Asymptotic flows: large Reynolds perturbation of Poiseuille channel flow.

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Abstract

This first chapter introduces on the heat entry problem in a channel flow the techniques that will be used thereafter for the flow near the wall in the "Lower Deck". Some asymptotic principles are presented on the so called Lévêque and Graetz heat problems. Next the same ideas are presented for the flow: first, the Couette flow with a small accident (a bump) is presented following the previous analysis. Second, the flow in a channel (or a in a pipe) with a small accident is presented. There are two "Lower Deck" layers (at the top and the bottom wall) which interact through the "Main Deck" consisting in the basic Poiseuille flow. The different scales arising are presented, some numerical experiments show the skin friction and pressure distributions. The upstream influence is then discussed.

Part I
Heat Flow in a Channel

1 Introduction the Lévêque/ Graetz problem

1.1 Introducing the problem of asymptotic expansions

The problem that we will tackle is depending of a small parameter (in fact the inverse of a large parameter). Even though now a lot of problems may be solved numerically, it is interesting to observe which terms are important in the equations. That is the aim of the method of Matched Asymptotic Expansion. It is a tool to analyze and understand the flow structure. One of the basic text book is Van Dyke one's [35], he introduced there the technique and the notations. A less centered of hydrodynamics text book is the Hinch one's [14]. It presents a large panel of the techniques on model equations.

More recently, Cousteix & Mauss [6] present a global survey of asymptotic techniques and compare them.

We will here use those theories to explain with very few mathematical details the ideas of the Triple Deck and Interactive Boundary Layer Theories. To start we introduce a very classical example which in fact contains most of the features.

1.2 Unit Step response of temperature in a Poiseuille steady flow

As an introduction let us consider the steady laminar incompressible flow between two parallel plates (in $y = 0$ and $y = h$). The flow solution is clearly the Poiseuille one:

$$u = U_0(y/h)(1 - y/h), \quad v = 0. \tag{1}$$

Let say that for $x < 0$ the temperature at the wall is T_0 and after $T = T_w$ (see figure 1 and 2 left for a sketch). We wish to compute the steady temperature profile with asymptotic analysis bearing in mind that convective effects are stronger than diffusive ones in this chosen case.

The first step is to adimensionalize the equations, this step is not so trivial. A first good guess is to use the channel height as scale $x = h\bar{x}$ and $y = h\bar{y}$. For the temperature, let write $T = T_0 + (T_w - T_0)\bar{T}$ (other choices are possible, this one is more simple to solve). The steady heat equation (for constant conductivity k, density ρ and specific heat capacity c_p and neglecting dissipation by viscosity) will be called $H_{(1/Pe)}$ it reads:

$$H_{(1/Pe)} \qquad \bar{y}(1-\bar{y})\frac{\partial \bar{T}}{\partial \bar{x}} = \frac{1}{Pe}\left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\right) \tag{2}$$

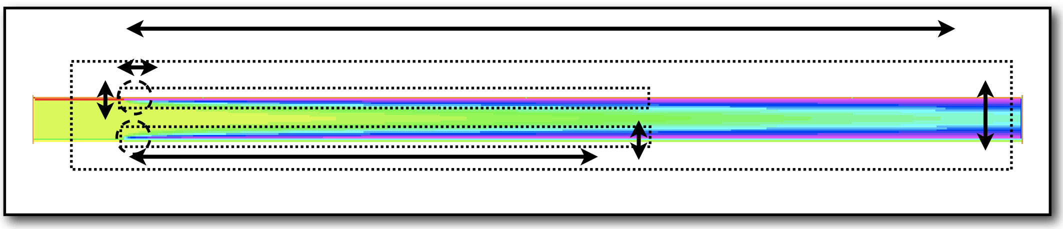

Figure 1: The Poiseuille flow in a pipe at temperature T_0 in $x < 0$ is experiencing a temperature discontinuity in $x > 0$ to T_w. Iso temperatures are presented. This example contains several distinct scales near the discontinuity, near the walls, etc.

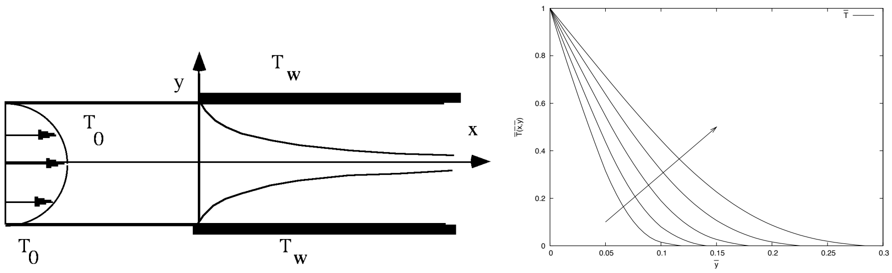

Figure 2: Left, the flow at temperature T_0 in $x < 0$ and experiencing a temperature discontinuity in $x > 0$ to T_w. Right, the numerically computed temperature profile $\bar{T}(\bar{x}, \bar{y})$ in the lower half of the flow, arrow in the direction of increasing x.

where $Pe = \frac{U_0 h}{k/(\rho c_p)}$ is the Péclet number (ratio of convective effects by diffusive effects). This number is not small. This is an elliptic equation and to solve it one has to impose boundary conditions. Those are:

$$\bar{T}(\bar{x} \to -\infty, \bar{y}) = 0 \text{ and } \bar{T}(\bar{x}, \bar{y} = \pm 1) = 0 \text{ and } \bar{T}(\bar{x} \to \infty, \bar{y}) = 1.$$

The problem may be solved numerically (here with FreeFem++ [12]). On figure 2 right the numerically computed temperature profile $\bar{T}(\bar{x}, \bar{y})$ is drawn near the lower wall for various values of $\bar{x}$. The more $\bar{x}$ increases, the more the flow is heated as it is indicated by the arrow in the direction of increasing $\bar{x}$. On figure 3 the iso temperature are plotted for several values of Pe showing that for increasing Pe there is a thin layer near the wall where the temperature increases abruptly.

In the sequel, the Péclet number Pe is assumed to be large.

1.3 the Lévêque (1928) problem

1.3.1 Singular problems

The PDE (2) as an heat equation problem is well posed and we guess that the solution is smooth enough except in the vicinity of $\bar{x} = 0$. For any fixed Pe even large, the solution is certainly continuous at fixed $\bar{x}$ when $\bar{y}$ goes to 0^+ or 1^-.

The inverse of the Péclet $(1/Pe)$ is assumed to be small, so the first problem consists to put $(1/Pe) = 0$ in the PDE (2). Let us call $\bar{\theta}$ the solution of this problem (H_0) which reads:

$$H_0 \qquad\qquad \bar{y}(1 - \bar{y})\frac{\partial \bar{\theta}}{\partial \bar{x}} = 0,$$

which solution is $\bar{\theta} = 0$ for $0 < \bar{y} < 1$. This is called the outer solution. The temperature is discontinuous at the wall where we should have $\theta(\bar{x} >$

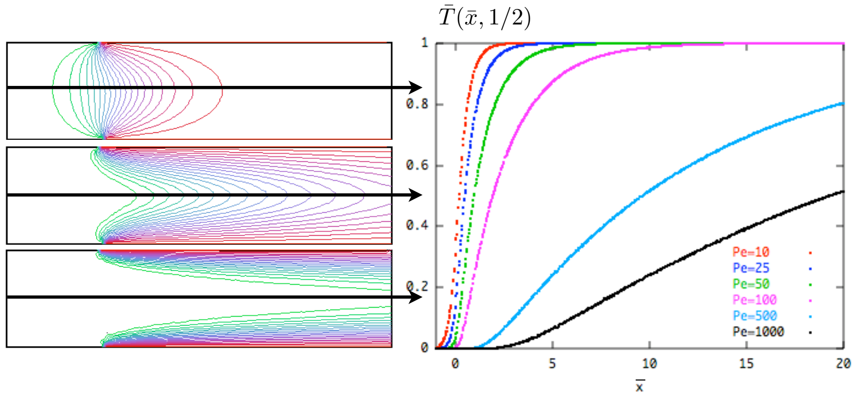

Figure 3: Left, iso temperatures of the numerical solution for various values of Pe. Right the numerical solution of the mid channel value $\bar{T}(\bar{x}, 1/2)$ for several values of Pe with $\bar{x}$ in abscissa.

$0, \bar{y} = 0, 1) = 0$. The highest derivative has disappeared, we can not fix the boundary conditions. The problem is said to be singular, the solution of the problem where $(1/Pe)$ is put to 0, is not the limit when $(1/Pe)$ becomes infinitely small to the full solution of the problem. The two limits are different:

$$\lim_{(1/Pe)\to 0} \big(Sol[H_{(1/Pe)}]\big) \neq Sol[\lim_{(1/Pe)\to 0} \big(H_{(1/Pe)}\big)] \tag{3}$$

One clue of the problem is that one as to look near the wall at small values of $\bar{y}$ (the same for the upper wall, that we will no more consider).
To solve the problem we follow Van Dyke [35] page 86, *"The guiding principles are that the inner problem shall have the least possible degeneracy, that it must include in the first approximation any essential elements omitted in the first outer solution, and that the inner and outer solutions shall match."*

i) The first step is the **Choice of inner variables**, this is done following Van Dyke first part of the sentence and more specifically the *"least possible degeneracy"*. We write $\tilde{y} = \bar{y}/\varepsilon$ meaning that we strech the variable. And take $\tilde{\theta}$ the temperature so that (2) is now:

$$\varepsilon\tilde{y}(1-\varepsilon\tilde{y})\frac{\partial\tilde{\theta}}{\partial\bar{x}} = \frac{1}{Pe}\left(\frac{\partial^2\tilde{\theta}}{\partial\bar{x}^2} + \frac{\partial^2\tilde{\theta}}{\varepsilon^2\partial\tilde{y}^2}\right) \tag{4}$$

the leading order of the left hand side is $\varepsilon\tilde{y}\frac{\partial\tilde{\theta}}{\partial\bar{x}}$, whereas the leading order of the right hand side is $\frac{1}{Pe\varepsilon^2}(\frac{\partial^2\tilde{\theta}}{\partial\tilde{y}^2})$. Using Van Dyke Principle, the best choice for the streching is $\varepsilon = Pe^{-1/3}$, with this choice, we have:

$$\tilde{y}\frac{\partial\tilde{\theta}}{\partial\bar{x}} = \frac{\partial^2\tilde{\theta}}{\partial\tilde{y}^2}. \tag{5}$$

We study the so called inner region which is near the wall where the effect of diffusion are strong enough to permit to ensure the boundary condition. In fact we see, that putting $(1/Pe) = 0$ in the problem (2) is not relevant as, in doing this we suppose that variations according to $\bar{y}$ are not fast, or are always at scale 1. This is not true near the wall where the derivatives are very large (of order $Pe^{2/3}$).

ii) The second important ingredient is the **Matching principle:** which is the last part of the Van Dyke sentence *"the inner and outer solutions shall match."*, he writes it as:

inner representation of (outer representation)
=
outer representationof (inner representation)

this gives the boundary condition that was missing in the preceeding problem. This reads

$$\lim_{\bar{y}\to 0}\bar{\theta} = \lim_{\tilde{y}\to\infty}\tilde{\theta} \tag{6}$$

In the bulk, the outer solution (of problem H_0) was always 0. So, far away from the wall, the inner solution $\tilde{\theta}$ matches to this value.

