Stationary Shear Flow of Granular Matter, a tentative continuum modelling

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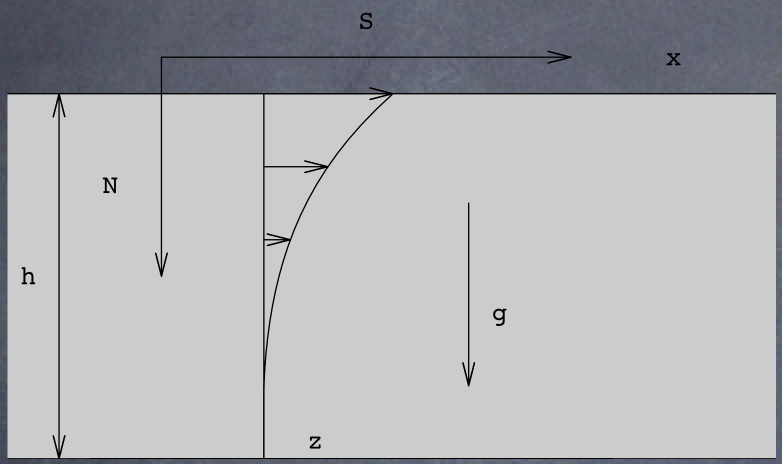
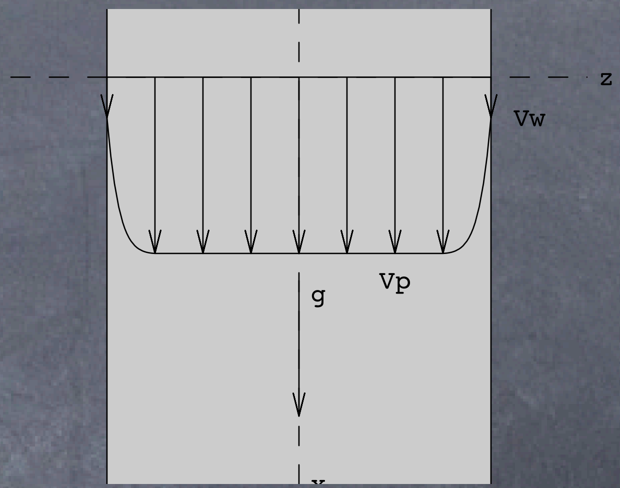
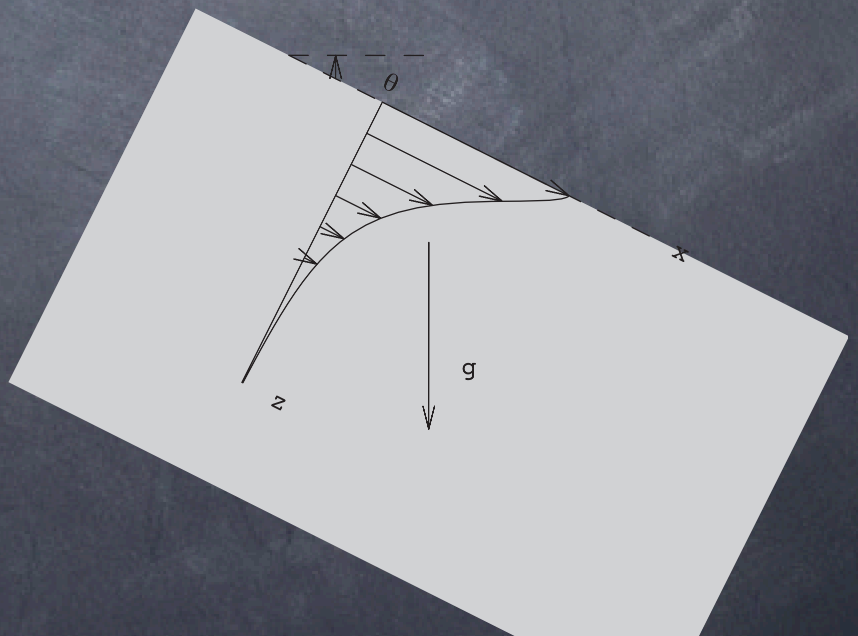
Topics

- Role of solid fraction
- Dry granular materials
- 2D steady shear flow
- main ingredients
  - solid fraction
  - dilatancy
  - granular pressure
  - Bagnold/ Coulomb
- Continuum mechanical model
- Predictions
  - free surface flows
  - confined flows


Shear Flows







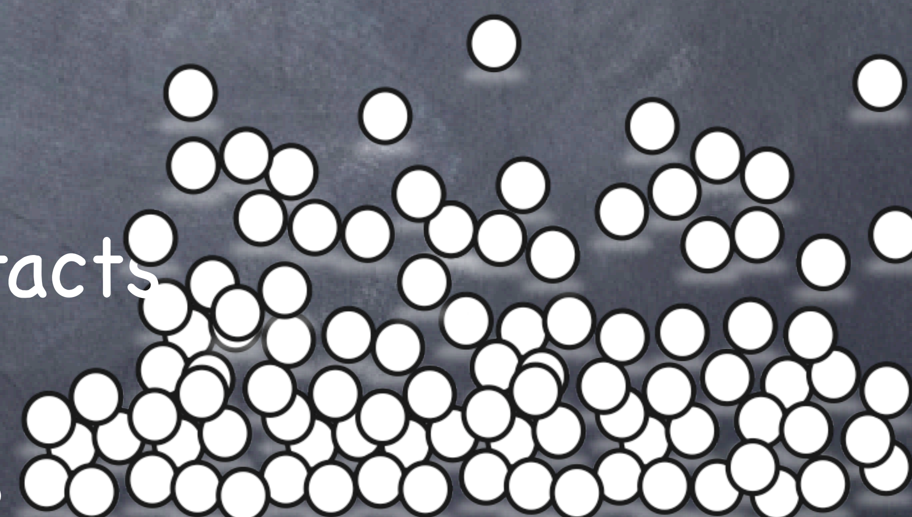
Solid Volume fraction ϕ

0.5 (2D) 0.55 (3D) $\phi_{min} < \phi < \phi_{Max}$ 0.8 (2D) 0.65 (3D

suspensions impacts

granular matter long lived contacts

solid-like matter elastic forces



Dry Granular Matter

$$\phi_{min} < \phi < \phi_{Max}$$

no interstitial fluid/ no cohesive forces

granular stress τ

static	$\tau = \text{contact stress}(\varepsilon, \phi_s)$ pseudo static preparation dependent
flow	$\tau = \text{contact pressure}(\phi)$ $+\text{rate-dep. stress}(\phi, (\nabla V)^S, \omega - \frac{1}{2}\nabla \times V)$

fully tensorial relation: too difficult!!!

simple 2D steady flows...

