SCIENCES SORBONNE UNIVERSITÉ

Multiscale Hydrodynamic Phenomena

M2, Fluid mechanics, MU5MEF15 2025/2026
Friday December 5 8 :30am - 12 :30pm AMPHI.43 Part I. : 75 minutes, NO documents

1. Quick Questions In few words and few formula :
1.1 Order of magnitude of drag on a sphere of radius L at small Re.
1.2 Order of magnitude of drag on a cylinder of radius L at small Re.
1.3 What is Oseen problem ?
1.4 ∂'Alembert equation : write the equation and the generic solution of it in 1D.
1.5 Solution of Laplace equation in upper half domain ($\forall x$ and $y \geq 0$) with an Neumann boundary contion in $y = 0$ and decay to 0 at infinity, what is the solution on the line $y = 0$?
1.6 Write 2D Boundary Layer equations (in x, y, u, v) in the case of Blasius problem, what is the scale of y compared to the scale of x ? What about second order ?
1.7 In which one of the 3 decks of Triple Deck is flow separation ?
1.8 Difference between Boussinesq approximation and Low Mach approximation ?
1.9 What is the KdV equation ? What balance is it ?
1.10 What are shallow water equations ?

2. Exercice
Consider the following equation (of course ε is a given small parameter)

$$(E_\varepsilon) \quad \varepsilon \frac{d^2u}{dx^2} + \frac{du}{dx} = x \text{ with } u(0) = 0 \quad u(1) = 1.$$

We want to solve this problem with the Matched Asymptotic Expansion method.
2.1 Why is this problem singular ?
2.2 What is the outer problem and what is the possible general form of the outer solution ?
2.3 What is the inner problem of (E_ε) and what is the inner solution ?
2.4 Suggest the plot of the inner and outer solution.
2.5 Composite expansion.
2.6 Plot of composite solution for different values of ε

3. Exercice
Let us look at the following ordinary differential equation : $(E_\varepsilon) \quad \dfrac{d^2y}{dt^2} + \varepsilon \dfrac{dy}{dt} + 9y = 0,$ valid for any $t > 0$ with boundary conditions $y(0) = 1$ and $y'(0) = 0$. Of course ε is a given small parameter.
3.1 Solve with Feynman averaging method.
3.2 We want to solve this problem with Multiple Scales Analysis. Introduce two time scales, $t_0 = t$ and t_1, what is the relation between t, t_1 and ε ?
3.3 Compute $\partial/\partial t$ and $\partial^2/\partial t^2$
3.4 Solve the problem.
3.5 Suggest the plot of the solution.
3.6 What is the exact solution for any ε, compare.

4. Exercice
Solve with WKB approximation the problem

$$\varepsilon^2 (y''(x) + y'(x)) + 9y = 0 \text{ with } y(0) = 1, y'(0) = 0$$

and compare with analytical solution

empty page

M2, Fluid mechanics 2024/2025
Friday December 5 AMPHI.43,
all documents.

Multiscale Hydrodynamic Phenomena

Part II : 1h15

Viscous collapse

This is a part of "The propagation of two-dimensional and axisymmetric viscous gravity currents over a rigid horizontal surface" By Herbert HUPPERT

We consider a 2D unsteady viscous collapse of an initially given heap of fluid over an horizontal flat plate ($z = 0$, $x > 0$). At $t = 0$, the initial data is a given shape $h(x, 0)$. The gravity is $\overrightarrow{g} = -g\overrightarrow{e}_z$, See figure 1. of the paper. The heavy viscous flow spreads at the bottom of the lighter one. Note that we will consider a case without injection so that the flow is symmetrical with respect to $x = 0$, and $\alpha = 0$. The aim of the problem is to write (2.1)-(2.10) with all the small parameters ε that you will identify in order to check all the hypothesis of the paper.

As all the results are more or less in the paper, be careful and rigorous to prove the results. Try to always give the estimates with an $O(\varepsilon)$, and define precisely each ε you identify.

Numbers refer to equations in the papers (Eq. X.) or questions (Q. X.).

