Couette Flow of Dry Granular Materials in the Proceedings of STAMM 2004

Christophe Josserand[1], Pierre-Yves Lagrée[1], and Daniel Lhuillier[1]

Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, France

Abstract. A continuum-mechanical model for the steady flows of dry granular materials is proposed, based on a modelling of the stresses generated by contacts and impacts between particles. The model takes the non-homogeneity of these materials into account via several transport coefficients depending on the solid fraction. When applied to the Couette flow between rotating coaxial cylinders, the model leads to compaction and velocity profiles which are in close agreement with those found experimentally.

1 Introduction

A Couette cell is certainly amongst the most convenient devices for determining the stationary flow behaviour of complex materials. The main specificity of dense granular media is their localized motion, the so-called shear localization. The main difficulty with granular media is their opacity which explains why the first experiments were mostly two-dimensional [11] or limited to the observation of the upper or bottom surfaces [2]. Magnetic resonance imaging and X-ray tomography allowed true three-dimensional measurements [7] [3]. The main conclusions of these $2D$ or $3D$ experiments were the exponential-like (or Gaussian-like) velocity profile in the shear band, and the depletion of the compaction close to the moving inner cylinder. To explain that behaviour, we will use a model based on a suggestion made long ago by Savage [8] and Johnson and Jackson [5] who split the stress tensor into a first part due to contacts and a second one due to impacts between particles. The resulting stress involves several transport coefficients depending on the grain volume fraction. We have renewed that approach by giving the transport coefficients better expressions, some of them inspired by the model of Bocquet et al. [2] and we tested them in flows over heaps or rough inclines [6]. Here we want to detail the predictions of the model for the Couette flow. The specific features of the motion in a Couette cell are briefly reviewed in section 2. The main lines of the model are presented in section 3. The predicted velocity and concentration profiles in Couette flow are the subject of section 4.

2 Pressure and shear in a Couette cell

A widely used apparatus to study the rheology of continuous media is the Couette cell, made of two co-axial vertical cylinders of radii r_1 and $r_2 > r_1$. The inner cylinder rotates and imparts momentum to the medium bounded by the two cylinders. Once in a stationary state, the medium moves with an angular velocity $\omega(r, z)$ an azimuthal velocity $r\omega(r, z)$ and a shear rate $r\partial\omega/\partial r$ for $r_1 < r < r_2$. Mass conservation is automatically satisfied. Momentum conservation simplifies considerably when assuming that the stress tensor τ is symmetric and has vanishing components τ_{zr} and $\tau_{z\theta}$. These assumptions are far from trivial in case of granular media : a rough boundary prevents the grains to rotate freely (meaning that the average rotation rate of the grains is possibly different from the local rotation rate ω) and the stress tensor is presumably not symmetric at distances less than a few diameters from it. Moreover the assumption of a vanishing τ_{zr} means the neglect of any Janssen-like effect in the Couette cell (i.e. no vertical friction forces on the cylinders). When these assumptions are taken for granted, momentum conservation reduces to

$$r\frac{\partial \tau_{rr}}{\partial r} + \tau_{rr} - \tau_{\theta\theta} = \rho r^2 \omega^2$$
$$\frac{\partial r^2 \tau_{r\theta}}{\partial r} = 0$$
$$\frac{\partial \tau_{zz}}{\partial z} = \rho g.$$

Here g is the gravity accelleration and ρ is the total mass per unit volume. For granular media, $\rho = \phi\rho_p + (1-\phi)\rho_f$ where ϕ is the volume fraction of the grains while ρ_p and ρ_f are the true mass densities of the grains and the interstitial fluid respectively.The momentum balance in the radial direction implies

$$\tau_{rr}(r_2, z) = \tau_{rr}(r_1, z) + \int_{r_1}^{r_2} (\tau_{\theta\theta} - \tau_{rr} + \rho r^2 \omega^2)\frac{dr}{r}.$$