rotation

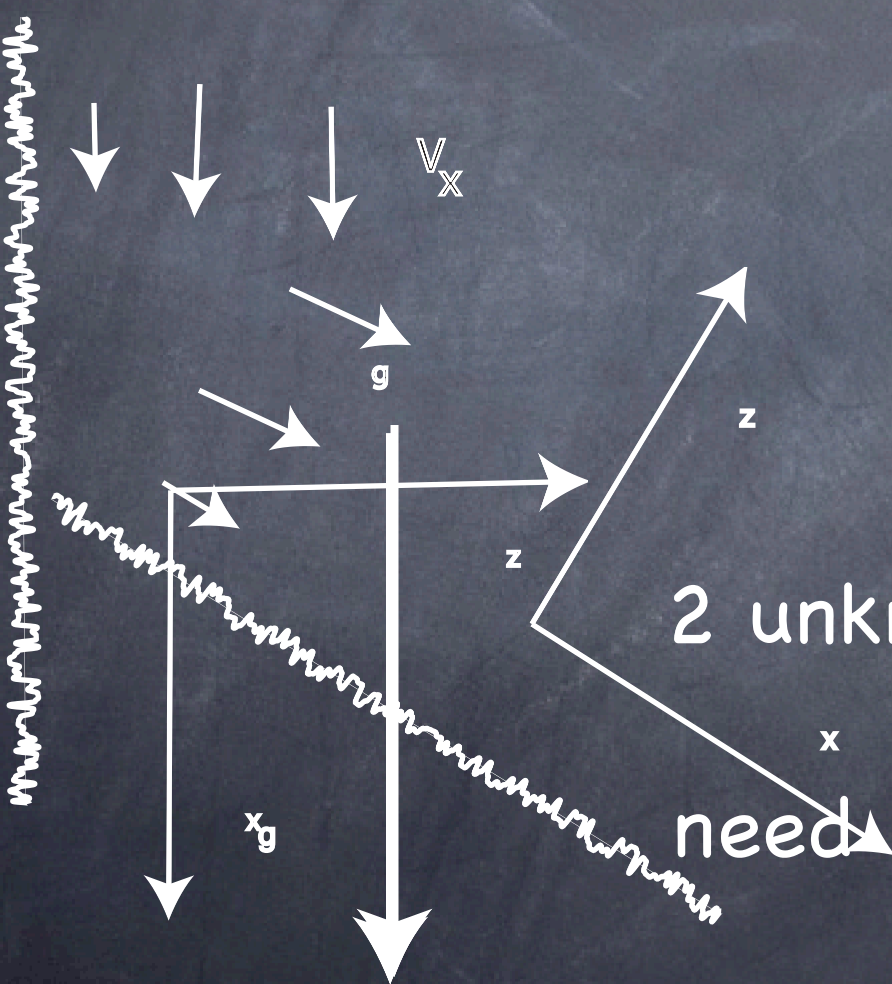
- Assumption: rotation = vorticity

$$\omega = \frac{1}{2}\nabla \times V$$

Warning ! Possible failure close to rough boundaries compression z

flow x

granular stress τ balance of forces



$$\frac{\partial \tau_{xz}}{\partial z} + \phi \rho g sin(\theta) = 0$$

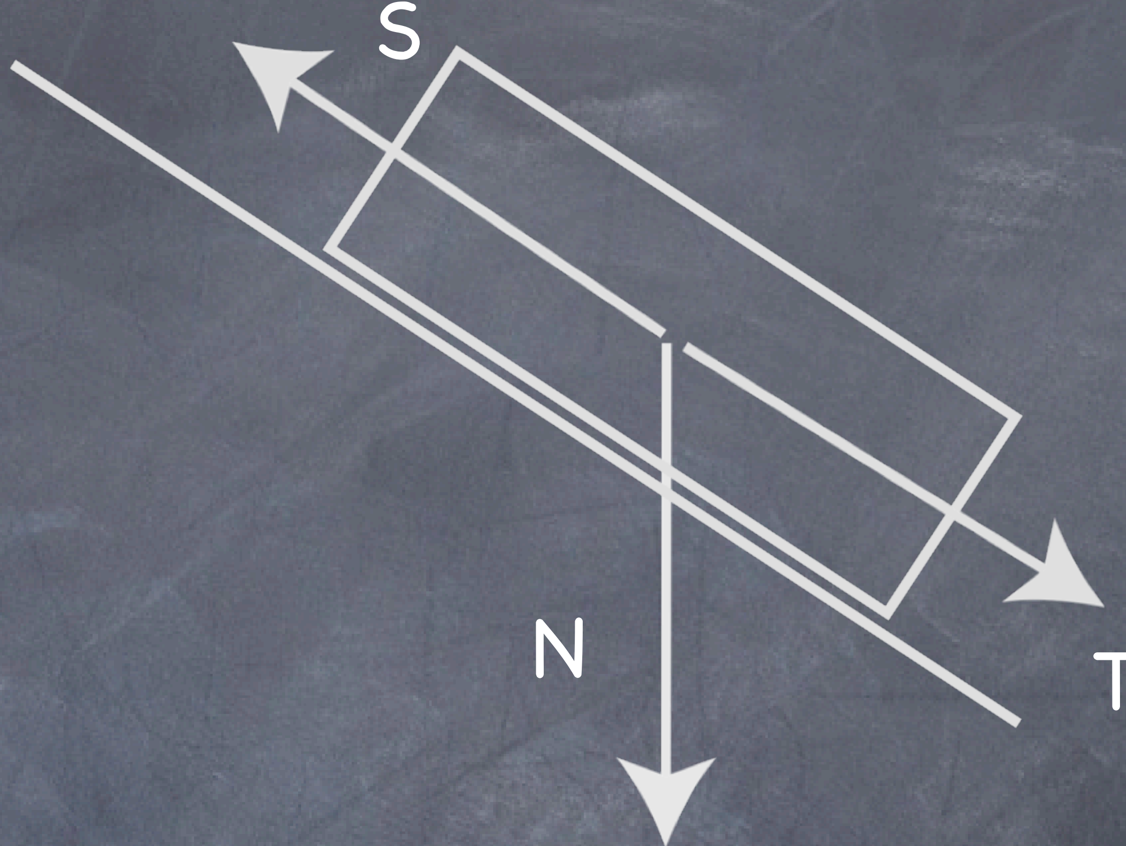
$$\frac{\partial \tau_{zz}}{\partial z} - \phi \rho g cos(\theta) = 0$$

2 unknows ϕ and $\frac{\partial V_x}{\partial z}$

need for: $\tau_{zz}(\phi, \frac{\partial V_x}{\partial z})$ and $\tau_{xz}(\phi, \frac{\partial V_x}{\partial z})$