1.3.2 Selfsimilar solution of Lévêque problem

Now, this problem (5) may be solved using the "self similar technique". This technique is based on the observation that lot of problems admit solutions with a shape which looks like always the same.

We have the numerical solution, it is plotted on the figure 2 right. This figure clearly shows that all the temperature profiles have nearly the same "shape" (a curve decreasing from 1 to 0) with increasing thickness in $\bar{x}$ say $\Delta(\bar{x})$. So we guess that maybe there is a unique temperature profile function of $\tilde{y}$ divided by this thickness such as $\tilde{\theta}(\bar{x}, \tilde{y}) = g(\tilde{y}/\Delta(\bar{x}))$ where g decreases from 1 to 0) .

The technique helps to find this dependance. We test whether the problem (5) is invariant trough stretching of the coordinates. It is the "method of invariance through a streching group", Bluman & Kumei [2]. Writing:

$$\bar{x} = X\hat{x},\ \tilde{y} = Y\hat{y} \text{ and } \tilde{\theta} = \Theta\hat{\theta}$$

we wish to obtain a PDE problem invariant under the rescaling X, Y, Θ. Clearly, we have $\Theta = 1$ to full fit the invariance of the boundary condition $\tilde{\theta}(\bar{x} > 0, 0) = 1$ or $\hat{\theta}(\hat{x} > 0, 0) = 1$. Starting form the original PDE we want it to be invariant after streching:

$$\tilde{y}\frac{\partial\tilde{\theta}}{\partial\bar{x}} = \frac{\partial^2\tilde{\theta}}{\partial\tilde{y}^2} \qquad \text{becomes after changing the scale: } (\frac{Y^3}{X})\hat{y}\frac{\partial\hat{\theta}}{\partial\hat{x}} = \frac{\partial^2\hat{\theta}}{\partial\hat{y}^2}. \quad (7)$$

so that $Y^3 = X$ allows the invariance of the PDE, it means that if we stretch with any $Y > 0$ the variables:

$$\bar{x} = Y^3\hat{x},\ \tilde{y} = Y\hat{y} \ \text{ and } \ \tilde{\theta} = \hat{\theta}$$

$$\tilde{y}\frac{\partial\tilde{\theta}}{\partial\bar{x}} = \frac{\partial^2\tilde{\theta}}{\partial\tilde{y}^2},\ \tilde{\theta}(\bar{x} > 0, 0) = 1 \text{ is after streching: } \hat{y}\frac{\partial\hat{\theta}}{\partial\hat{x}} = \frac{\partial^2\hat{\theta}}{\partial\hat{y}^2},\ \hat{\theta}(\hat{x} > 0, 0) = 1.$$

The next step is to take advantageof this invariance. If we have a solution f for the temperature dependance in $\bar{x}$ and $\tilde{y}$ then say $\tilde{\theta}(\bar{x}, \tilde{y}) = f(\bar{x}, \tilde{y})$ we may write it in an implicit way rather than in a usual explicit one: $\tilde{\theta} - f(\bar{x}, \tilde{y}) = F(\bar{x}, \tilde{y}, \tilde{\theta})$ so that

$$F(\bar{x}, \tilde{y}, \tilde{\theta}) = 0, \text{ with the invariance } F(Y^3\hat{x}, Y\hat{y}, \hat{\theta}) = 0$$

this is true for any $Y > 0$, so we may imagine to change the function F, and introduce another one, where we just changed

$$F(\bar{x}, \tilde{y}, \tilde{\theta}) = 0, \text{ changed into } G(Y^3\hat{x}, \hat{y}/\hat{x}^{1/3}, \hat{\theta}) = 0$$

as this is valid for any Y, we guess that the first slot is empty, so that $\hat{\theta} = g(\eta)$ with $\eta = \hat{y}/\hat{x}^{1/3}$, this reduced variable is called the selfsimilar variable and by definition $\eta = \hat{y}/\hat{x}^{1/3} = \tilde{y}/\bar{x}^{1/3}$. Looking to a self similar solution $\tilde{\theta}(\bar{x}, \tilde{y}) = g(\eta)$, the transformed problem is $\eta\bar{x}^{1/3}\frac{g'(\eta)\eta}{-3\bar{x}} = g''(\eta)\bar{x}^{-2/3}$ so

$$-\frac{\eta^2}{3} = \frac{g''}{g'} \text{ with } g(0) = 1,\ g(\infty) = 0.$$

The solution is written as:

$$g(\eta) = 1 - \frac{\int_0^\eta exp(-\xi^3/9)d\xi}{\int_0^\infty exp(-\xi^3/9)d\xi}$$

where we recognise the incomplete gamma function $\Gamma(a, z) = \int_z^\infty t^{a-1}e^{-t}dt$. So that

$$g(\eta) = \tilde{\theta}(\bar{x}, \tilde{y}) = \Gamma(\frac{1}{3}, \frac{\tilde{y}^3}{9\bar{x}})/\Gamma(\frac{1}{3}).$$

The flux at the wall will then be $\tilde{\theta}'(\bar{x}, 0) = -3^{1/3}/\Gamma(1/3)\bar{x}^{-1/3} = -0.538366\bar{x}^{-1/3}$

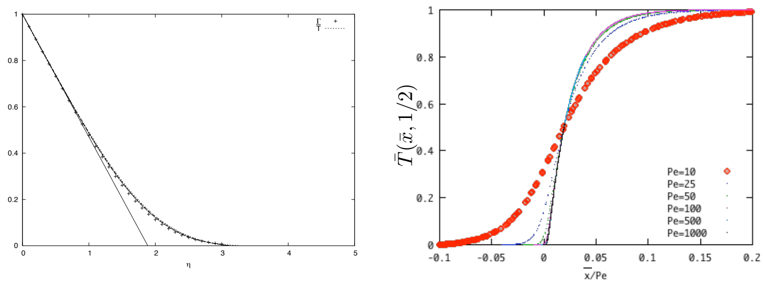

Figure 4: Left, the numerical solution $\bar{T}$ written with the selfsimilar variable $\eta = \tilde{y}/\bar{x}^{1/3}$ collapsing on the selfsimilar solution labelled Γ and the slope at origin: $1 + g'(0)\eta$. Right the numerical solution of the mid channel value $\bar{T}(\bar{x}, 1/2)$ for several values of Pe with $\bar{x}/Pe$ in abscissa, the curves collapse on the Graetz solution.

Note

To be convinced on an example for the F to G:

Suppose	$f(x, y, z) = (x^2 + y^2)sin(z)$
We may write it	$f(x, y, z) = x^2(1 + (y/x)^2)sin(x(z/x))$
So that	$f(x, y, z) = g(x, y/x, z/x)$
with g the function	$g(\xi, \eta, \zeta) = \xi^2(1 + \eta^2)sin(\xi\zeta)$.
or	$g(x, y, z) = x^2(1 + y^2)sin(xz)$.

1.3.3 Fourier solution of Lévêque problem

One other useful tool is the Fourier transform that we will use extensively in numerical studies. One may try to find solutions of problem (5) in term of Fourier series:

$$TF[\phi](k) = \frac{1}{\sqrt{2\pi}} \int \phi(x)e^{ikx}dx,$$

looking for each mode in e^{-ikx}:

$$(-ik)\tilde{y}TF[\tilde{\theta}] = \frac{\partial^2 TF[\tilde{\theta}]}{\partial \tilde{y}^2},$$

so that we see that $TF[\tilde{\theta}]$ is solution of the Airy equation defined by

$$Ai''(\xi) - \xi Ai(\xi) = 0$$

with $Ai(+\infty) = 0$, after changing the variable in $\xi = y(-ik)^{1/3}$ and by definition $Ai(0) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})}$ and $Ai'(0) = -\frac{1}{\sqrt[3]{3}\Gamma(\frac{1}{3})}$, there is another solution of this equation the $Bi(\xi)$ function which is not bounded in ∞, see Abramowitz & Stegun p 446 [1] for details). Then, as the unit step function has $\frac{i}{k\sqrt{2\pi}} + \delta(k)\sqrt{\frac{\pi}{2}}$ as Fourier transform, we can evaluate:

$$TF[\tilde{\theta}] = (\frac{i}{k\sqrt{2\pi}} + \delta(k)\sqrt{\frac{\pi}{2}})\frac{Ai(y(-ik)^{1/3})}{Ai(0)}$$

and we then obtain the flux at the wall as:

$$TF[\tilde{\theta}_0'] = (-ik)^{1/3}(\frac{i}{k\sqrt{2\pi}} + \delta(k)\sqrt{\frac{\pi}{2}})\frac{Ai'(0)}{Ai(0)}$$

going back in Real space, we reobtain the selfsimilar result:

$$\tilde{\theta}_0' = -\frac{\sqrt[3]{3}}{\Gamma\left(\frac{1}{3}\right)}x^{-1/3} \quad \text{if } x > 0, \quad \text{else } \tilde{\theta}_0' = 0$$