The normal stress difference $\tau_{\theta\theta} - \tau_{rr}$ is possibly non-zero in granular media but since the free (upper) surface of a granular medium in a Couette cell is close to perfectly horizontal, we conclude hat both the normal stress difference and the centrifugal pressure give negligible contributions to the above integral [10] . As a consequence, the conservation of the momentum of slowly moving and dense granular media is expressed as

$$\tau_{rr} \approx P(z) \tag{1a}$$
$$\tau_{r\theta} = (\frac{r_1}{r})^2\ S(z) \tag{1b}$$

where $P(z)$ and $S(z)$ are the confining pressure and shear on the inner cylinder respectively.

3 Constitutive relations for dense and dry granular media

A dense granular medium is one in which contacts between grains are long-lived and lead to a contact network spanning all over the sample. This requires a minimum volume fraction ϕ_m which is of order 0.5 for spherical grains and represents the loosest random packing. When ϕ is larger than ϕ_m , the medium displays a compressibility which is not linked to the elasticity of the grains but stems from the loose packing and the possibility of spatial reorganisation of the grains when submitted to a pressure step. This irrelevance of the elastic properties of the grains is an approximation which holds up to the tightest random packing ϕ_M of order 0.65 for spherical grains. The constitutive relations to be presented in what follows are those prevailing in the narrow compaction range $\phi_m \leq \phi \leq \phi_M$. In a Couette device the main pressure load is exerted along the radial direction and one can refer to τ_{rr} as the granular pressure. This granular pressure is the result of two distinct phenomena, long-lived contacts and impacts between grains. The contact pressure depends on the solid fraction and we write it as $\rho_p g D F(\phi)$ where D is the grain diameter and $\partial F/\partial\phi$ represents the rigidity of the granular medium. Besides the contact pressure, the normal stress also includes the effects of the impacts between grains. The name "impact" is used to stress on the difference between the many-body and reboundless collisions that prevail in dense media as opposed to the more traditional two-body collisions of dilute media ($\phi < \phi_m$).

At variance with the contact pressure, the impact pressure only exists when the medium is in motion. It is thus rate-dependent and on dimensional grounds must be written in the Bagnold-like form [1] so that the overall granular pressure appears in the form

$$\tau_{rr} = \rho_p g D F(\phi) + \rho_p D^2 \mu_N(\phi)(r\partial\omega/\partial r)^2 \ . \tag{2}$$

In this expression $\mu_N(\phi)$ represents Reynolds' dilatancy : shearing the medium at a constant volume fraction creates a larger pressure or conversely shearing the medium at constant pressure induces a decrease of the granular concentration. Considering now the shear stress, we assume it to be the sum of a Coulomb-like contribution and a viscous-like contribution

$$\tau_{r\theta} = -\mu(\phi)\tau_{rr} sign(\partial\omega/\partial r) - \rho_p D^2 \mu_T(\phi) r^2 \parallel \partial\omega/\partial r \parallel \partial\omega/\partial r \ .$$

Because the local rotation rate ω decreases when one moves away from the inner cylinder, the above expression is simplified into

$$\tau_{r\theta} = \mu(\phi)\tau_{rr} + \rho_p D^2 \mu_T(\phi)(r\frac{\partial\omega}{\partial r})^2 \ . \tag{3}$$

In this expression $\mu(\phi)$ is a compaction-dependent friction coefficient and $\mu_T(\phi)$ depicts the extra dissipation due to the friction developed by sliding contacts. Because the normal stress difference $\tau_{rr} - \tau_{\theta\theta}$ appears to be negligible, the modelling of the two components τ_{rr} and $\tau_{r\theta}$ is enough for the description of the Couette-like flow. Expressions (2) and (3) contain four positive scalar functions of the compaction. If we admit an infinite rigidity and the impossibility of any motion for $\phi > \phi_M$, then F, μ_N and μ_T are expected to diverge for $\phi = \phi_M$. If we take for granted the absence of contact pressure and dilatancy phenomena for $\phi < \phi_m$, then F, μ_N and μ_T are expected to be very small and perhaps to vanish for $\phi = \phi_m$. The friction coefficient μ has a much smoother behaviour since it is expected neither to vanish nor to become infinite in the whole range $\phi_m \leq \phi \leq \phi_M$. In fact F, μ_N, μ_T and μ are functions of the *reduced compaction*