Warning ! Possible failure close to rough boundaries

Solid Friction



S=μN   motion

present model:
at incipient motion

$$\tau_{xz} = \mu(\phi)\tau_{zz}$$

present model:
Bagnold Scaling is only part of the stress

$$\tau_{zz}, \tau_{xz} \;>\; \rho D^2 \left(\frac{\partial V_x}{\partial z}\right)^2$$

Contact Pressure

normal stress

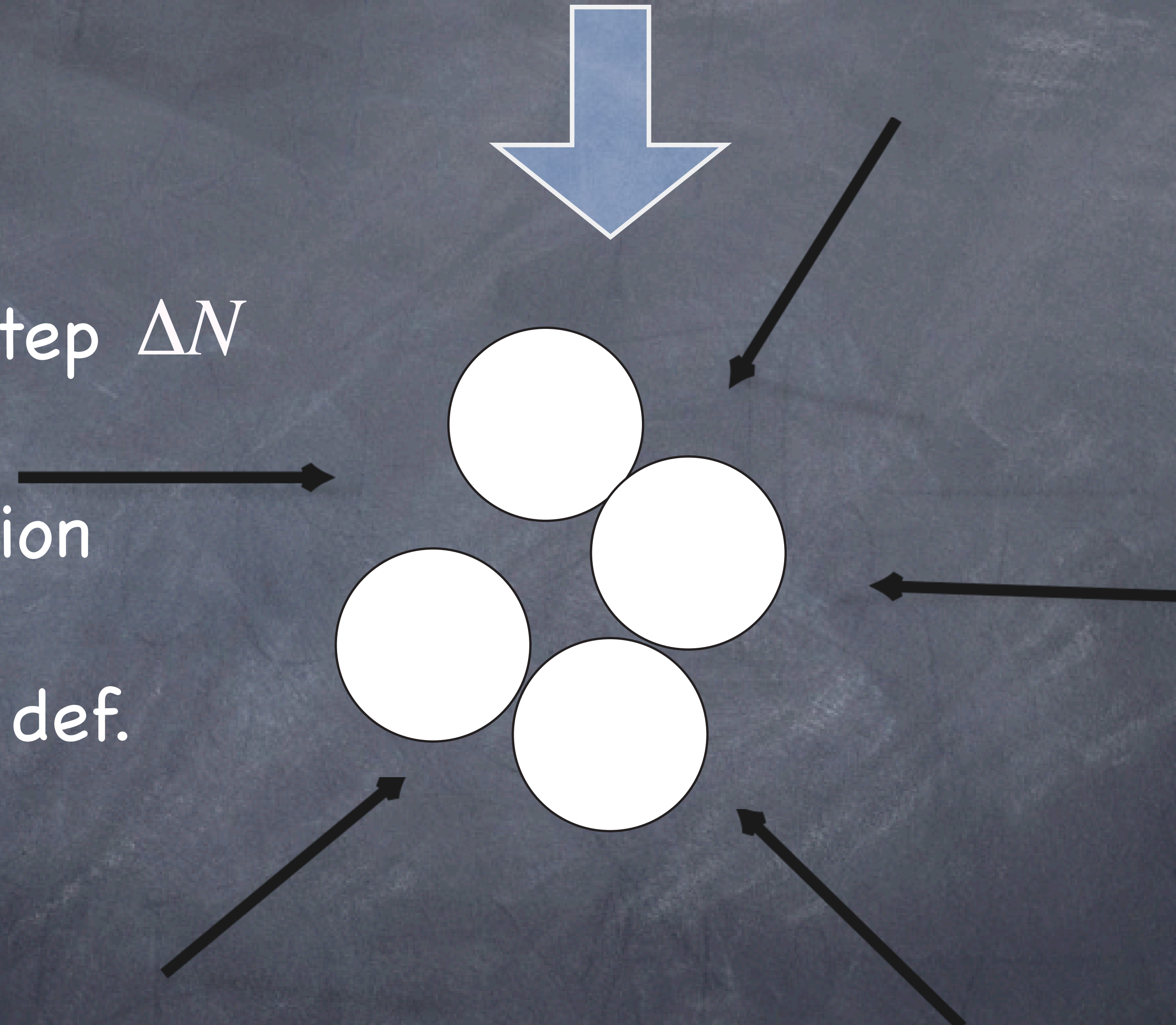
compaction ϕ_s

normal stress step ΔN

elastic deformation

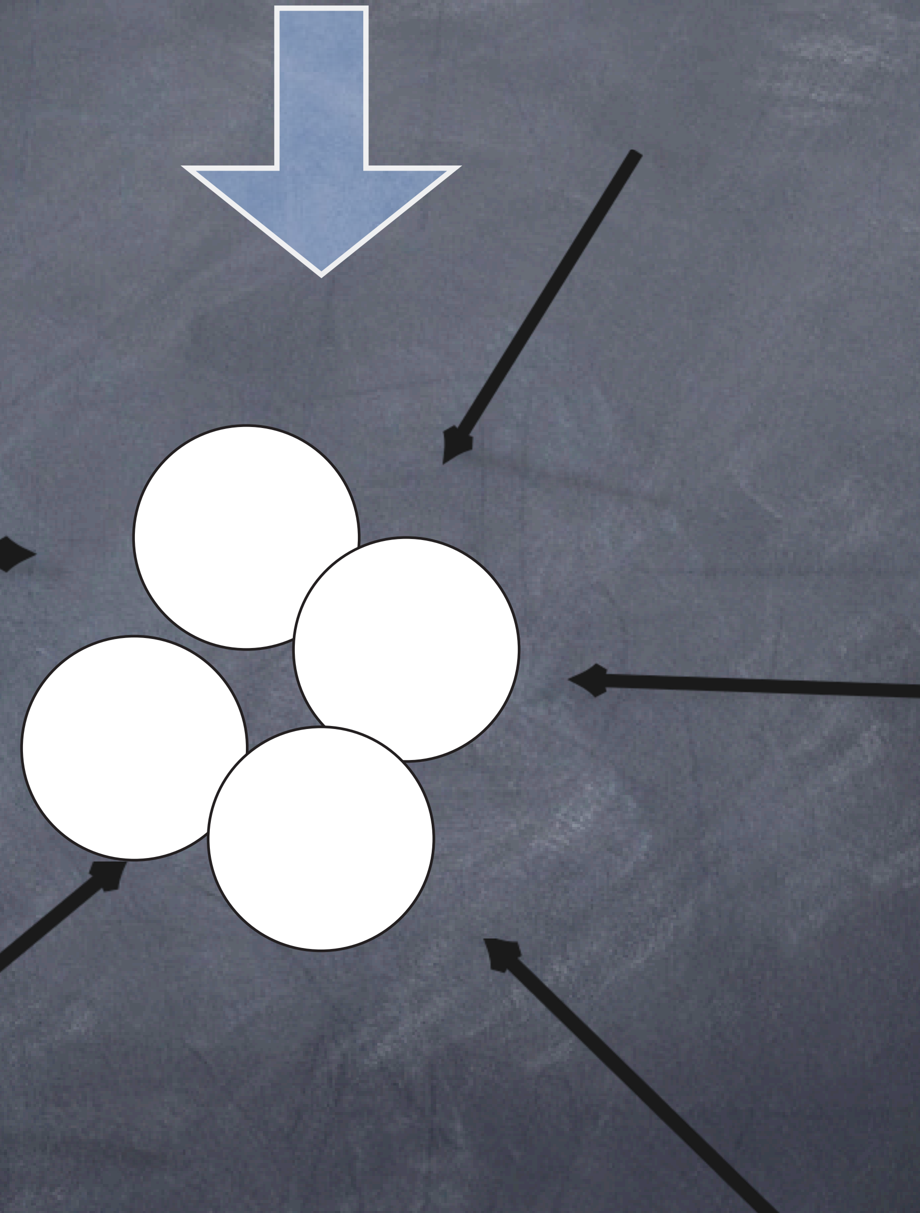
sliding contacts

relax. of elastic def.



$$\tau_{zz} = \rho g D F(\phi)$$

Contact Pressure



normal stress step ΔN

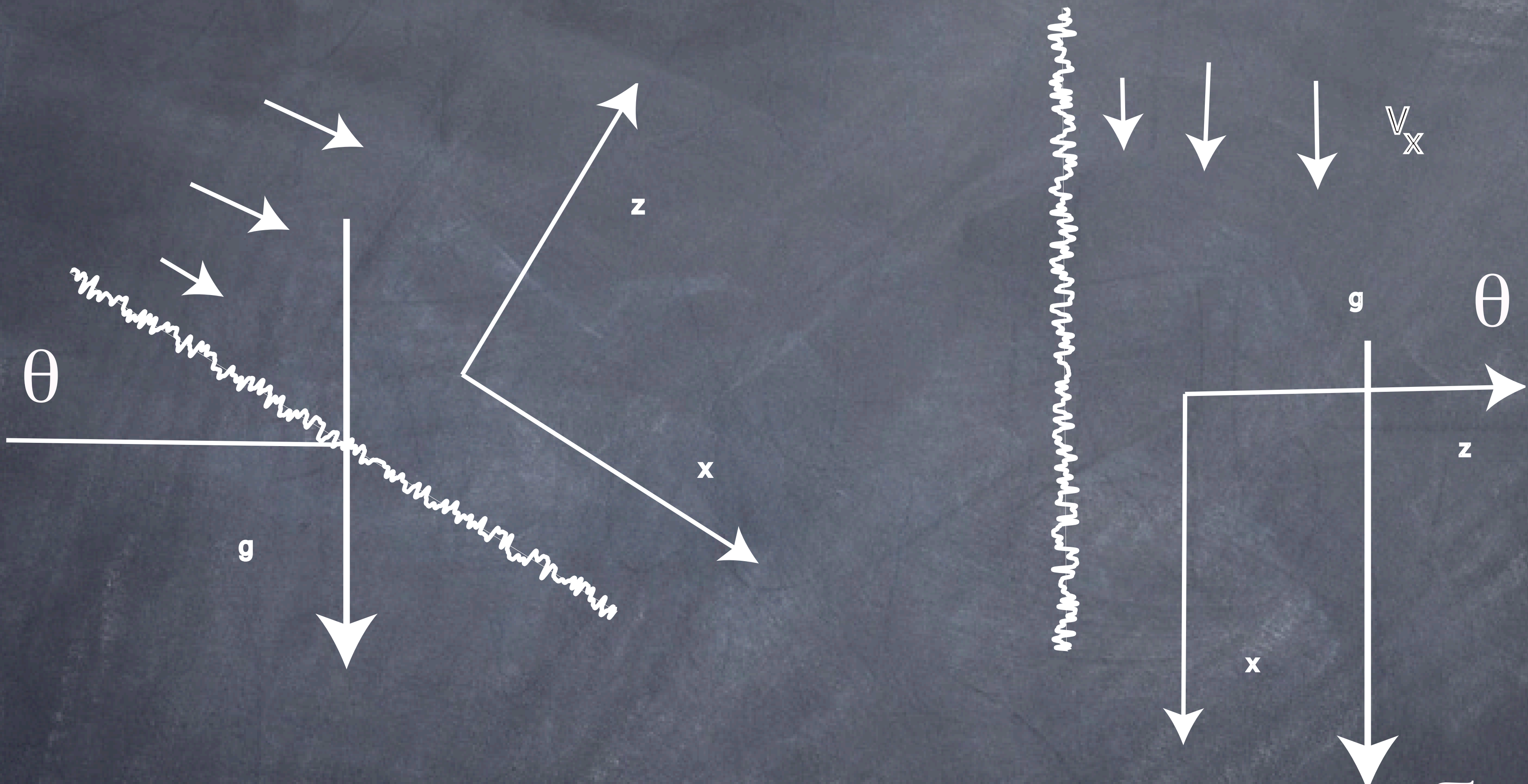
elastic deformation
sliding contacts
relax. of elastic def.

Normal stress $N + \Delta N$

compaction $\phi + \Delta\phi$

$$\tau_{zz} = \rho g D F(\phi)$$

gravity induced compressive stress



$$\theta = \frac{\pi}{2}$$

$$\tau_{zz} = \rho g D F(\phi) f(\theta) \text{ with } f(0) = 1 \text{ and } f(\frac{\pi}{2}) = 0$$

$$\tau_{zz} = \rho g D F(\phi) cos(\theta)$$