First, with no flow. Suppose that the layers H and h are constant and infinite in x.
1.1 Starting form Navier Stokes at rest show and explain that the equilibrium solution for pressure is indeed (Eq. 2.1). What is the name of this dependance of p in z ?
1.2 If we suppose that the upper fluid (the "ambiant fluid") of density $\rho - \Delta\rho$ is far less dense than the fluid at bottom of density ρ, how degenerates (Eq. 2.1) and (Eq. 2.3) ?

Second, with flow but $(1 - \Delta\rho/\rho) \to 0$ (neglect the "ambiant fluid")
2.1 Write full incompressible NS equations for the lower fluid in this limit.
2.2 What are the boundary conditions at the interface of the lower fluid and the neglected "ambiant fluid".
2.3 We write the shape of the interface without dimension : use say $h(x, t) = h_0\bar{h}(\bar{x}, \bar{t})$ and $x = L_0\bar{x}$, (time $t = \tau\bar{t}$), find a sentence saying that $h_0/L_0 \ll 1$ in the text.
2.4 Define $u = U_0\bar{u}$, and $w = W_0\bar{w}$ (give W_0 with U_0, h_0, L_0), use for pressure $p = \Pi\bar{p}$. From transverse equation show that the pressure is hydrostatic at dominant order, value of Π ? Find again (Eq.1) in the $(1 - \Delta\rho/\rho) \to 0$ limit
2.5 Give a pertinent scale for τ
2.6 Explain the balance (Eq. 2.2).
2.7 Balance (Eq. 2.2) gives U_0, write its value in terms of ν, g, h_0, L_0...
2.8 We consider $\alpha = 0$, what is (Eq. 2.10), what is the relation between q, L_0, h_0
2.9 Once the $O(1)$ viscous term is identified in (Q. 2.7), what is the parameter in front of $d\bar{u}/d\bar{t}$ in the longitudinal momentum ? Same for $d\bar{w}/d\bar{t}$?
2.10 Discuss the smallness of these parameters (of Q.2.9). If the parameter in front of $d\bar{u}/d\bar{t}$ is of order one, show that this is a special scale $L_0 = L_s$, identify this length with q, ν etc. What does it mean ? Compare L_s and L_0 to discuss the order of magnitude of longitudinal inertia.
2.11 Show that at dominant order (Eq. 2.6) is one of the boundary condition.
2.12 What are the other boundary conditions ?
2.13 Prove (Eq. 2.8), where does it come from ? Check that the scale of τ of (Q.2.5) is indeed the good one.
2.14 Write the final asymptotic problem (without dimension) with all the equations to be solved with all boundary conditions. 2.15 Find (Eq .2.7) as a solution of (Q.2.14)
2.16 Explain how to obtain (Eq 2.9) and the condition (2.10).

Third, Solution of the PDE (Eq 2.9)-(2.10) (without dimension)
3.1 Show that (Eq 2.9) without dimension admits a self similar solution. Explain all the steps of the method leading to (Eq.2.11) and (2.12). 3.2 Find the equation (2.13) (with $\alpha = 0$), integrate it once
3.3 find the selfsimilar (2.16) and (2.17) solution using the identity
$\int_0^1 (1-x^n)^m dx = \left[\frac{\Gamma(m+1)\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(m+\frac{1}{n}+1\right)}\right]$. where $\Gamma : z \mapsto \int_0^{+\infty} t^{z-1}\, e^{-t}\, \mathrm{d}t$ which satisfies $\Gamma(z+1) = z\,\Gamma(z)$.
3.4 Find the position of the front as function of time, (Eq. 2.18), with dimensions. Find the maximal height as function of time.
3.5 Plot the shape of the surface of the spreading fluid as function of time.

Fourth, $\alpha \neq 0$
4.1 If you have time (as this problem is very classical) show that in the case of $\alpha \neq 0$ the orders of magnitudes of the scaling gives indeed (Eq.2.18) for the run out (i.e. $(gq^3/\nu)^{1/5} t^{(3\alpha+1)/5}$).
4.2 Paragraph at the bottom of page 44, looking at the scale of inertial terms versus viscous terms, show that the "Julian number" $J \gg 1$.