$$\varphi = \frac{\phi - \phi_m}{\phi_M - \phi_m} \tag{4}$$

which can be considered as some order parameter with $\varphi = 0$ for the loosest random packing and $\varphi = 1$ for the tightest one. For flows over heaps as well as over rough inclines, a satisfactory fit with experimental results could be obtained with [6]

$$F(\varphi) = F_0 \ log\frac{1}{1-\varphi} \tag{5}$$

$$\mu_N(\varphi) = \frac{\mu_{N0}}{(1-\varphi)^2} \tag{6}$$

$$\mu_T(\varphi) = \frac{\mu_{T0}}{(1-\varphi)^2} \tag{7}$$

$$\mu(\varphi) = \mu_0 \ . \tag{8}$$

The expression for $F(\varphi)$ is reminiscent of the configuration pressure in the lattice-gas model and it was already adopted by Savage [9]. The other three functions are more debatable. From numerical simulations of the plane shear flow, Da Cruz [3] suggested the linear variation $\mu(\varphi) = \mu(0) - [\mu(0) - \mu(1)]\varphi$ with the interesting but rather non-intuitive result $\mu(0) > \mu(1)$. Because of a dispute on the velocity profile close to the free-surface of heap flows (linear or Bagnold-like or other) the present authors [6] have explored the potentialities of $\mu_T(\varphi) = \mu_{T0}\ \varphi^\gamma/(1-\varphi)^2$ with an exponent γ in the range $0 \le \gamma \le 3$. This suggests that a more general expression for μ_N could be $\mu_N(\varphi) = \mu_{N0}\ \varphi^\delta/(1-\varphi)^2$, with a positive exponent δ possibly different from γ. However, not to confuse the issue, we will use the simpler expressions (5) to (8) in what follows.

4 Compaction and Velocity Profiles

Let us consider some horizontal plane (z fixed) somewhere between the upper and bottom surface of the granular medium. Combining equations (1) and the constitutive relations (2) and (3) results in

$$\rho_p g D F(\varphi) + \rho_p D^2 \mu_N(\varphi)(r\frac{\partial\omega}{\partial r})^2 = P$$
$$\mu(\varphi)P + \rho_p D^2 \mu_T(\varphi)(r\frac{\partial\omega}{\partial r})^2 = (\frac{r_1}{r})^2\ S\ .$$

The compaction profile is obtained after elimination of ω between the two above expressions. Taking expressions (5) to (8) for granted, one obtains

$$F_0\ log[1-\varphi(r)] = \frac{\mu_{N0}}{\mu_{T0}}\ [(\frac{r_1}{r})^2 S^* - \mu_0 P^*] - P^* \tag{9}$$

where $P^* = P/\rho_p g D$ and $S^* = S/\rho_p g D$ are the non-dimensional pressure and shear on the inner cylinder. Once the reduced compaction $\varphi(r)$ is obtained from (9), it is introduced in the equation

$$\sqrt{\frac{\mu_{T0} D}{g}}\ r\frac{\partial\omega}{\partial r} = -(1-\varphi(r))\sqrt{(\frac{r_1}{r})^2 S^* - \mu_0 P^*} \tag{10}$$

