Boundary layer flows in large vessels

Simplified set of Navier Stokes Equations:
Application in Biomecanics

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Aim

- simplification of Navier Stokes equations
- thanks to asymptotic theory:

  “Boundary Layer”

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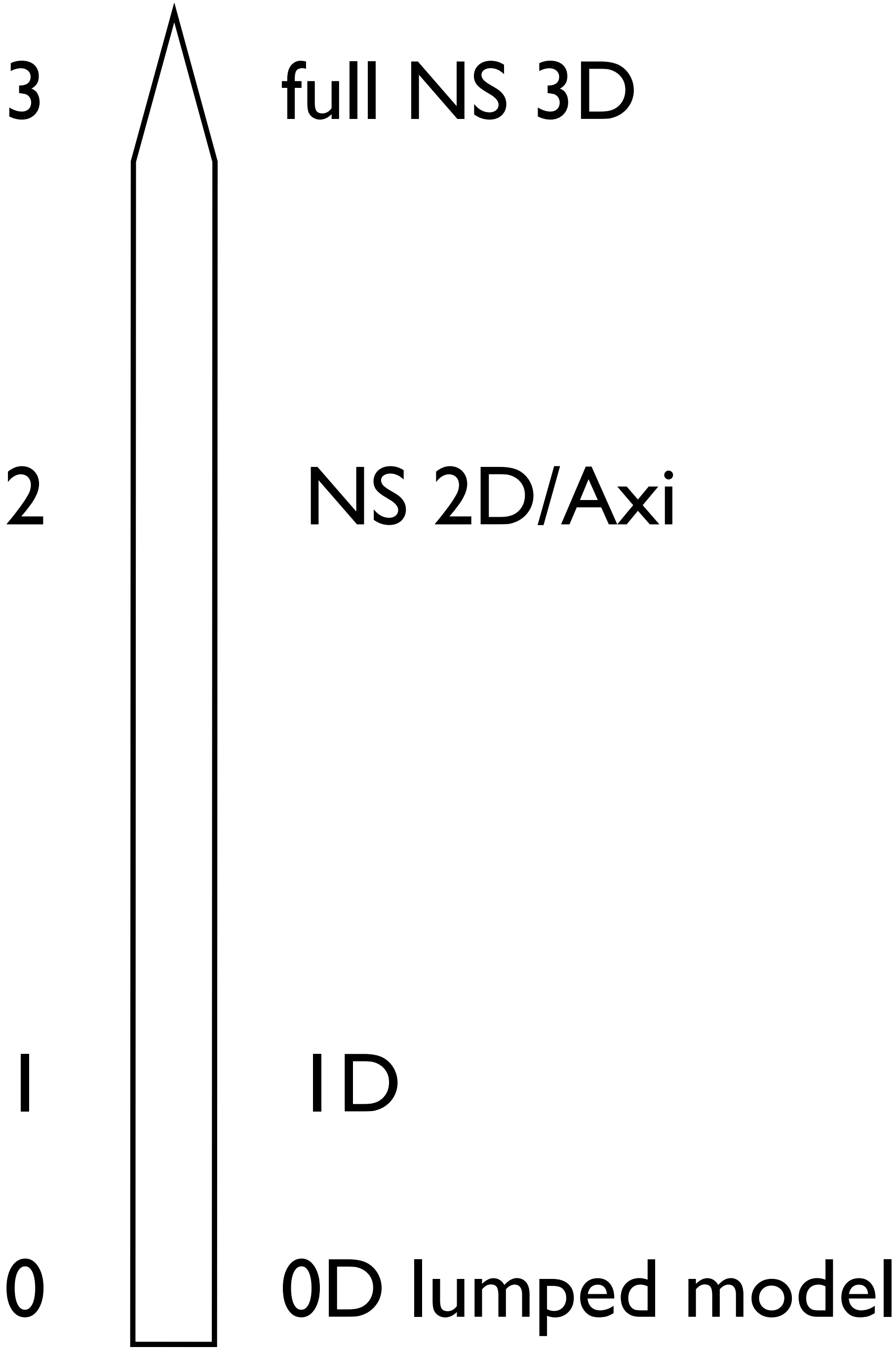
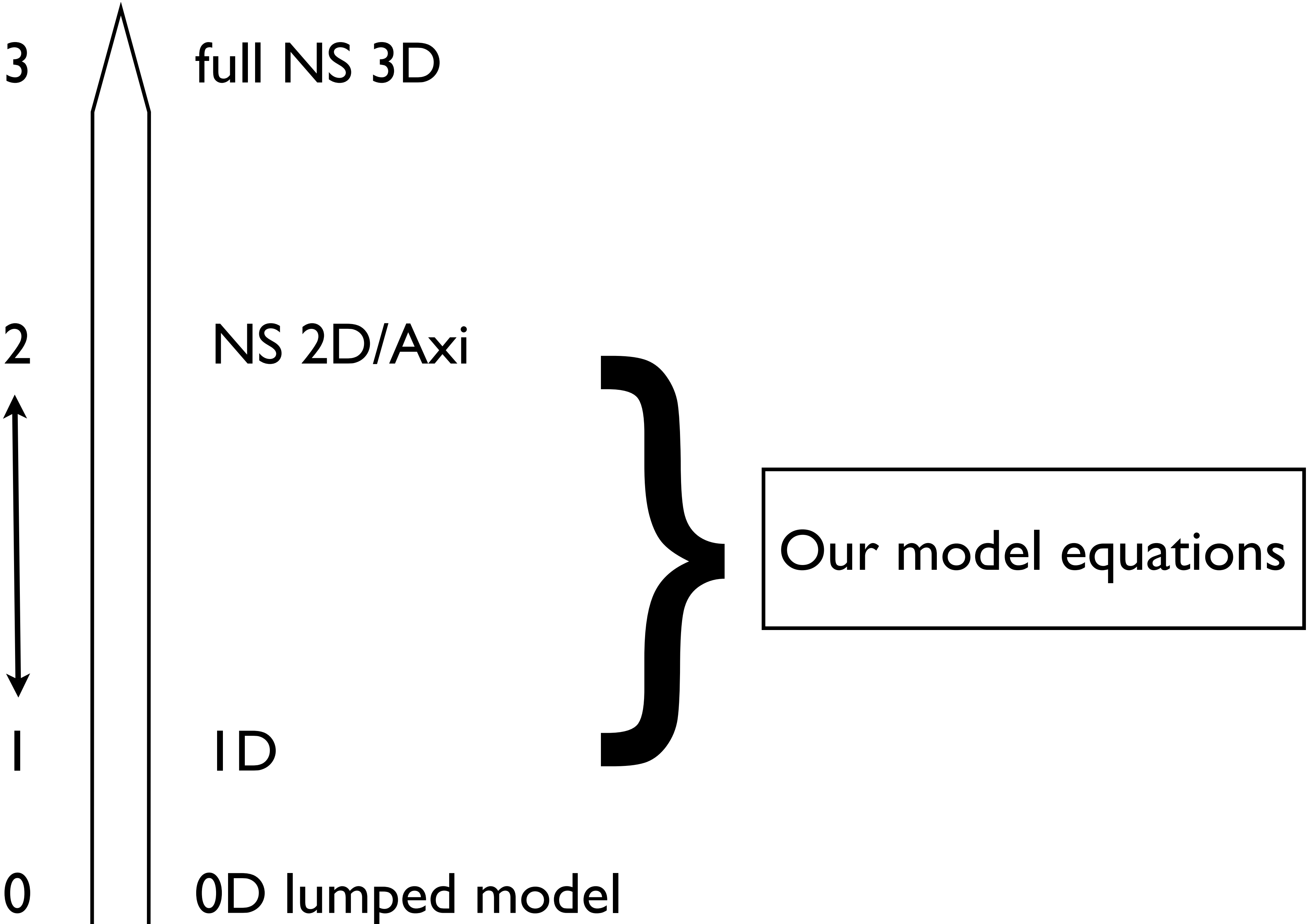
Starting from Navier Stokes (Axi)

- we simplify NS to a Reduced set of equations
  - which contains the physical scales,
  - the most important phenomena
- much more simple set of equations: Integral equations (1D)
- cross comparisons in some cases of NS/ RNSP/ Integral


3 full NS 3D
2 NS 2D/Axi
1 1D
0 0D lumped model


3
full NS 3D
2
NS 2D/Axi
1
1D
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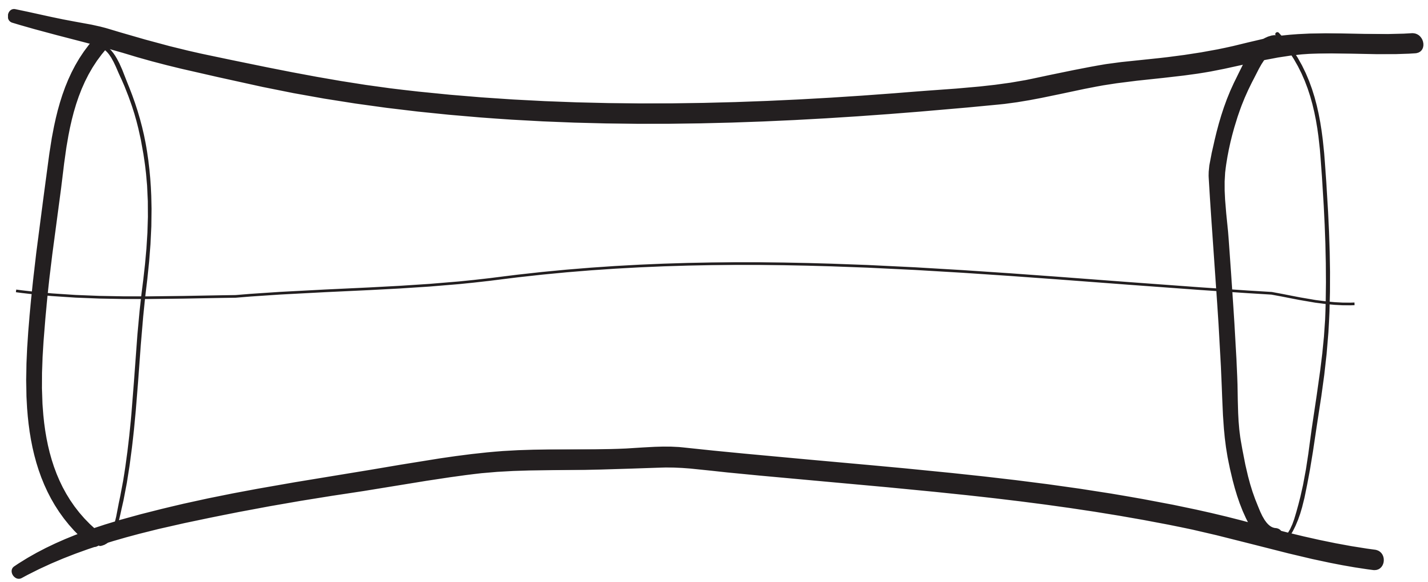
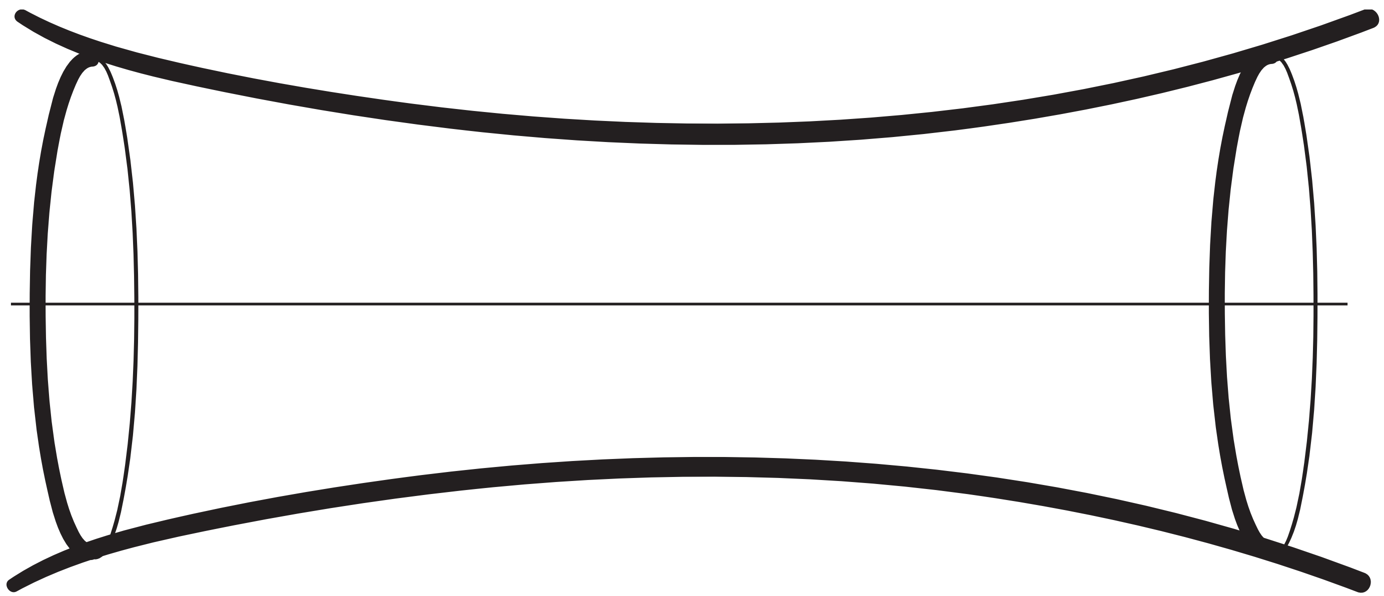
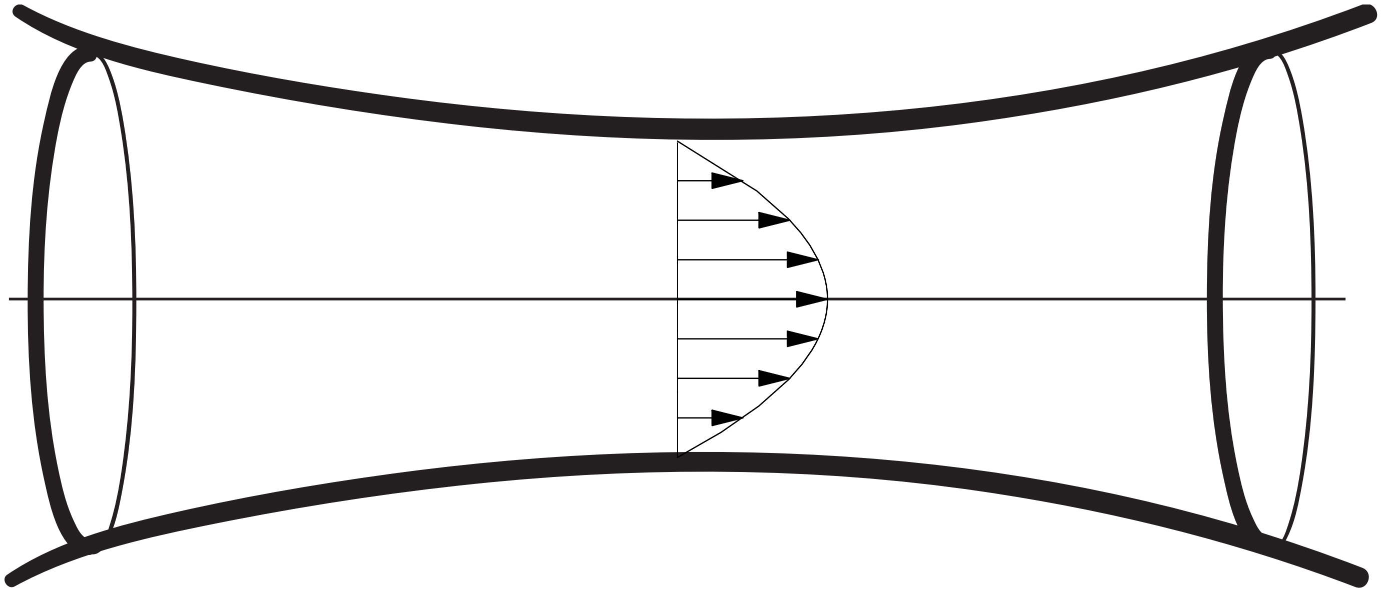
0D lumped model
Our model equations



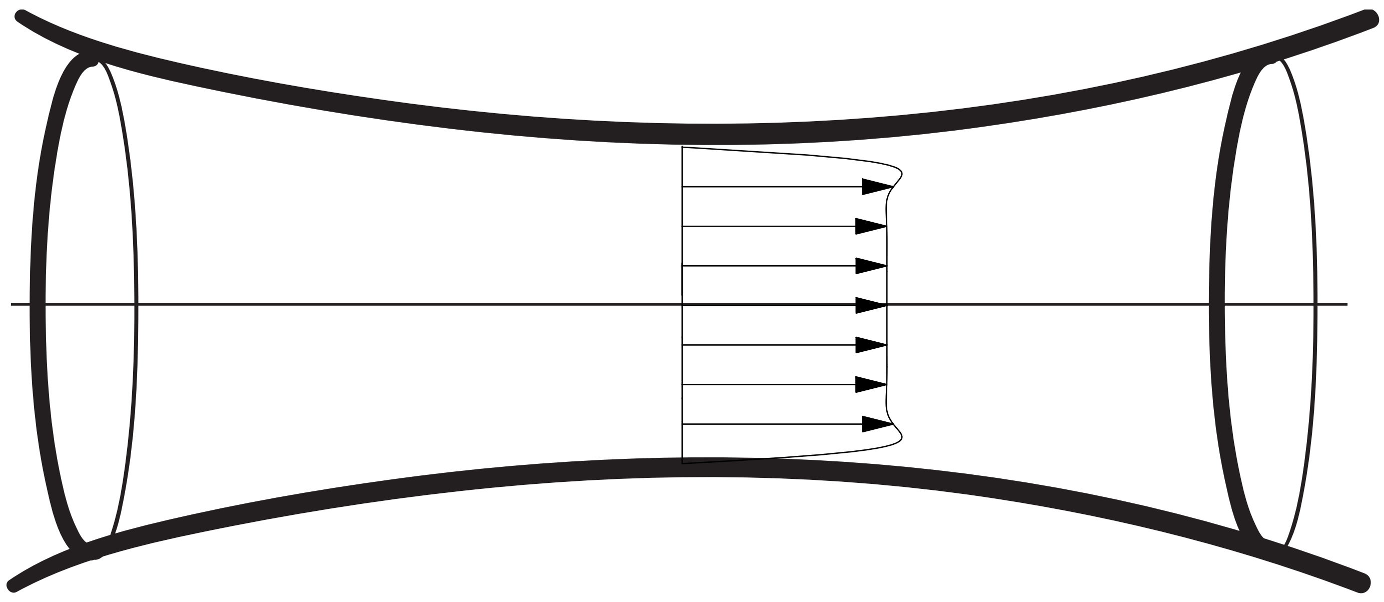
reality?