Fifth, the paper continues in axi ... if you have still time, do the same for an axisymmetric current.

Biblio

The propagation of two-dimensional and axisymmetric viscous gravity currents over a rigid horizontal surface By HERBERT E. HUPPERT J . Fluid Mech. (1982), vol. 121, p p . 43-58

J. Fluid Mech. (1982), *vol.* 121, *pp.* 43–58
Printed in Great Britain

The propagation of two-dimensional and axisymmetric viscous gravity currents over a rigid horizontal surface

By HERBERT E. HUPPERT
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW

(Received 2 September 1981 and in revised form 23 March 1982)

The viscous gravity current that results when fluid flows along a rigid horizontal surface below fluid of lesser density is analysed using a lubrication-theory approximation. It is shown that the effect on the gravity current of the motion in the upper fluid can be expressed as a condition of zero shear on the unknown upper surface of the gravity current. With the supposition that the volume of heavy fluid increases with time like t^{α}, where α is a constant, a similarity solution to the governing nonlinear partial differential equations is obtained, which describes the shape and rate of propagation of the current. The viscous theory is shown to be valid for $t \gg t_1$ when $\alpha < \alpha_c$ and for $t \ll t_1$ when $\alpha > \alpha_c$, where t_1 is the transition time at which the inertial and viscous forces are equal, with $\alpha_c = \frac{7}{4}$ for a two-dimensional current and $\alpha_c = 3$ for an axisymmetric current. The solutions confirm the functional forms for the spreading relationships determined for $\alpha = 1$ in the preceding paper by Didden & Maxworthy (1982), as well as evaluating the multiplicative factors appearing in the relationships. The relationships compare very well with experimental measurements of the axisymmetric spreading of silicone oils into air for $\alpha = 0$ and 1. There is also very good agreement between the theoretical predictions and the measurements of the axisymmetric spreading of salt water into fresh water reported by Didden & Maxworthy and by Britter (1979). The predicted multiplicative constant is within 10% of that measured by Didden & Maxworthy for the spreading of salt water into fresh water in a channel.

1. Introduction

Open the front door of a centrally-heated house and a gravity current of cold air immediately flows in. The essential feature of such gravity currents is that fluid of one density flows primarily horizontally into fluid of a different density and the motion is driven by gravitational forces. Gravity currents originate in many different situations and have recently been subjected to extensive investigation. Interest in these flows spans the fields of geophysics, industrial engineering and geology, as is evident from the examples of thunderstorm outflows, polluted discharges into rivers or lakes, spreading lava domes and pyroclastic flows from volcanic explosions. Further examples are cited by Simpson (1982) in a recent review of the field.

Most investigators have studied high-Reynolds-number gravity currents, for which the motion is governed by a balance between buoyancy and inertial forces. In the preceding paper, Didden & Maxworthy (1982) consider low-Reynolds-number gravity currents propagating over a rigid horizontal surface, for which the motion is governed by a balance between buoyancy and viscous forces. Didden & Maxworthy

studied the rate of propagation of the front of two-dimensional and axisymmetric gravity currents due to a constant inflow rate of new fluid into fluid of much greater depth. From order-of-magnitude evaluations of the buoyancy forces, inertial forces and viscous forces, they obtained an estimate of the time, t_1 say, beyond which viscous forces overwhelm inertial forces. They also determined for $t \gg t_1$ dimensional relationships between the position of the front as a function of time and the external parameters. In a series of experiments they verified the functional forms of the position-versus-time relationships and evaluated the constants of proportionality from their experimental measurements.

Previous studies of low-Reynolds-number gravity currents include those by Fay (1969) and Hoult (1972), who analysed the release of a fixed volume of relatively light fluid flowing under a free surface. The viscous forces in this case are quite different to those relevant for the case of a relatively heavy current flowing along a rigid bottom. One can hence anticipate that the spreading relationships and shapes of the currents in the two cases will be entirely different. A pertinent result for viscous gravity currents propagating above a horizontal surface is briefly presented by Britter (1979). Britter performed a series of experiments using an axisymmetric geometry, in which a fixed flux of salt water was released into a very much deeper layer of fresh water. The experiments were designed mainly to investigate large-Reynolds-number gravity currents. However, in the last appendix of his paper, Britter reports data on the spreading relationships valid for times very much longer than the transition times t_1. We compare our theoretically determined results with these experimental ones in §4.