which then gives the velocity profile. It is clear from (10) that to move the granular medium requires $S^* > \mu_0 P^*$ and that the motion is restricted to a band $r_1 < r < r^*$ close to the inner cylinder with $(r^*/r_1)^2 = S^*/\mu_0 P^*$. Such a result suggests that increasing the shear will increase the thickness of the shear band. This is true if the shear satisfies the inequality $1 < S^*/\mu_0 P^* < 1 + \mu_{T0}/\mu_0\mu_{N0}$. When $S^* > (\mu_0 + \mu_{T0}/\mu_{N0})P^*$, the right-hand side of equation (9) is positive for $r = r_1$ which implies that the compaction at the inner cylinder is smaller than ϕ_m while our model holds for $\phi > \phi_m$ only. This means that for very large shears the contact network breaks down close to the moving cylinder and that the grains must develop another way to receive impulse from it. In case the grains are surrounded by a gas, collisions with the moving cylinder is the only efficient way to get momentum. A thin layer where concentration is smaller than ϕ_m will then develop around the inner cylinder, with a comparatively high velocity and large velocity fluctuations. Outside this thin collisional layer, the granular medium is compact (its concentration is everywhere larger than ϕ_m) and its motion is localized in a shear band of thickness r^*_{max} where

$$r^*_{max} = r_1\ \sqrt{1 + \mu_{T0}/\mu_0\mu_{N0}}. \tag{11}$$

It is worthy to note that this maximum thickness is proportional to the radius of the moving cylinder r_1 and not to the particle size. This is not a complete surprise because in the limit of very large r_1, the Couette cell is transformed into a plane shear cell for which there is no shear localization [4]. The thickness of the shear layer at very large shears is thus the sum of r^*_{max} and the small and shear-dependent thickness of the collision layer. We are not interested in these very high shears which are quite difficult to obtain experimentally and will consider $r^*_{max} - r_1$ as the *maximum width* of the shear band. We now define the compaction ϕ^*, or reduced compaction φ^*, related to the magnitude of the pressure load P by $P = \rho_p g D F(\phi^*)$ or equivalently $P^* = F_0 \; log[1/(1-\varphi^*)]$. The compaction ϕ^* (or φ^*) increases with the pressure load and is the one prevailing in the motionless part of the granular medium. Any moving part has a compaction smaller than ϕ^*. From (9) one easily deduces the generic form of the compaction profile in the shear band

$$\phi(r) = \phi^* - (\phi_M - \phi^*)[exp^{\lambda(r*^2/r^2-1)} - 1] \tag{12}$$

where $r_1 \leq r \leq r^* \leq r^*_{max}$ while λ is proportional to the confining pressure and defined as

$$\lambda = \frac{\mu_0 \mu_{N0}}{F_0 \mu_{T0}} \, P^* \, .$$

The generic form of the strain rate profile is deduced from (10) and appears as

$$\begin{aligned} \sqrt{\frac{\mu_{N0} D}{F_0 g}} \, r \frac{\partial \omega}{\partial r} &= -(1-\varphi) \sqrt{log \frac{1-\varphi}{1-\varphi^*}} \\ &= -(1-\varphi^*) \sqrt{\lambda(r^{*2}/r^2 - 1)} \; exp^{\lambda(r^{*2}/r^2-1)} \, . \end{aligned} \tag{13}$$

Typical compaction, strain rate and velocity profiles, deduced from (12) and (13), are represented in Figures 1 to 3 for a fixed confining pressure and variable shears. It is worthy to note that the velocity profile is neither exponential nor Gaussian but somewhere "in between" as seen on Figures 4 and 5, to be compared with the experimental results reported in Figures 6.d and 6.e of [12]. It must be understood that all the existing experiments have being performed with relatively small shears, and we have not represented the collision layer which would develop close to the inner cylinder for larger shears, i.e. for $S > (\mu_0 + \mu_{T0}/\mu_{N0})P$. According to result (11), in a Couette cell with $r_2 < r^*_{max}$ and for large enough shears, the grains are moving all over the gap between the two cylinders, giving the (wrong) impression that the localization phenomena have disappeared. Conversely, since localization has been observed in all Couette cells with $r_2 > 1.3 r_1$ [2,3,7,11] we deduce that the ratio μ_{T0}/μ_{N0} is of order 0.3 - 0.7 for most granular media. The same order of magnitude was adopted in [6] to explain the maximum angle of chute observed in experiments on heap flows.