straight pipe, smooth walls, symmetry

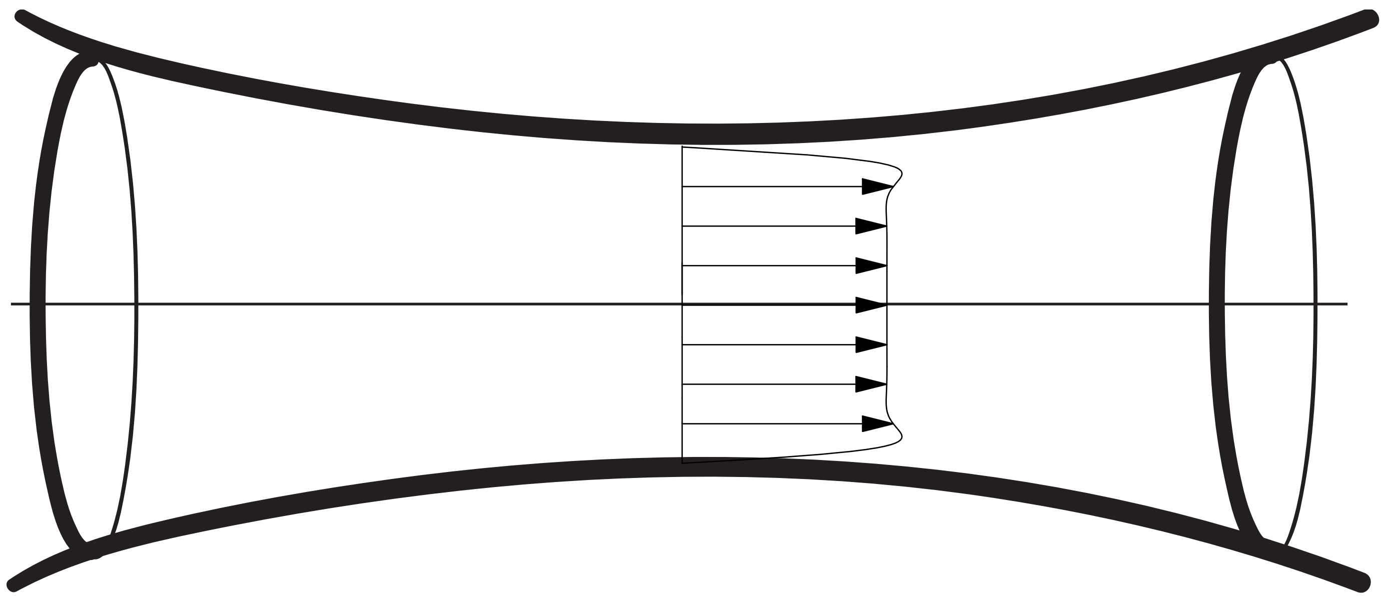


velocity profile

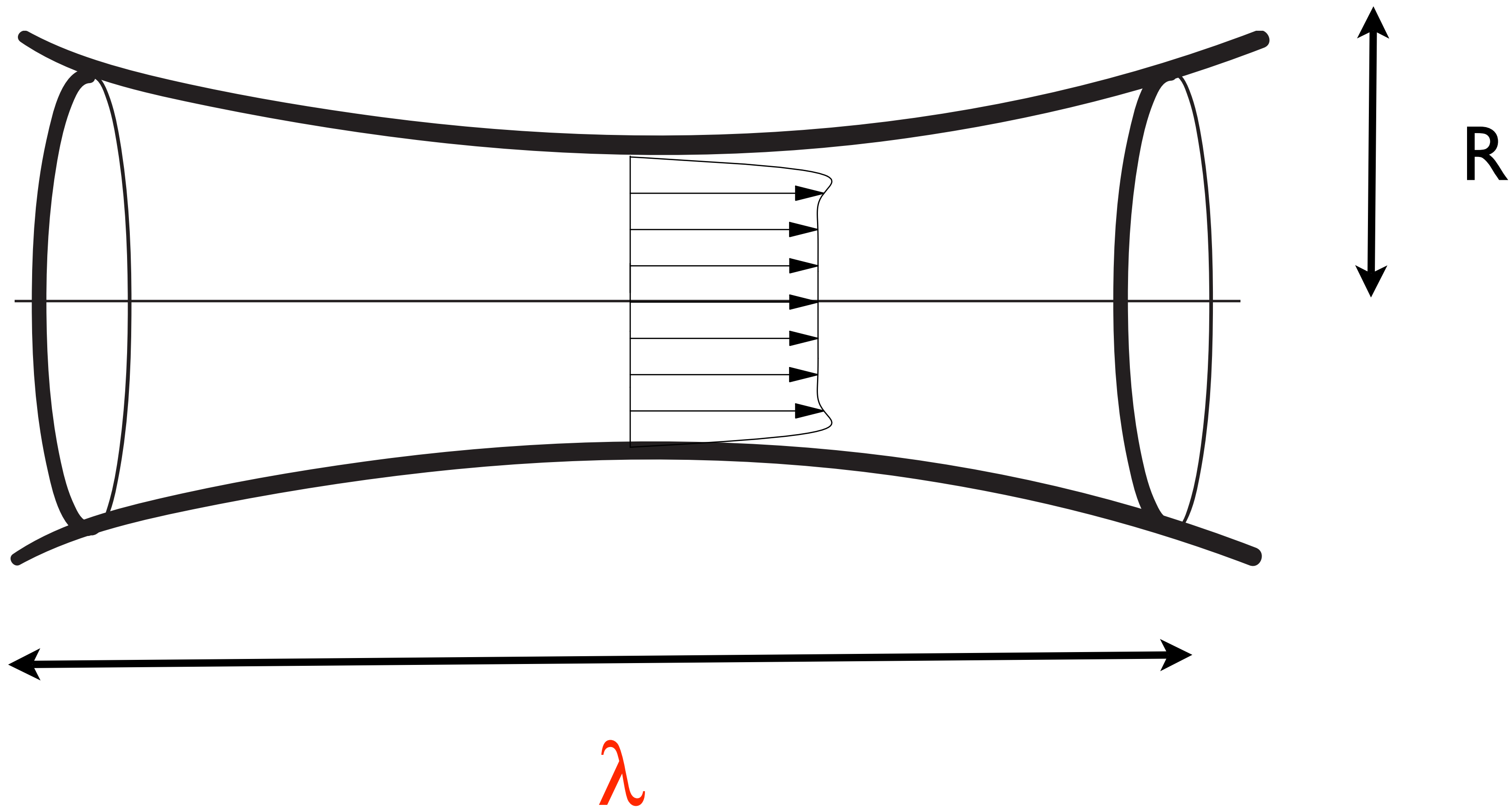


velocity profile

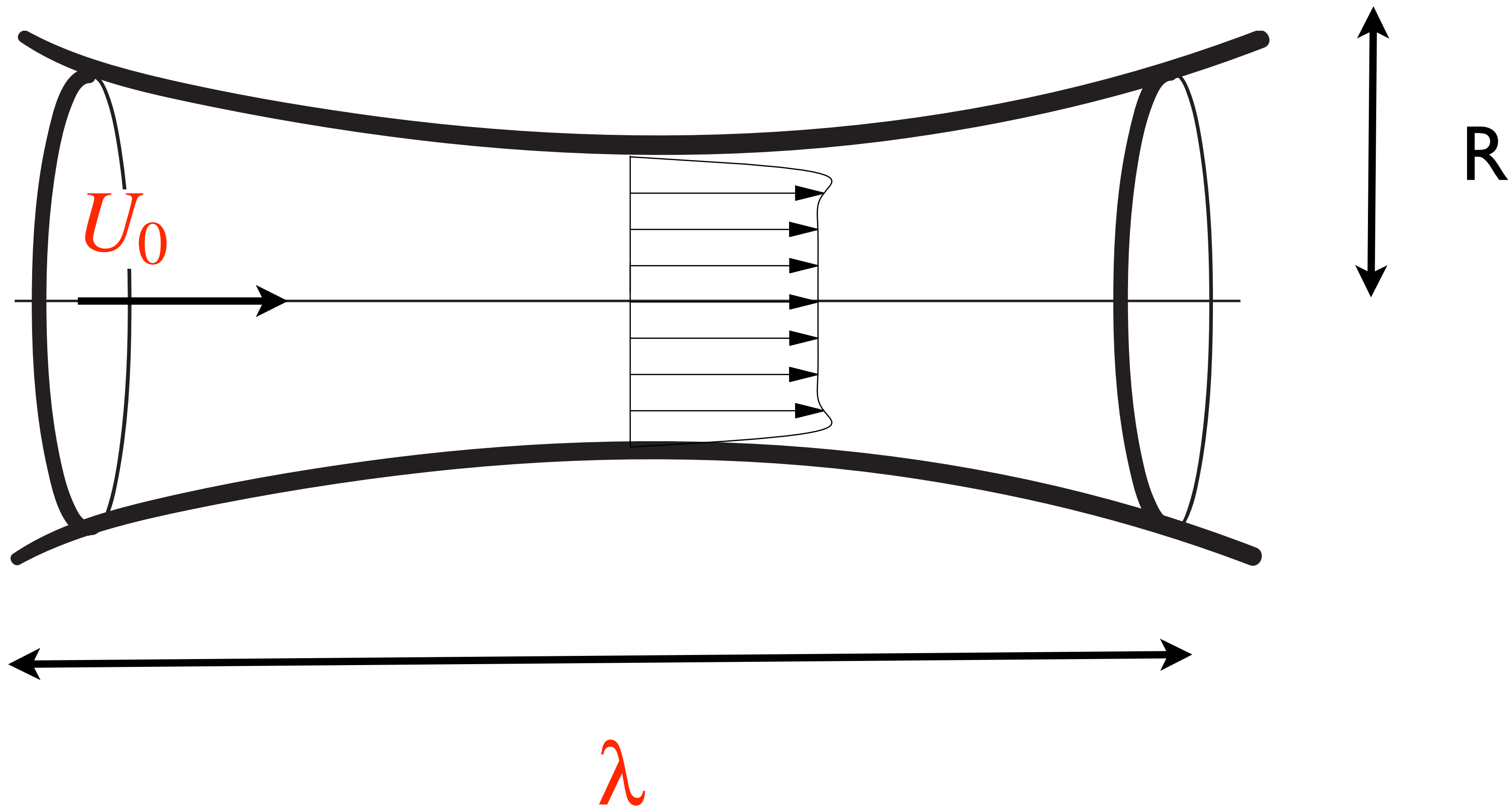
Equations



- simplified set
- deduced from orders of magnitude

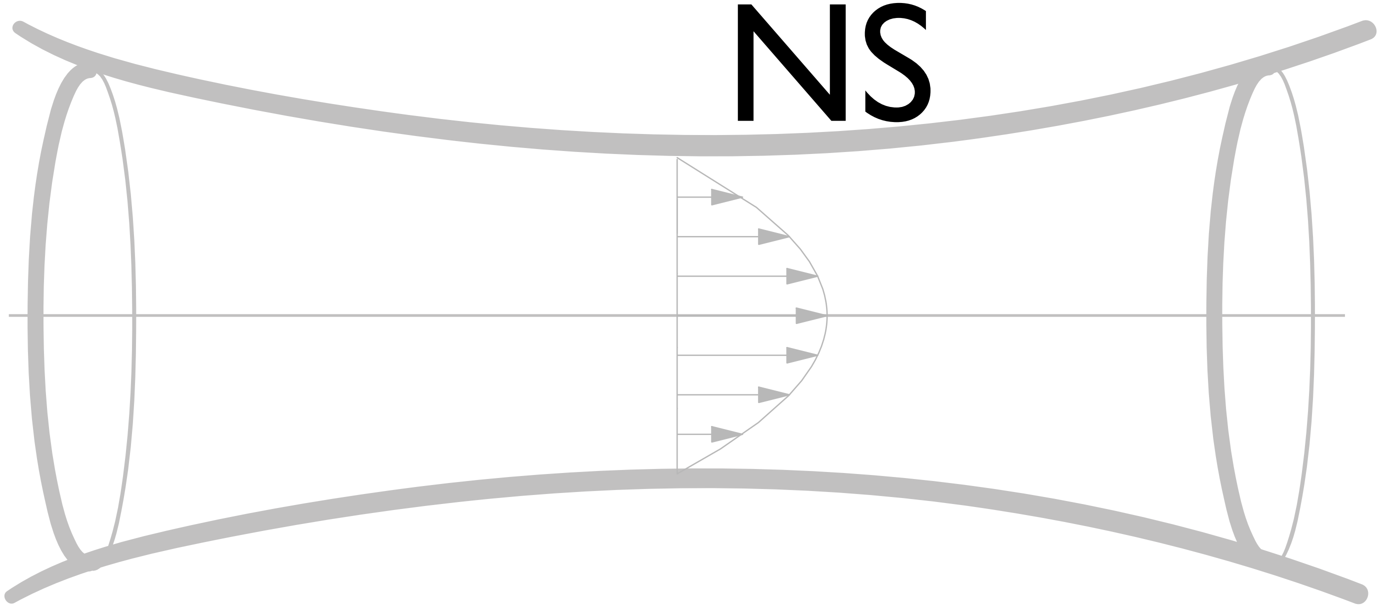


$R << \lambda$



U_0
R
λ

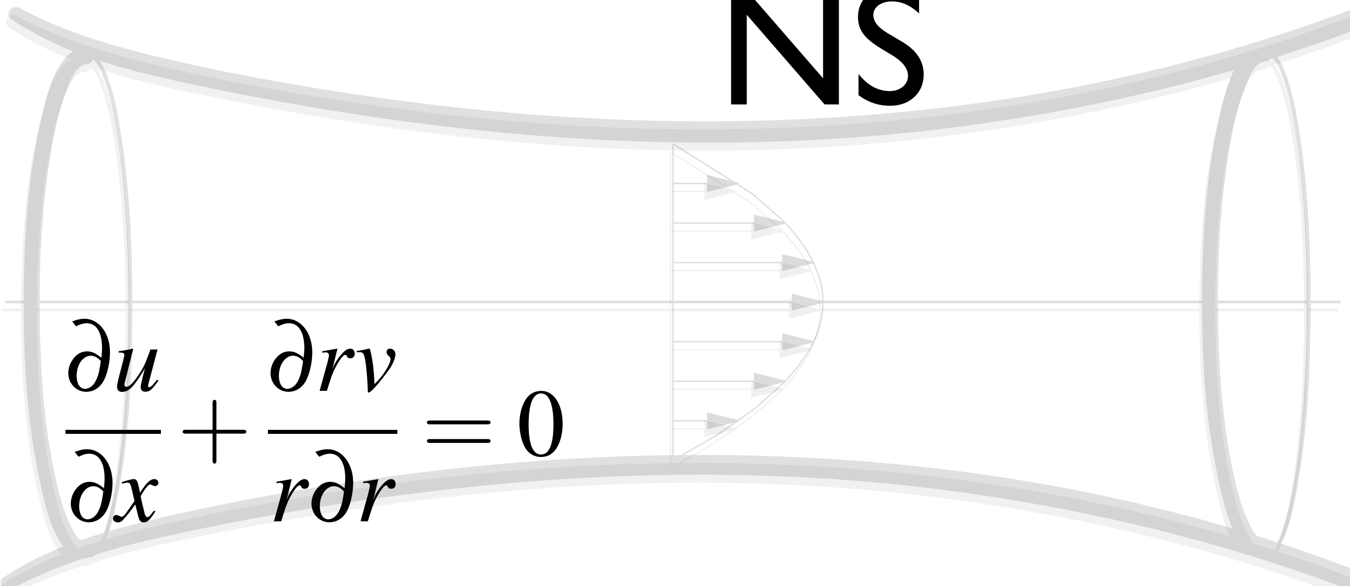
NS



$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu\frac{\partial^2}{\partial x^2}u + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial r}v = -\frac{\partial p}{\rho \partial r} + \nu\frac{\partial^2}{\partial x^2}v + \nu\frac{\partial}{r\partial r}r\frac{\partial v}{\partial r}$$

NS



$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu\frac{\partial^2}{\partial x^2}u + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

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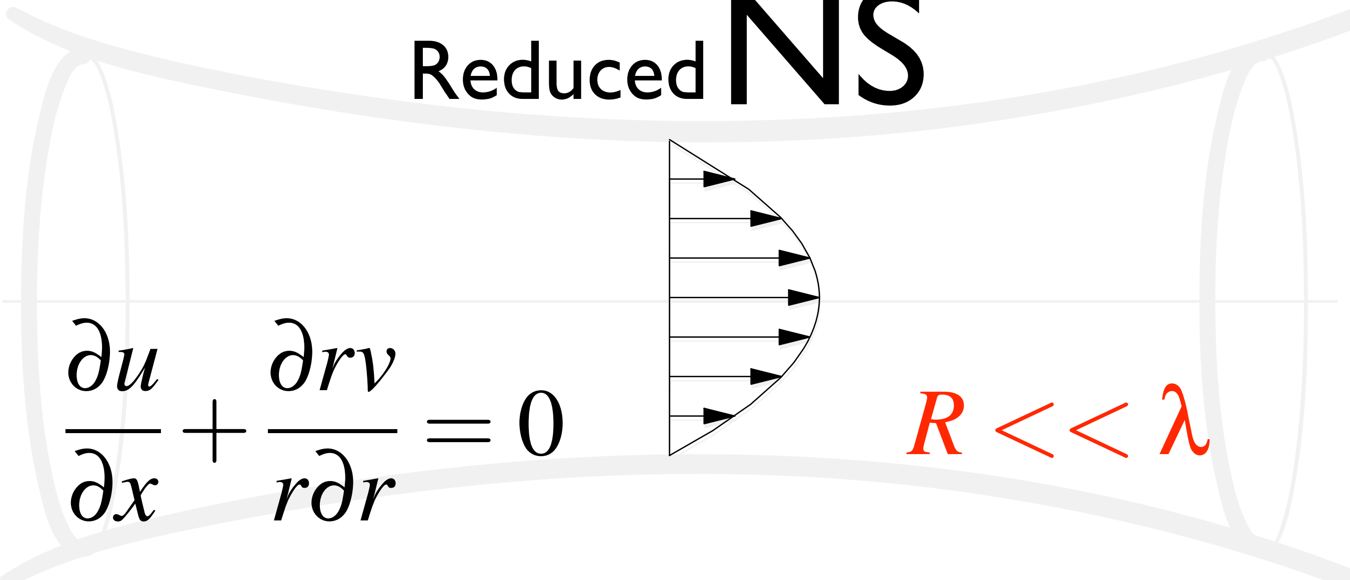
Reduced NS



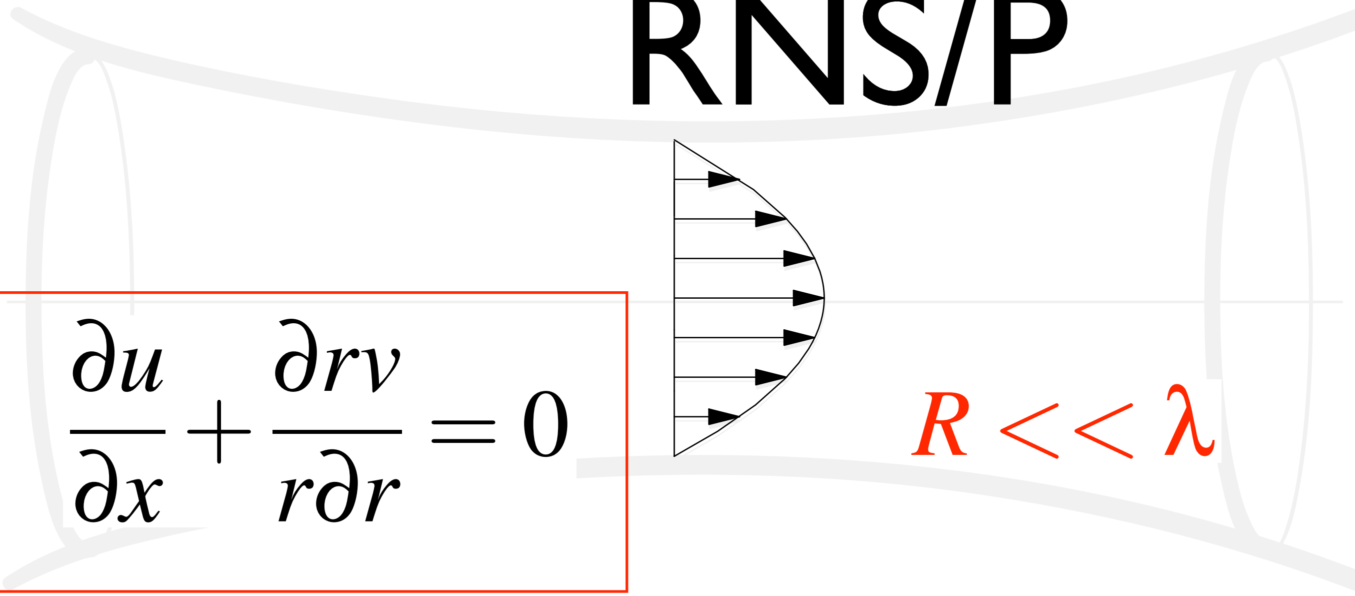
$$\frac{\partial u}{\partial x}+\frac{\partial r v}{r\partial r}=0$$

$$R << \lambda$$

$$\frac{\partial u}{\partial t}+u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial r}u=-\frac{\partial p}{\rho\partial x}+\nu\cancel{\frac{\partial^2}{\partial x^2}}u+\nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial}{\partial x}v+v\frac{\partial}{\partial r}v=-\frac{\partial p}{\rho\partial r}+\nu\cancel{\frac{\partial^2}{\partial x^2}}v+\nu\frac{\partial}{r\partial r}r\frac{\partial v}{\partial r}$$

RNS/P



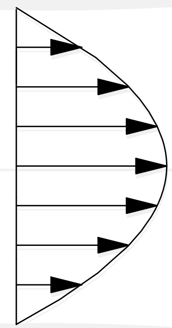
$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$R << \lambda$ $\qquad V \sim U_0 \frac{R}{\lambda}$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \cancel{\frac{\partial^2}{\partial x^2}}u + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial r}v = -\frac{\partial p}{\rho \partial r} + \nu \cancel{\frac{\partial^2}{\partial x^2}}v + \nu\frac{\partial}{r\partial r}r\frac{\partial v}{\partial r}$$

RNS/P



$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

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$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \cancel{\frac{\partial^2}{\partial x^2}} u + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r}$$

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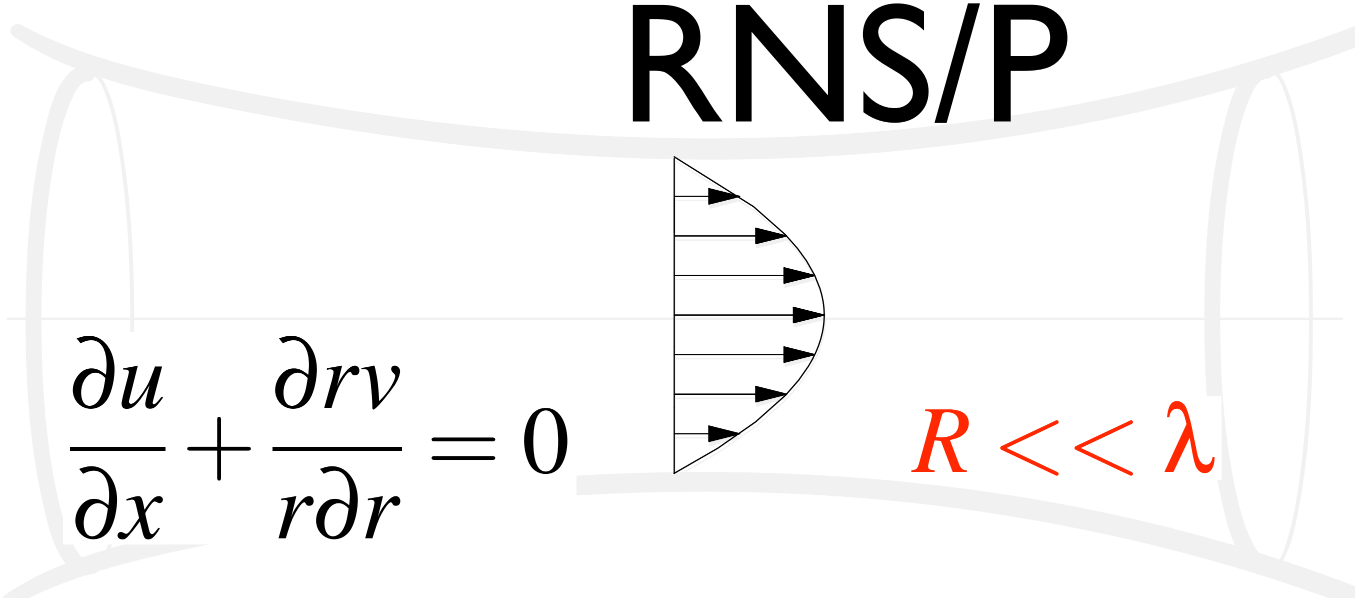
RNS/P