constitutive relations

- Normal stress $$\tau_{zz} = \rho g D F(\phi) cos(\theta) + \rho D^2 \mu_N(\phi) (\frac{\partial V_x}{\partial z})^2$$

- shear stress $$\tau_{xz} = \mu(\phi)\tau_{zz} + \rho D^2 \mu_T(\phi) (\frac{\partial V_x}{\partial z})^2$$

Equations of motion

$$\frac{\partial \tau_{xz}}{\partial z} + \phi \rho g sin(\theta) = 0$$

$$\frac{\partial \tau_{zz}}{\partial z} - \phi \rho g cos(\theta) = 0$$

for given F, μ, μ_N, μ_T, we obtain $\phi(z)$ and $V_x(z)$

solution:

$$D\frac{d\phi}{dz} = \frac{\phi}{\frac{\partial}{\partial\phi}[\frac{F}{1-(\mu_N/\mu_T)(\tan(\theta)-\mu)}]}$$

$$\left(\frac{D}{g}\right)^{1/2}\frac{dV}{dz} = -\left(\frac{F(sin(\theta)-\mu cos(\theta))}{\mu_T(1-(\mu_N/\mu_T)(\tan(\theta)-\mu))}\right.$$

$$\mu(\phi) \leq \tan(\theta) \leq \mu(\phi) + \mu_T(\phi)/\mu_N(\phi)$$

special model

μ and μ_T/μ_N independant of compaction

$$F = F_0 Log\left(\frac{\phi_M - \phi_m}{\phi_M - \phi}\right)\ and\ \mu_T = \mu_{T0}\left(\frac{\phi_M - \phi_m}{\phi_M - \phi}\right)^2$$