The primary aim of this paper is to determine entirely from theoretical considerations the spreading relationships for and shapes of two-dimensional and axisymmetric viscous gravity currents propagating over rigid horizontal surfaces. As sketched in figure 1, we consider fluid of density ρ intruding into fluid of density $\rho - \Delta\rho$ ($\Delta\rho > 0$) and depth H so that the volume at any time is proportional to t^{α}, where α is some non-negative constant. Thus $\alpha = 0$ corresponds to the release of a constant volume of fluid and $\alpha = 1$ to a constant flux. In developing the theory we assume that the two fluids are not of greatly different viscosities and neglect the effects of mixing between them. We also neglect the effects of surface tension at the interface and at the front, or contact line. This requires that the Bond number $B = \rho g' a^2/T \gg 1$, where g' is the reduced gravity, a the length, or radius, of the current and T the surface tension.

Only viscous and buoyancy effects are taken into consideration, so the theory can only be expected to be valid when the viscous forces are much larger than the inertial forces. This occurs when $\alpha < \alpha_c$ for $t \gg t_1$, where t_1 as a function of α and the other external parameters is estimated in appendix A, (equation (A 7)) and $\alpha_c = \frac{7}{4}$ for a two-dimensional current and $\alpha_c = 3$ for an axisymmetric current. If $\alpha = \alpha_c$, the viscous forces greatly exceed the inertial forces, and do so for all time, provided that the Julian number $J = \nu^3 g'^2/q^4 \gg 1$ for two-dimensional currents or that $\nu g'/Q \gg 1$ for axisymmetric currents, where q or Q is the constant of proportionality between the volume of the current and t^{α}. If the inequalities are reversed the current propagates under the influence of inertial forces. For $\alpha > \alpha_c$ the viscous forces initially dominate the inertial forces and so the viscous theory will be valid only for $t \ll t_1$; thereafter the influence of the inertial forces becomes important.

In all cases, except possibly at the initiation of the current, its length will greatly exceed its thickness, which allows for the boundary-layer approximation common to lubrication theory to be used. We argue in §2, which presents the theoretical development, that the effect on the gravity current of the motion in the upper fluid can be entirely taken into account by applying the boundary condition of zero stress at the upper surface of the current. The current is thus identical with one propagating with a free surface beneath fluid of negligible inertia. The resulting nonlinear partial differential equations have similarity solutions which yield the shape and rate of propagation of the current.

For $\alpha = 1$, the only case considered by Didden & Maxworthy, the calculated functional forms of the position of the front of the current as a function of time are the same as evaluated by them. Further, our calculated value for the constant of proportionality is in excellent agreement with those presented for axisymmetric currents by Britter (1979) and by Didden & Maxworthy, but 10% higher than that presented by Didden & Maxworthy for two-dimensional currents. We discuss in more detail below a plausible explanation for this discrepancy which incorporates an estimate of the effects of drag at the sidewalls in the experiments.

Following the derivation and solution of the equations in §2, we present the results of a series of experiments with viscous silicone oils spreading axisymmetrically into air from either a fixed volume or fixed flux release. We find that there is extremely good agreement between our theoretically calculated and experimentally measured rate of spreading for both types of release. Accurate measurements of the thickness at the centre of the current for a fixed volume release are difficult to make, but the results of such measurements are shown to be in quite good agreement with the theory.