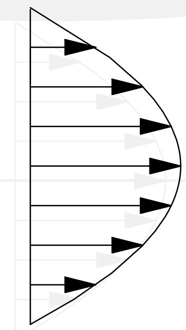
$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$R << \lambda \qquad V \sim U_0 \frac{R}{\lambda}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u = -\frac{\partial p}{\rho \partial x} + \nu \cancel{\frac{\partial^2}{\partial x^2}} u + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}$$

$$\cancel{\frac{\partial v}{\partial t}} + u \cancel{\frac{\partial}{\partial x}} v + v \cancel{\frac{\partial}{\partial r}} v = -\frac{\partial p}{\rho \partial r} + \nu \cancel{\frac{\partial^2}{\partial x^2}} v + \nu \frac{\partial}{r \partial r} \cancel{r} \frac{\partial v}{\partial r}$$

RNS/P

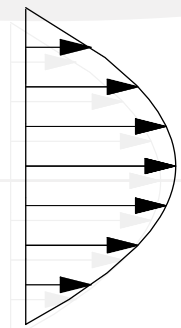


$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\boxed{\frac{\partial u}{\partial t}} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \boxed{\nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r}}$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

RNS/P



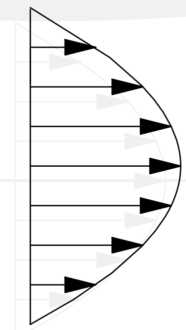
$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\nu \frac{1}{\omega R^2}$$

$$\boxed{\frac{\partial u}{\partial t}} + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u = -\frac{\partial p}{\rho \partial x} + \boxed{\nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}}$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

RNS/P



$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\alpha = R\sqrt{\frac{\omega}{\nu}}$$

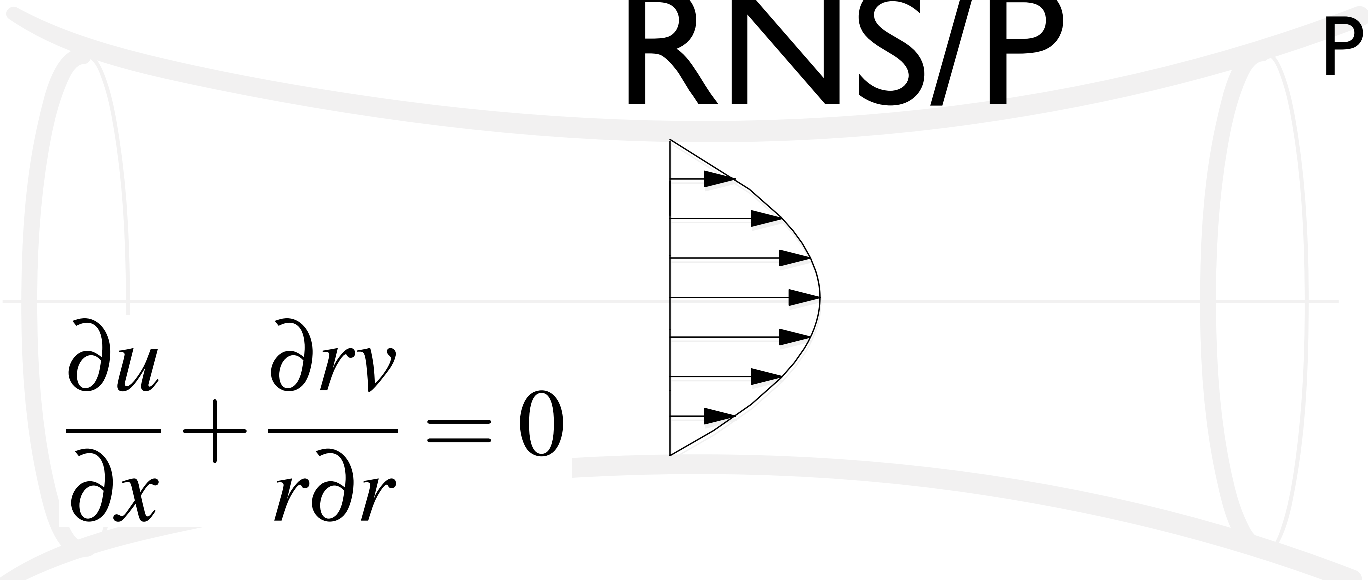
$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

1/(Womersley)^2

RNS/P

Prandtl

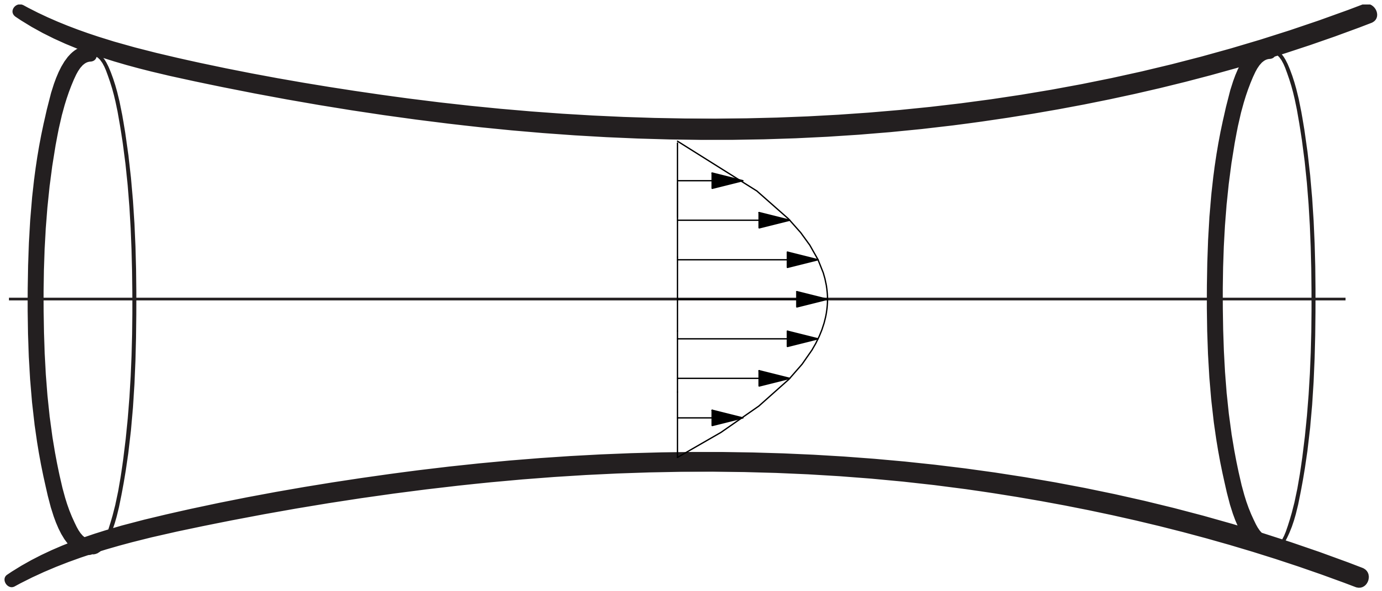


$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}$$

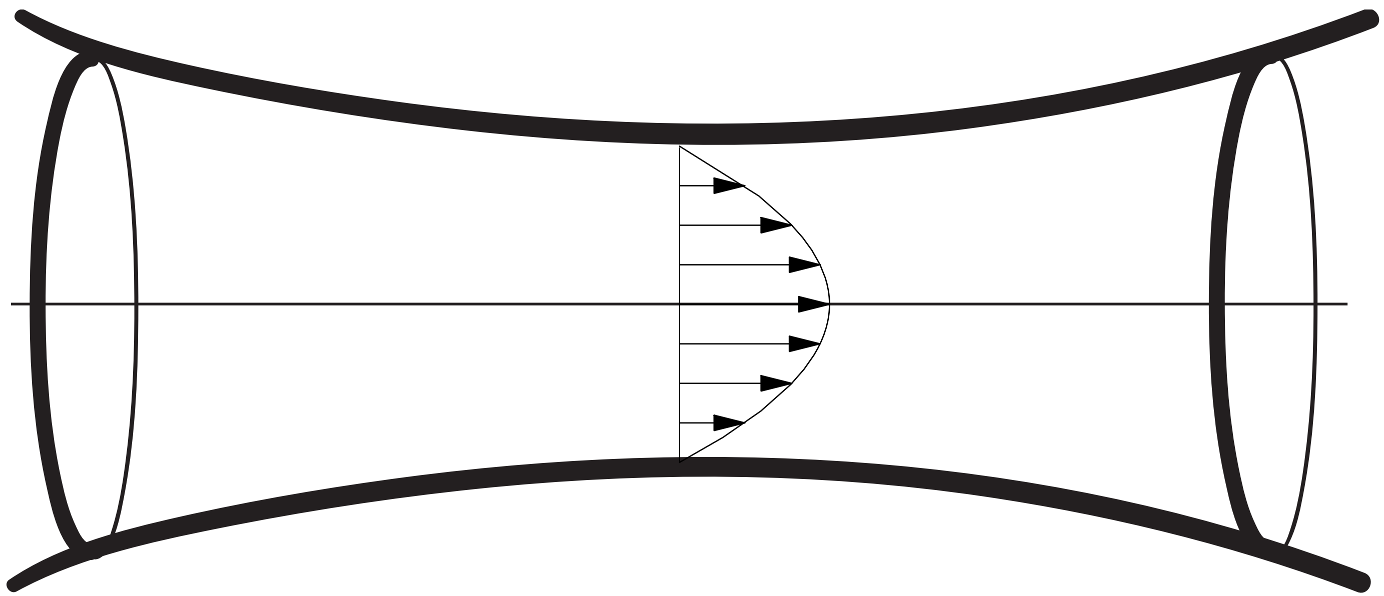
$$0 = -\frac{\partial p}{\rho \partial r}$$