Remark All those Fourier transform are not so trivial to compute, and there is some magick that Mathematica [33] handles well. To be convinced, we have to evaluate

$$\varphi(x) = \int k^n e^{-ikx} dk \text{ here, we have } n = -2/3$$

so changing the variable $k = \lambda k'$ gives $\varphi(x) = \lambda^{n+1} \int k'^n e^{-ik'\lambda x} dk'$ taking $\lambda = 1/x$ we have the expected power dependence (here, we have $-(n+1) = -1/3$) so

$$\varphi(x) = x^{-(n+1)} \int k'^n e^{-ik'} dk'.$$

Fowler [10] proposes to look at Gradshteyn and Ryzhik 1980 to compute those integrals and remarks that we recover a Γ function:

$$\int_0^\infty k'^n e^{ik'} dk' = \Gamma(n+1)e^{i\pi(n+1)/2}.$$

1.4 The Graetz problem

Now, let us look at what happens for $\bar{x}$ large. On figure 3 we saw that at fixed value Pe, there is always a position where the two thermal boundary layers meet ultimately. So we study what happens for very large value of $\bar{x}$, let define $\check{x}$ a long variable (of scale say $1/\varepsilon$, it is here a new ε) so that:

$$\varepsilon\bar{x} = \check{x}$$

Now at this large scale, the temperature changes all across the flow so we do not change the transverse scale $\bar{y}$. The temperature with the new scale $\check{x}$ is denoted as $\check{T}$ and the heat equation is now:

$$\bar{y}(1-\bar{y})\frac{\partial\check{T}}{\varepsilon^{-1}\partial\check{x}} = \frac{1}{Pe}(\frac{\partial^2\check{T}}{\varepsilon^{-2}\partial\bar{x}^2} + \frac{\partial^2\check{T}}{\partial\bar{y}^2}) \quad (8)$$

the left hand side is $\varepsilon\bar{y}(1-\bar{y})\frac{\partial\check{T}}{\partial\check{x}}$ and the dominant right hand side is $\frac{1}{Pe}(\frac{\partial^2\check{T}}{\partial\bar{y}^2})$, so the *the least possible degeneracy* choice is $\varepsilon = Pe^{-1}$.

$$\bar{y}(1-\bar{y})\frac{\partial\check{T}}{\partial\check{x}} = \frac{\partial^2\check{T}}{\partial\bar{y}^2} \quad (9)$$

This problem has been solved by Nuβelt and is solved using separation of variables as a infinite sum of terms like:

$$\check{T} = \sum_{n=0}^{N} \psi_n(\check{y}) exp(-\lambda_n^2 \check{x}),$$

each of the modes n verifies the eigen value equation:

$$-\lambda_n^2(1-\bar{r}^2)\psi_n(\check{y}) = \psi_n(\check{y})'', \;\; \psi_n(\check{0}) = \psi_n(1) = 0.$$

We do not here solve this problem (a master piece of heat transfer theory text Book), but on figure 4 right, we plot the numerical resolution of the full problem 2 for various values of Pe with the $\check{x}$ variable. We observe that as Pe increases the solution goes on the same master curve corresponding to the solution of the Graetz problem.

1.5 Local scaling near the discontinuity

Up to now, we always neglect the longitudinal variation in the temperature, it should come back somewhere. We did not study what happens just at the point where the temperature changes, at this place $x = 0$, $y = 0$ there is a huge longitudinal variation in the temperature. This place is a good candidate to reintroduce the always removed second order derivative.

Then following the *"least possible degeneracy"*. We write $\tilde{x} = \bar{x}/\varepsilon$, $\tilde{y} = \bar{y}/\varepsilon$ meaning that we strech the variable with same scale. And take $\tilde{\theta}$ the temperature so that (2) is now:

$$\varepsilon\tilde{y}(1-\varepsilon\tilde{y})\frac{\partial\tilde{\theta}}{\varepsilon\partial\bar{x}} = \frac{1}{Pe}(\frac{\partial^2\tilde{\theta}}{\varepsilon^2\partial\bar{x}^2} + \frac{\partial^2\tilde{\theta}}{\varepsilon^2\partial\tilde{y}^2}) \quad (10)$$

the leading order of the left hand side is $\tilde{y}\frac{\partial\tilde{\theta}}{\partial\bar{x}}$, whereas the right hand side is the complete Laplacian. $\frac{1}{Pe\varepsilon^2}(\frac{\partial^2\tilde{\theta}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\theta}}{\partial\tilde{y}^2})$. The local scale is then:

$$\varepsilon = Pe^{-1/2}.$$

This is the convenient scale to study a local accident, with this scale we have the exact equilibrium between the convection and diffusion.

In Pedley [22] one may find that this solution matches with the Lévêque one at infinity.

It very important to notice here that at this scale, there is some upstream influence. It means that at a given point before $\tilde{x} = 0$, the flow "feels" the heat produced in $\tilde{x} > 0$. That what we see at local scale on figure 1.5. In the tilde variable, the Laplacian gives informations against the flow, the

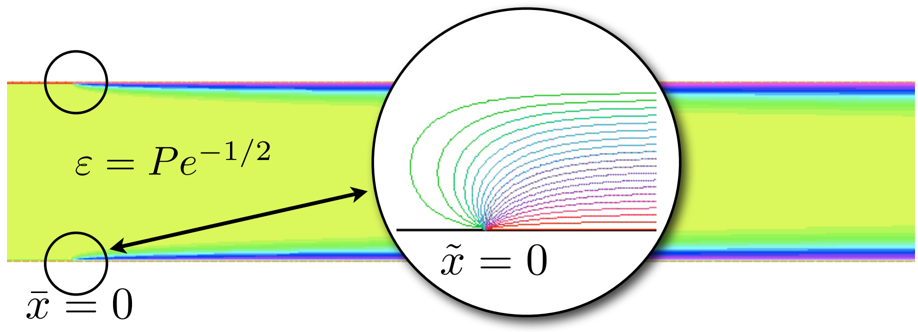

Figure 5: The iso temperature near the point where the flow is heated. Note that the flow is heated upstream at scale $Pe^{-1/2}$.

problem is "elliptic", the downstream influences the upstream.
At all the other scales the convection is too strong at a given point before $\tilde{x} = 0$, the flow does not "feel" the heat produced in $\tilde{x} > 0$. The equation is "parabolic". The downstream no more influences the upstream.

We may note that in this case the smaller interesting scale is

$$hPe^{-1/2} = h(U_0 h/\kappa)^{-1/2} = ((U_0/h)/\kappa)^{-1/2}$$

it means that when we are near the lower wall and near the temperature discontinuity, only matter the shear of the velocity say $U_0' = (U_0/h)$. The scale which is then natural is

$$\sqrt{\frac{\kappa}{U_0'}}$$

it is the sole scale that we can construct in a shear viscous flow.

Of course, if we scale the flow at a scale which is smaller, the convective term becomes negligible. We have only a Laplacian to solve:

$$\frac{\partial^2 \tilde{\theta}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\theta}}{\partial \tilde{y}^2} = 0. \tag{11}$$

with $\tilde{\theta}(\tilde{x} < 0) = 0$ and $\tilde{\theta}(\tilde{x} > 0) = 1$. Note that this problem is not simple to solve and that it implies a logarithmic term, but that is another story...

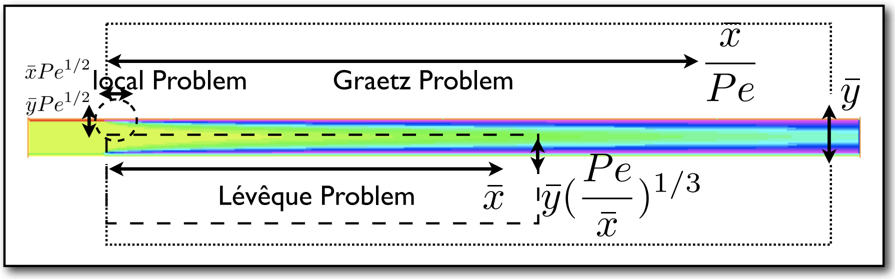

Figure 6: The final scales in the thermal pipe flow which allow in each case a peculiar convective diffusive equilibrium. First, the entrance where the two scales are the same $\bar{x}Pe^{1/2}, \bar{y}Pe^{1/2}$. Second the thin thermal boundary layer where we have $\bar{x}$, and a thin $\bar{y}(Pe/\bar{x})^{1/3}$. Third the long longitudinal final scale $\bar{x}/Pe$ and $\bar{y}$ where the boundary layers have merged.

1.6 Conclusion

•This simple example allows us to introduce the salient ingredients of the asymptotics:
- non dimensional equations with small parameters,
- the least degeneracy principle,
- matching principle.

• We introduced some techniques and remarks that we will see again:
- variety of scales which can be intricated
- self similar/ Fourier solutions
- parabolic equations/ upstream influence

Part II
Flow in a Channel

2 Pipe/ Channel flow: perturbation of a linear shear flow.

Now, in fact we have seen most of the ideas that will be used in the sequel. We will apply them to a slightly perturbed shear flow. We may consider the flow in a pipe and study the flow near the wall where the Poiseuille profile $u = U_0(y/h)(1-y/h)$ reduces to $u = (U_0/h)y$ as it does for the Lévêque problem, we then obtain a basic Couette profile. We will define the shear as $U_0' = U_0/h$. The aim is to put a small perturbation in this simple shear flow and evaluate the perturbation of the skin friction ($\tau_w = \mu U_0'$) and the perturbation of the pressure at the wall.

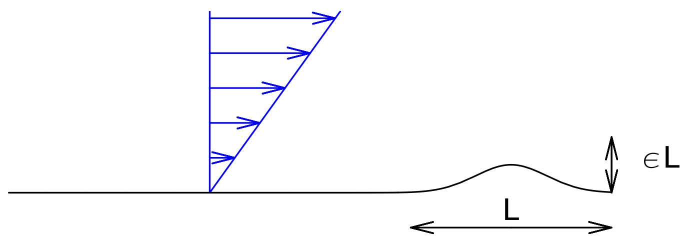

Figure 7: A small perturbation (a bump) in a simple shear flow $u = U_0'y$. The flow is not bounded in $y > 0$.

2.1 Simple shear flow problem

So, before looking at the "Triple Deck" problem it self, let first look at a simple case which is the hearth of the problem. Let imagine a pure shear flow over a flat plate in $y = 0$ and extending to infinity. The basic velocity is pure Couette flow $u = U_0'y$. We note that in this special case, only a parameter dimensional to a frequency is given, we have no length scale.

