \frac{K^{1/3}P}{Ai'(0)}\int_0^y Ai(\xi)d\xi$ so that we deduce, because $\int_0^y Ai(\xi)d\xi = 1/3$:

$$\phi'(\infty) = \frac{K^{1/3}}{3Ai'(0)}P$$

• The supersonic case allows then an eigen solution

$$K = (-3Ai'(0))^{3/4}$$

with $K = 0.827$
This exponential is the rational explanation of the observed self induced separation.
• The supercitical case and the hypersonic case and the mixed convection case allow then an eigen solution

$$K = (-3Ai'(0))^3$$

with $K = 0.47$

• The jet case (or pipe) allows then an eigen solution

$$K = (-3Ai'(0))^{3/7}$$

with $K = 0.89$

The incompressible, the fluvial and the couette or symmetrical pipe cases do not allow this self induced solution.

3.8   The Prandtl transposition theorem

There is a trick called "Prandtl tranform" or "Prandtl transposition theorem" which allows to change the bumpy wall into a flat one. One writes $\tilde{y} = y - f(x)$ and keeps $\tilde{x} = x$. Then, as $\partial_x = \partial_{\tilde{x}} - f'(x)\partial_{\tilde{y}}$ and $\partial_y = 0 + \partial_{\tilde{y}}$ continuity equation becomes

$$\frac{\partial}{\partial\tilde{x}}u + \frac{\partial}{\partial\tilde{y}}(v - f'u) = 0$$

and as the total derivative:

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = u\frac{\partial}{\partial\tilde{x}}u + (v - f'u)\frac{\partial}{\partial\tilde{y}}u$$

so the Prandtl transform or "Prandtl transposition theorem" is: $\tilde{y} = y - f(x)$, $\tilde{x} = x$, $\tilde{u} = u$ and $\tilde{v} = (v - f'u)$ so that system is invariant:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\[2mm] u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{dp}{dx} + \dfrac{\partial^2 u}{\partial y^2}. \end{cases} \tag{2}$$

$u = v = 0$ on $y = 0$, $u \to y$ when $x \to -\infty$, and $u \to y + f(x)$ when $y \to \infty$.

The sole difference lies in the boundary condition at the top.

3.9 Linearised solution explicit solution in Fourier space

We can look at a linearised solution of the canonical system. The linearised system is in Prandtl transform:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad y\frac{\partial u_1}{\partial x} + v_1 = -\frac{dp_1}{dx} + \frac{\partial^2 u_1}{\partial y^2}.$$

With at the wall ($u_1 = v_1 = 0$), at the entrance $u_1(x \to -\infty, y) = 0$, and at the infinity $u_1(x, y \to \infty) = A_1 + f_1$.

We are looking for solutions in Fourier space so we test e^{ikx} solutions on the linearized system.

$$u_1 = e^{ikx}\phi'(y), \quad v_1 = -(ikx)e^{ikx}\phi(y), \quad p_1 = e^{ikx}P_k \quad f_1 = f_k e^{ikx}, \quad A_1 = a_k e^{ikx}$$

with $\phi(0) = \phi'(0) = 0$ and then $\phi'(\infty) = a_k + f_k$ so that; as the incompressibility is fulfilled, the momentum is

$$iky\phi'(y) - \phi(y) = -ikP_k + \frac{\partial^2 \phi'(y)}{\partial y^2}, \tag{3}$$

so $\frac{\partial^2 \phi''(y)}{\partial y^2} = iky\phi''(y)$, and as $\phi'''(0) = ikP_k$, so ϕ'' is $(ik)^{2/3}Ai((ik)^{1/3}y)P_k/Ai'(0)$ and $\phi' = \frac{(ik)^{1/3}P}{Ai'(0)}\int_0^y Ai(\xi)d\xi$ so that we deduce $\phi'(\infty) = a_k + f_k = \frac{(ik)^{1/3}}{3Ai'(0)}P$

The relation between the perturbation of pressure and the displacement is then in Fourier space:

$$\beta^* FT[p] = FT[(A + f)]$$

where $\beta^* = (3Ai'(0))^{-1}(ik)^{1/3}$.

- The supersonic case $\beta_{pf} = 1/(-ik)$

- The subsonic case the pressure displacement relation is $\beta_{pf}FT[p] = FT[(A)]$ with $\beta_{pf} = 1/|k|$

- The supercitical case is such that $\beta_{pf} = -1$

- The fluvial case is such that $\beta_{pf} = 1$

- The no displacement case $\beta_{pf} = 0$.