Allows numerical computation

Special Form of F

- like entropy for lattice gaz

$$F = F_0 Log\left(\frac{\phi_M - \phi_m}{\phi_M - \phi}\right)$$

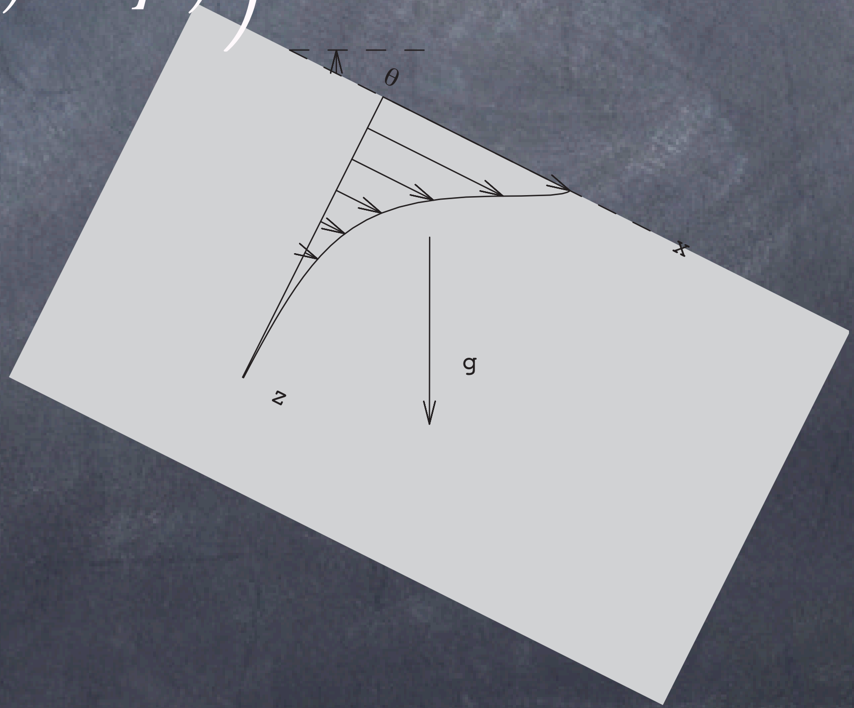
explicit expression

$$\phi_{heap}(z,\theta) = \frac{\phi_M}{1 + (\frac{\phi_M}{\phi_m} - 1)e^{-z/L(\theta)}}$$

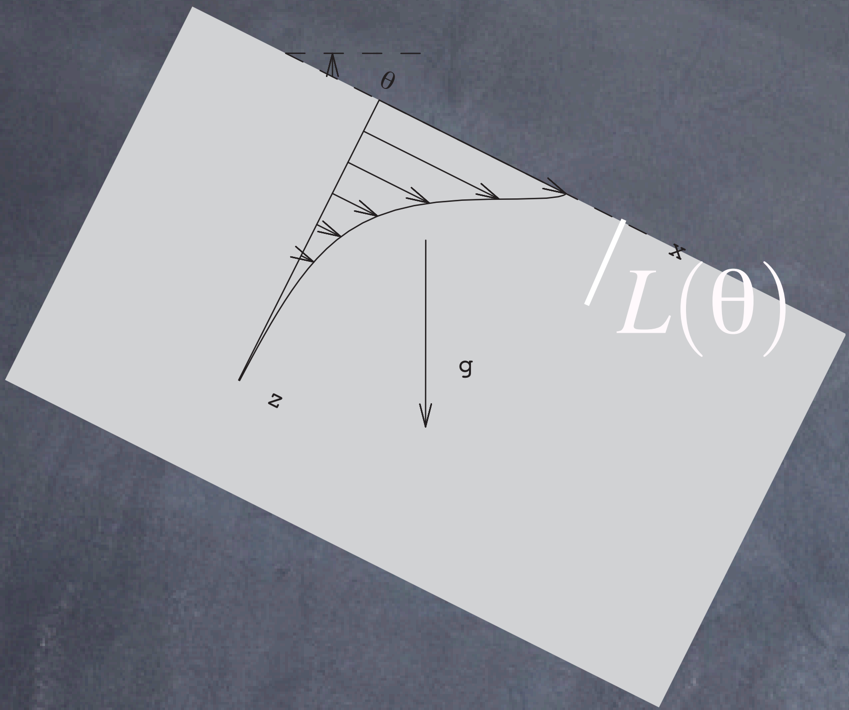
$$L(\theta) = \frac{F_0 D}{\phi_M(1 - \frac{\mu_N}{\mu_T}(\tan(\theta) - \mu))}$$

Heap Flow

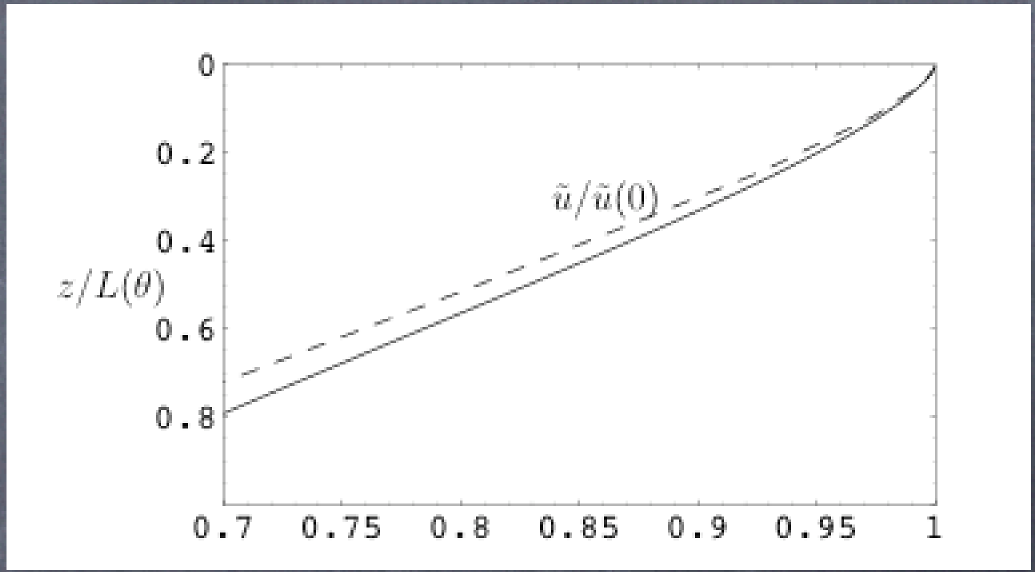
$$\frac{Q_{heap}}{D\sqrt{gD}} = \frac{(sin(\theta) - \mu cos(\theta))^{1/2}}{\left(1 - \frac{\mu_N}{\mu_T}(\tan(\theta) - \mu)\right)^{5/2}} \int_{\phi_m}^{\phi_M} \left(\frac{F^3}{\mu_T}\right)^{1/2} \frac{\partial F}{\partial \phi}\frac{d\phi}{\phi}$$