We suppose now, that the wall is not flat but there is an "accident" such as a small bump. Let us use the longitudinal scale of the bump say L as scale to adimensionalise the Navier Stokes Equations (with Reynolds number $Re = U_0' L^2/\nu$):

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{\partial p}{\partial x} + \dfrac{1}{Re}(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}), \\ u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{\partial p}{\partial y} + \dfrac{1}{Re}(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}). \end{cases} \tag{12}$$

Large Reynolds number analysis of the problem using the least degeneracy principle forces us to balance the viscous terms and the convective ones. This is necessary to recover the second order derivative which disappears at large Reynolds. So near the wall, we introduce a thin layer of relative size $\varepsilon << 1$, of longitudinal scale 1; in this layer the longitudinal velocity is of scale ε as the basic profile is linear in y, so that the viscous convective balance:

$$\varepsilon^2 \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} \sim \frac{\varepsilon}{\varepsilon^2 Re}(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2})$$

so we obtain the scale $\varepsilon = Re^{-1/3}$; we recover exactly, with no surprise, the Lévêque scaling. Then with the following least degeneracy scales $u = \varepsilon\bar{u}$, $v = \varepsilon^2\bar{v}$, $x = \bar{x}$, $y = \varepsilon\bar{y}$ and $p = \varepsilon^2\bar{p}$ we obtain in fact the Prandtl equations with bars over all the variables, with then remove the bars:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{\partial p}{\partial x} + \dfrac{\partial^2 u}{\partial y^2}, \\ 0 = -\dfrac{\partial p}{\partial y}. \end{cases} \tag{13}$$

To be self consistent, if the accident is a bump, its size must scale with ε. We therefore have a full non linear problem.

The boundary condition are the no slip conditions, the initial linear velocity profile far upstream. At scale 1 in y the solution is not disturbed, so a matching with the scale ε gives that there are no perturbation conditions at infinity (it will be changed).

$$u = v = 0 \text{ on } y = f(x),\ u \to y \text{ when } x \to -\infty, \text{ and } u \to y \text{ when } y \to \infty. \tag{14}$$

Note that the problem is such that the pressure is a result of the computation.

2.2 linear Fourier solution

We notice that in this case we have a simple analytical solution obtained by linearisation: say the bump is small $f(x) = \alpha f_1$ with $\alpha << 1$, then

$u = y + \alpha u_1$, $v = \alpha v_1$ and $p = \alpha p_1$. The boundary condition is linearised by *"transfer of boundary condition"* [35]): in $y = \alpha f_1$ the velocity is zero: $u(x, \alpha f_1) = 0$, but as the velocity may be Taylor expanded at the wall $u(x, \alpha f_1) = u(x, 0) + (\alpha f_1)\frac{\partial}{\partial y}u(x, 0) + ...$ as $\partial_y u(x, 0) = 1$ we have $u(x, \alpha f_1) = u(x, 0) + (\alpha f_1)1 + ...$ and as for $y = 0$ the development of the velocity is $u(x, 0) = 0 + \alpha u_1$ then $u_1 = -f_1$. We will see thereafter another trick known as Prandtl transform which gives the same result (but which is more general).

We then go in Fourier space (∂_x is $-ik$) and write for the Fourier Transform

$$TF[\phi](k) = \hat{\phi} = \frac{1}{\sqrt{2\pi}} \int \phi(x) e^{ikx} dx$$

of the function (in fact $\hat{u}_1$ is $TF[u_1]$):

$$\begin{cases} -ik\hat{u}_1 + \dfrac{\partial \hat{v}_1}{\partial y} = 0, \\ -iky\hat{u}_1 + \hat{v}_1 = ik\hat{p}_1 \quad + \dfrac{\partial^2 \hat{u}_1}{\partial y^2}, \end{cases} \tag{15}$$

Let define $\hat{\tau}_1 = \frac{\partial \hat{u}_1}{\partial y}$, then the second equation after derivation by y is: $-iky\hat{\tau}_1 = \frac{\partial^2 \hat{\tau}_1}{\partial y^2}$ (with help of the first and as the pressure does not depend on y). This equation is again an Airy equation, so that: $\hat{\tau}_1$ is proportional to $Ai((-ik)^{1/3}y)$. Integrating the shear, gives the velocity. Integrating from 0 to infinity gives: $\int_0^\infty (\hat{\tau}_1)dy = \hat{u}_1(\infty) - \hat{u}_1(0)$, with $\hat{u}_1(\infty) = 0$ and $\hat{u}_1(0) = -\hat{f}_1$. But as $\int_0^\infty Ai(\xi)d\xi = 1/3$ (Abramowitz and Stegun [1]) we obtain coefficient of proportionality and then the friction at the wall:

$$\hat{\tau}_1(0) = 3(-ik)^{1/3} Ai(0) \hat{f}_1.$$

The pressure follows from the second equation written at the wall

$$0 = ik\hat{p}_1 + \frac{\partial \hat{\tau}_1(0)}{\partial y}$$

so the pressure is:

$$\hat{p}_1 = 3Ai'(0)(-ik)^{-1/3} \hat{f}_1.$$

$$\tau = U'_S + U'_S(3Ai(0))(U'_S)^{1/3} TF^{-1}[(-ik)^{1/3} TF[f]] \tag{16}$$

and the pressure over the bump is

$$p = (U'_S)^2 (3Ai'(0))(U'_S)^{-1/3} TF^{-1}[(-ik)^{-1/3} TF[f]]. \tag{17}$$

where $Ai(x)$ is the Airy function, $Ai(0) = 0.355028$ and $Ai'(0) = -0.258819$.

This result is relevant as long as $f(x)$ has a physical height lesser than $LRe^{-1/3}$

2.3 linear integral solution

The solution involves $TF^{-1}[(-ik)^n TF[f]]$, this may be written in an explicit way. It is convenient to use the derivative of the bump, so the integral is $TF^{-1}[(-ik)^{n-1}TF[f']]$ which will be a convolution of the Heaviside function $H(x)$ times x^{-n} and the slope of the wall $f'(x)$. This is for the skin friction:

$$\tau = U'_S\big(1 + \big(\frac{3^{2/3}}{\Gamma(2/3)^2}\big)(U'_S)^{1/3}\int_0^\infty \frac{f'(x-\xi)}{\xi^{1/3}}d\xi\big).$$

and for pressure

$$p = (U'_S)^2\big(-\big(\frac{3^{2/3}\sqrt{\pi}}{2^{11/3}\Gamma(1/3)^2}\big)(U'_S)^{1/3}\int_0^\infty \frac{f(x-\xi)}{\xi^{2/3}}d\xi\big).$$

With this point of view it is clear that the solution presents no influence of the bump upstream the bump. Perturbation exist at the first position of the beginning of the bump. We will see that it is an important feature in the flows.

2.4 Scale invariance of the problem

It is straightforward to see that if we change all the scales in :
If $x \to Y^3 x$, $y \to Yy$, $f \to Yf$, $v \to Y^{-1}v$, $y \to Yy$, and $p \to Y^2 p$,
then the transformed equations are invariant. It means that, as in the Lévêque problem, self similar solutions with $yx^{-1/3}$ are relevant.

2.5 The Prandtl transform

There is a trick called "Prandtl tranform" which allows to change the bumpy wall in a flat one. One writes $\tilde{y} = y - f(x)$ and keeps $\tilde{x} = x$. Then, as $\partial_x = \partial_{\tilde{x}} - f'(x)\partial_{\tilde{y}}$ and $\partial_y = 0 + \partial_{\tilde{y}}$ continuity equation becomes

$$\frac{\partial}{\partial \tilde{x}}u + \frac{\partial}{\partial \tilde{y}}(v - f'u) = 0$$

and as the total derivative:

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = u\frac{\partial}{\partial \tilde{x}}u + (v - f'u)\frac{\partial}{\partial \tilde{y}}u$$

so the Prandtl transform is: $\tilde{y} = y - f(x)$, $\tilde{x} = x$, $\tilde{u} = u$ and $\tilde{v} = (v - f'u)$ so that system is invariant:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{dp}{dx} + \dfrac{\partial^2 u}{\partial y^2}. \end{cases} \tag{18}$$

$u = v = 0$ on $y = 0$, $u \to y$ when $x \to -\infty$, and $u \to y + f(x)$ when $y \to \infty$.

The sole difference lies in the boundary condition at the top.

2.6 Numerical Solution of the problem

2.6.1 The "Keller Box" technique

The problem is here solved with the "Keller Box" technique [4] or [15]. The baseline is to solve the problem as an heat equation $u\frac{\partial u}{\partial x} + ... = \frac{\partial^2 u}{\partial y^2} + ...$ As the problem is a kind of heat equation, it seems natural that it is a marching procedure in x such as the heat equation is solved in marching in time. The system is "parabolic", Navier Stokes with is Laplacian is "elliptic".
The equations are written in introducing only first order derivatives:

$$\frac{\partial u}{\partial y} = G, \;\; \frac{\partial G}{\partial y} = u\frac{\partial u}{\partial x} + ...$$

then, the derivatives are centered in the "box" of corners $(i-1, j-1)$ $(i-1, j)$ $(i, j-1)$ and (i, j). Values in $(i-1, j-1)$ $(i-1, j)$ been known. for examples $\frac{\partial u}{\partial y} = G$, reads $\frac{u(i,j)-u(i,j-1)}{\Delta y} = \frac{G(i,j)+G(i,j-1)}{2}$.

In fact we need four variables, ψ the stream function, G the shear and W a fictionous variable such as $\frac{\partial p}{\partial x} = -\frac{\partial (W^2/2)}{\partial x}$ (denoted as *Mechoul* approach by Cebeci & Keller) so that Prandtl equations are:

$$\begin{cases} \dfrac{\partial \psi}{\partial y} = u, \\ \dfrac{\partial u}{\partial y} = G, \\ \dfrac{\partial G}{\partial y} = -\dfrac{\partial (W^2/2)}{\partial x} + u\dfrac{\partial u}{\partial x} - G\dfrac{\partial \psi}{\partial x}, \\ \dfrac{\partial W}{\partial y} = 0. \end{cases} \tag{19}$$

As among others there are non linear terms, so $u\frac{\partial u}{\partial x}$ is discretized in

$$\frac{(\frac{u(i,j)+u(i-1,j)}{2} + \frac{u(i,j-1)+u(i,j-1)}{2})}{2} \frac{(\frac{u(i,j)-u(i-1,j)}{\Delta x} + \frac{u(i,j-1)-u(i-1,j-1)}{\Delta x})}{2},$$

and then a Newton iteration is necessary. Writing the new step $n+1$ as a small increase of the preceding: $u^{n+1}(i,j) = u^n(i,j) + \delta u^n(i,j)$, we obtain a block tridiagonal system the $\delta u^n(i,j)$, $\delta G^n(i,j)$ *etc* solved by Thomas algorithm.