- The no displacement case $\beta_{pf} = 0$.

To plot the figures 3.10 one uses then the following relations:

$$FT[p] = \frac{FT[f]}{\beta^* - \beta_{pf}} \quad \text{and} \quad FT[\tau] = \frac{(ik)^{2/3}}{Ai'(0)}Ai(0)FT[p]. \tag{4}$$

and the linearized perturbation of the skin friction (τ)

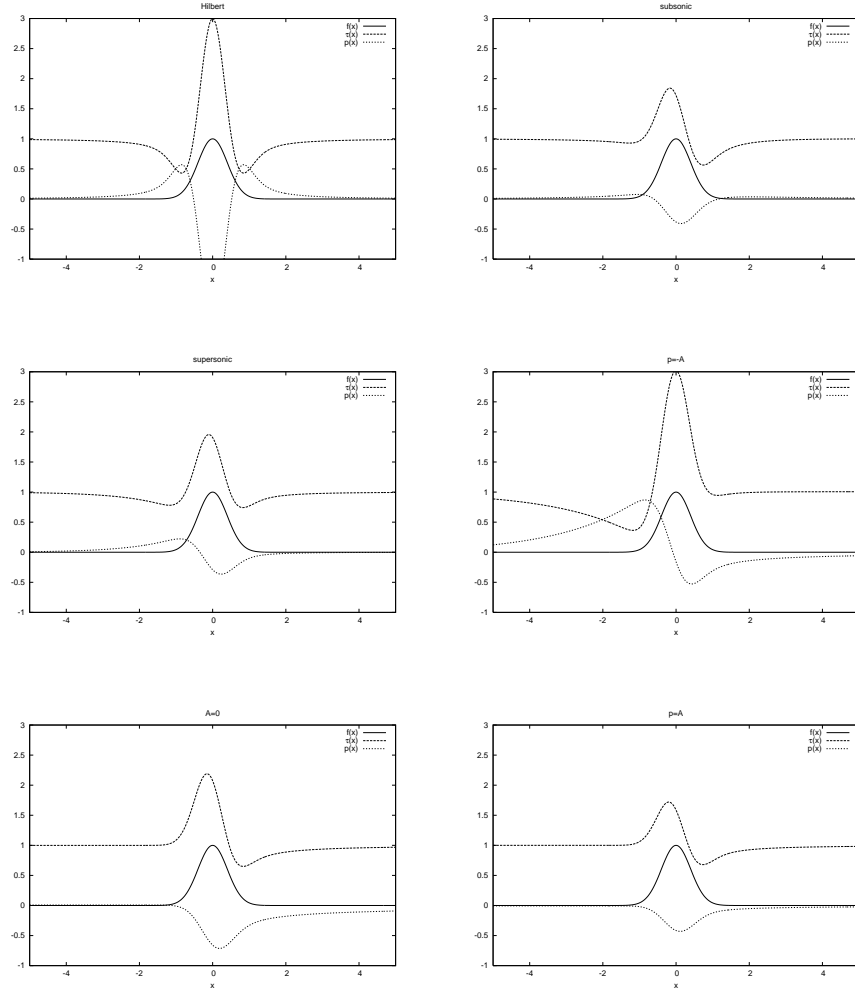## 3.10   Plots of linearised solutions

Figure 3:   Friction distribution and pressure over a bump in 6 cases, linear solution. Top left the Hilbert case, just to compare. Top right the subsonic case $p = \frac{-1}{\pi} \int - \frac{\frac{dA}{dx}}{x-\xi} d\xi$. Middle left, the supersonic $p = -A'$ case. Middle right, $p = -A$ case. Bottom left, the $A = 0$ case. Bottom right, the $p = A$ case.