5 Conclusions

We have tested the potentialities of a frictional-collisional model for describing the shear localization in a Couette cell. The results are quite encouraging. Among the predictions of the model are i) the presence of a fully motionless medium at the boundary of the shear band (contrasting with the exponential decrease of the velocity for flows over heaps that were predicted with the same model equations (2) and (3) in [6]) ii) the proportionality (11) of the maximum width of the shear band with the radius of the inner cylinder, independently of the particle size and iii) a local relationship (13) between the shear rate and the compaction. Future work will include an extension of the above model to "wet" granular media.

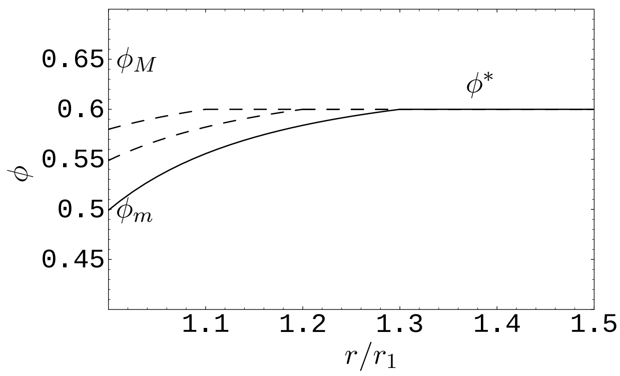

Fig. 1. Compaction profile $\phi(r)$ for fixed pressure and variable shear. The two dotted profiles correspond to two small shears. The third profile is common to all shears larger than $(\mu_0 + \mu_{T0}/\mu_{N0})P$. The maximum width of the shear layer was arbitrarily chosen as $r^*_{max} = 1.3r_1$.

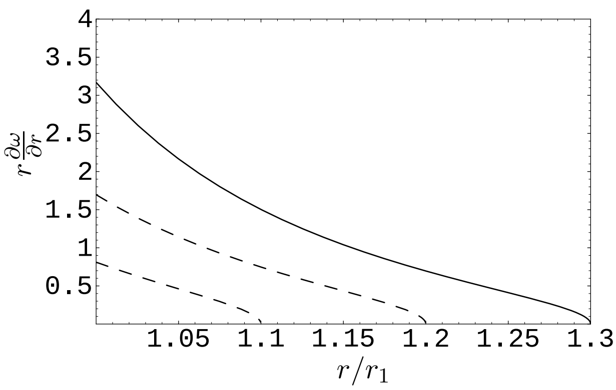

Fig. 2. Shear rate $r\frac{\partial\omega}{\partial r}$ under conditions stated in caption of figure 1.

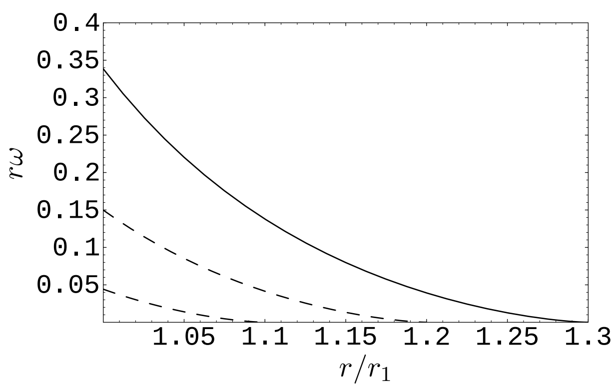

Fig. 3. Velocity profile $r\omega$ under conditions stated in caption of figure 1.