Boundary condition at the wall and at the entrance are simple. At the top of the domain the velocity is equal to y so that the third equation of the sytem (19) becomes $0 = -\frac{\partial (W^2/2)}{\partial x} + u\frac{\partial u}{\partial x} - \frac{\partial \psi}{\partial x}$. His integral $-W^2 + u^2 - 2\psi$ is then a constant at the top of the domain. This last expression is linearised to obtain the relation in $j = J$ the last line

$$J\Delta y \delta u^n(i,J) - \delta\psi^n(i,J) - W^n(i,J)\delta W^n(i,J) = 0.$$

A last important trick is to introduce the so called FLARE (introduced in [8]) approximation: $u\frac{\partial u}{\partial x}$ is put to 0 when $u < 0$. We will see latter that this approximation is not so strong, and that the upstream influence that it introduces is very small.

2.6.2 Results

Examples of skin friction and pressure distribution are plotted on figure 8 for the linear response. We plot left the perturbation of the skin friction $\frac{\partial u_1}{\partial y}$ at the wall defined by a bump (arch of cosine: $cos(\pi x/2)^2$, for $-1 < x < 1$) and right the perturbation of pressure p_1. On this bump we clearly see that before $x = -1$ there is no response. The same feature is observed on the non linear solution on figure 9, before the bump, there is no response by the flow. We will say that there is no "upstream influence".

To emphasize the influence of the non linearities, we plot the non linear solution on the next graph. We plot the skin friction $\frac{\partial u}{\partial y}$ at the wall defined by a bump $\alpha e^{-(2x/3)^2}$ (for various values of α). We notice that the skin friction increases a lot due to the acceleration introduced by the bump. The extremum is obtained before the crest. Next, the skin friction decreases and may become negative. In those case, there is a small eddy after the bump. If the bump increases a bit more the length of the separated region increases. On the right part of the figure, the pressure is plotted. The minimum of the pressure is a bit after the crest. Next it increases to a final negative value indicating the small pressure drop.

An important result of the non linear resolution is that this asymptotic model can compute reverse flow regions. Those are region where $\tau < 0$ and where the velocity is reversed. The region of reverse flow is called the "separated bulb". What is called separation is described by the system of equations. This is a very important remark.

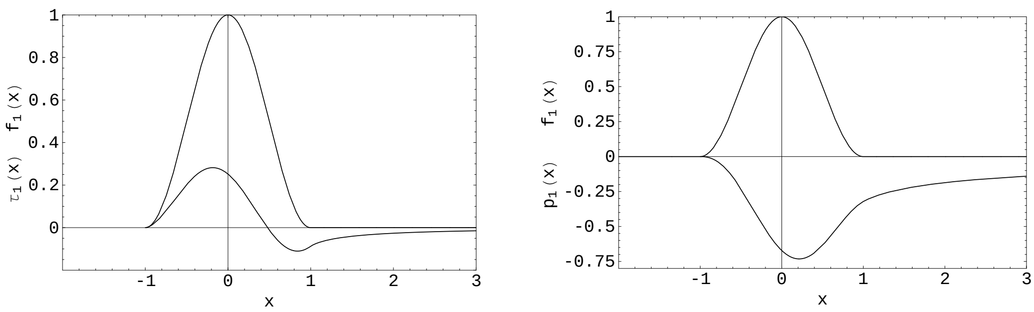

Figure 8: Linearised response over a bump defined by $f_1(x) = cos(\pi x/2)^2$, for $-1 < x < 1$, linear perturbation of the skin friction τ_1 (left) and linear perturbation of pressure p_1 (right).

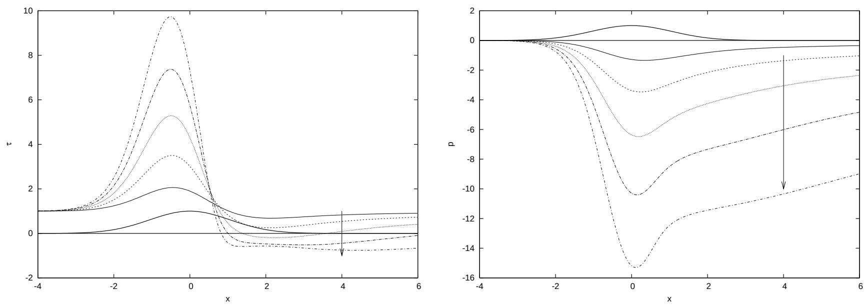

Figure 9: Non linear computation, left, the skin friction over a bump, the pressure over the same bump $\alpha e^{-(2x/3)^2}$ for $\alpha = 1, 2, 3, 4, 5$. Notice that the skin friction becomes negative indicating a reverse flow on the lee side of the bump. In the separated bulb, the pressure is nearly linear.

2.7 Smaller scale

As we mentioned, there is no length scale in the flow, the only one that we can build is based on U_0' and ν. It is worth to note here that with this scale $\ell = \sqrt{\nu/U_0'}$, we have $Re = 1$ so $x = \ell x$, $y = \ell y$, $u = U_0'\ell u$ etc is exactly the scale at which both terms in the Laplacian and in the Navier Stokes Equation are present

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}), \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + (\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}). \end{cases} \tag{20}$$

This is the full problem with all the terms. We then have a lower bound:

$$\ell << L$$

for the size of the bump. This small scale in reminiscent of what happened in the heat problem.

2.8 3D solution

In this section we present the 3D counterpart of this problem. The bump was of length L and thickness εL with $\varepsilon = Re^{1/3}$. To extend it to 3D, we note that the convective diffusive balance is always relevant $u\frac{\partial}{\partial x}w \sim \frac{\partial^2}{\partial y^2}w$. But, we wish to reintroduce the pressure gradient $-\frac{\partial}{\partial z}p$ and the $\frac{\partial}{\partial z}w$ term in the continuity equation. So we take $u = \varepsilon u$, $v = \varepsilon^2 v$, $w = \varepsilon w$, $x = x$, $y = \varepsilon y$ $z = \varepsilon z$, and $p = \varepsilon^2 p$.

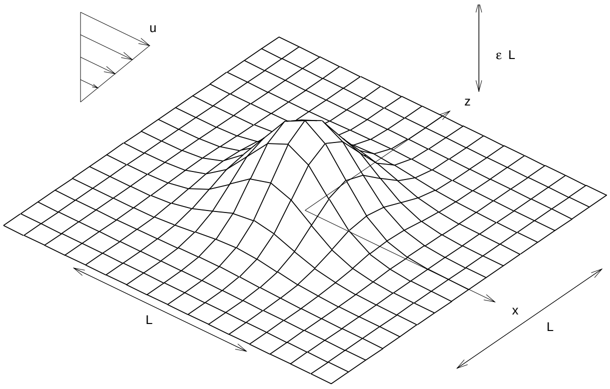

Figure 10: A bump in a shear flow

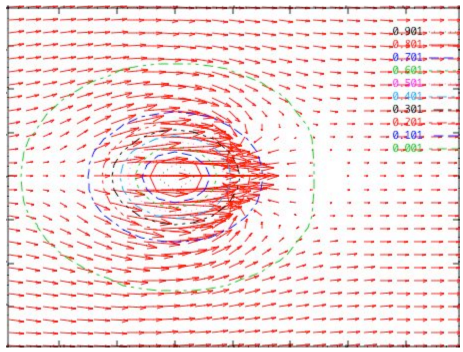

Figure 11: Field of skin friction.

2.8.1 Equations of the lower Deck

Equations are now:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0, \tag{21}$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u = -\frac{\partial}{\partial x}p + \frac{\partial^2}{\partial y^2}u, \tag{22}$$

$$u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w = -\frac{\partial}{\partial z}p + \frac{\partial^2}{\partial y^2}w, \tag{23}$$

with boundary conditions:
$u = v = w = 0$ in $y = f(x, z)$,
$y \to \infty$, $u = y$, $w = 0$
$x \to -\infty$, $u = y$, $v = 0$, $w = 0$.

First we do the Prandtl transform which transform the wall in a flat one: $y = y - f(x, z)$ so boundary conditions now becomes:
$u = v = w = 0$ in $y = 0$,
$y \to \infty$, $u = y + f(x, z)$, $w = 0$

$x \to -\infty$, $u = y$, $v = 0$, $w = 0$.
The system (21-23) being inchanged.

2.8.2 linearisation

We look at a linearized solution: $u = y + au_1$, $v = av_1$, $w = aw_1$, $p = ap_1$ with $a << 1$. The system (21-23) becomes:

$$\frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial y}v_1 + \frac{\partial}{\partial z}w_1 = 0, \tag{24}$$

$$y\frac{\partial}{\partial x}u_1 + v_1 = -\frac{\partial}{\partial x}p_1 + \frac{\partial^2}{\partial y^2}u_1, \tag{25}$$

$$y\frac{\partial}{\partial x}w_1 = -\frac{\partial}{\partial z}p_1 + \frac{\partial^2}{\partial y^2}w_1, \tag{26}$$

with boundary conditions:
$u_1 = v_1 = w_1 = 0$ in $y = f(x, z)$,
$y \to \infty$, $u_1 = +f(x, z)$, $w_1 = 0$
$x \to -\infty$, $u_1 = 0$, $v_1 = 0$, $w_1 = 0$.
Looking at solutions in Fourier space:

$$(u_1, v_1, w_1, p_1) = (\hat{u}(y), \hat{v}(y), \hat{w}(y), \hat{p}(y))e^{-ik_x x - ik_z z}$$

we obtain for the two last (25-26) after substitution and after elimination of the pressure by y derivation:

$$-ik_x y(k_x\frac{d}{dy}\hat{u} + k_z\frac{d}{dy}\hat{w}) = \frac{d^2}{dy^2}(k_x\frac{d}{dy}\hat{u} + k_z\frac{d}{dy}\hat{w})$$

we define a total skin friction $\tau_{uw} = (k_x\frac{d}{dy}\hat{u} + k_z\frac{d}{dy}\hat{w})$ and its value at the wall $\tau'_{uw}(0) = -i(k_x^2 + k_z^2)\hat{p}$ so the equation is an Airy one:

$$-ik_x y\tau_{uw} = (\tau_{uw})''$$

hence the solution of :

$$Ai''(z) = zAi(z), \quad \text{is } Ai(z) = \frac{1}{\pi}\int_0^\infty cos(\frac{t^3}{3} + zt)dt.$$

we have:

$$\tau_{uw} = -i(k_x^2 + k_z^2)\hat{p}Ai(y(-ik_x)^{1/3})/((-ik_x)^{1/3}Ai'(0))$$

$(k_x\hat{u}+k_z\hat{w}) = -i(k_x^2+k_z^2)(\int_0^y Ai(y(-ik_x)^{1/3})dy)\frac{\hat{p}}{((-ik_x)^{1/3}Ai'(0))}$ as $\hat{w}(\infty) = 0$
we obtain; $k_x f = \frac{-i(k_x^2+k_z^2)\hat{p}}{3((-ik_x)^{2/3}Ai'(0))}$.
The perturbation of pressure of mode (k_x, k_z) is then:

$$\hat{p} = \frac{3((-ik_x)^{2/3}Ai'(0))k_x f}{-i(k_x^2 + k_z^2)}$$

The final expression for the total skin friction:

$$(k_x \hat{u} + k_z \hat{w})' = 3((-ik_x)^{1/3} Ai(0)) k_x f$$

To solve equation for $\hat{w}$:

$$-ik_x y \hat{w} = ik_z \hat{p} + \frac{d^2}{dy^2} w$$

We notice that the solution of :

$$L''(z) = zL(z) - 1 \text{ is } L(z) = -\tfrac{2}{\sqrt{3}} \int_0^\infty cos(\tfrac{t^3}{3} + zt - \tfrac{\pi}{6}) dt$$

This allows to have $\hat{w}$ as $\hat{w} = -ik_y(-ik_x)^{-2/3} L((-ik_x)^{1/3} y)$.