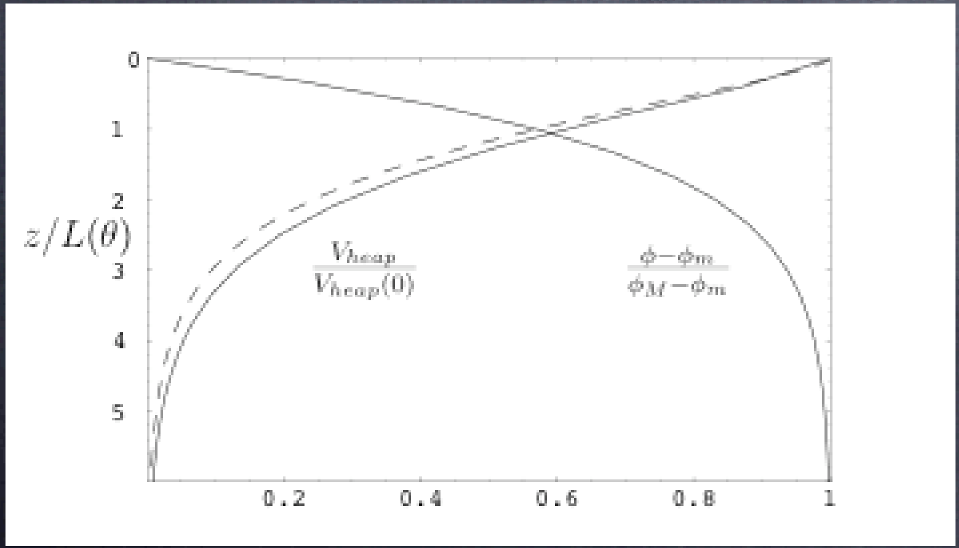


heap flows



Flow thickness $L(\theta)$



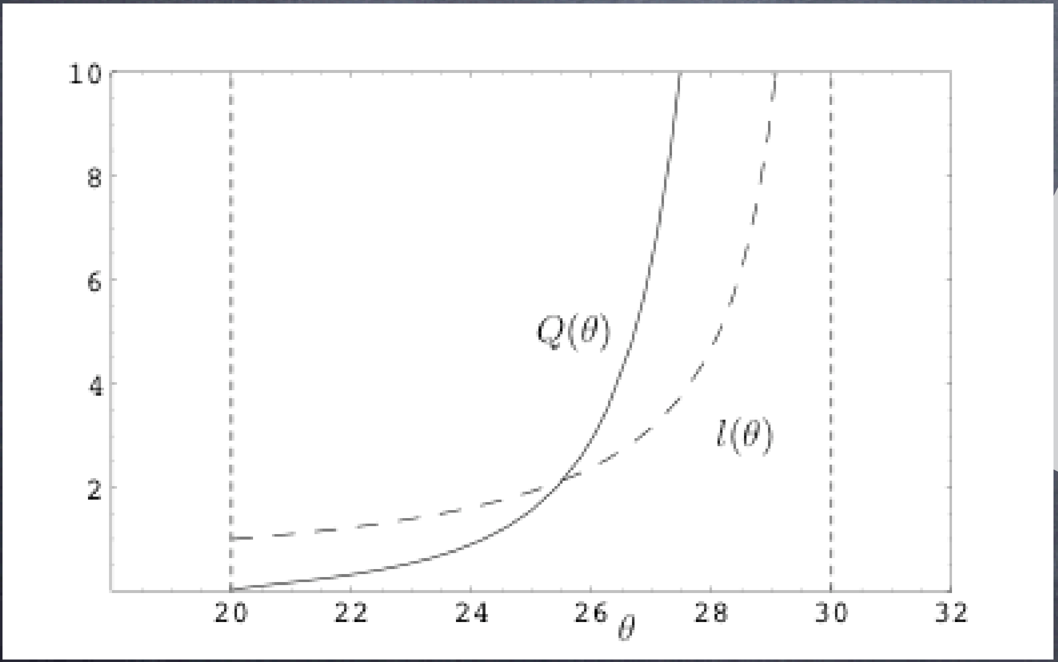


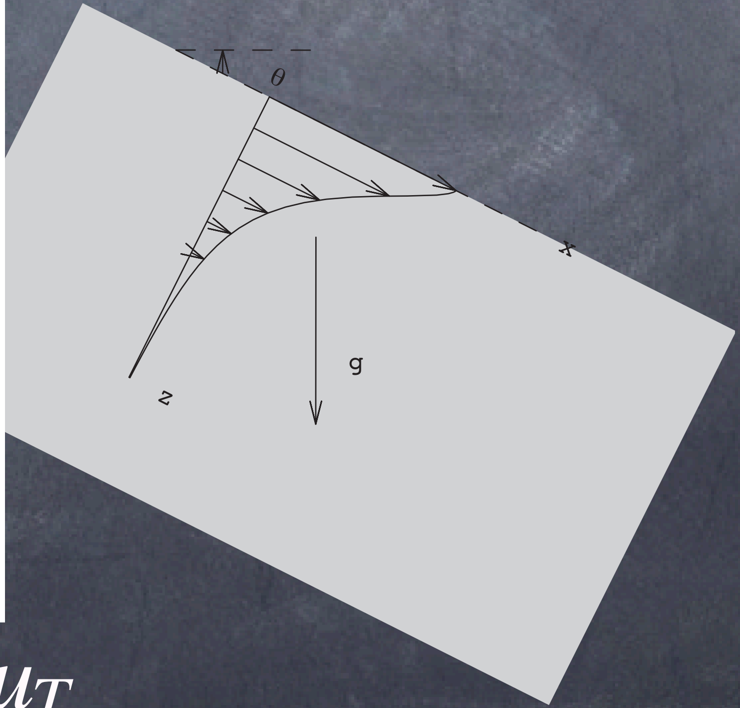
Relative velocity

Reduced compaction

Heap Flow

$$\frac{Q_{heap}}{D\sqrt{gD}} = \frac{(sin(\theta) - \mu cos(\theta))^{1/2}}{\left(1 - \frac{\mu_N}{\mu_T}(\tan(\theta) - \mu)\right)^{5/2}} \int_{\phi_m}^{\phi_M} \left(\frac{F^3}{\mu_T}\right)^{1/2} \frac{\partial F}{\partial \phi}\frac{d\phi}{\phi}$$