Boundary conditions



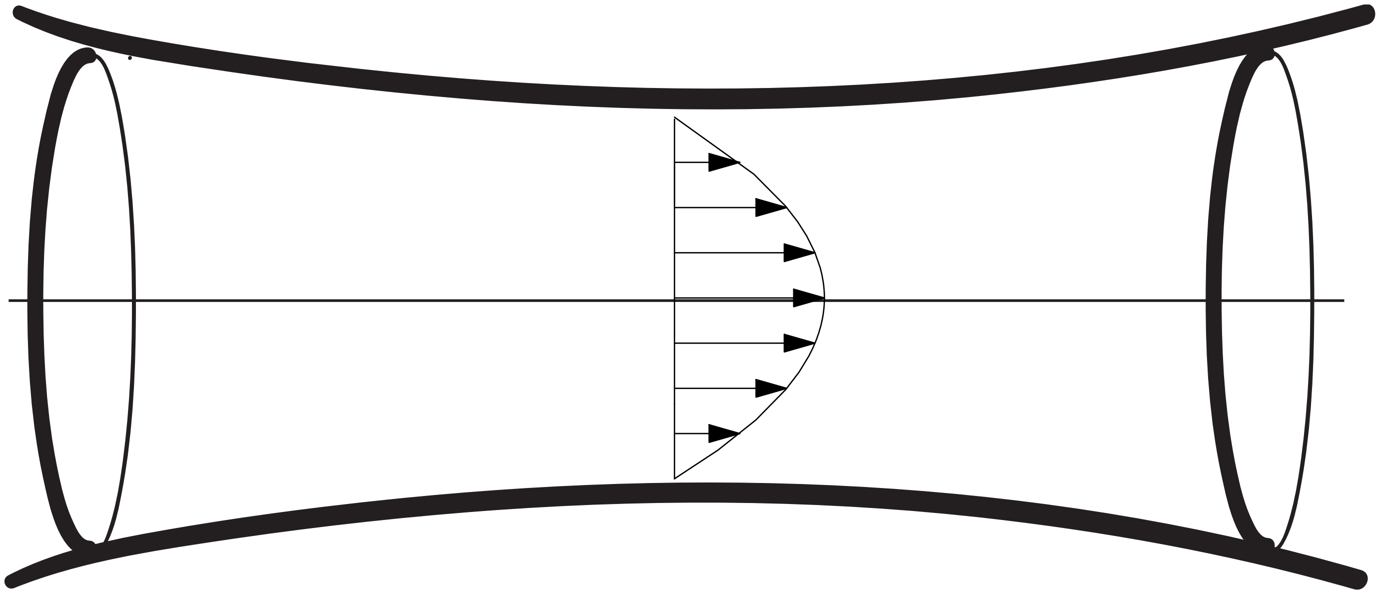
Rigid wall: $u = v = 0$

Boundary conditions



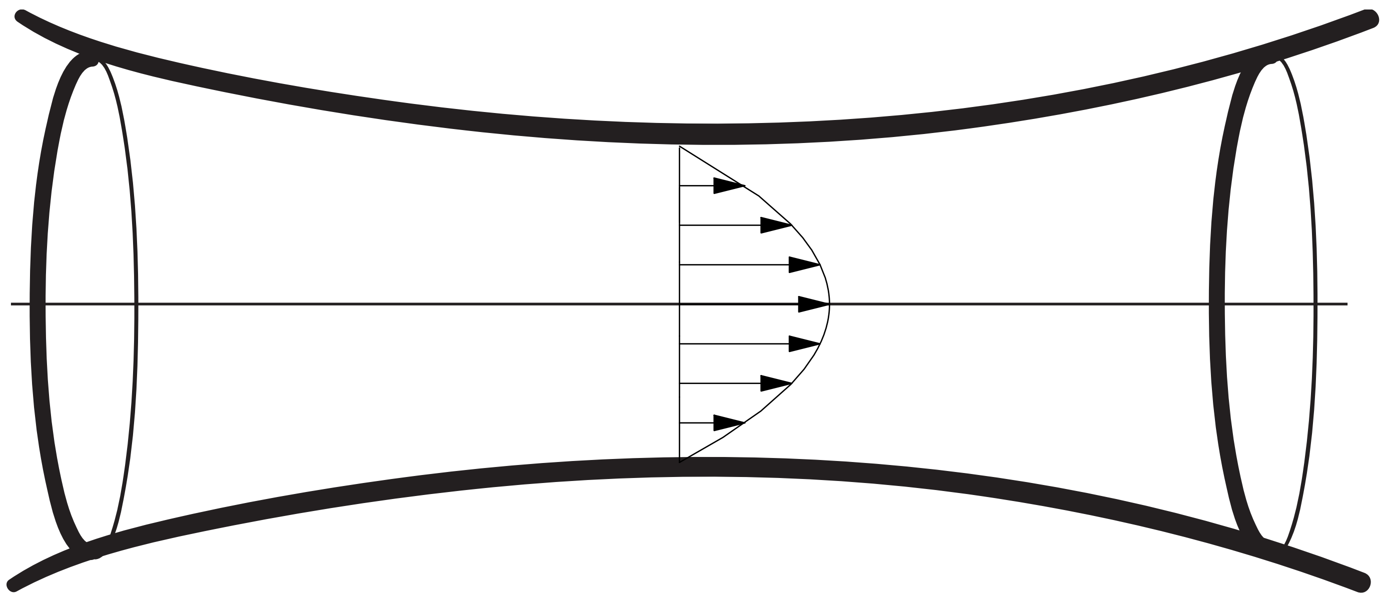
moving wall

Boundary conditions



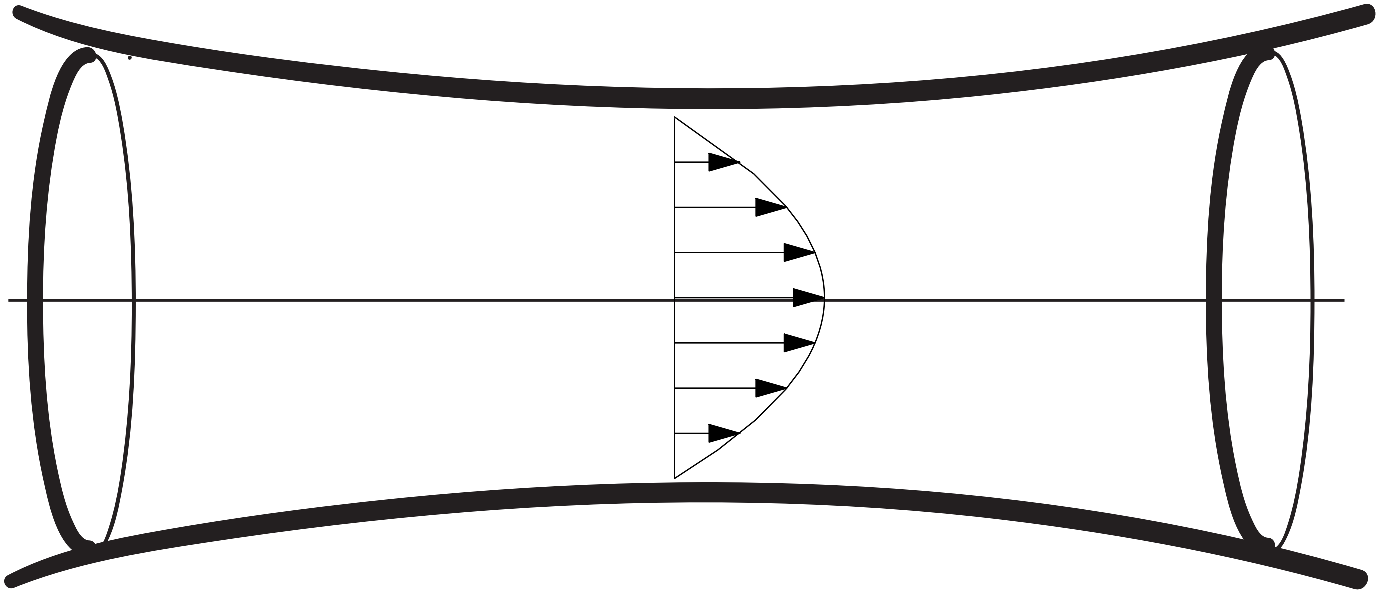
moving wall

Boundary conditions



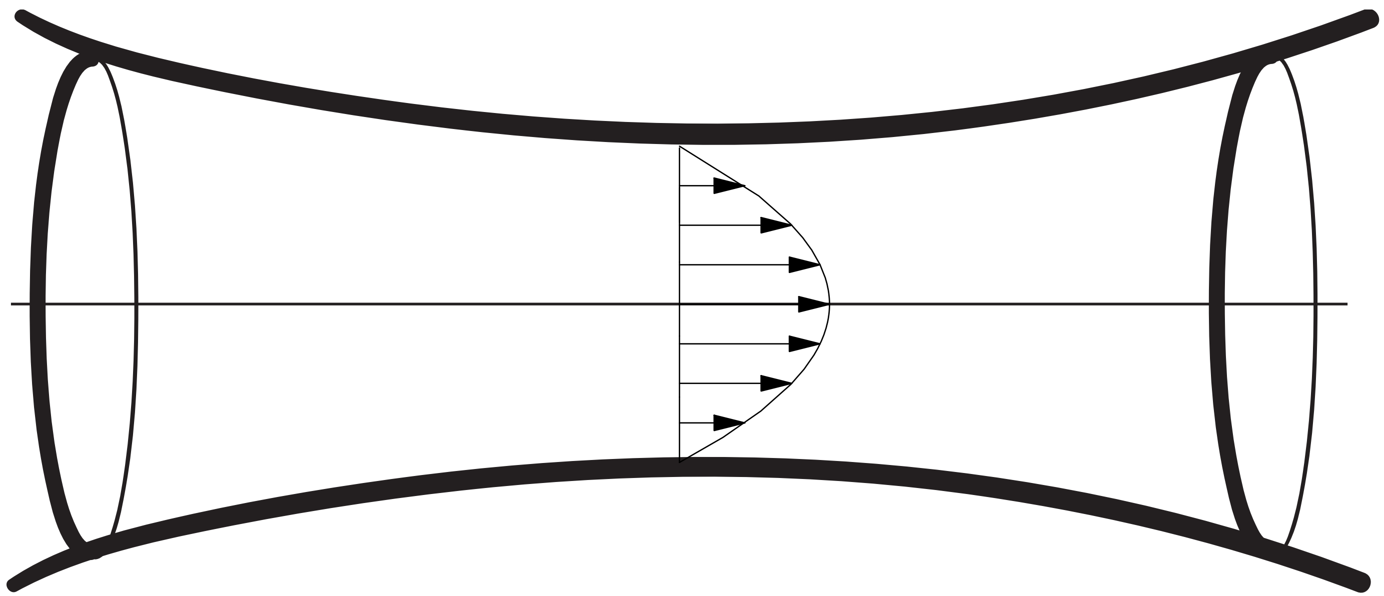
moving wall

Boundary conditions



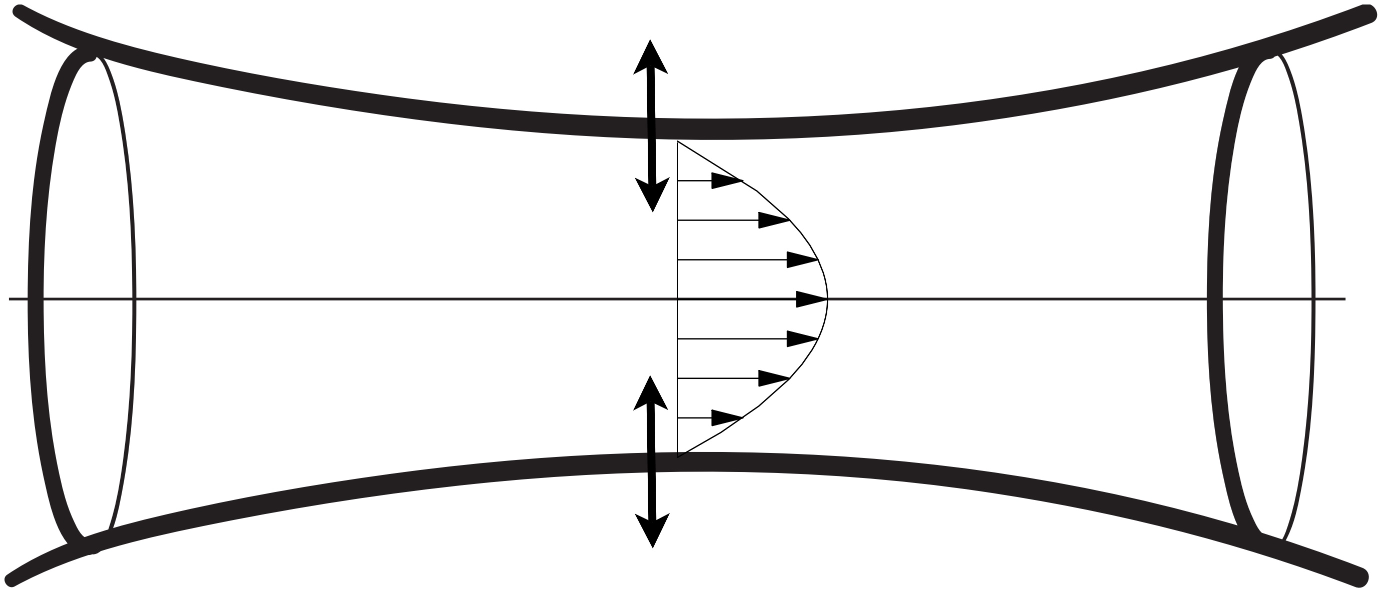
moving wall

Boundary conditions



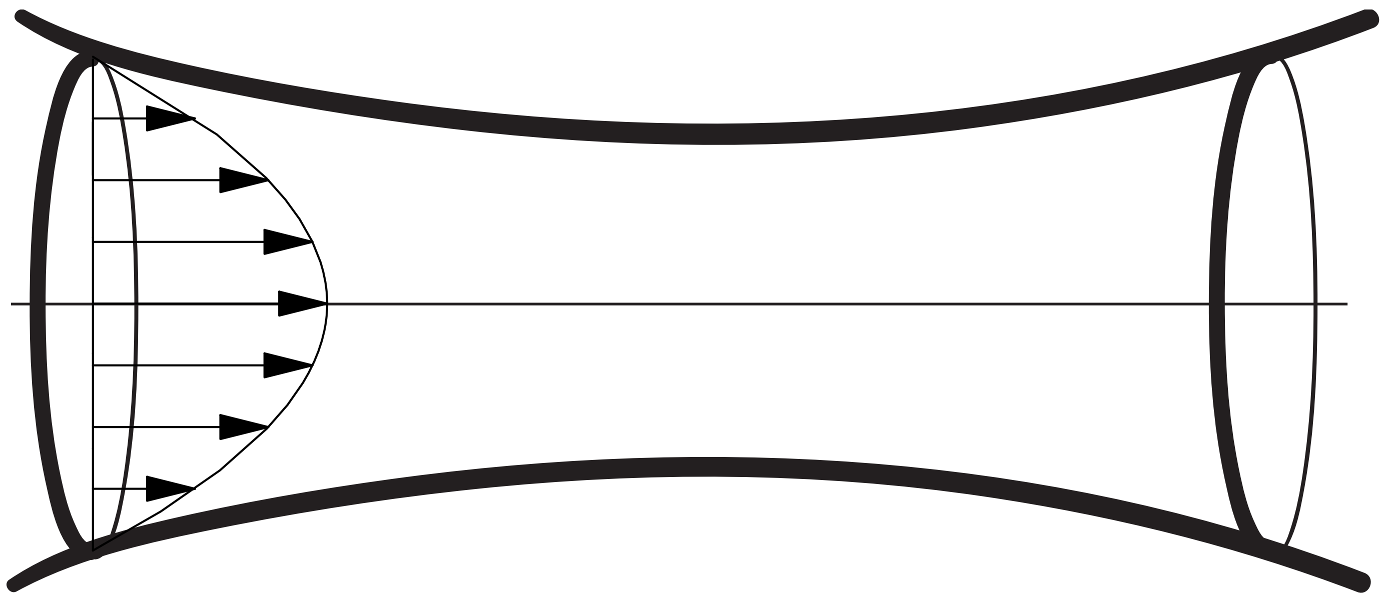
moving wall

Boundary conditions



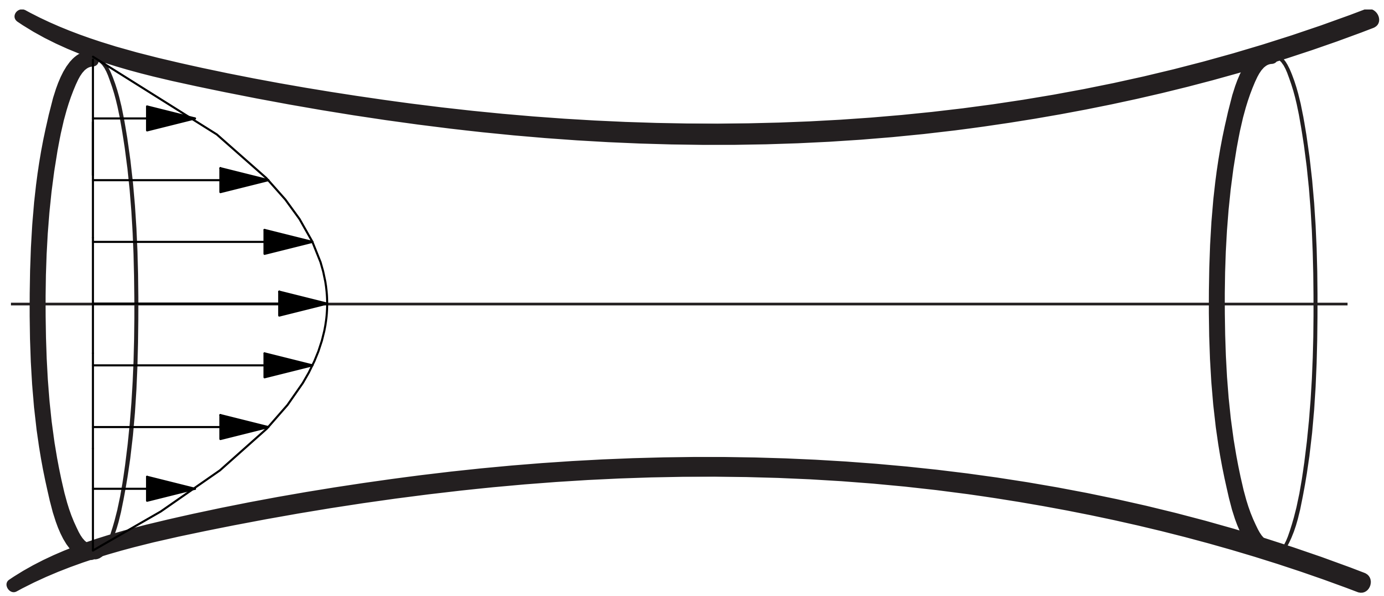
moving wall $$v = \frac{\partial R}{\partial t}$$

Boundary conditions



First given profile:

Boundary conditions

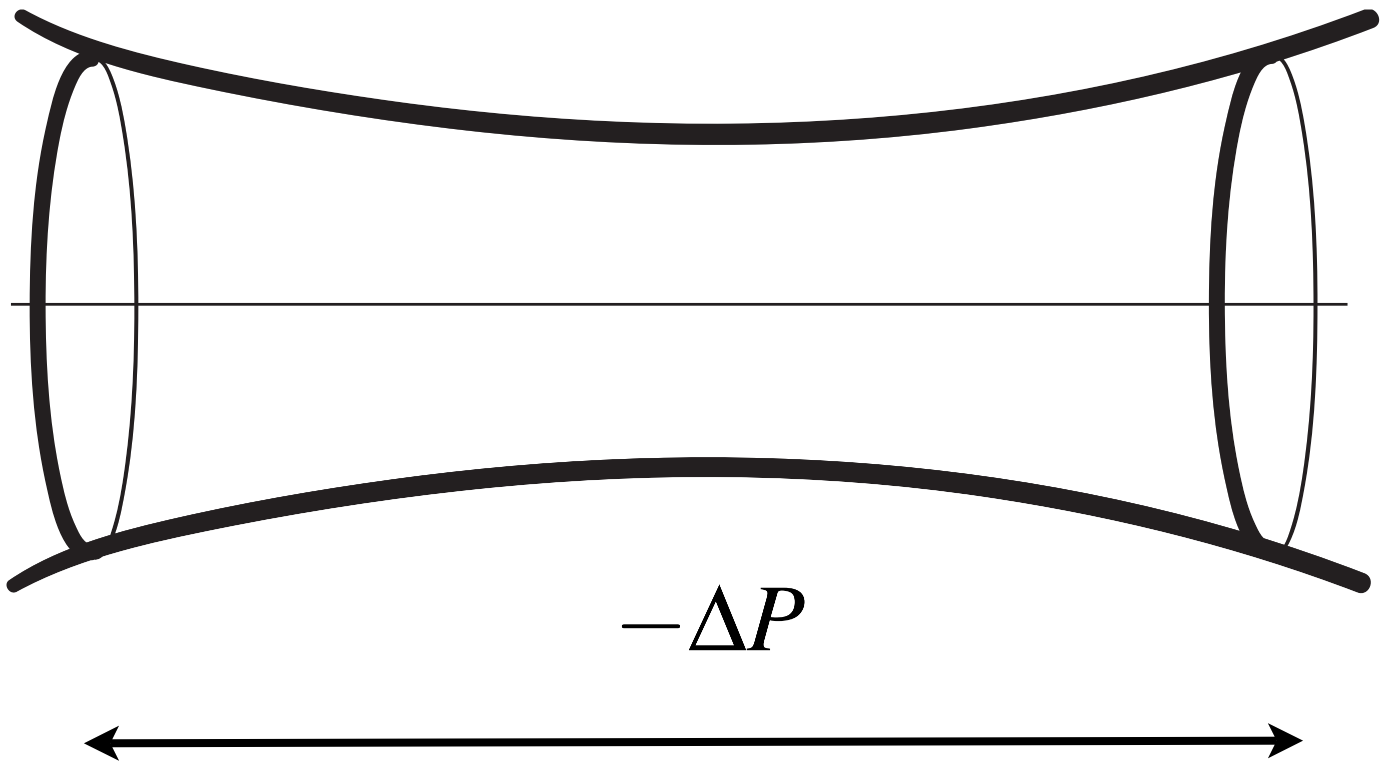


First given profile:

marching procedure

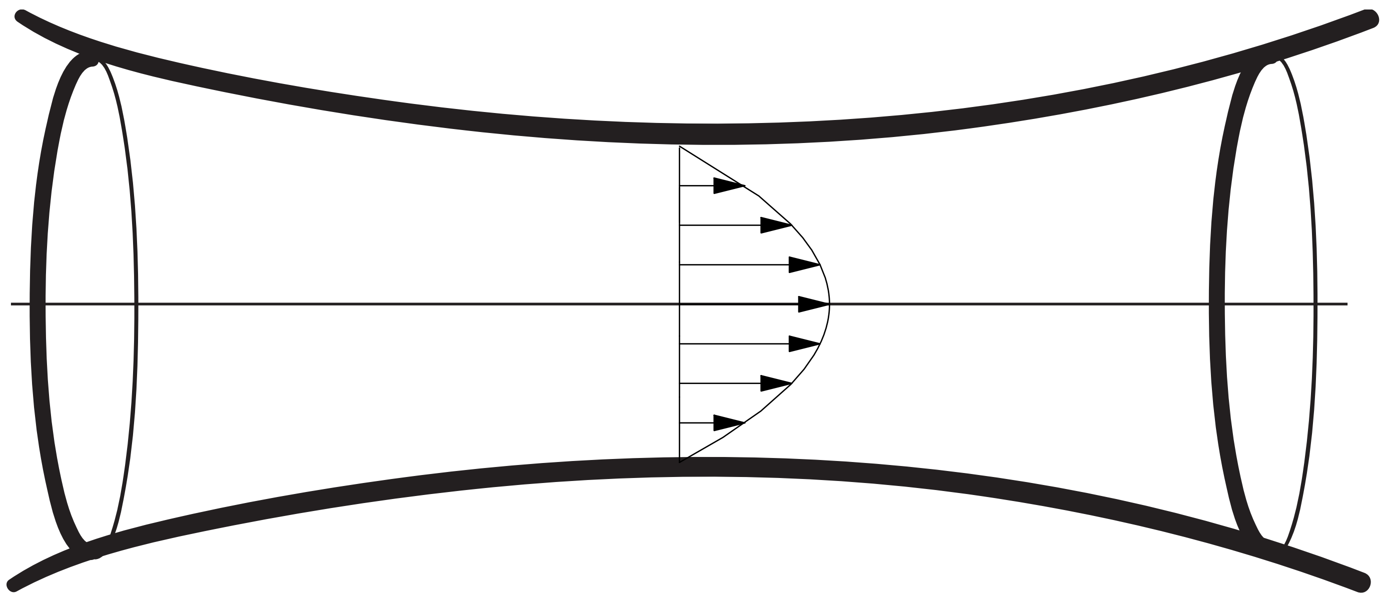
distribution of pressure is a result

Boundary conditions

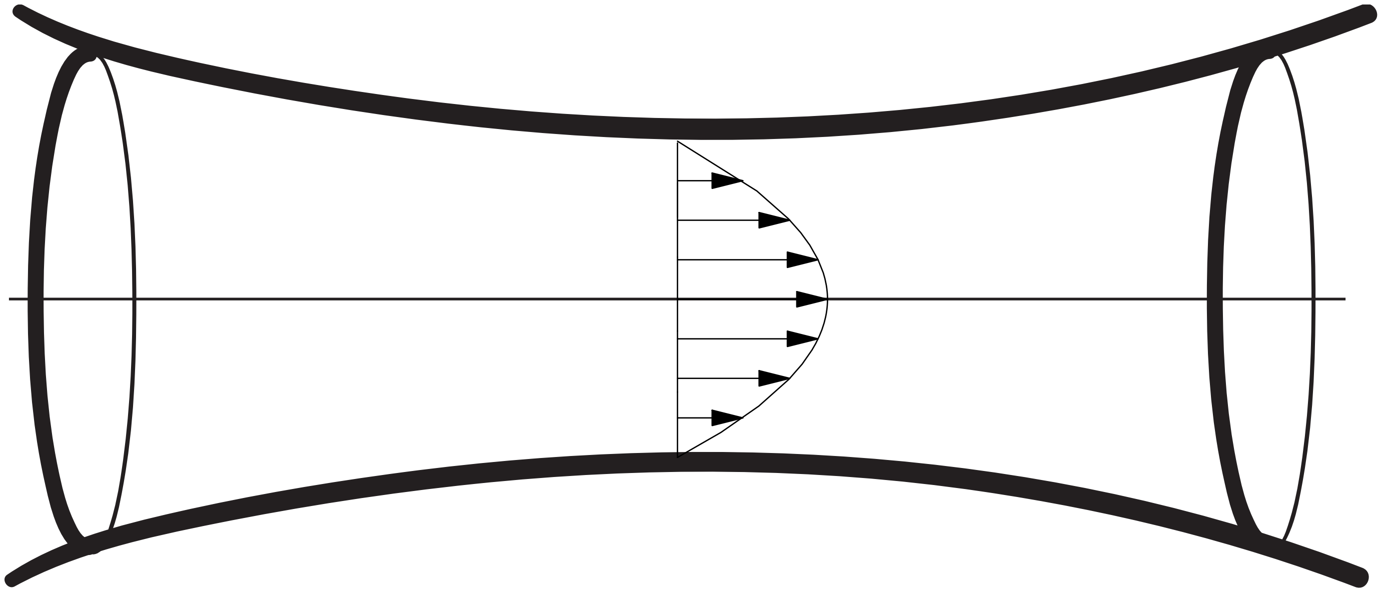


or given pressure drop
by Newton iteration on the entrance flux

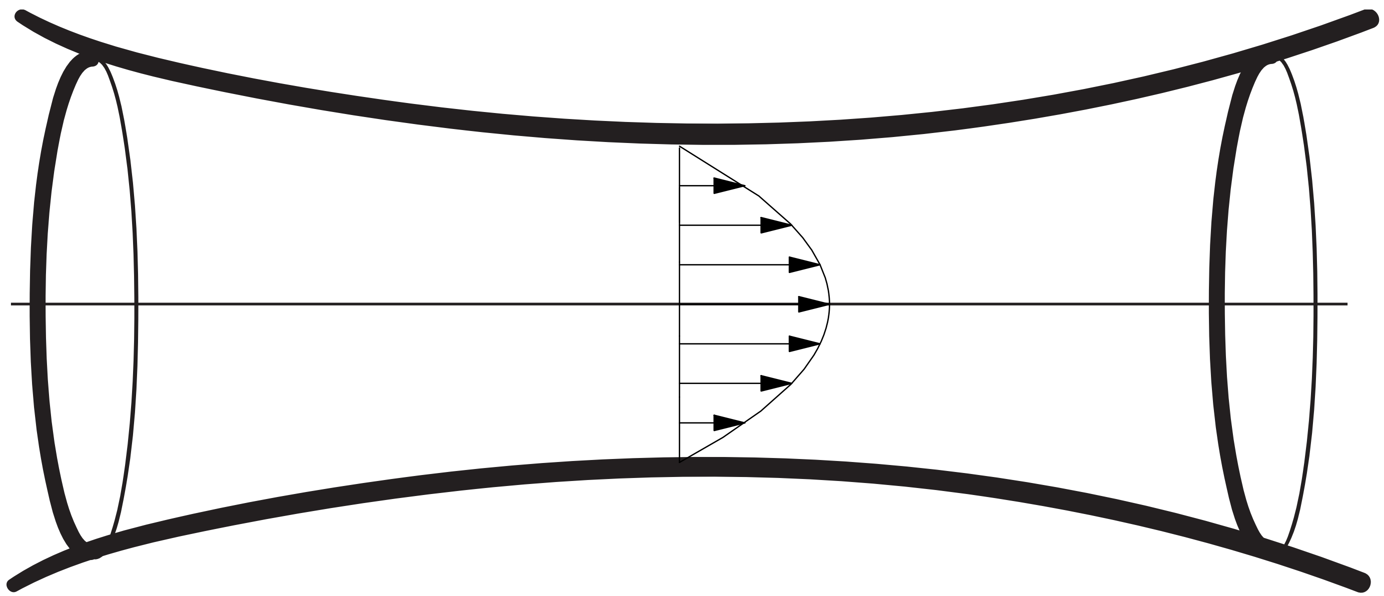
Numerical resolution



finite differences,
implicit in time

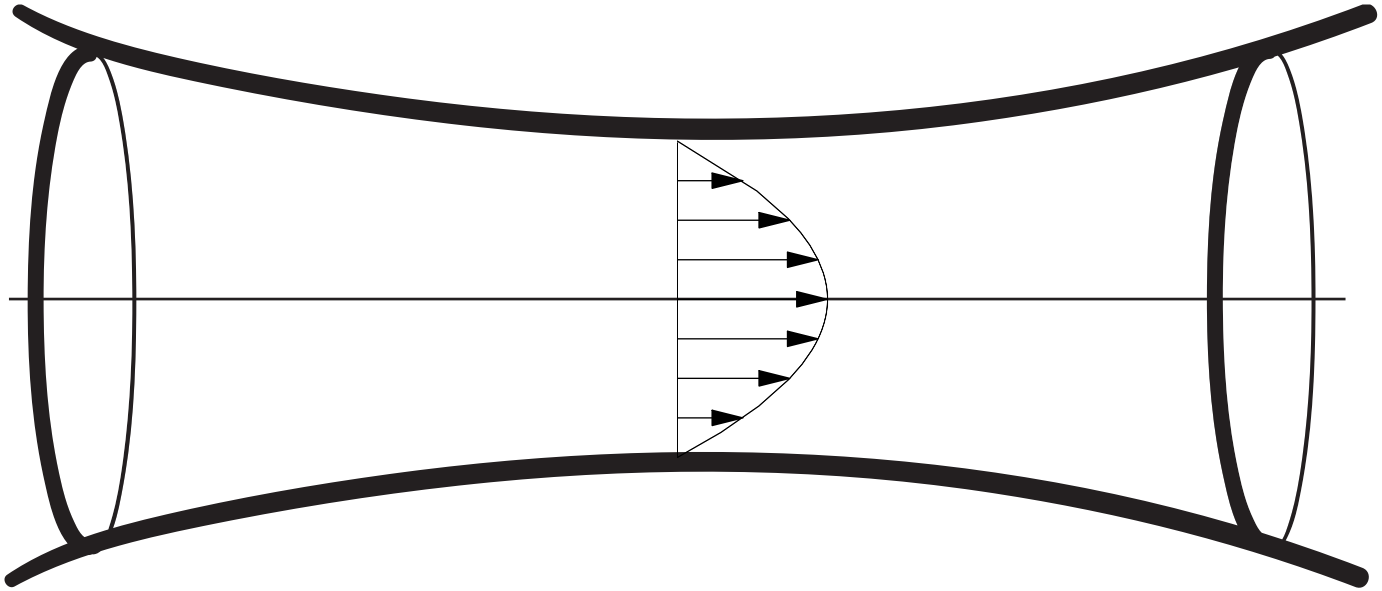


$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$



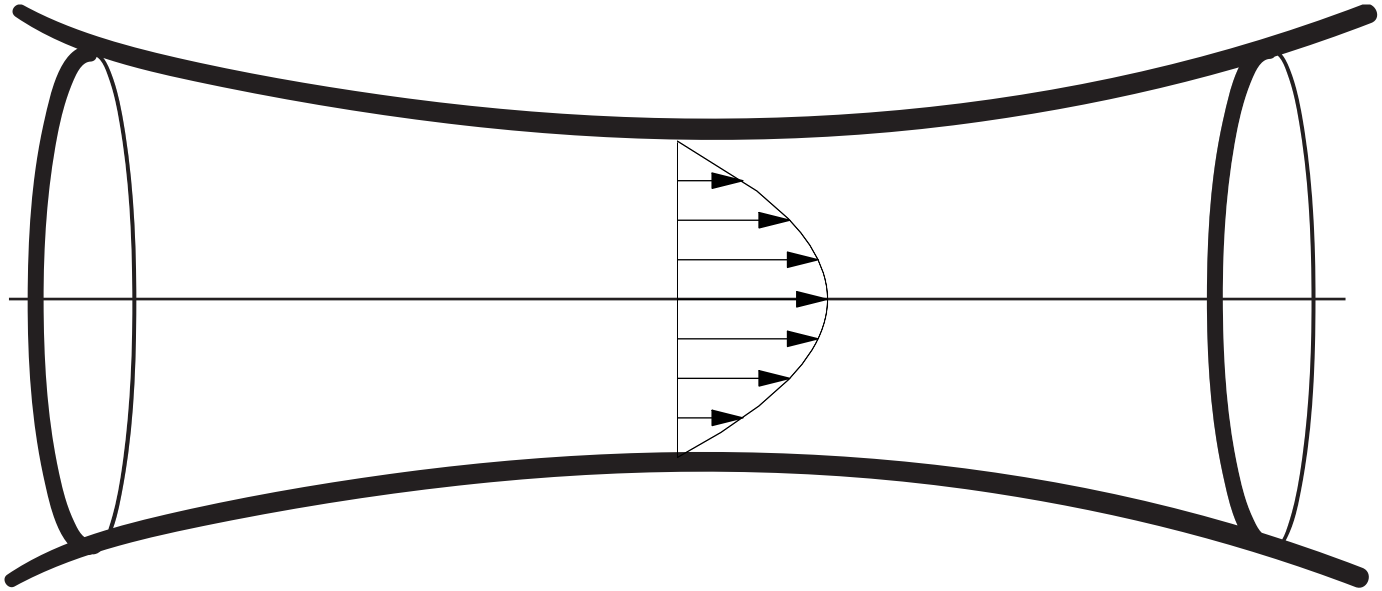
$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

$$p^{given} \rightarrow u^*$$



$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

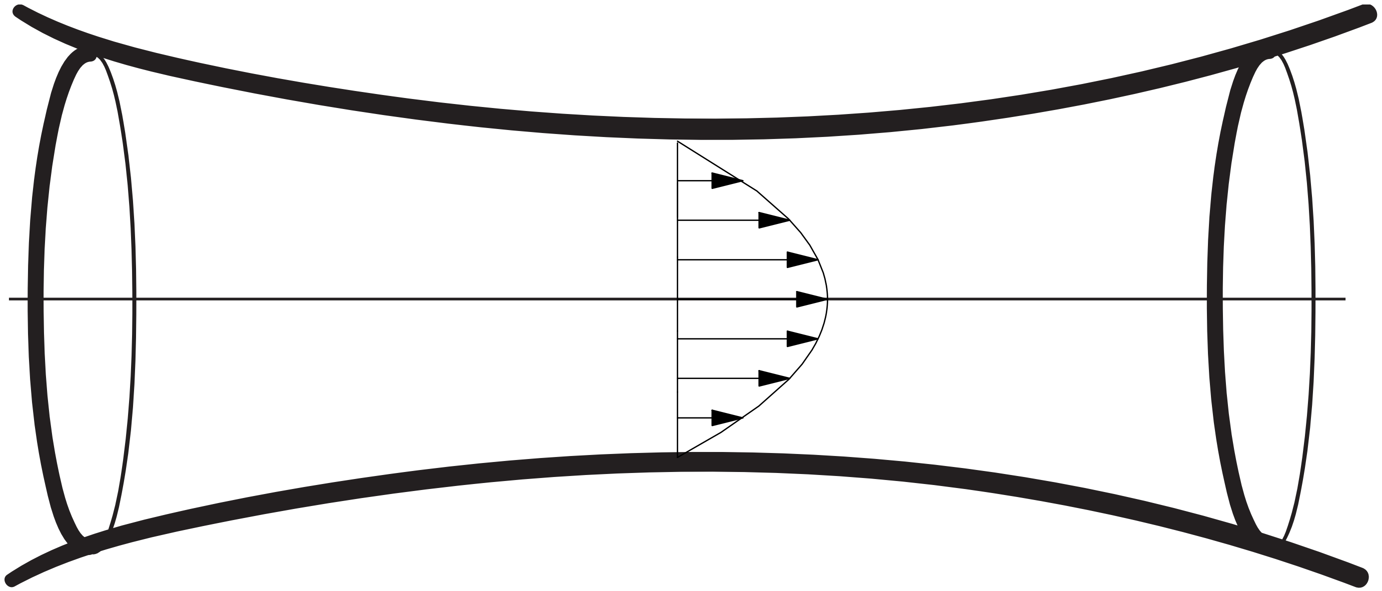
$$p^{given} \rightarrow u^* \qquad rv^*(R) = -\int_0^R r\frac{\partial u^*}{\partial x} dr$$



$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

$$p^{given} \rightarrow u^* \qquad rv^*(R) = -\int_0^R r\frac{\partial u^*}{\partial x} dr \Bigg| \frac{\partial R}{\partial t}\ ?$$

0?



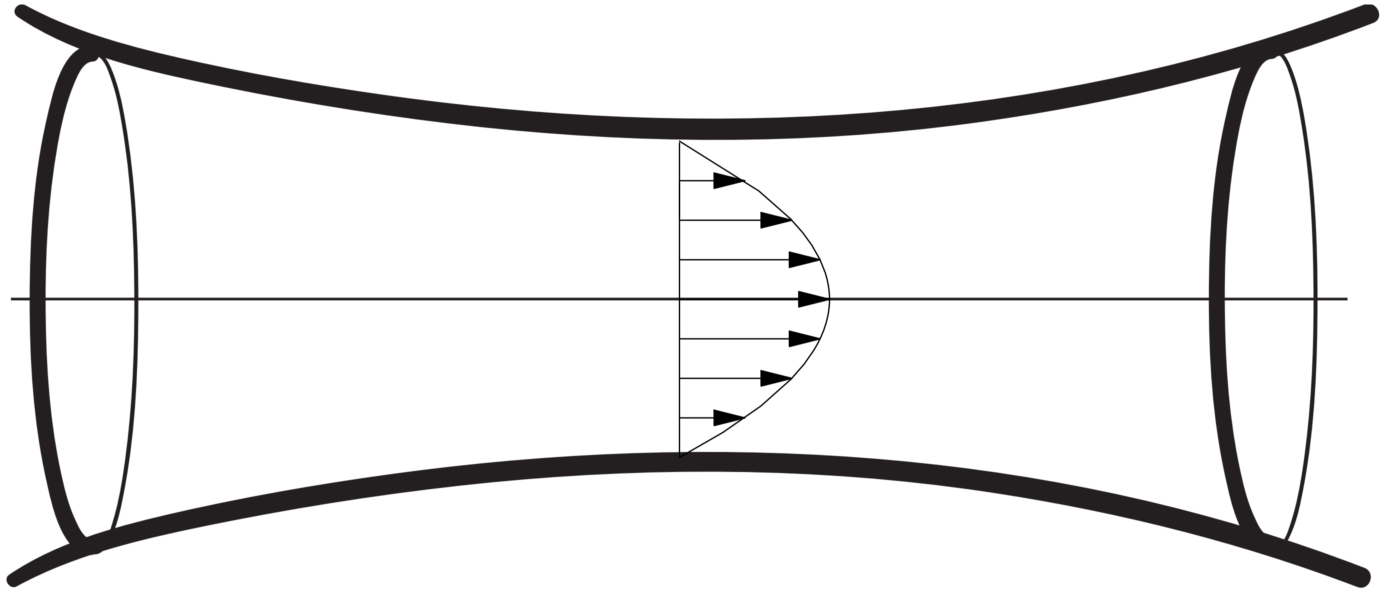
Newton on the pressure to obtain the boundary condition

$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

$$\downarrow$$

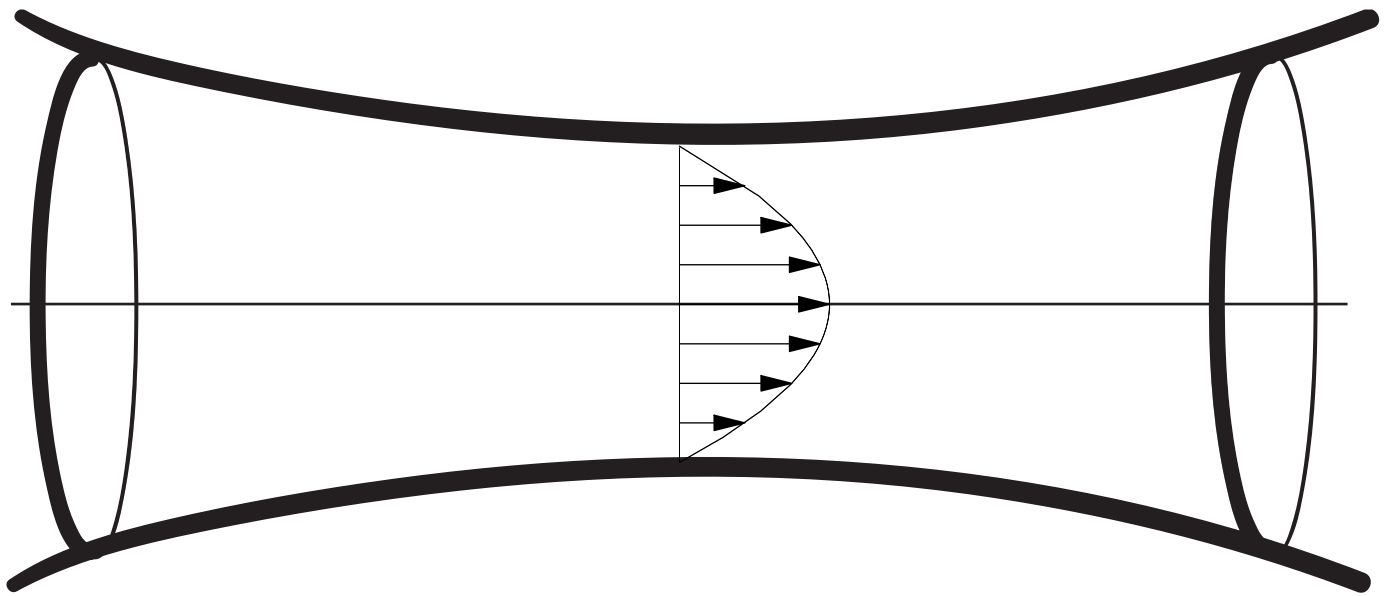
$$p^{given} \rightarrow u^* \longrightarrow rv^*(R) = -\int_0^R r \frac{\partial u^*}{\partial x} dr \Big| \quad \frac{\partial R}{\partial t} \ ?$$

0?