3.11 Supersonic case

Supersonic case $p = -\frac{dA}{dx}$, flow over a wedge, non linear case.
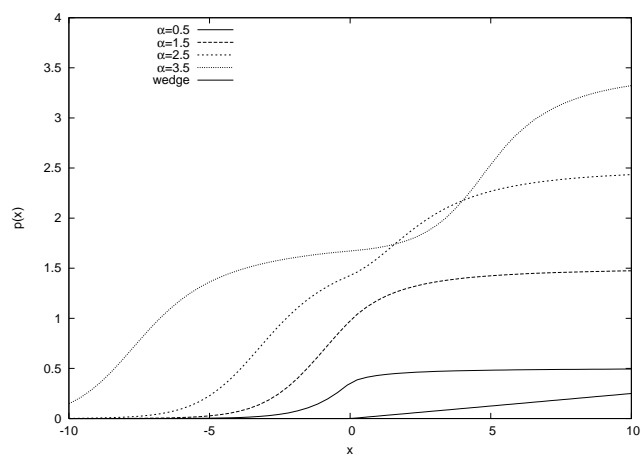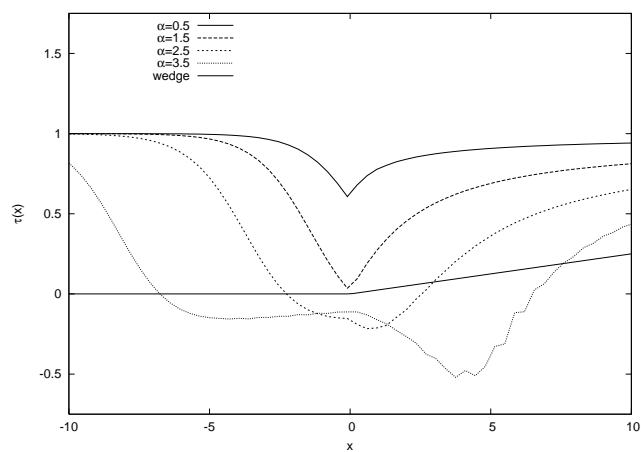

Figure 4:   pressure distribution over a wedge

Figure 5:   Friction distribution over a wedge

3.12   Subsonic case

$p = \frac{-1}{\pi} \int - \frac{\frac{dA}{dx}}{x - \xi} d\xi$ subsonic/ incompressible case with the Hilbert integral, non linear case.
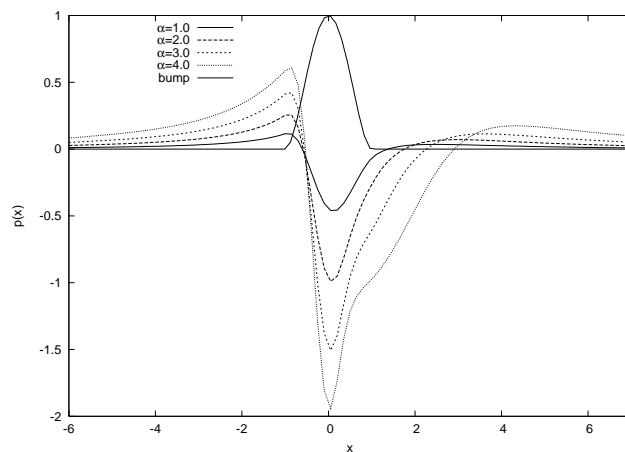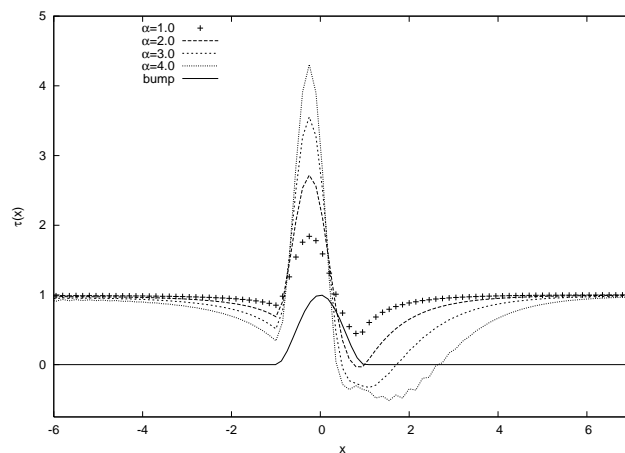