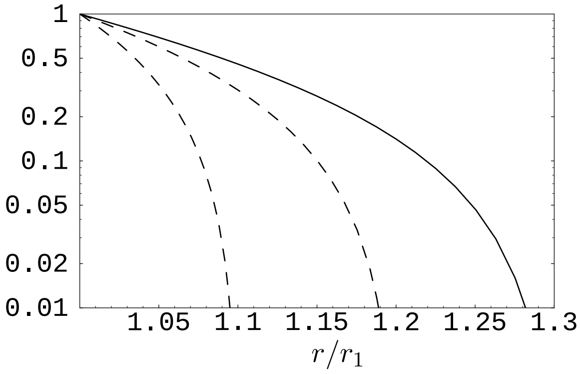

Fig. 4. Log plot of reduced velocity profile under conditions stated in caption of figure 1.

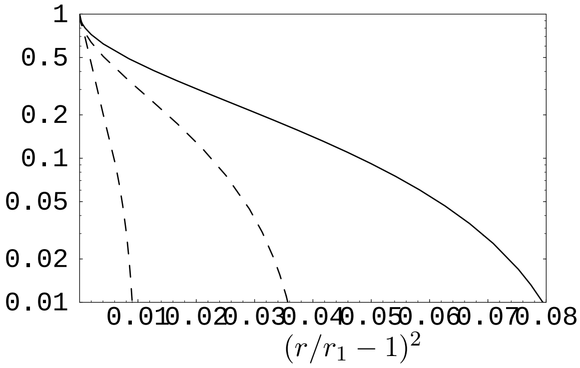

Fig. 5. Log plot of reduced velocity profile as function of $(r/r_1 - 1)^2$ under conditions stated in caption of figure 1.

References

1. R.A. Bagnold : *The Physics of Sediment Transport by Wind and Water : A Collection of Hallmarks Papers by R.A. Bagnold*, ed. by C.R. Thorne, R.C. MacArthur and J.B. Bradley (The American Society of Civil Engineers, New-York 1988).
2. L. Bocquet, W. Losert, D. Schalk, T.C. Lubensky and J.P. Gollub, Granular shear flow dynamics and forces : Experiments and continuum theory, Phys Rev **E 65** , 011307 (2002).
3. F. Da Cruz, Ecoulement de grains secs : frottement et blocage, Ph.D. Thesis, ENPC (2004).
4. F. Da Cruz, F. Chevoir, J.-N. Roux and I. Iordanoff, Macroscopic friction of dry granular materials, *Proceedings of the 30th Leeds-Lyon Symposium on Tribology* ed. by G. Dalmaz (2003).
5. P.C. Johnson and R. Jackson, Frictional-collisional constitutive relations for granular materials, with application to plane shearing, J. Fluid Mech. **176** 67-93 (1987).
6. C. Josserand, P.-Y. Lagrée and D. Lhuillier, Stationary shear flows of dense granular materials : a tentative continuum modelling, Eur. Phys. J. **E 14** 127-135 (2004).
7. D.M. Mueth, G.F. Debregeas, G.S. Karczmar, P.J. Eng, S.R. Nagel and H.M. Jaeger, Signatures of granular microstructure in dense shear flows, Nature **406** 385-389 (2000).

8. S.B. Savage, Granular flows down rough inclines : review and extension, *Proc. of US-Japan Seminar on New Models and Constitutive Relations in the Mechanics of Granular Materials* ed. by J.T. Jenkins and M. Satake (Elsevier Science Publishers, Amsterdam, 1982) pp.261-282.

9. S.B. Savage, Analysis of slow high-concentration flows of granular materials, J. Fluid Mech. **377** 1-26 (1998).

10. S.B. Savage and M. Sayed, Stresses developed by dry cohesionless granular materials sheared in an annular shear cell, J. Fluid Mech. **142** 391-430 (1984).

11. C.T. Veje, D.W. Howell and R.P. Behringer, Kinematics of a two-dimensional granular Couette experiment at the transition to shearing, Phys. Rev. **E 59** 739-745 (1999).

12. Groupement de Recherche Milieux Divisés, On dense granular flows, Eur. Phys. J. **E 14** 341-365 (2004).