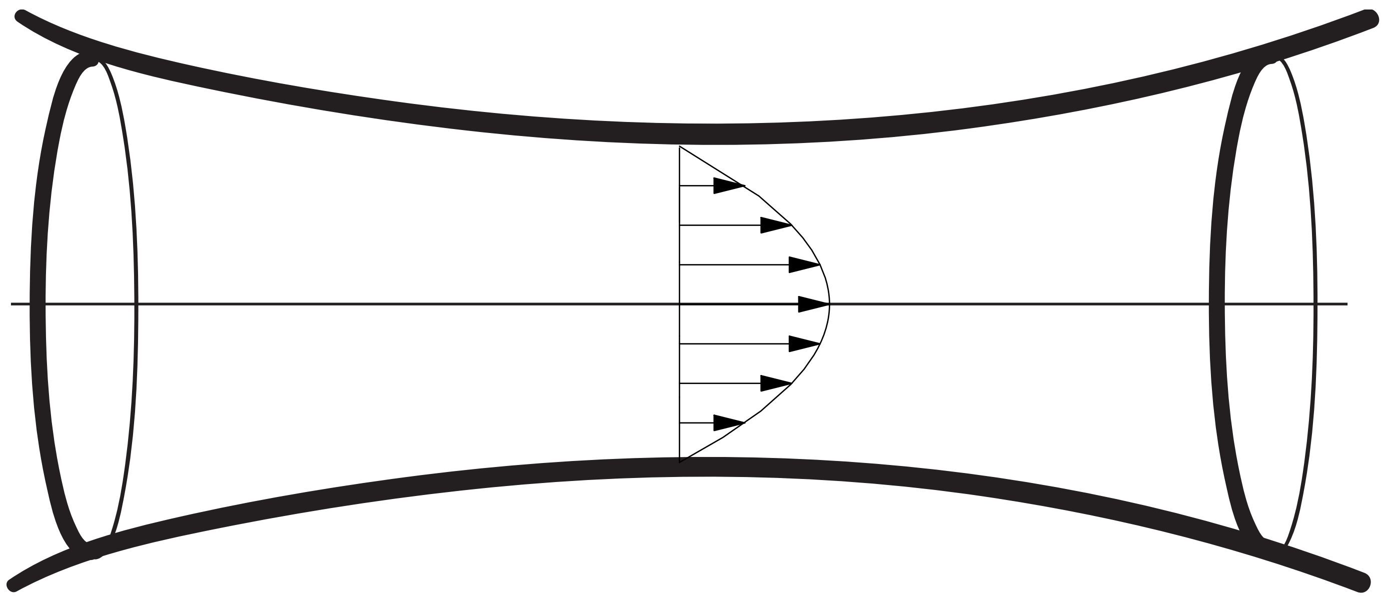


Pressure is a result of the computation

Integral resolution

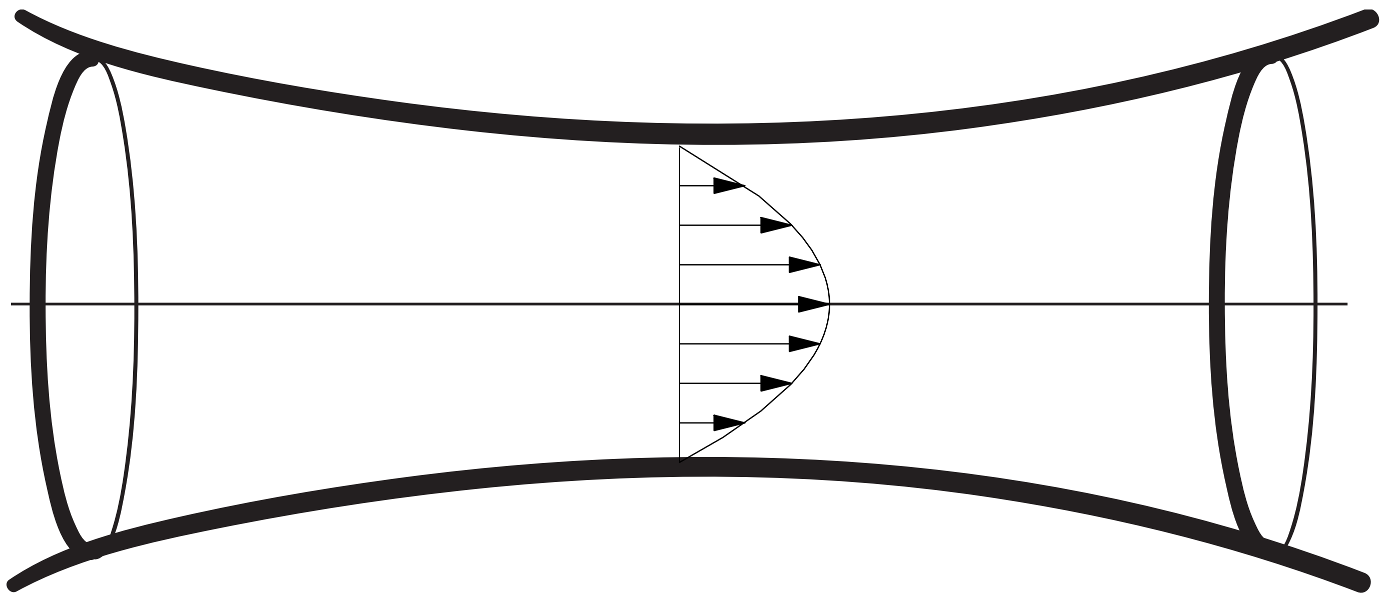


Integral resolution

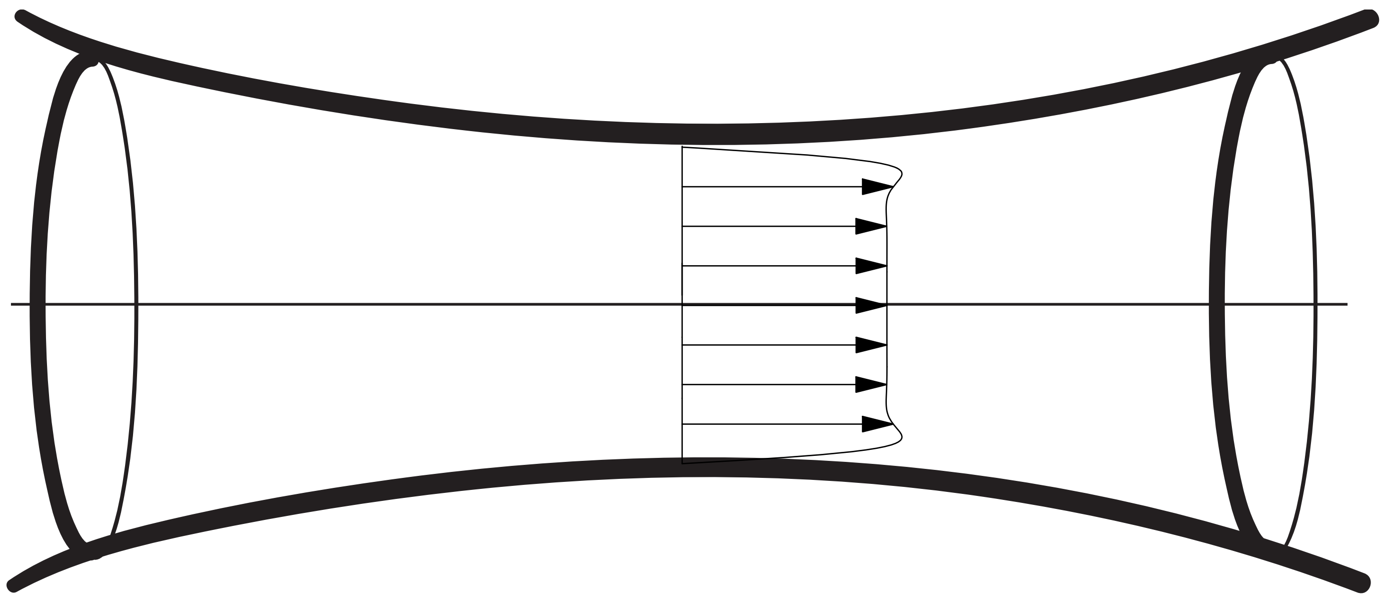


- integral system (1D) is included in RNSP
- we compute a more real profile

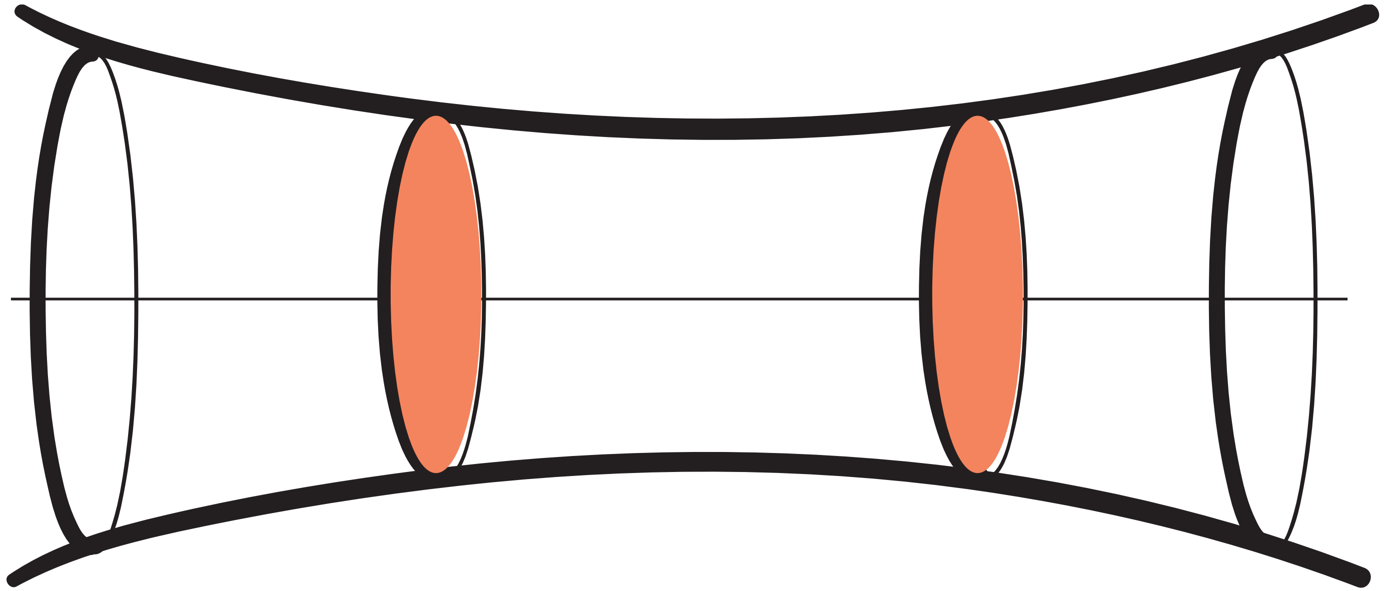
Integral resolution



Integral resolution

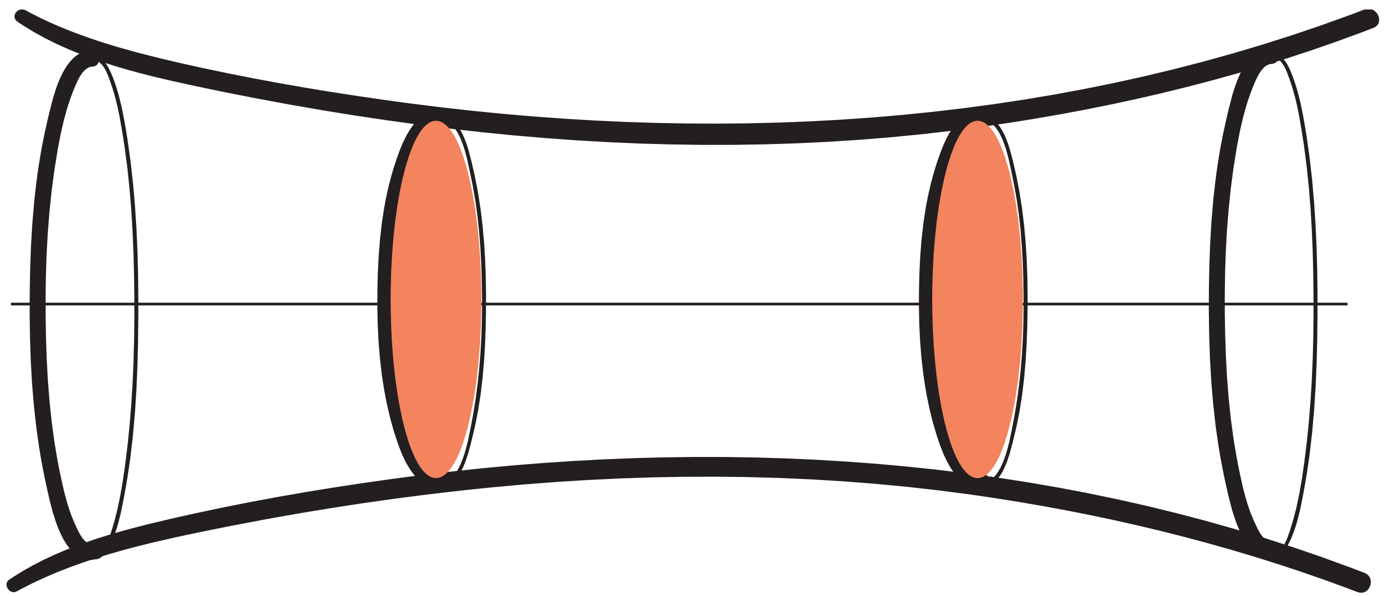


Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

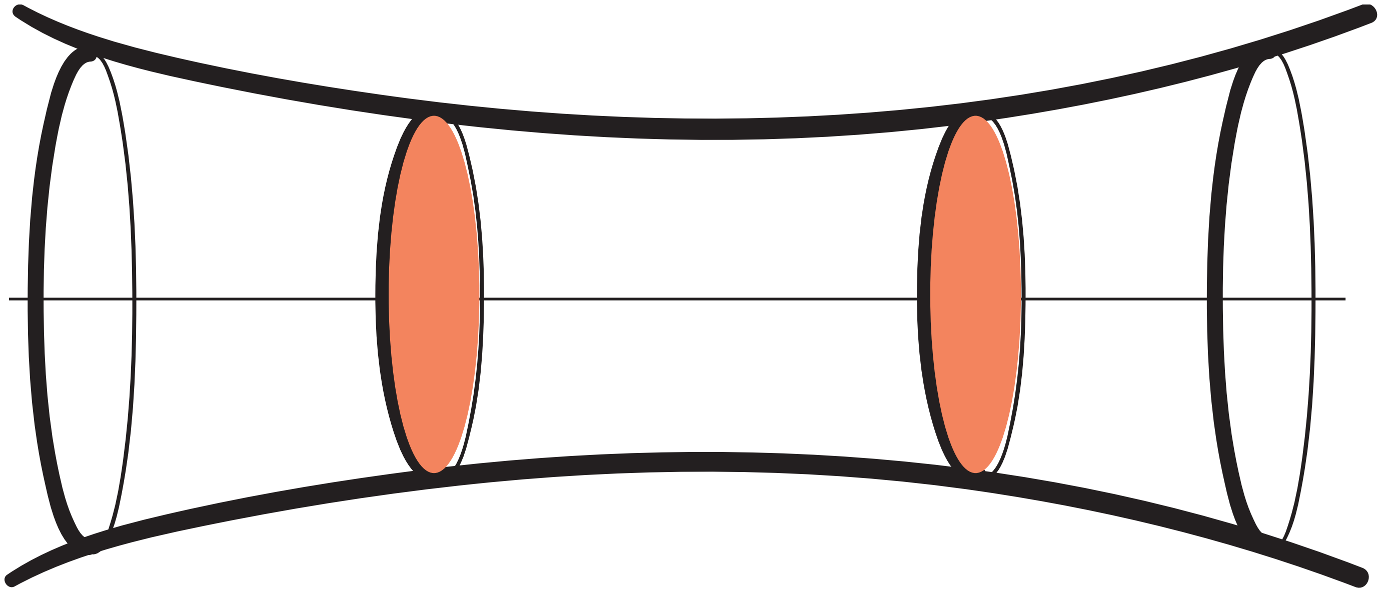
Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

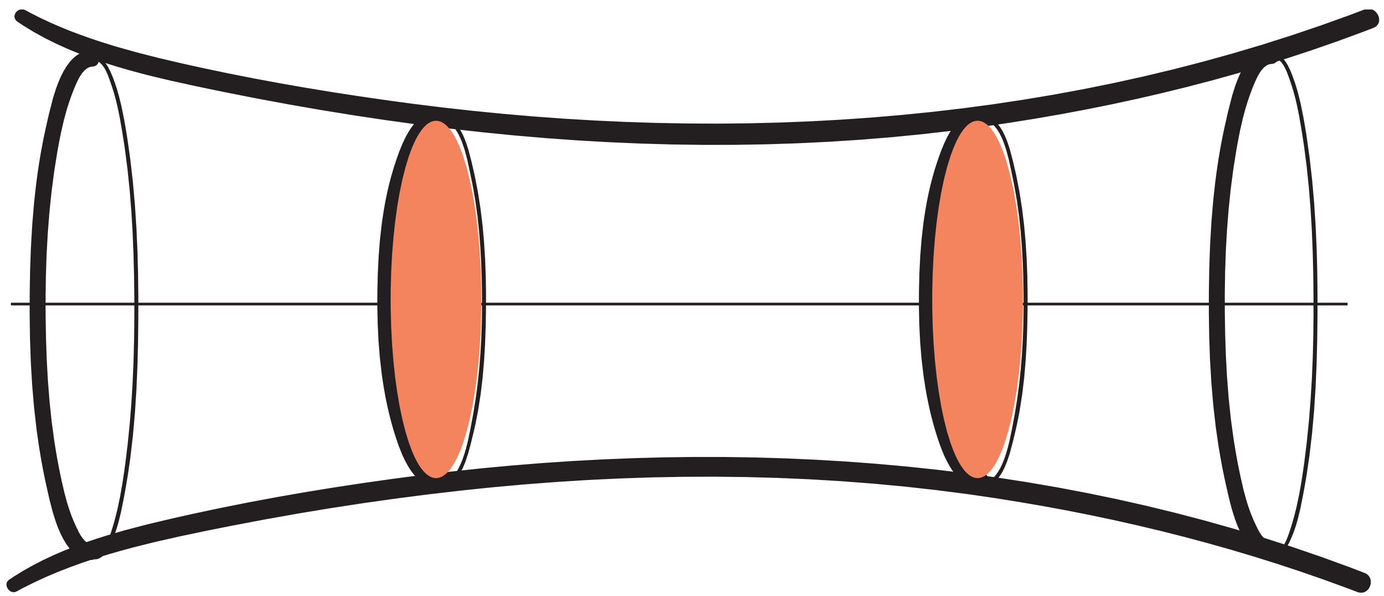
Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

$$\int_0^R 2\pi r dr \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} \right) = 0$$

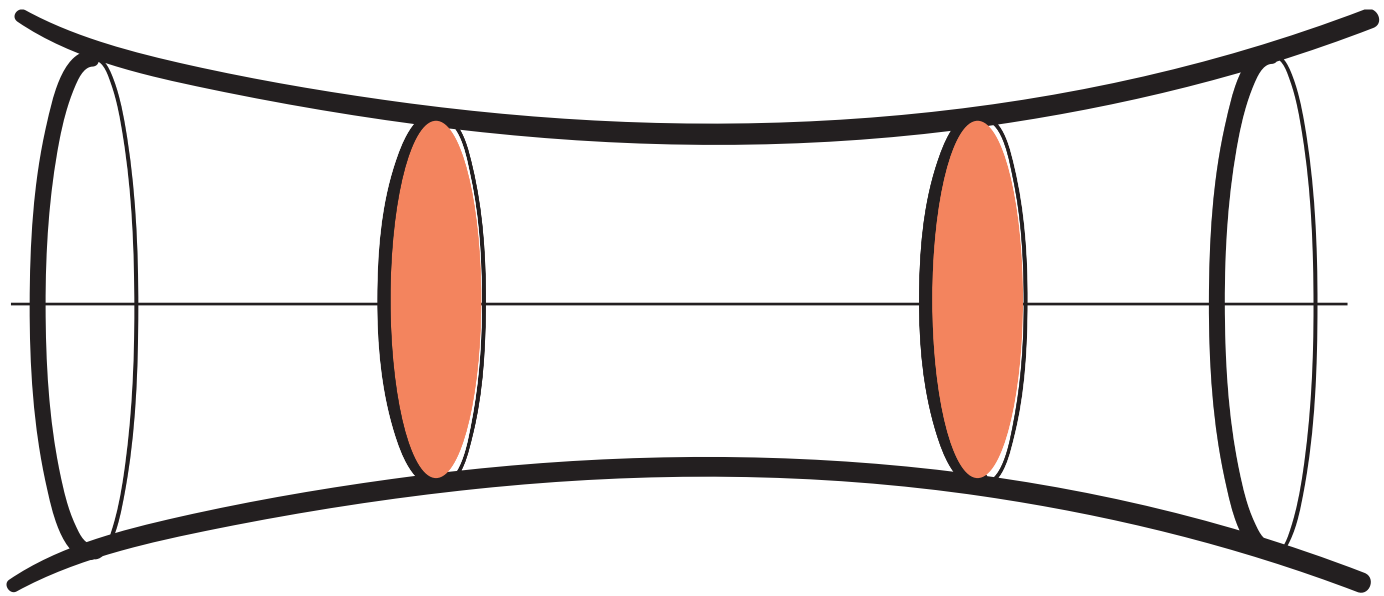
Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

$$\int_0^R 2\pi r dr \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} \right) = 0 \rightarrow \frac{\partial (2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

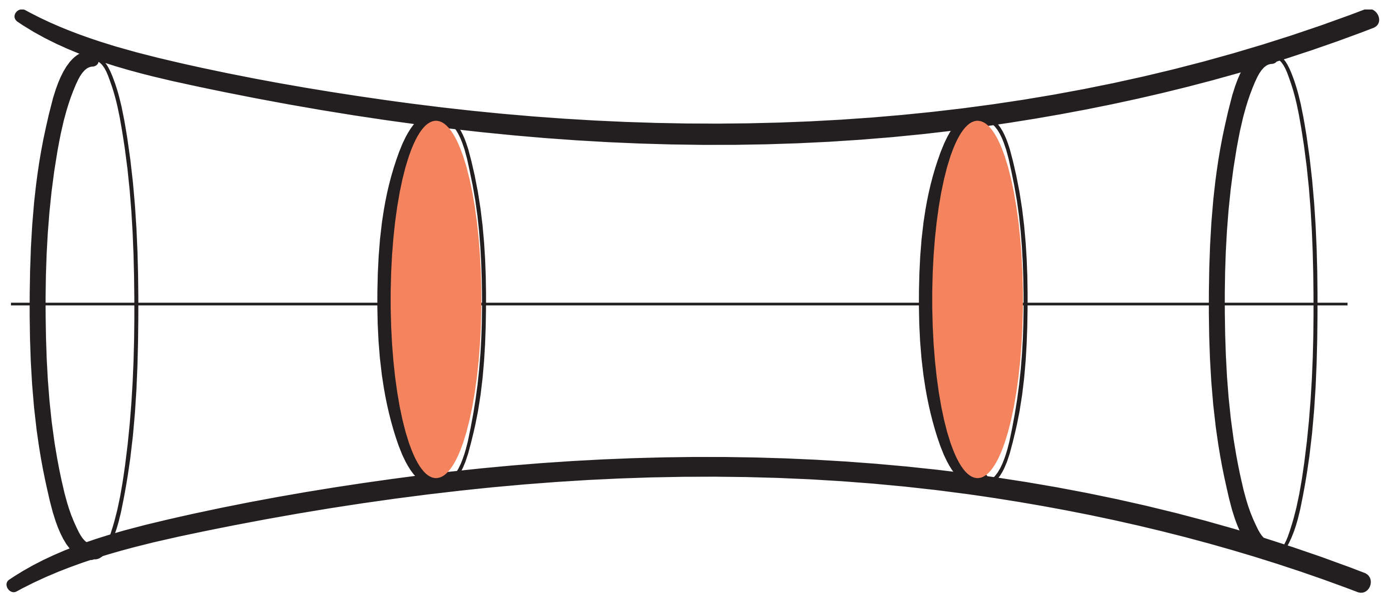
Integral resolution



$$Q = \int_0^R 2\pi r u \, dr$$

$$\frac{\partial (2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

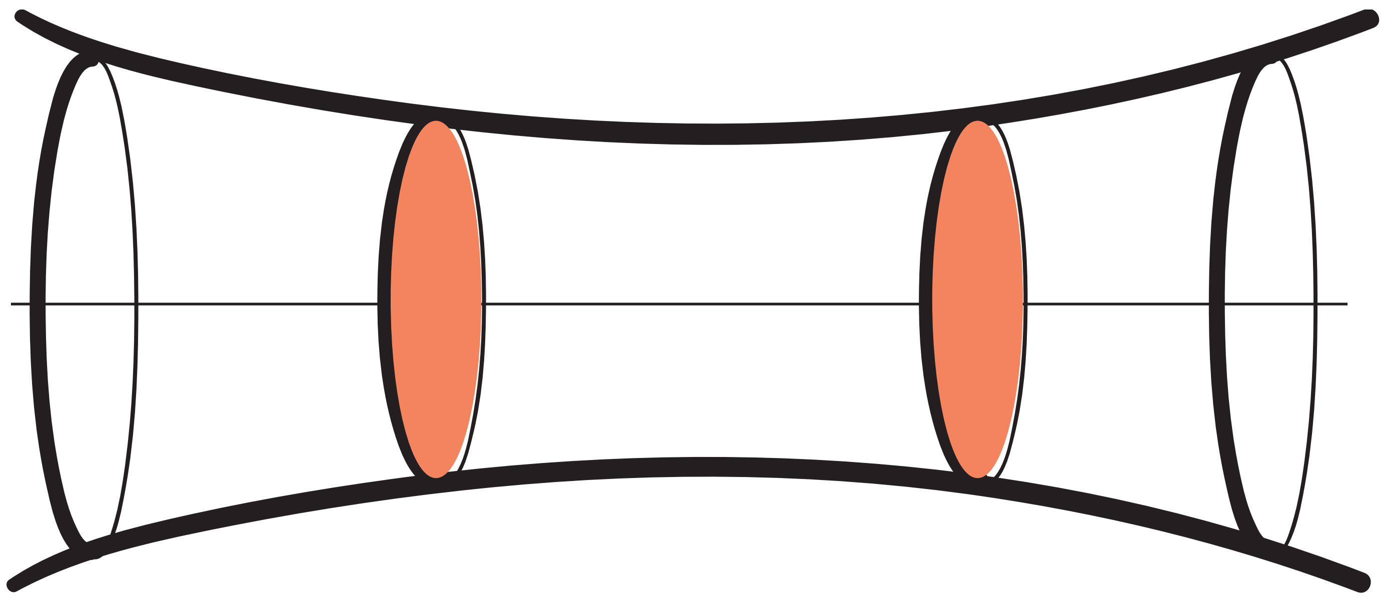
Integral resolution



$$Q = \int_0^R 2\pi r u \, dr$$

$$\tau = \frac{\partial u}{\partial r}$$

Integral resolution

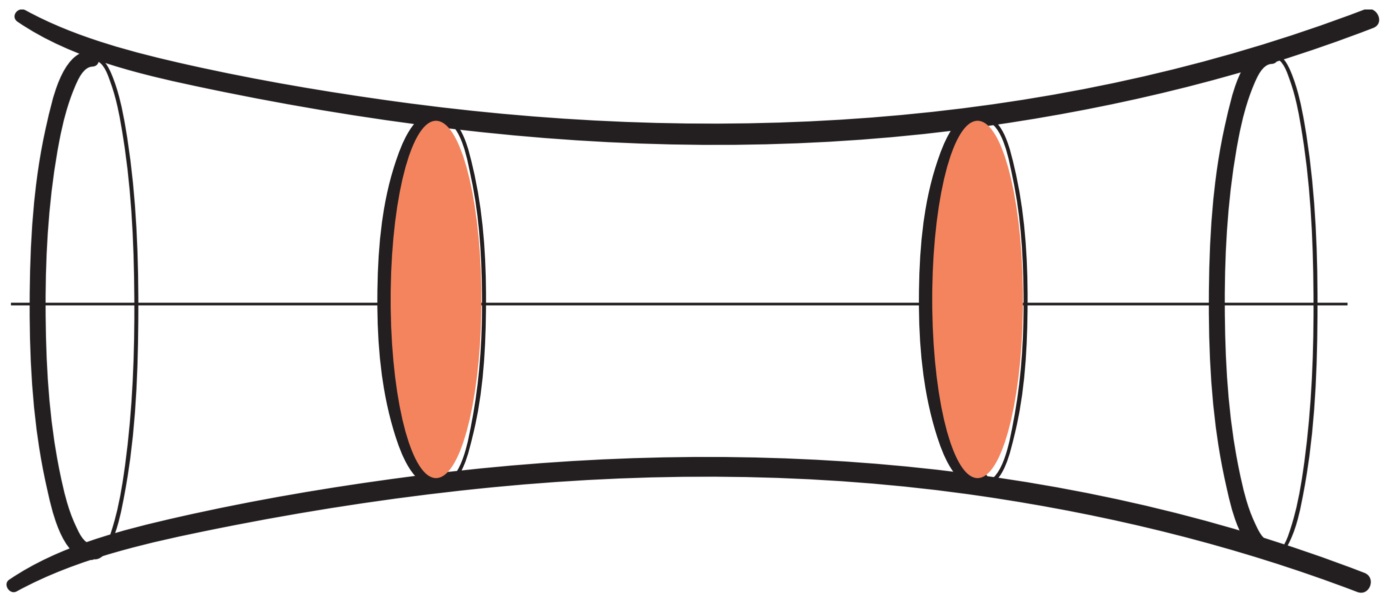


$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\int \left(\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r} \right)$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