This finally gives the perturbation for the skin friction

$$\frac{d\hat{u}}{dy} = 3((-ik_x)^{1/3} Ai(0)) k_x (1 - c(k_x, k_z)) \hat{f} \quad (27)$$

$$\frac{d\hat{w}}{dy} = 3((-ik_x)^{1/3} Ai(0)) \frac{k_x c(k_x, k_z)}{k_z} \hat{f} \quad (28)$$

$$c(k_x, k_z) = \frac{(-3Ai'(0)) k_z^2}{9Ai(0)^2 (k_x^2 + k_z^2)} \quad (29)$$

It is striking to observe that there is upstream influence in the 3D case see figures 12 and 13.

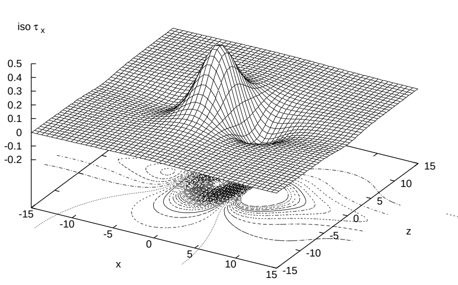

Figure 12: Longitudinal perturbation of skin friction $\partial u_1/\partial y$ on a gaussian bump

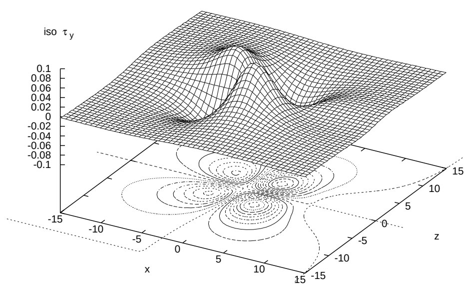

Figure 13: Transversal perturbation of skin friction $\partial w_1/\partial y$ on a gaussian bump

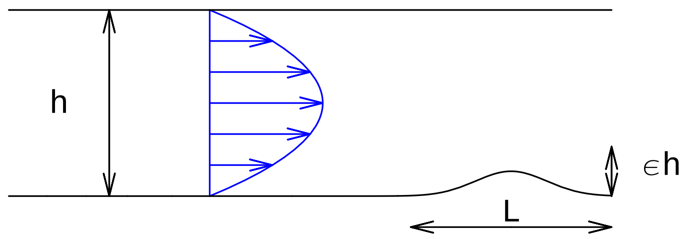

Figure 14: A small perturbation (a bump)of length $L = x_3 h$ and of height εh in a simple Poiseuille flow $u/U_0 = y/h(1 - y/h)$.

3 Pipe/ Channel Flow

3.1 Core flow

The previous section was devoted to the perturbation of a Couette profile. We showed the nonlinear response of the flow to a perturbation: in practice a bump. This Couette flow was either a pure Couette flow either a Poiseuille profile taken very near the wall. Now, in the sequel, we present what happens when perturbations of the flow near the wall are enough strong to disturb the core flow itself.

Of course, flow in pipes are very important and very common in all industrial devices and in biological flows. But, we will focus on flow between parallel plates which are more simple to write. The pioneering work is from Smith [26] and has been reexamined by Saintlos & Mauss [25].

The problem is defined on figure 14, there is a channel of width h. As we introduce explicitly the channel section h, we will use this scale to construct the Reynolds number. The basic solution is always the fully developped Poiseuille profile

$$(U_p(y), V_p(y)) = (U_0(y/h)(1 - y/h), 0), \quad \text{and} \quad p(x) = p_0 - 2U_0^2 \frac{x/h}{Re}$$

that we may writte without dimension $y = \bar{y}h$... The pressure gradient is then a constant $-dp/dx = 2$. We then introduce a bump in this flow and look at what happens. This bump is of small height εh and of lenght L. We will see what is the relation between L and εh in order to obtain a non linear problem.

3.2 Lower Deck

The Navier Stokes System (12) is always the same but with a different Reynolds number $Re = U_0 h/\nu$. In the viscous lower layer of height εh the velocity is always of order of magnitude εU_0 so the viscous inviscid balance is

$$\frac{1}{x_3}\varepsilon^2\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} \sim \frac{\varepsilon}{\varepsilon^2 Re}(\frac{\partial^2\bar{u}}{\partial\bar{y}^2})$$

the scale is then

$$x_3 = \varepsilon^3 Re$$

so a bump of length $h\varepsilon^3(U_0 h/\nu)$ and height εh will produce significant perturbation. In the Couette flow we used the length of the bump as scale L and found that height $L(U_0' L^2/\nu)^{-1/3}$ will produce significant perturbation. Subsitution of $L = h\varepsilon^3(U_0 h/\nu)$ in $L(U_0' L^2/\nu)^{-1/3}$ (remembering that $U_0' = U_0/h$ in a Poiseuille flow) gives of course εh.

So near the wall we have a thin layer that we will call the "lower Deck".

Then with the following least degeneracy scales:
$u = \varepsilon\bar{u}$, $v = \varepsilon^2\bar{v}$, $x = \varepsilon^3 Re\bar{x}$, $y = \varepsilon\bar{y}$ and $p = \varepsilon^2\bar{p}$ we obtain in fact the Prandtl equations with bars over all the variables, we then remove the bars:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{\partial p}{\partial x} + \dfrac{\partial^2 u}{\partial y^2}, \\ 0 = -\dfrac{\partial p}{\partial y}. \end{cases} \tag{30}$$

The boundary condition are the no slip conditions, the initial linear velocity profile far upstream.

$$u = v = 0 \text{ on } y = f(x),\ u \to y \text{ when } x \to -\infty, \tag{31}$$

We now no more suppose that disturbances are zero at infinity but that they still exist.

3.3 Main Deck

In the wall layer, the velocity is of order ε and the pressure is of order ε^2, using the matching principle, the velocity should match to the velocity when $y \to \infty$ in the lower Deck to the velocity $\tilde{y} \to 0$ (with $\tilde{y}$ is mesured by h) in the core flow. The same for the pressure.

We have the basic non dimensional profile $U_p(\tilde{y}) = \tilde{y}(1-\tilde{y})$ and $V_p = 0$ in the Pipe, now suppose that at longitudinal scale say x_3 there is a perturbation of this basic profile. We will call "Main Deck" the region considered which is of scale x_3 but which is of scale 1 in the transverse direction. So, suppose that at longitudinal scale say x_3 there is a perturbation of this basic profile of magnitude ε, then:

$$\tilde{u} = U_p(\tilde{y}) + \varepsilon\tilde{u}_1$$

as $\tilde{y} \to 0$, we see that the Poiseuille profile is linear $U_p(\tilde{y}) \to \tilde{y}$ and then the velocity is $\tilde{y} + \varepsilon\tilde{u}_1(x,0)$, written in the inner variables this is (as $\tilde{y} = \varepsilon y$)

$$\varepsilon(y + \tilde{u}_1(x,0))$$

so we deduce that in the lower deck

$$\lim_{y\to+\infty} u = y + \tilde{u}_1(x,0)$$

In order to retain all the terms in the incompressibility and in the total derivative equation,

$$\tilde{u} = U_p(\tilde{y}) + \varepsilon\tilde{u}_1, \;\; \tilde{v} = V_p(\tilde{y}) + \frac{\varepsilon}{x_3}\tilde{v}_1$$

longitudinal equation of momentum ($U_p\frac{\partial\tilde{u}_1}{\partial x} + \tilde{v}_1 U_p'$), is of order ε/x_3 which is larger than the pressure term in ε^2/x_3 and the small viscous terms.

A first system to solve is then

$$\frac{\partial\tilde{u}_1}{\partial x} + \frac{\partial\tilde{v}_1}{\partial\tilde{y}} = 0, \; (U_p\frac{\partial\tilde{u}_1}{\partial x} + \tilde{v}_1 U_p') = 0.$$

By elimination we find

$$U_p^2\frac{\partial}{\partial\tilde{y}}(\frac{\tilde{v}_1}{U_p}) = 0$$

the classical notation is then to introduce a function of x say $A(x)$, such as

$$\tilde{u}_1 = A(x)U_p'(\tilde{y}) \text{ and } \tilde{v}_1 = -A'(x)U_p(\tilde{y})$$

is solution of the system.

The velocity is then

$$\tilde{u} = U_p(\tilde{y}) + \varepsilon A(x)U_p'(\tilde{y}) \text{ or } \tilde{\psi} = \psi_p + A\varepsilon d\tilde{\psi}_p/d\tilde{y}$$

so the function $-A(x)$ may be understood as a displacement of the stream function. So it would have been a good idea to define this function with a minus sign, but Stewartson choose the reverse sign (remember that everything is reversed in Great Britain... French joke!). We then obtain the matching between the lower deck and the main deck as:

$$\lim_{y\to+\infty} u = y + A(x)$$

So when $-A$ increases the velocity decreases.