Figure 6:   pressure distribution over a bump

Figure 7:   Friction distribution over a bump

3.13 Transsonic case
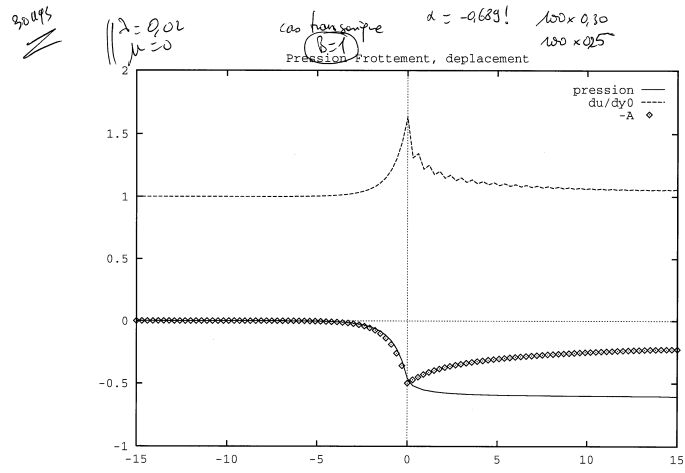
$$p = -(\frac{3}{2\sqrt{\gamma+1}}A')^{2/3}$$

Figure 8:   Friction distribution pressure distribution over a expansive wedge

3.14 Various scales on a bump

Smith et al. [12] and Roget et al. [7] showed that a bump which is at the triple deck size is at a kind of interaction of several effects.
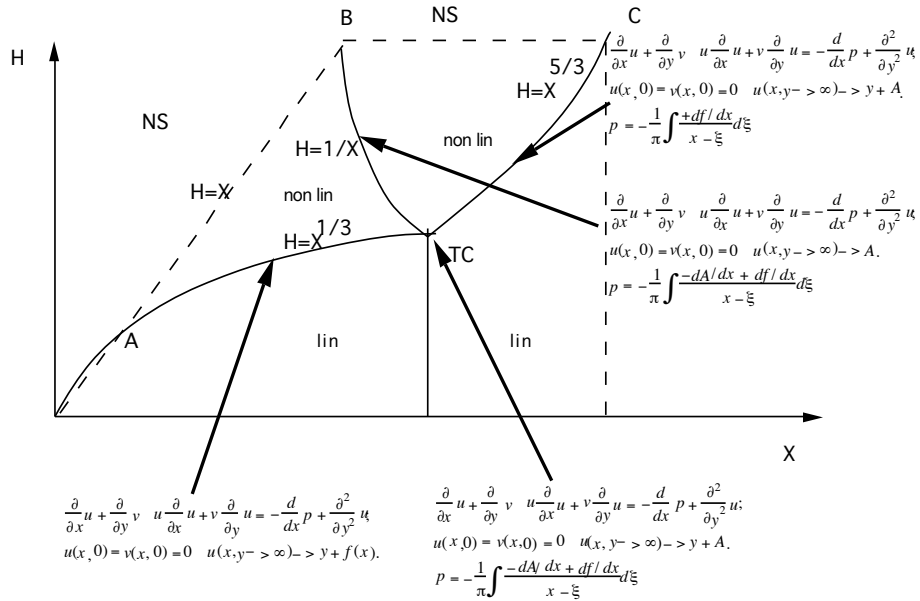

Figure 9: various scales converging in the Triple Deck

We saw that the triple deck corresponds to a very special size for a bump. So we may ask what happens if we change the size of the bump. Imagine that the bump is no more of size $O(1)$ in the Triple Deck scale but is smaller or larger. How changes teh longitudinal scale if we want to have always the maximum number of terms in the equations.

Let say that y is changed by Yy, so, in the triple deck if we change $y \rightarrow Yy$ then in order to have the non linear viscous balance, we have: $u \rightarrow Yu$, $x \rightarrow Y^3$, $p \rightarrow Y^2 p$ and $A \rightarrow YA$. With this transformation lower deck equations are the same.

Let use H the size of the bump as the parameter of height. H is not Y, and $f \rightarrow Hf$

The ideal fluid relation (using Prandtl transform):

$$p = \frac{-1}{\pi} \int -\frac{A' - f'}{x - \xi} d\xi$$

has the following rescaling:

$$(Y^2)p = (Y^{-2})\frac{-1}{\pi}\int -\frac{A'}{x-\xi}d\xi + (HY^{-3})\frac{-1}{\pi}\int -\frac{-f'}{x-\xi}d\xi$$

• If now Y is large (this is a large bump $X = Y^3$), and if H is large, the displacement contribution decreases. for $Y^2 = HY^{-3}$ i.e. $H = Y^5$ or $H = X^{5/3}$.

The largest size of the bump is the boundary layer itself, so it gives a maximum size of $Re^{-3/10}$.

• If now Y is small (small bump $X = Y^3$), and if H is large, the displacement contribution decreases.

$$(Y^4)p = \frac{-1}{\pi}\int -\frac{A'}{x-\xi}d\xi + (HY^{-1})\frac{-1}{\pi}\int -\frac{-f'}{x-\xi}d\xi$$

so $H = Y$ and we have $A' + f' = 0$. this is the no displacement case.