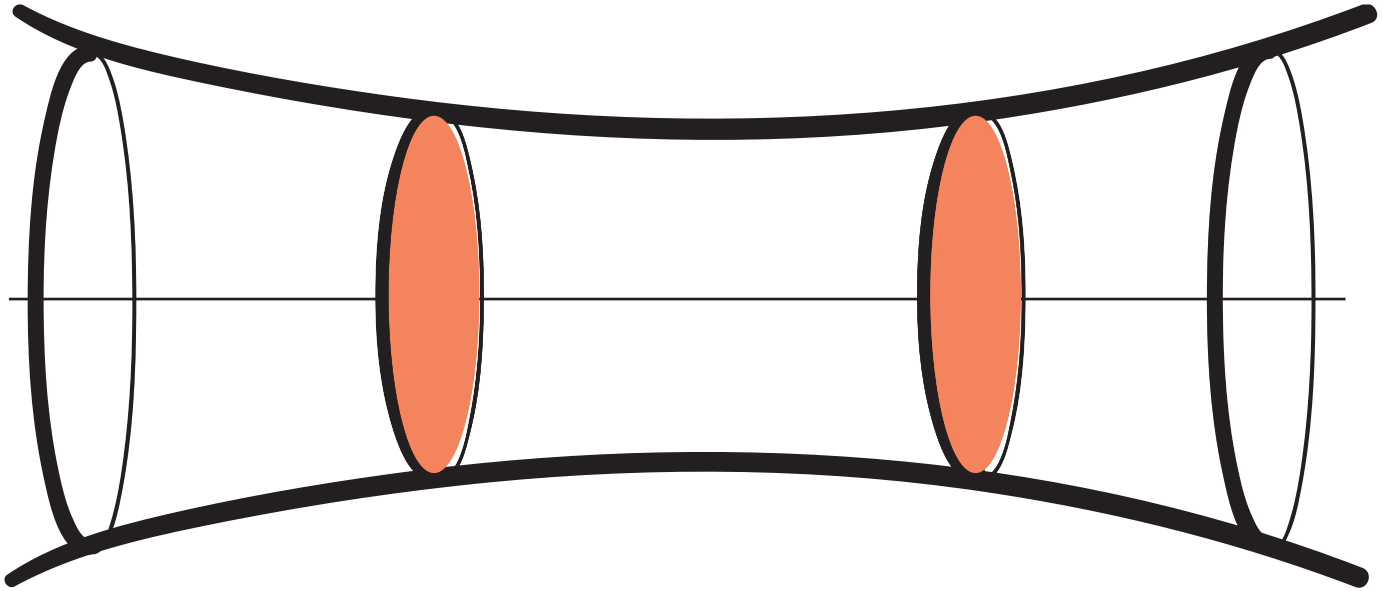
Integral resolution



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2)\frac{\partial p}{\partial x} - \tau$$

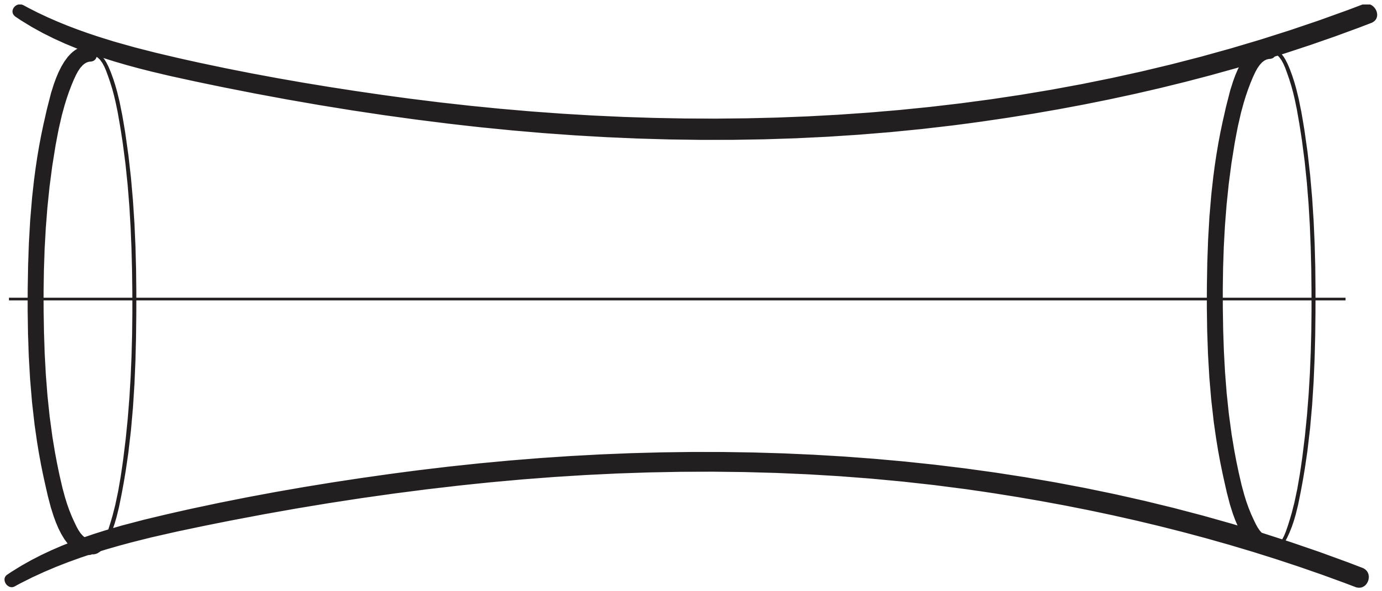
Integral resolution 1D equations



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\frac{\partial(2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2)\frac{\partial p}{\partial x} - \tau$$

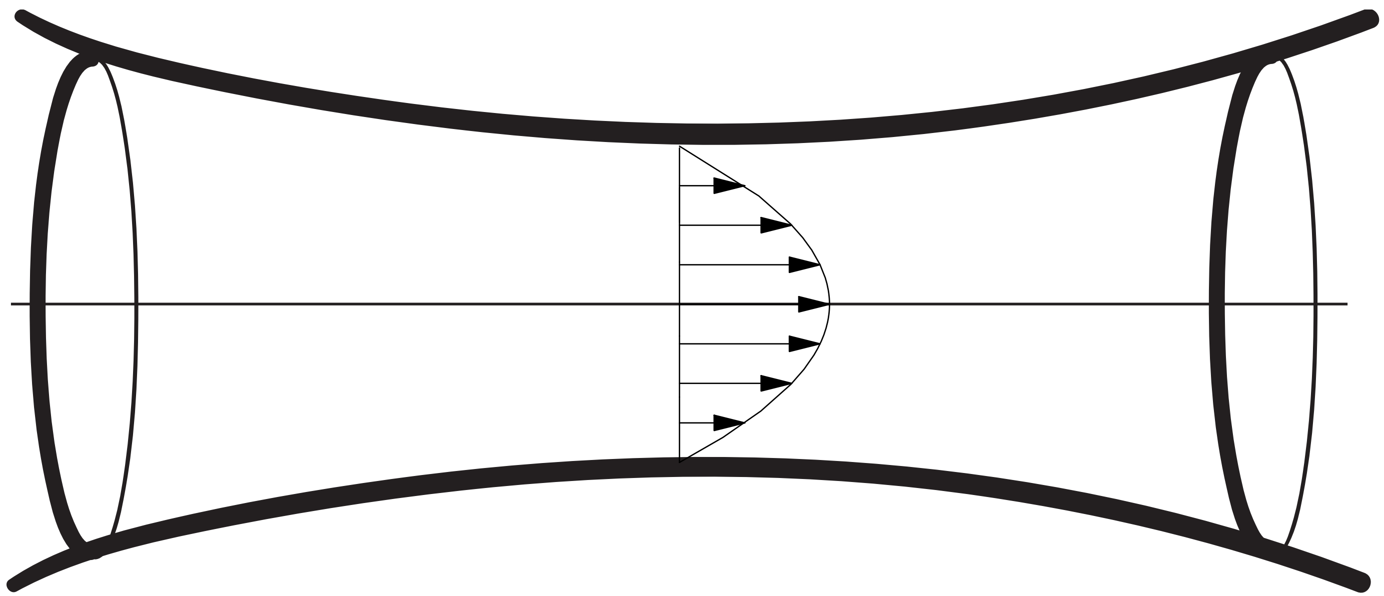
Integral resolution 1D equations



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

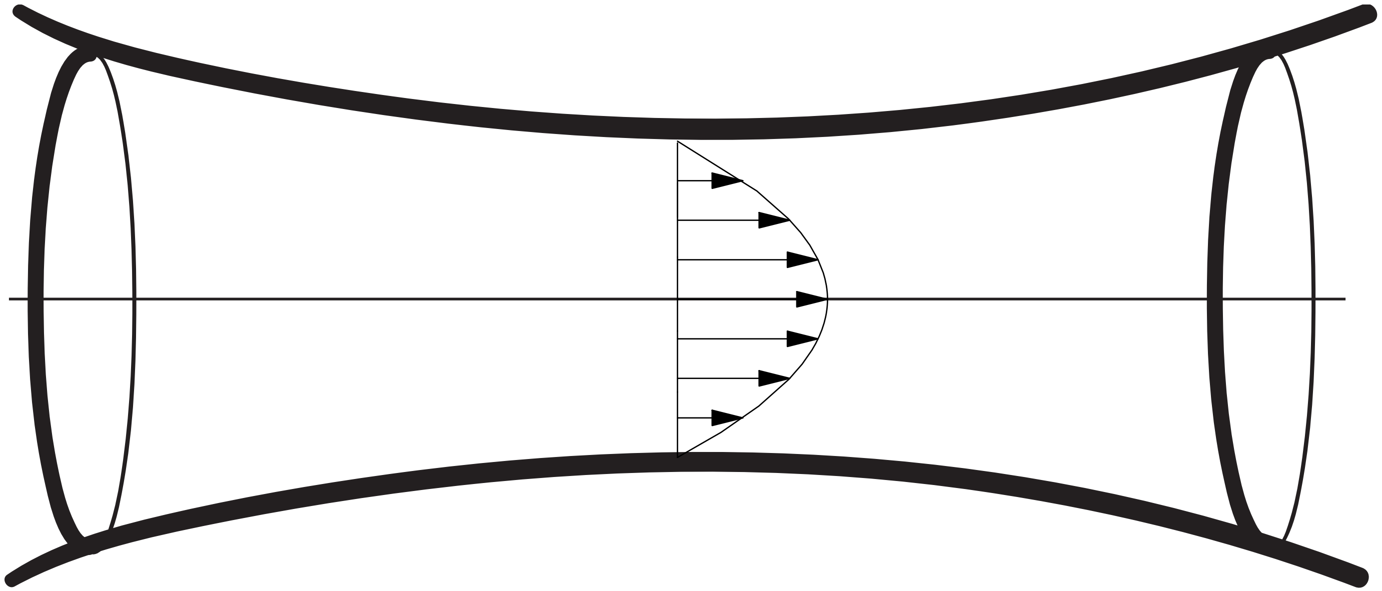
gives Q_2 as function of Q an τ as function Q

Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

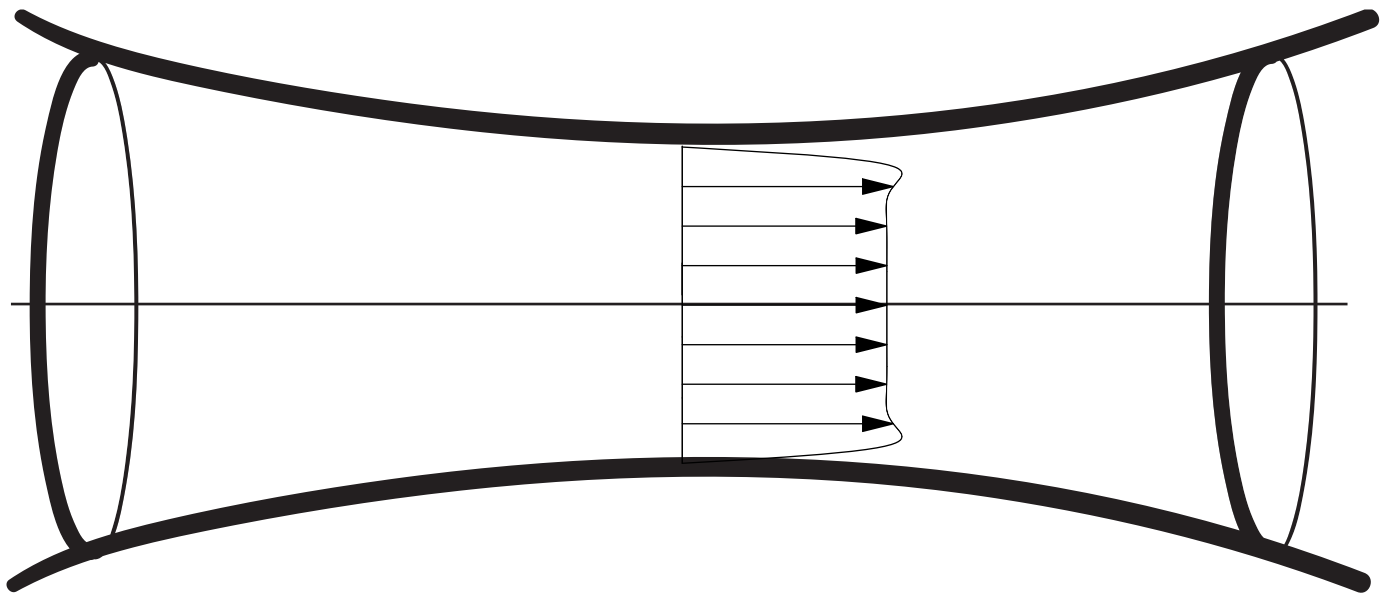
Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

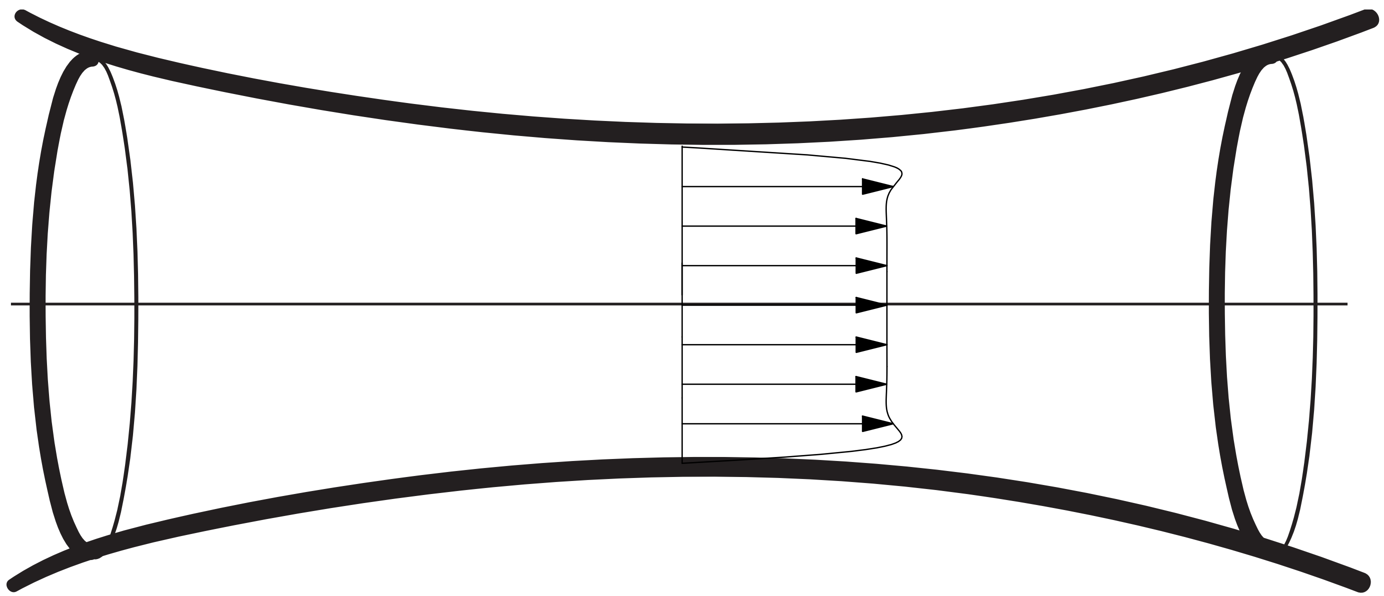
$$Q_2 = (\frac{4}{3})\frac{Q^2}{\pi R^2} \qquad \tau = (8\pi)\frac{Q}{\pi R^2}$$

Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

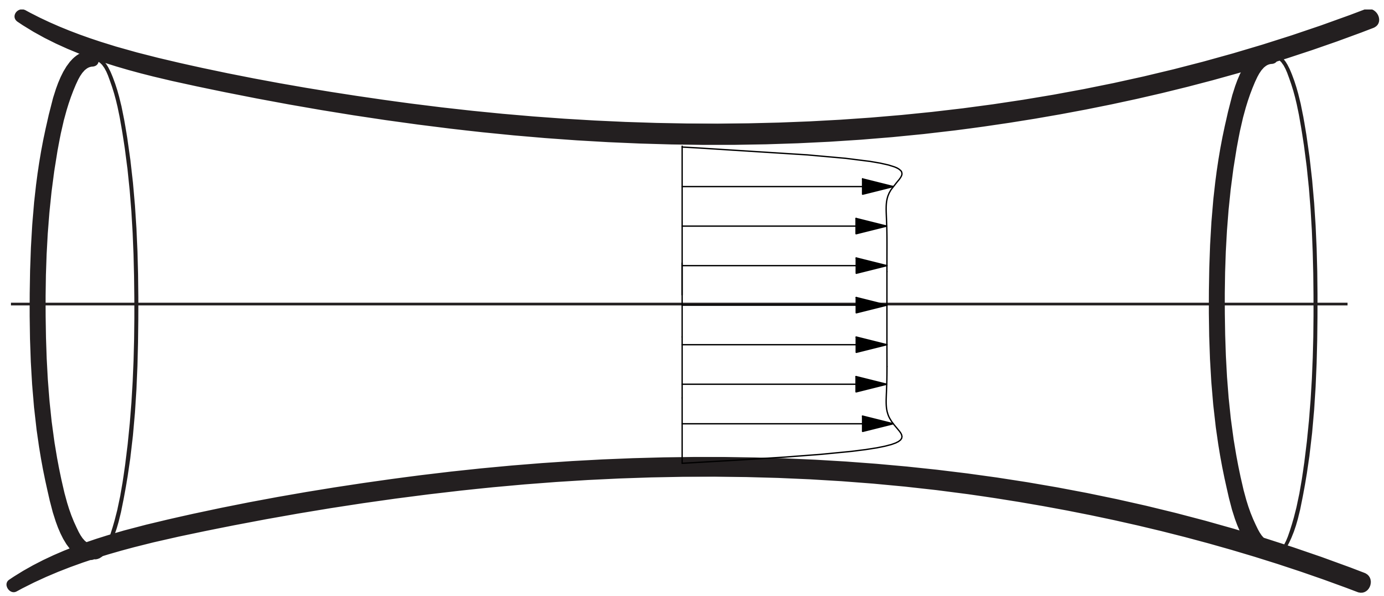
Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

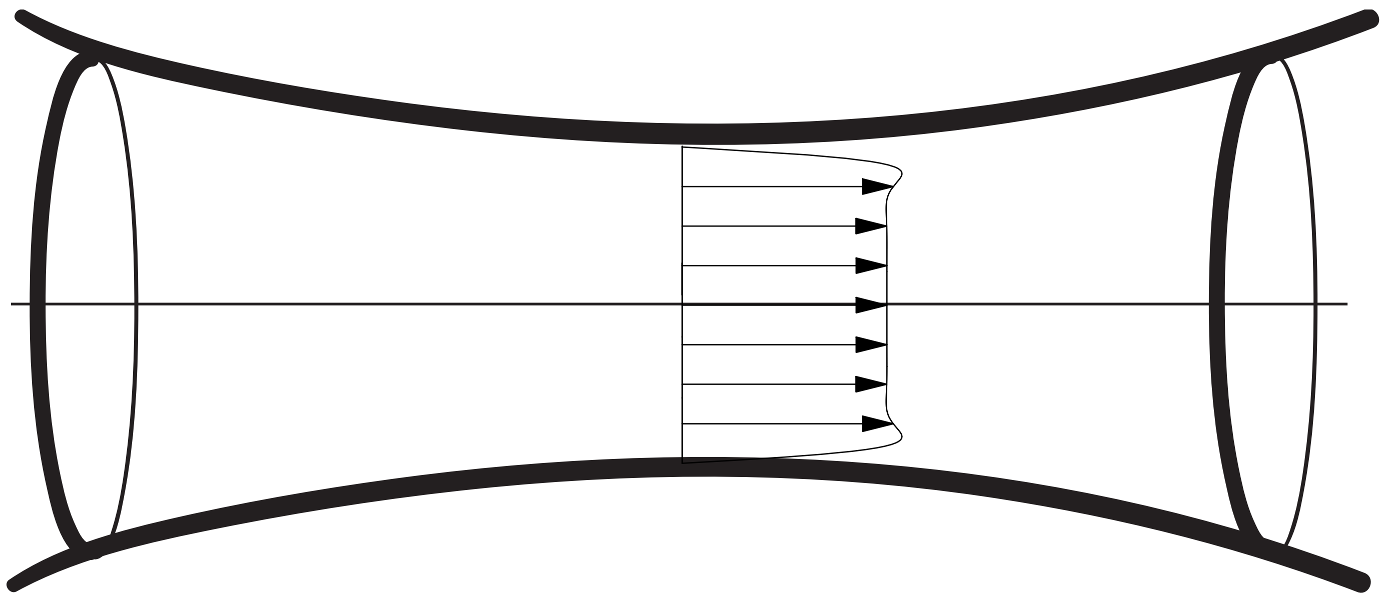
$$Q_2 = \frac{Q^2}{\pi R^2} \qquad \tau = F(Q)$$