2. Theory

2.1. *Two-dimensional currents*

Consider a current of density ρ intruding into a fluid of density $\rho - \Delta\rho$ and depth H. Since the vertical velocities are negligibly small, the pressure is hydrostatic and, with the co-ordinate system sketched in figure 1, the pressure in the gravity current is given by

$$p = p_0 + (\rho - \Delta\rho)\, g(H-h) + \rho g(h-z), \tag{2.1}$$

where p_0 is the (constant) pressure at $z = H$. The balance between the pressure gradient and the viscous forces is thus expressed by

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = g'\frac{\partial h}{\partial x} = \nu \frac{\partial^2 u}{\partial z^2}, \tag{2.2}$$

where

$$g' = (\Delta\rho/\rho)\, g, \tag{2.3}$$

and horizontal derivatives have been neglected in comparison with vertical derivatives on the right-hand side of (2.2) because the length of the current is very much greater than its thickness. At the base of the current

$$u(x, 0, t) = 0, \tag{2.4}$$

and at the top of the current the shear stress is continuous and thus

$$\mu \frac{\partial u}{\partial z} = \mu_a \frac{\partial u_a}{\partial z} \quad (z = h), \tag{2.5}$$

where the subscript a refers to quantities in the ambient fluid. The motion in this

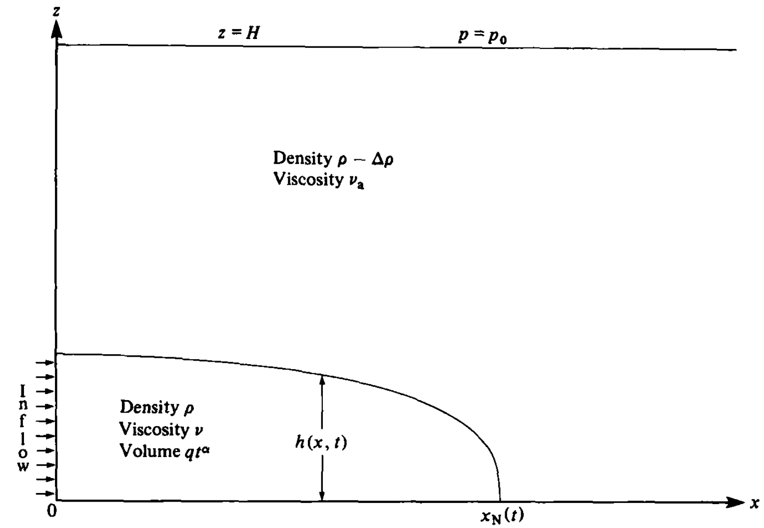

FIGURE 1. A sketch of the flow field and co-ordinate system.

fluid is due to viscous diffusion, and hence the ratio of the term on the right-hand side of (2.5) to that on the left is given, in order of magnitude, by $h_0/(\nu_a t)^{\frac{1}{2}}$, where h_0 is a representative thickness of the current. When the current is propagating under a buoyancy–viscous balance this ratio, as calculated in appendix B, is very much less than unity. The shear stress at the top of the current is hence very much less than its value within the current, and thus (2.5) can be approximated as

$$\frac{\partial u}{\partial z}(x, h, t) = 0. \tag{2.6}$$

The solution of (2.2), (2.4) and (2.6) is

$$u(x, z, t) = -\frac{1}{2}\frac{g'}{\nu}\frac{\partial h}{\partial x}\, z(2h - z). \tag{2.7}$$

Exactly the same velocity profile occurs if the motion in the ambient fluid is neglected, and at the top of the current the pressure is constant and the shear stress is zero. The flow is thus identical with that produced by a current spreading into fluid of negligible inertia. The physical explanation of this result is that the viscous dissipation in the current dominates over the dissipation in the fluid above it. A further relation between the unknowns u and h is

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^h u\, dz\right) = 0, \tag{2.8}$$

which is the depth-integrated version of the equation of continuity. Substituting (2.7) into (2.8), we obtain

$$\frac{\partial h}{\partial t} - \frac{1}{3}\frac{g'}{\nu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) = 0, \tag{2.9}$$