• there is a more subtle case, as the matching relation is $u(x,\infty) = y + A + f$, then if H is large, we may imagine that A is large (change $A \to HA$) and as $y \to Yy$ (with $Y << H$), the lower Deck is broken in two parts one where u goes from 0 to A and another one where $u = A + f$ (u is of order $H >> Y$) so if we change $A \to HA$, $x \to Xx$, $u \to Hu$ and $p \to H^2p$. The ideal fluid relation (using Prandtl transform):

$$p = \frac{-1}{\pi}\int -\frac{A' - f'}{x-\xi}d\xi$$

has the following rescaling:

$$(H^2)p = (H/X)\frac{-1}{\pi}\int -\frac{A'}{x-\xi}d\xi + (H/X)\frac{-1}{\pi}\int -\frac{-f'}{x-\xi}d\xi$$

so that $H = 1/X$.

We then have to solve:

$$A\frac{\partial A}{\partial x} = \frac{1}{\pi}\frac{\partial}{\partial x}\int -\frac{A' - f'}{x-\xi}d\xi$$

4   Link with IBL

The IBL formulation emphasizes on the displacement thickness,

$$\delta_1 = (Re^{-1/2}) \int_0^\infty (1 - u(x, \tilde{y})) d\tilde{y}$$

we have to decompose it into two parts as we cross the lower and the main decks. Let introduce $\tilde{Y}$

$$\delta_1 = (Re^{-1/2})(\int_0^{\tilde{Y}} (1 - \tilde{u}(\bar{x}, \tilde{y})) d\tilde{y} + \int_{\tilde{Y}}^\infty (1 - \tilde{u}(\bar{x}, \tilde{y})) d\tilde{y})$$

the first integral is estimated near the wall, so the Lower Deck description ($\tilde{y} = \varepsilon y$) is valid there, but a good idea is to write the velocity $u(x, y) = U'_B(0)(y + A) + u_c$ where u_c is a correction:

$$(\int_0^{\tilde{Y}} (1 - \tilde{u}(\bar{x}, \tilde{y})) d\tilde{y}) = \varepsilon(\int_0^{\tilde{Y}/\varepsilon} (1 - \varepsilon(U'_B(0)(y + A))) dy - \int_0^{\tilde{Y}/\varepsilon} \varepsilon u_c dy)$$

the second one is in the Main Deck

$$\int_{\tilde{Y}}^\infty (1 - u(x, \tilde{y})) d\tilde{y} = \int_{\tilde{Y}}^\infty (1 - U_B(\tilde{y}) - \varepsilon A(x) U'_B(\tilde{y})) d\tilde{y}.$$

Re summing the two integrals and changing the order of the terms allows then write:

$$\delta_1 = (Re^{-1/2})\{[\varepsilon(\int_0^{\tilde{Y}/\varepsilon} (1 - \varepsilon(U'_B(0)(y))) dy + \int_{\tilde{Y}}^\infty (1 - U_B(\tilde{y})) d\tilde{y}] +$$

$$+ [\varepsilon(\int_0^{\tilde{Y}/\varepsilon} (-\varepsilon(U'_B(0)(A))) dy + \int_{\tilde{Y}}^\infty (-\varepsilon A(x) U'_B(\tilde{y})) d\tilde{y}] - \varepsilon^2 \int_0^{\tilde{Y}/\varepsilon} u_c dy)\}$$

so that we recognize :

$$\delta_1 = (Re^{-1/2})\{\int_0^\infty (1 - U_B(\tilde{y})) d\tilde{y} + \int_0^\infty (-\varepsilon A(x) U'_B(\tilde{y})) d\tilde{y} - \varepsilon^2 \int_0^{\tilde{Y}/\varepsilon} u_c dy)\}.$$

or

$$\delta_1 = (Re^{-1/2})\{\int_0^\infty (1 - U_B(\tilde{y})) d\tilde{y} - \varepsilon A(x) - O(\varepsilon^2)\}.$$

the $-\varepsilon A$ contribution of the triple deck is the perturbation of the displacement thickness $\int_0^\infty (1 - U_B(\tilde{y})) d\tilde{y}$. So the IBL technique based on δ_1 is justified by the triple deck analysis.