Then, in the transverse momentum equation $U_p\frac{\partial\tilde{v}_1}{\partial x}$ is of order $\frac{\varepsilon}{x_3^2}$ the transverse momentum is then of magnitude ε^2:

$$\frac{\varepsilon}{x_3^2}(-A''(x))U_p^2 \sim -\varepsilon^2\frac{\partial\tilde{p}_1}{\partial\tilde{y}}$$

so the least degeneracy principle gives $\frac{\varepsilon}{x_3^2} = \varepsilon$ i.e.

$$\varepsilon = Re^{-2/7} \text{ and } x_3 = R^{1/7}.$$

and by integration we have

$$\tilde{p}_1(x,1) - \tilde{p}_1(x,0) = A''(x)\int_0^1 U_p^2(\tilde{y})d\tilde{y}$$

There is a transverse pressure gradient across the flow.

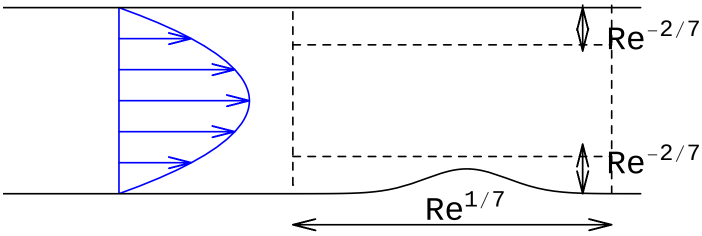

Figure 15: A bump at the lower will disturb the core flow, the pressure changes across the core flow, perturbations are induced at the upper wall.

3.4 upper lower deck

We have just obtained that the flow in the lower viscous layer matches with the bulk, but as there is an upper wall, perturbation are induced at this wall. So, writting $\tilde{y} = 1 - \varepsilon Y$, we have a viscous problem at the upper wall.

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial Y} = 0, \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial Y} = -\dfrac{dp}{dx} + \dfrac{\partial^2 u}{\partial Y^2}. \end{cases} \tag{32}$$

The boundary condition are the no slip conditions, the initial linear velocity profile far upstream.

$$u = v = 0 \text{ on } Y = 0, \; u \to Y \text{ when } x \to -\infty, \tag{33}$$

and the perturbation of stream lines due to the $-A$ function imposes a matching condition $u \to Y - A$. The pressure in this layer is $\tilde{p}_1(x,1)$.

3.5 Full interacting problem

Near the wall we have :

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{dp}{dx} + \dfrac{\partial^2 u}{\partial y^2} \end{cases} \tag{34}$$

with

$$u = v = 0 \text{ on } y = f(x),\ u \to y \text{ when } x \to -\infty, \text{ and } u \to y + A \text{ for } y \to \infty. \tag{35}$$

the pressure is in fact $\tilde{p}_1(x, 0)$. We have exactly the same problem at the upper layer, but the pressure is $\tilde{p}_1(x, 1)$ and we have the pressure jump across the pipe

$$\tilde{p}_1(x, 1) - \tilde{p}_1(x, 0) = A''(x) \int_0^1 U_p^2(\tilde{y}) d\tilde{y}$$

this gives the link between the pressure at the top and the bottom of the Main Deck.

3.6 The various scales in the Pipe

3.6.1 The $x_3 = R^{1/7}$ scale

We looked at a channel with a bump on a lower wall, we saw that there is pressure variation across the channel and an interaction with the upper wall layer even if the wall was flat.

Now we examine what happens if there are two indentations, one at the lower wall defined by $y = \varepsilon f(x)$ and another at the upper wall defined by $y = 1 + \varepsilon g(x)$. The problem is at the lower wall, in the lower layer after having performed a Prandtl Transform, one solves: the problem (34) with

$$u = v = 0 \text{ in } y = 0 \text{ and } u \to y + f(x) + A \text{ for } y \to +\infty$$

this gives the pressure which is matched to $p(x, 1)$ at the top of the main deck. At the upper one:

$$u = v = 0 \text{ in } Z = 0 \text{ and } u \to Z - g(x) - A \text{ for } Z \to +\infty$$

this gives the pressure which is matched to $p(x, 0)$ at the bottom of the Main Deck, the relation between the pressures is

$$\tilde{p}_1(x, 1) - \tilde{p}_1(x, 0) = A''(x) \int_0^1 U_p^2(\tilde{y})d\tilde{y}.$$

Notice that in the case here of the Poiseuille flow $\int_0^1 U_p^2(\tilde{y})d\tilde{y} = \frac{1}{30}$.

Linear solution

A linear perturbation of this system may be obtained in Fourier space, In the Lower Deck we have for the pressure $3Ai'(0)(-ik)^{-1/3}(\hat{f}_1 + \hat{A}_1)$ and in the upper lower Deck we have $3Ai'(0)(-ik)^{-1/3}(-\hat{g}_1 - \hat{A}_1)$ so as the difference between this two pressures is $-k^2\frac{1}{30}\hat{A}_1$, then, for example one can compute

$$\hat{A}_1 = \frac{\hat{f}_1 + \hat{g}_1}{2(1 - \frac{1/30}{6Ai'(0)}(-ik)^{7/3})} \tag{36}$$

the pressure in the lower deck as:

$$\hat{p}_1 = (3Ai'(0)(-ik)^{-1/3})(\hat{f}_1 + \frac{\hat{f}_1 + \hat{g}_1}{2(1 - \frac{1/30}{6Ai'(0)}(-ik)^{7/3})})$$

There is a unique problem canonical problem to solve:

$$u = v = 0 \text{ in } y = 0 \text{ and } u \to y + \tfrac{1}{2}(f(x) - g(x)) \text{ for } y \to +\infty$$

If the constriction is symmetrical, we have $A = 0$:

$$u = v = 0 \text{ in } y = 0 \text{ and } u \to y \text{ for } y \to +\infty$$

3.6.2 The $x_3 >> R^{1/7}$ scale

In this case the bumps are larger than the interacting scale, coming back to the transverse equation of momentum in the Main Deck:

$$\frac{\varepsilon}{x_3^2}U_p\frac{\partial \tilde{v}_1}{\partial x} = -\varepsilon^2\frac{\partial \tilde{p}_1}{\partial \tilde{y}}$$

If x_3 is larger than $R^{1/7}$ we then have $\frac{\partial \tilde{p}_1}{\partial \tilde{y}} = 0$. There is no pressure drop across the channel. At the lower wall, in the lower layer after having performed a Prandtl Transform, one solves: the problem (34) with

$$u = v = 0 \text{ in } y = 0 \text{ and } u \to y + f(x) + A \text{ for } y \to +\infty$$

this gives the pressure which is matched to $\tilde{p}_1(x, 1)$ at the top of the main deck. At the upper one, as the displacement A is the same through the Main Deck, it will reenter in the equation but with a minus sign (we are upside down), then the upper wall is defined upside down by $-g(x)$, so the Prandtl transform gives:

$$u = v = 0 \text{ in } Z = 0 \text{ and } u \to Z - g(x) - A \text{ for } Z \to +\infty$$

this gives the pressure which is matched to $\tilde{p}_1(x, 0)$ at the bottom of the Main Deck, so we deduce that as the two pressure must be equal, then the displacement function is such as the two displacements from the linear profile must be the same $-g - A = f + A$ so:

$$-A = \frac{1}{2}(f(x) + g(x)).$$

There is a unique problem to solve:

$$u = v = 0 \text{ in } y = 0 \text{ and } u \to y + \tfrac{1}{2}(f(x) - g(x)) \text{ for } y \to +\infty$$

re applying the backwards the Prandtl transform in f an g gives the solution in each lower and upper lower decks.

Note

Notice that if the constriction is symmetrical ($f(x) = -g(x)$), we have $A = 0$, the problem to solve is:

$$u = v = 0 \text{ in } y = 0 \text{ and } u \to y + f(x) \text{ for } y \to +\infty$$

3.6.3 The $1 << x_3 << R^{1/7}$ scale

We look now at short bumps, coming back to the transverse equation of momentum in the Main Deck:

$$\frac{\varepsilon}{x_3^2}U_p\frac{\partial \tilde{v}_1}{\partial x} = -\varepsilon^2\frac{\partial \tilde{p}_1}{\partial \tilde{y}}$$

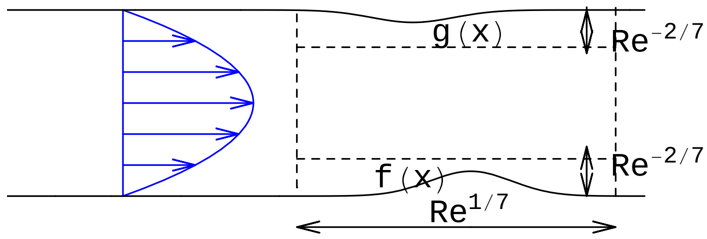

Figure 16: Two indentations in a channel at scales with interaction across the Poiseuille flow. On this graph f is positive, and g is negative. A symmetrical indentation is such as $f(x) = -g(x)$

we deduce that now that is $\frac{\partial \tilde{v}_1}{\partial x}$ which is zero, so that $A = 0$ is the solution in the Main Deck.

The problem is then well known: it is the same as in the unbounded Couette flow that we first examined.

3.6.4 The $x_3 = 1$ scale

This scale is interesting as it is the same behaviour $A = 0$, so at order one, so that there is no perturbation at order one. But, there is a perturbation at order 2, which is driven by the pressure and which does not interact with the wall layers.

$$\tilde{u} = U_p(\tilde{y}) + \varepsilon^2 \tilde{u}_2, \quad \tilde{v} = \varepsilon^2 \tilde{v}_2,$$

so we have to solve the problem at order 2:

$$\frac{\partial \tilde{u}_2}{\partial x} + \frac{\partial \tilde{v}_2}{\partial \tilde{y}} = 0, \quad (U_p \frac{\partial \tilde{u}_2}{\partial x} + \tilde{v}_2 U_p') = -\frac{\partial \tilde{p}_1}{\partial x}, \quad U_p \frac{\partial \tilde{v}_2}{\partial x} = -\frac{\partial \tilde{p}_1}{\partial \tilde{y}}.$$

It does not retroact on the wall layers.