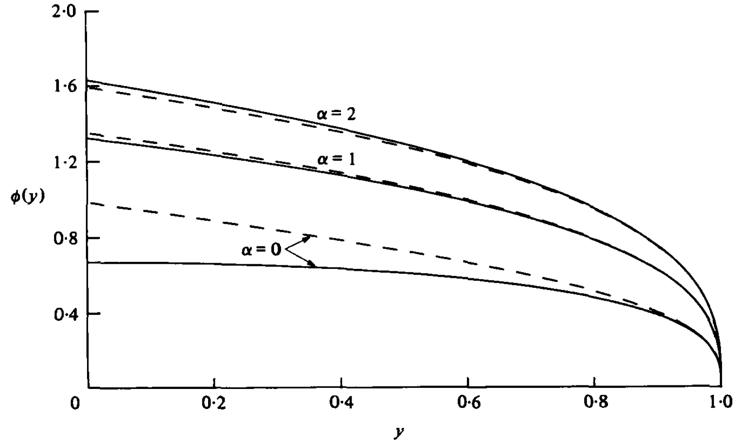

FIGURE 2. The shape of a two-dimensional viscous gravity current for $\alpha = 0$, 1 and 2. The dashed lines represent the approximate shapes (2.15).

as the governing nonlinear partial differential equation for $h(x, t)$. It is the global continuity equation

$$\int_0^{x_N(t)} h(x, t)\, dx = qt^\alpha \tag{2.10}$$

that completes the mathematical specification of the problem.

While numerical solutions to (2.9) and (2.10) for any given initial conditions could be obtained, it is much more fruitful to determine the similarity solutions of (2.9) and (2.10), to which all solutions with sufficiently regular initial conditions will tend except possibly at $x = 0$. The similarity solution can be expressed in terms of the similarity variable

$$\eta = (\tfrac{1}{3} g' q^3/\nu)^{-\frac{1}{5}} x t^{-(3\alpha+1)/5} \tag{2.11}$$

and the solution form

$$h(x, t) = \eta_N^{\frac{2}{3}} (3q^2\nu/g')^{\frac{1}{5}} t^{(2\alpha-1)/5} \phi(\eta/\eta_N), \tag{2.12}$$

where η_N is the value of η at $x = x_N(t)$. Substituting (2.11) and (2.12) into (2.9) and (2.10), we find that $\phi(y)$, where $y = \eta/\eta_N$, satisfies

$$(\phi^3\phi')' + \tfrac{1}{5}(3\alpha+1)\, y\phi' - \tfrac{1}{5}(2\alpha-1)\,\phi = 0, \tag{2.13}$$

and that

$$\eta_N = \left(\int_0^1 \phi\, dy\right)^{-\frac{3}{5}}. \tag{2.14}$$

Incorporating the condition that $\phi(1) = 0$ in (2.13), we find that about $y = 1$

$$\phi(y) = \left[\frac{3}{5}(3\alpha+1)\right]^{\frac{1}{3}} (1-y)^{\frac{1}{3}} \left[1 - \frac{3\alpha-4}{24(3\alpha+1)}(1-y) + O(1-y)^2\right], \tag{2.15}$$

which is sufficient to specify a unique solution of (2.13).

For arbitrary α the solution can only be obtained by numerically integrating (2.13) inward from $y = 1$ and using (2.15) as a starting condition. However, for $\alpha = 0$ the

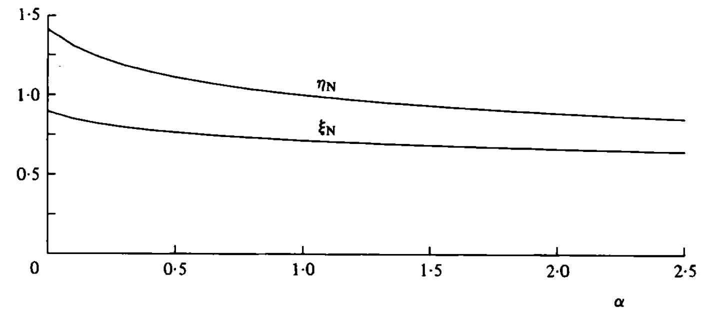

FIGURE 3. The values of the constants η_N and ξ_N as functions of α.