Remember that the ideal fluid velocity is

$$\bar{u}_e = 1 + \frac{1}{\pi} \int \frac{\bar{f}'(\bar{x})\bar{u}_e + Re^{-1/2}\frac{d(\tilde{\delta}_1 \bar{u}_e)}{d\bar{x}}}{x - \xi} d\xi + O(1/Re)$$

in this framework $\frac{d(\tilde{\delta}_1 \bar{u}_e)}{d\bar{x}}$ is no more small, it is large. Hence one can not neglect $Re^{-1/2}\frac{d(\tilde{\delta}_1 \bar{u}_e)}{d\bar{x}}$. The order of magnitude of the term $Re^{-1/2}\frac{d(\tilde{\delta}_1 \bar{u}_e)}{d\bar{x}}$ is $Re^{-1/2}\varepsilon/x_3$, but Triple Deck gives $x_3 = \varepsilon^3$ and that the perturbation of velocity in the Upper Deck is ε^2. Hence $Re^{-1/2}\varepsilon/x_3 = \varepsilon^2$, and again $\varepsilon = Re^{-1/8}$.

5 Full flat plate problem

The leading edge problem, velocity is of order one, x and y are of same order of magnitude so

$$u\frac{\partial u}{\partial x} \sim Re^{-1}\frac{\partial^2 u}{\partial x^2}$$

with the scales

$$\frac{1}{X} \sim Re^{-1}\frac{1}{X^2}$$

hence

$$X = Re^{-1}$$

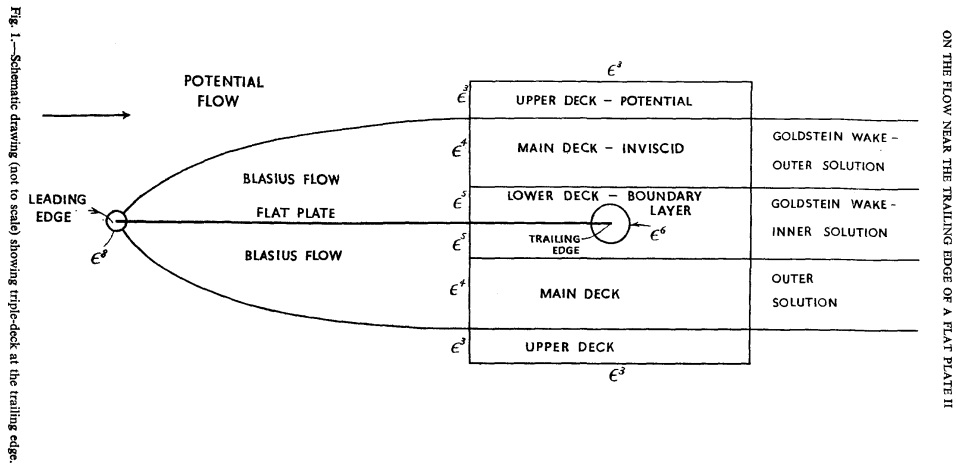written $\varepsilon^8 = Re^{-1}$ on the figure.

The trailing edge problem.

Figure 10: The scales on the plate, from K. Stewartson, On the flow near the trailing edge of a flat plate, Mathematica 16 1969

6 ∂'Alembert Paradox and Kutta condition

At this point, it is time to introduce the famous d'Alembert Paradox 1752.

Remember what it is exactly, it states that in Ideal fluid there is no drag. To remove this, Prandtl introduced the boundary layer in 1904. But discussion is still active on the large Reynolds number wake (separation on a cylinder).

The triple deck structure is a possible response to solve it. The Kutta condition does not exist, the flow can turn round the trailing edge, but this is at a $Re^{-3/8}$ scale, it is so small when $Re \to \infty$ that this gives the Kutta condition.

7   Conclusion

In this chapter we presented the Triple Deck scales and equations. We showed that there is an interactive problem between a thin layer near the wall and a layer of ideal fluid through the displacement of the stream lines $-A$. In the thin layer, Prandtl equations are valid with new scales, and a different matching condition involving this displacement function $-A$. The upper layer Euler small disturbance theory applies, the layer in between is the boundary layer which is passive and only transmits the perturbations of $-A$ and pressure p. This framework allows to understand boundary layer separation and self induced separation. The pressure deviation relation pressure p displacement $-A$ allows a large variety of various coupled problems...

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$\mathscr{P}$.-$\mathscr{Y}$. $\mathscr{L}$agrée' s contribution is:
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