3.6.5 The $x_3 = Re^{-1/2}$ scale

This structure in valid up to the smallest scale which is the scale such as the convective and both diffusive terms (transversal and longitudinal) are equal, i.e. $x_3 = Re^{-1/2}$. This scale is in fact the viscous scale constructed on the shear $U_0' = U_0/h$:

$$x_3 h = \sqrt{\frac{\nu}{U_0'}}.$$

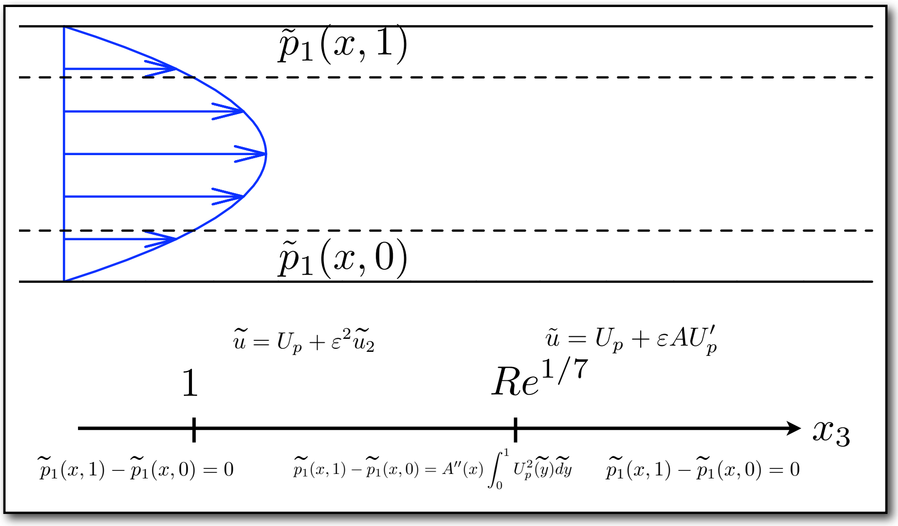

Figure 17: the scales in the channel

3.6.6 Upstream influence

We saw that the $A = 0$ system described in the case of Couette flow or in the case of no displacement (symmetrical channel etc) presents no perturbation of the flow before the beginning of the bump. This behavior was clear because:

• the 'Lower Deck" equations are a kind of heat equation ($\partial_t = \partial_y^2 T$) and are parabolic in x as the heat is in time t.

• as for the heat equation there is no influence of the future on the present, in the Lower Deck equation there is no influence of the positions downstream on a given position.

• this is clear on the expression with integrals, for example, the perturbation of the skin friction involves: $\displaystyle\int_0^\infty \frac{f'(x-\xi)}{\xi^{1/3}} d\xi$. So when there is no bump, there is no linear response.

• FLARE approximation reintroduces some influence on the downstream to to upstream but, it will be done at a very small scale and after the beginning of the bump.

We have seen on the numerical examples that when there is a pressure variation across the Main Deck, before the very birth of the bump, the flow is perturbed. So in this case, there is "upstream influence", it means that the downstream experience of the flow retroacts on the upstream.

The striking feature associated to this is that a self induced interaction may appear (we will see this again in the thermal flows in hypersonic régime and in mixed convection régime). This interaction is due to the deflection of the stream line and to the difference of pressure due to the curvature:

$$p_{up} - p_{down} = A''(\bar{x}) \int_0^1 U_p^2(\tilde{y}) d\tilde{y}.$$

In fact, suppose that there is a small perturbation of pressure in the flow near the lower wall. Suppose that this small perturbation is a small increase of pressure (say p_{down}), the response of the flow is a deceleration so that $-A$ start to zero and becomes positive, so $-A'$ is positive (or A' negative). This deceleration promotes a positive displacement $-A$ in the Lower Deck which affects the stream lines in the Main Deck and in the Upper Lower Deck as well. In this Upper Lower Deck, it is a A displacement (sign reversed) so the pressure (say p_{up}) the exact opposite of the small increase of pressure of the Lower Deck, it decreases: $p_{up} = -p_{down} < 0$.

Then the second derivative of A is negative, so $-A$ has a positive curvature, so that A' is more negative, A decreases a bit more, the displacement $-A$ increases a bit more, the velocity decreases a bit more so that the pressure increases .... this increase of pressure increases again the displacement $-A$, so it is a selfinduced interaction of the flow.

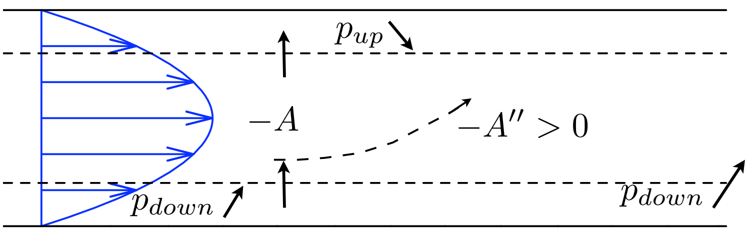

Figure 18: Self induced interaction: "upstream influence", increase of p_{down} increases $-A$, which in turn decreases p_{up}, as $p_{up} = -p_{down}$, the cross pressure variation $p_{up} - p_{down} = A'' \int_0^1 U_p^2 dy$ gives an positive curvature for $-A$ so $-A$ increases again increasing p_{down}.

To put formulas on words, we look whether we can obtain e^{Kx} solutions on the linearized system, with $K > 0$. $u_1 = e^{Kx}\phi'(y)$ $v_1 = -e^{Kx}\phi(y)$, $p_1 = e^{Kx}P$ with $\phi(0) = \phi'(0) = 0$ and say $\phi'(\infty)$ is the value of the perturbation of A; as the incompressibility is fulfilled, the momentum is

$$Ky\phi'(y) - \phi(y) = -KP + \frac{\partial^2 \phi'(y)}{\partial y^2}, \tag{37}$$

so $\frac{\partial^2 \phi''(y)}{\partial y^2} = Ky\phi''(y)$, and as $\phi''(0) = KP$, so ϕ'' is $K^{2/3}Ai(K^{1/3}y)P/Ai'(0)$ and $\phi' = \frac{K^{1/3}P}{Ai'(0)} \int_0^y Ai(\xi)d\xi$ so that we deduce $\phi'(\infty) = \frac{K^{1/3}}{3Ai'(0)}P$ For the upper layer, the same will be done with a change of sign, the pressure relation is then

$$-\frac{3Ai'(0)}{K^{1/3}}-\frac{3Ai'(0)}{K^{1/3}}=K^2(\frac{1}{30})\ \text{hence}\ K=\big(-180Ai'(0)\big)^{3/7}.$$

This is an eigen solution of the system. $K \sim 5.187 > 0$. So self induced interactions may appear in a pipe flow, in fact those solutions are the influence of the downstream accident on the upstream as we will see on figure 20.

This upstream influence is clear as well from linearised expression of eq (36) which is: $\hat{A}_1 = \dfrac{\hat{f}_1+\hat{g}_1}{2(1-\frac{1/30}{6Ai'(0)}(-ik)^{7/3})}$, as one clearly see as well that if the channel is symmetrical, $\hat{A}_1 = 0$. Further more in going back in real space there is a pole $(1-\frac{1/30}{6Ai'(0)}(-ik)^{7/3})$ which is of course the value $-ik = (\big(-180Ai'(0)\big)^{3/7})$. So the upstream influence may be interpreted as the existence of this pole.

3.6.7 Example of linear computation

On figure 19 we plot the perturbation of skin friction in the linear case for the $A = 0$ case and the $A'' \int_0^1 U_p^2 d\tilde{y}$ case. We see the that the case $A = 0$ presents no upstream influence as already mentioned, but we clearly see that the case with A'' promotes upstream response of the flow (before the first position of the bump, the pressure has increased and the skin friction has decreased). On figure 20 we plot the eigen solution in $exp(Kx)$ and the A'' response.

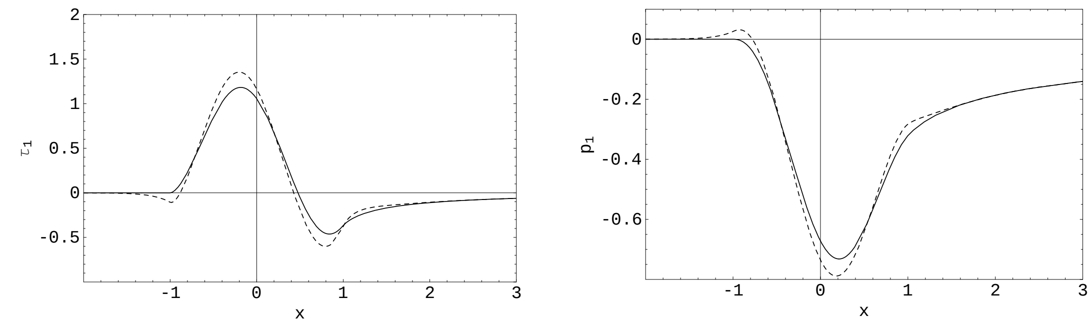

Figure 19: Linear perturbation of skin friction τ_1 (left) and pressure p_1 (right) over a bump $f_1(x) = cos(\pi x/2)^2$, for $-1 < x < 1$. The $A = 0$ is in plain line and the A'' case is in dashed line.

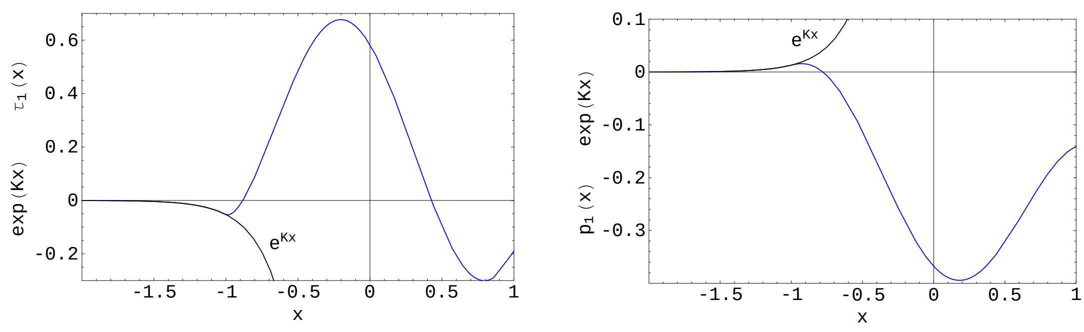

Figure 20: Linear perturbation of skin friction τ_1 (left) and pressure p_1 (right) over a bump $f_1(x) = cos(\pi x/2)^2$, for $-1 < x < 1$ In the A'' case. The eigen solution $exp(Kx)$ is plotted as well.

4 Conclusion

One important scaling is the 1/3 one which comes from the balance

$$y\frac{\partial}{\partial x} \sim \frac{\partial^2}{\partial y^2}.$$

This balance was first seen by Lévêque. This scaling represents a balance between convection and diffusion, it is valid for momentum as well.
One second point is the possibility of separation of the flow in the thin layer near the wall where the basic flow is linear.
One third point is the parabolicity of the flow in the lower layer.
One fourth feature is the upstream influence when there are interactions with other part of the flow.

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