solution of (2.13) and (2.14) can be obtained analytically, and is

$$\phi(y) = \left(\frac{3}{10}\right)^{\frac{1}{3}} (1-y^2)^{\frac{1}{3}}, \tag{2.16}$$

$$\eta_N = \left[\frac{1}{5}\left(\frac{3}{10}\right)^{\frac{1}{3}} \pi^{\frac{1}{2}}\Gamma(\tfrac{1}{3})/\Gamma(\tfrac{5}{6})\right]^{-\frac{3}{5}} = 1{\cdot}411\ldots, \tag{2.17}$$

as first determined by Pattle (1959). The existence of this analytic solution is convenient as it allows for a check to be made on the numerical integration schemes used to solve (2.13) and (2.14). Solutions of (2.13) for $\alpha = 0$, 1 and 2 are presented in figure 2, along with the first term in the expansion of the solution about $y = 1$ as given in (2.15). These approximate solutions are seen to agree quite well with the exact solutions. Note that, since at any fixed x the mass flux is linearly proportional to the surface slope $\partial h/\partial x$, except for $\alpha = 0$ the slope of the exact solutions at $x = 0$ is non-zero, as is consistent with the fact that fluid is being continually introduced here. In figure 3 a graph of η_N as a function of α is presented, from which the length of the current

$$L \equiv x_N(t) = \eta_N (\tfrac{1}{3} g' q^3/\nu)^{\frac{1}{5}} t^{(3\alpha+1)/5} \tag{2.18}$$

can be determined.

Note that there are no undetermined parameters in the solution, in contrast to the case of a relatively light current propagating under a free surface, analysed quantitatively by Hoult (1972). Hoult gives a clear description of the inadequacies of his approximations and an explanation of why he needed to resort to an experimental determination in order to complete the calculation. The important difference between the two studies is that in Hoult's case the motion of, and dissipation in, the ambient fluid determines the rate of propagation of the current. This is in contrast to our case, where the motion in the ambient fluid may be neglected.

2.2. *Axisymmetric currents*

The radial spreading of an axisymmetric current can be analysed in a similar manner to the spreading of a two-dimensional current. With a co-ordinate system as in figure 1 except that the radial co-ordinate r replaces x, the velocity profile in the current is

$$u(r, z, t) = -\frac{1}{2}\frac{g'}{\nu}\frac{\partial h}{\partial r} z(2h-z). \tag{2.19}$$

1.1 hydostatic
1.2 trivial
2.1 see
2.2
2.3 "The length is very much greater than its thickness"
2.4 $\Pi = \rho g \varepsilon$
2.5 $\tau = L/U_0$
2.6 pressure gradient is balanced by viscous terms
2.7 $U_0 = g\varepsilon^3 L^2/\nu$
2.8 $q = g\varepsilon^4 L^3/\nu$
2.9 Full final system with all terms

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial \tilde{x}} + \dfrac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \\ \frac{gL^3}{\nu^2}\varepsilon^5 \left(\dfrac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\dfrac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{w}\dfrac{\partial \tilde{u}}{\partial \tilde{z}} \right) = -\dfrac{\partial \tilde{p}}{\partial \tilde{x}} + (\varepsilon^2 \dfrac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \dfrac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}) \\ \varepsilon\frac{gL^3}{\nu^2}\varepsilon^5 \left(\dfrac{\partial \tilde{w}}{\partial \tilde{t}} + \tilde{u}\dfrac{\partial \tilde{w}}{\partial \tilde{x}} + \tilde{v}\dfrac{\partial \tilde{w}}{\partial \tilde{z}} \right) = -\dfrac{\partial \tilde{p}}{\partial \tilde{z}} + (\varepsilon^3 \dfrac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} + \varepsilon^2 \dfrac{\partial^2 \tilde{w}}{\partial \tilde{y}^2}) - 1. \end{cases} \tag{1}$$

the coefficient $\frac{gL^3}{\nu^2}\varepsilon^5$ is in front of $d\bar{u}/d\bar{t}$ in the longitudinal momentum.

$q = \frac{gL^3}{\nu}\varepsilon^4$

2.10 if $\frac{gL^3}{\nu^2}\varepsilon^5 = 1$