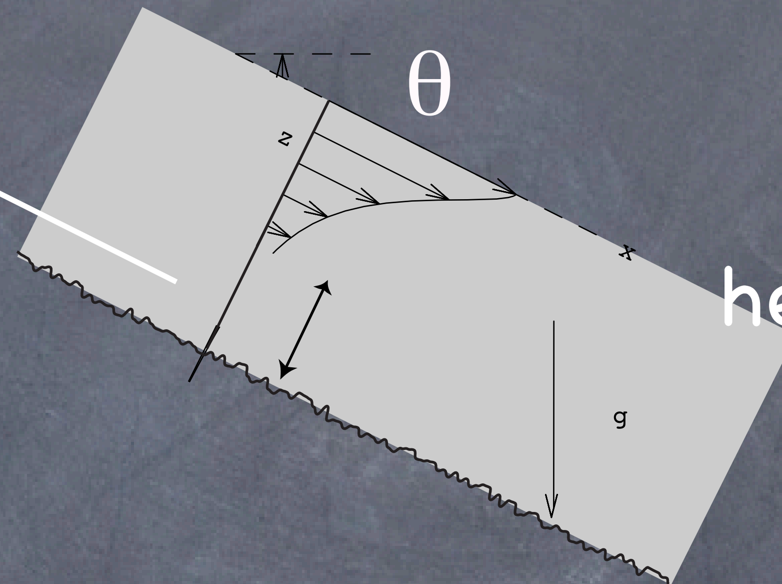


μ $\mu + \mu_N/\mu_T$

flows on rough inclined plates

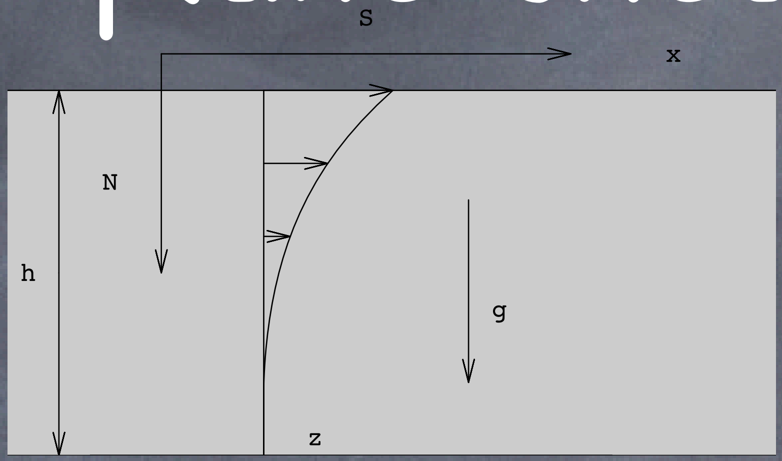
θ and Q imposed

influence of roughness

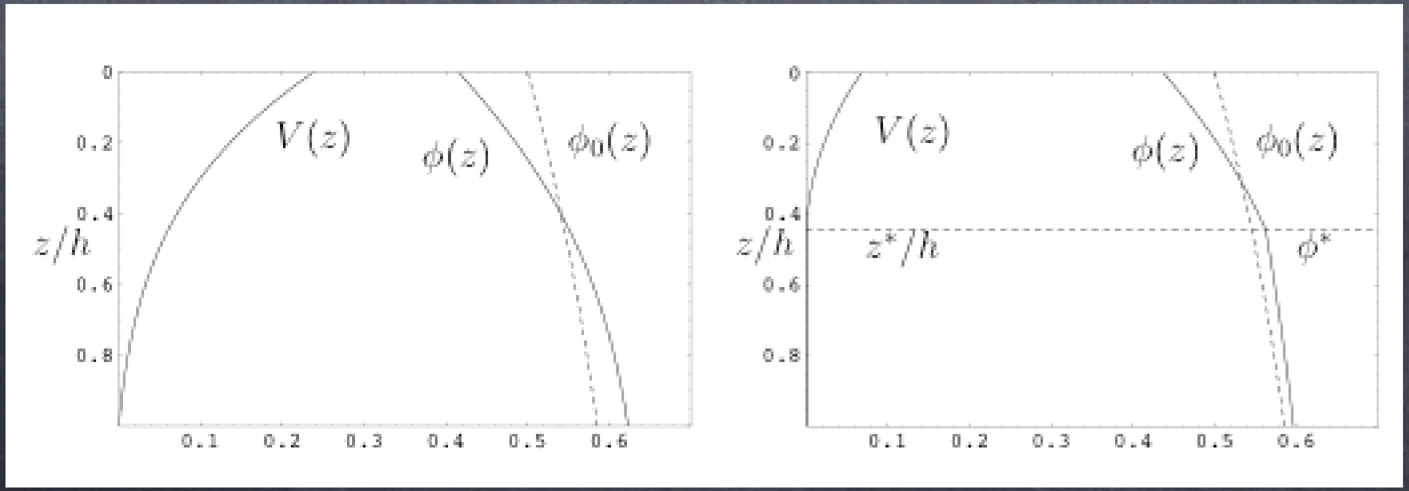


heap like

plane shear

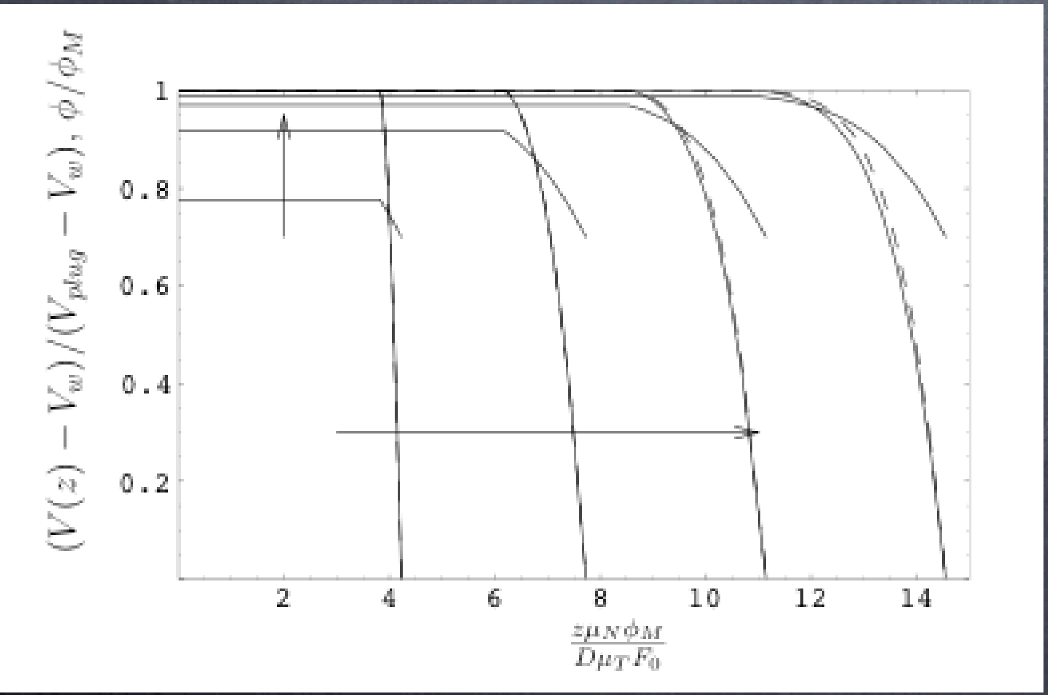


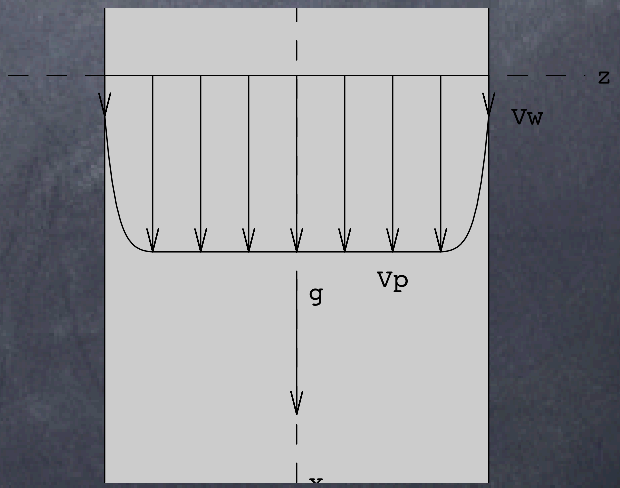
$$\rho D^2 \mu_T(\phi) \left(\frac{\partial V}{\partial z}\right)^2 = S - \mu(\phi)P(0) - \mu(\phi)\rho g \int_0^z \phi(\xi) d\xi$$



vertical chute flow

$$\rho D^2 \mu_T(\phi)\left(\frac{\partial V}{\partial z}\right)^2 = \rho g \int_0^z \phi(\xi)d\xi - \mu(\phi)P$$





h stop

- Just after stopping
- we know the compaction profile

$$0 = -\frac{\partial p}{\partial z} + \phi \rho_p g cos\theta \qquad p = P_0 Log\left(\frac{\phi_M - \phi_m}{\phi_M - \phi}\right)$$

$$\varphi(z) = 1 - \frac{1}{1 + \frac{\phi_m}{\phi_M}(e^{\frac{z}{L}} - 1)} \qquad L/D = P_0/(\phi_M cos(\theta))$$

h stop

- We know the compaction profile $\varphi(z)$
- Da Cruz dependance of μ

$$\mu = tan\theta_{max} - (tan\theta_{max} - tan\theta_{min})\ \varphi$$

h stop

- We know the compaction profile $\varphi(z)$
- Da Cruz dependance of μ

$$tan\theta = tan\theta_{max} - (tan\theta_{max} - tan\theta_{min})\,\varphi(h_{stop})$$

h stop

- as we have an explicit dependance

$$\frac{h_{stop}}{D} = \frac{P_0}{\phi_M cos\theta} Log\left[1 + \frac{\phi_M}{\phi_m}\frac{\tan\theta_{max} - \tan\theta}{\tan\theta - \tan\theta_{min}}\right].$$

Pouliquen $$\frac{h_{stop}}{D} = B\,Log\left[\frac{\tan\theta_{max} - \tan\theta_{min}}{\tan\theta - \tan\theta_{min}}\right] \text{ or } \quad B\,\frac{\tan\theta_{max} - \tan\theta}{\tan\theta - \tan\theta_{min}}$$

h stop

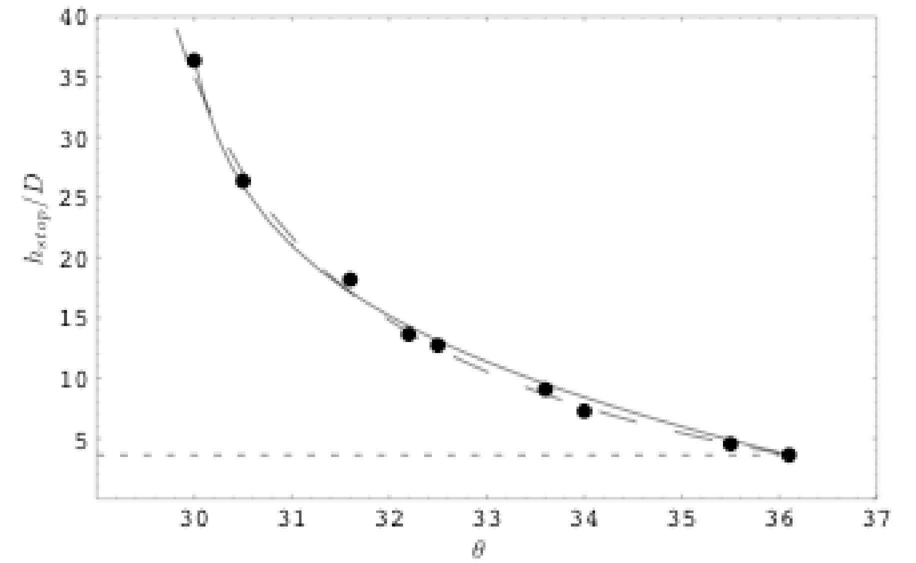


Figure 2: Dependence of h_{stop}/D on the inclination θ. Comparison between experimental results (points) for sand over carpets (from [16]) and fitting curves deduced from the granular pressure (21) (plain) and (19) (dashed). Experimental results with $h_{stop}/D \leq 3.6$ were discarded and the value $\theta_{max} \approx 36.1$ was adopted.

$$\frac{h_{stop}}{D} = \frac{P_0}{\phi_M cos\theta} Log \left[1 + \frac{\phi_M}{\phi_m} \frac{\tan\theta_{max} - \tan\theta}{\tan\theta - \tan\theta_{min}} \right].$$

exp. from Da Cruz PhD

summary

- Local non local continuum description
- Blend of "old" ideas: Savage, Bagnold...
- Granular pressure
- some simple examples
- h stop
- perspectives?
  - changing the relations (F, mu...)
  - introducing time?
  - omega near the boundary
- closure relation for Saint-Venant...