Integral resolution 1D equations



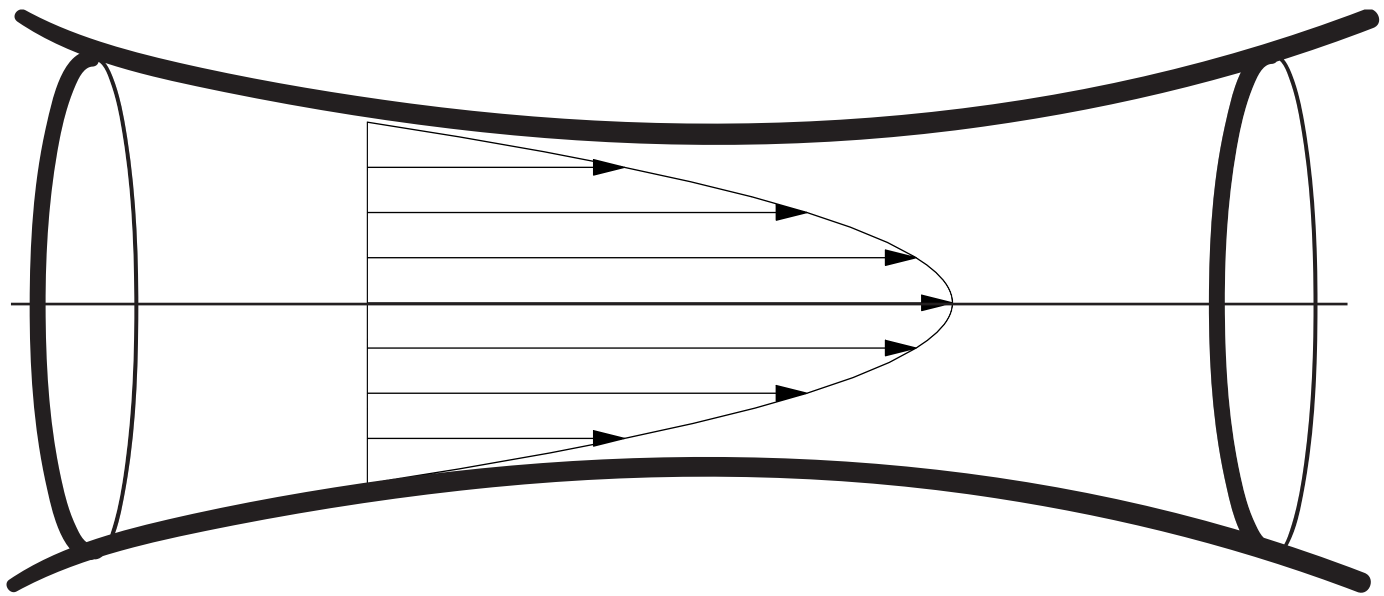
need of profile

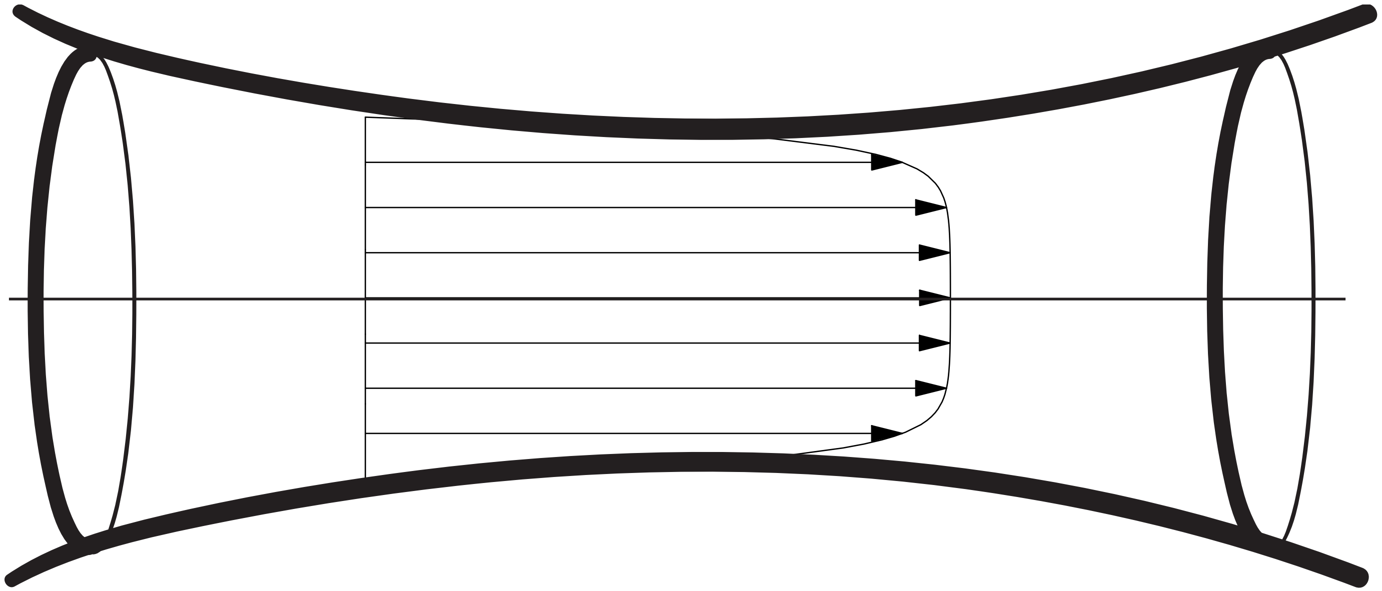
Integral resolution 1D equations

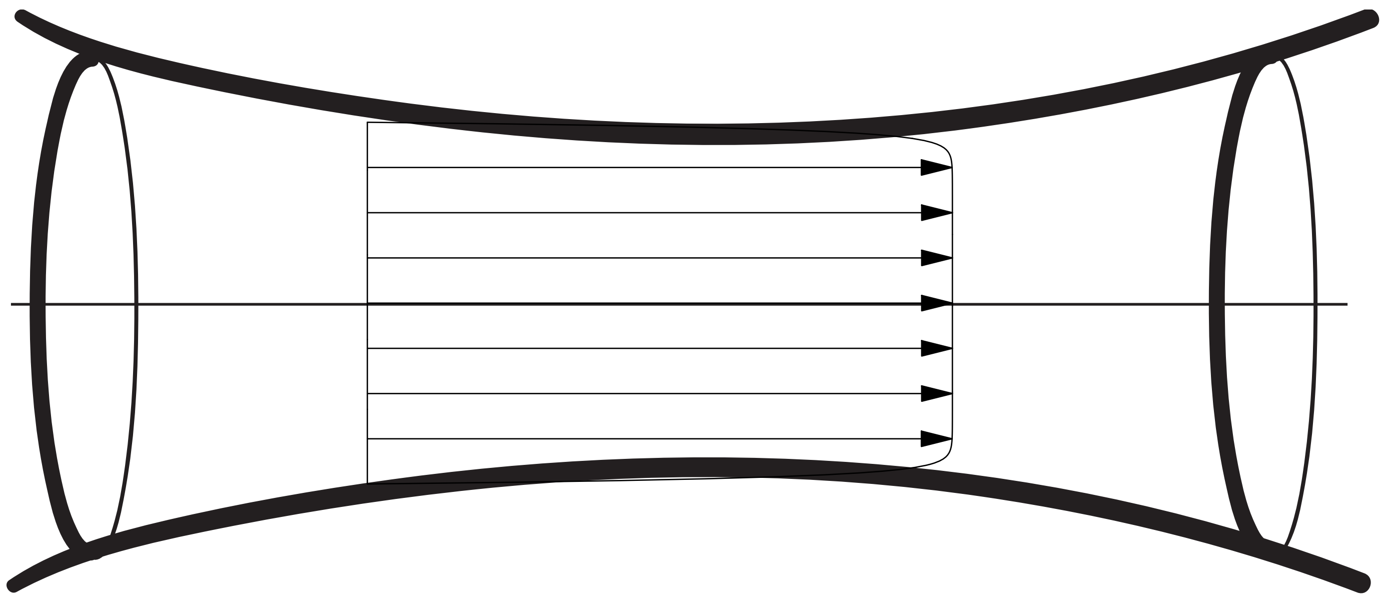


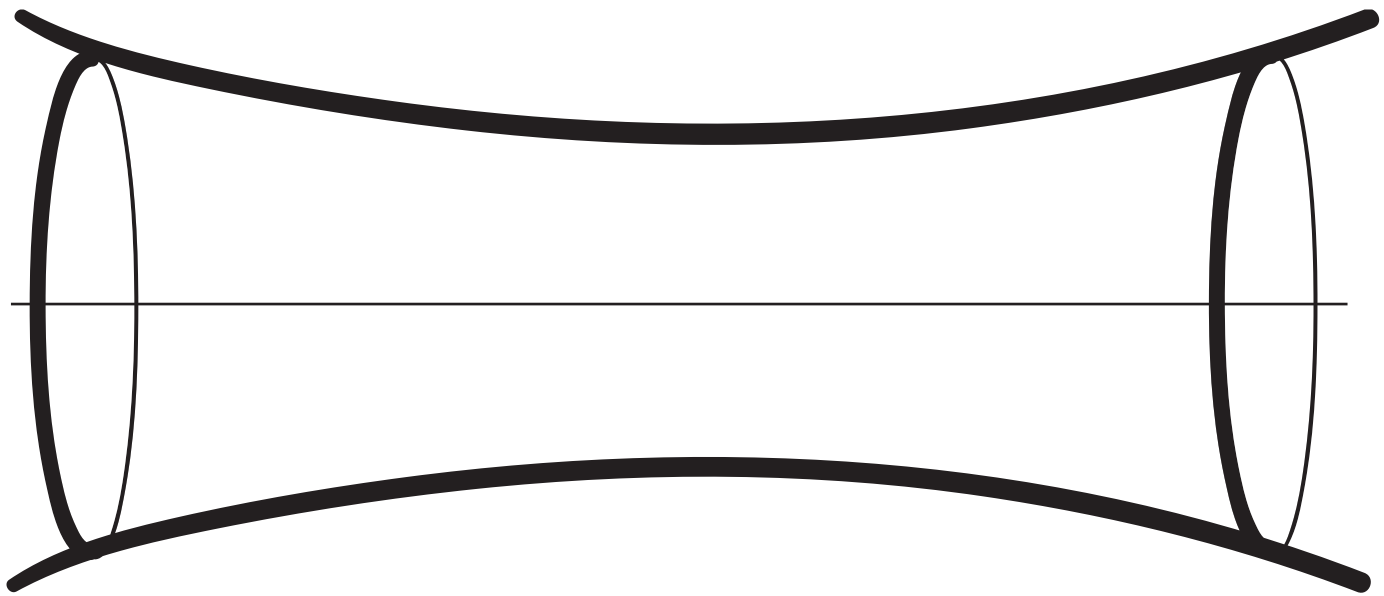
“usual” 1D equations are a simplification of RNSP

Choice of profiles

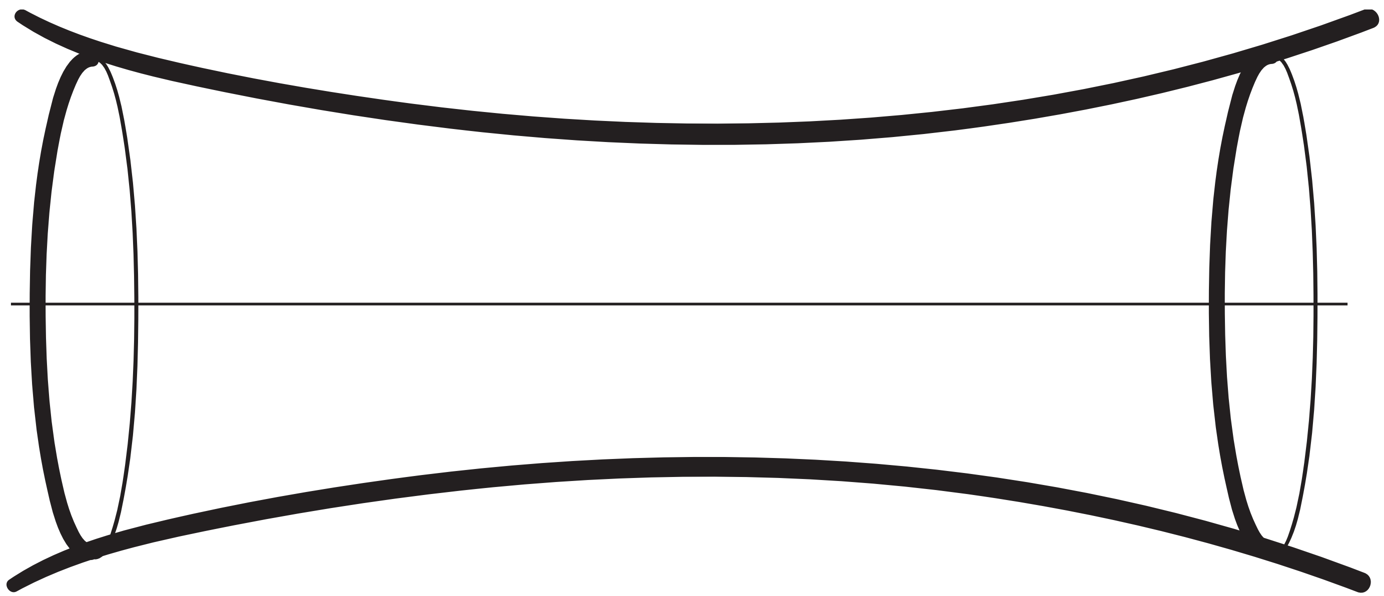








Choice of the family of simple profiles

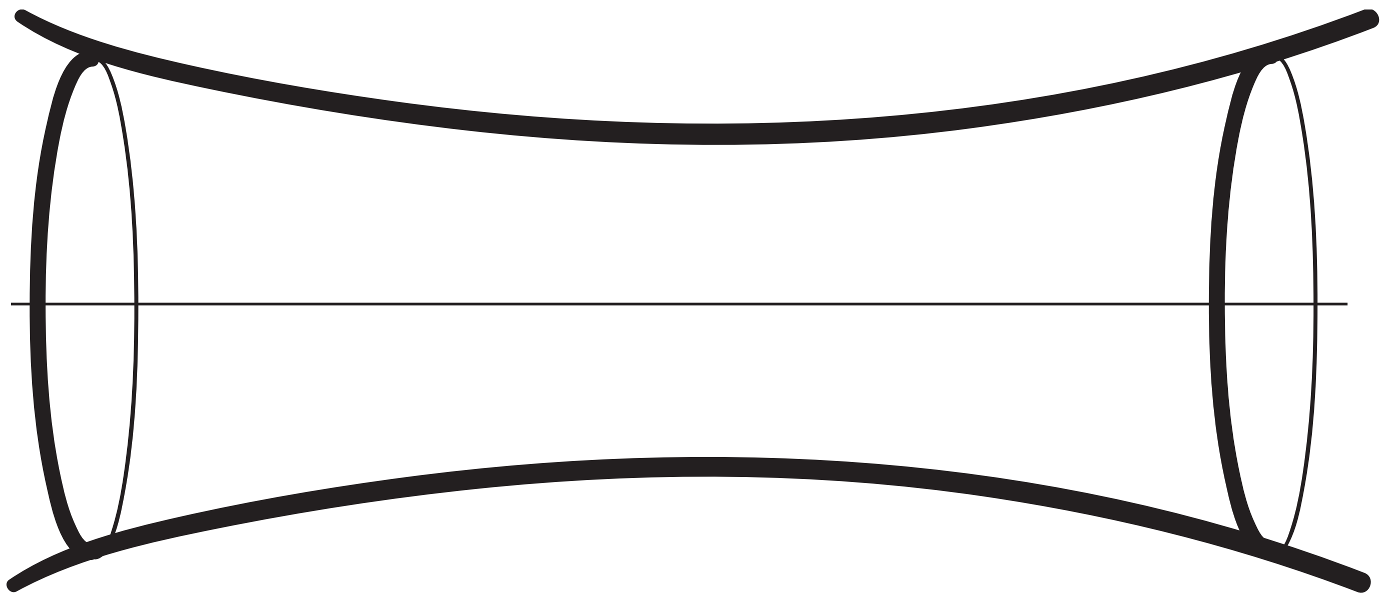


Choice of the family of simple profiles

In an unsteady flow it is natural to use Womersley

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho\partial x} + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

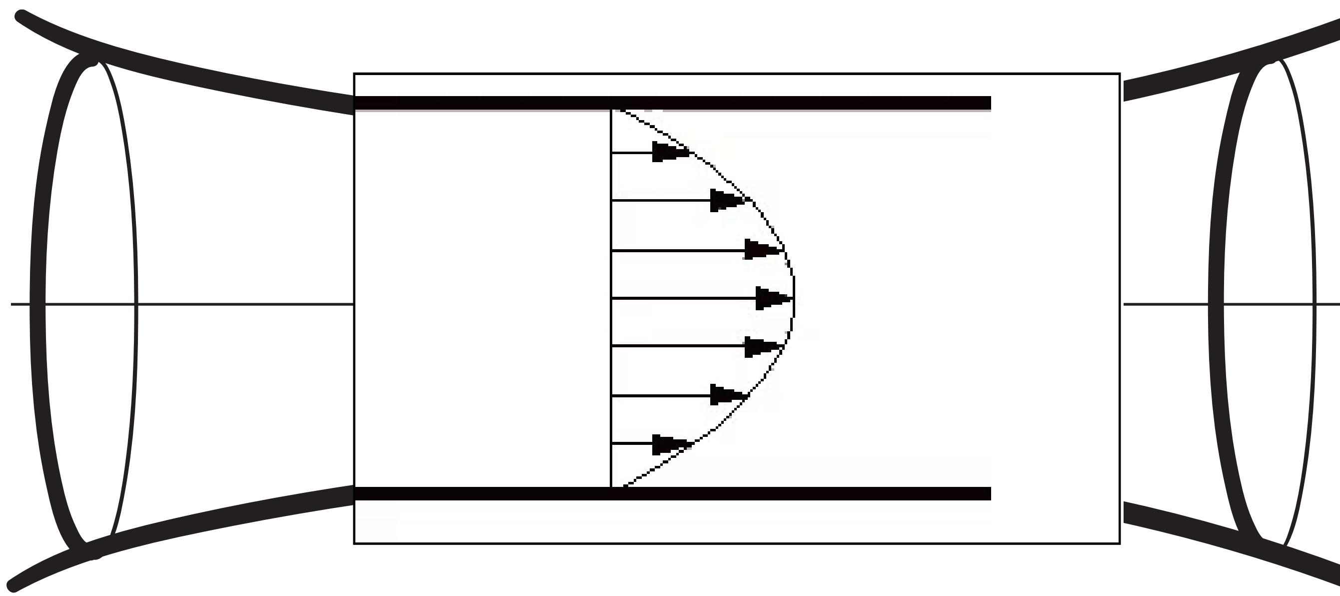
$$0 = -\frac{\partial p}{\rho\partial r}$$



Choice of the family of simple profiles

In an unsteady flow it is natural to use Womersley

Womersley profiles are solution of RNSP



Choice of the family of simple profiles

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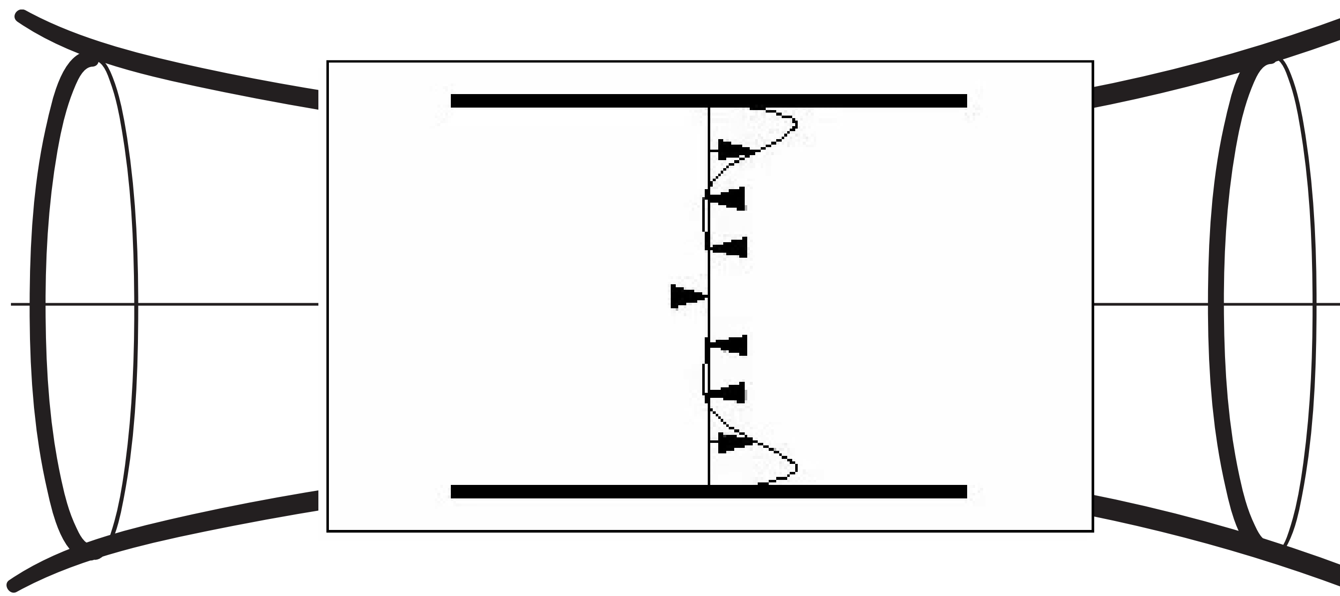

Choice of the family of simple profiles

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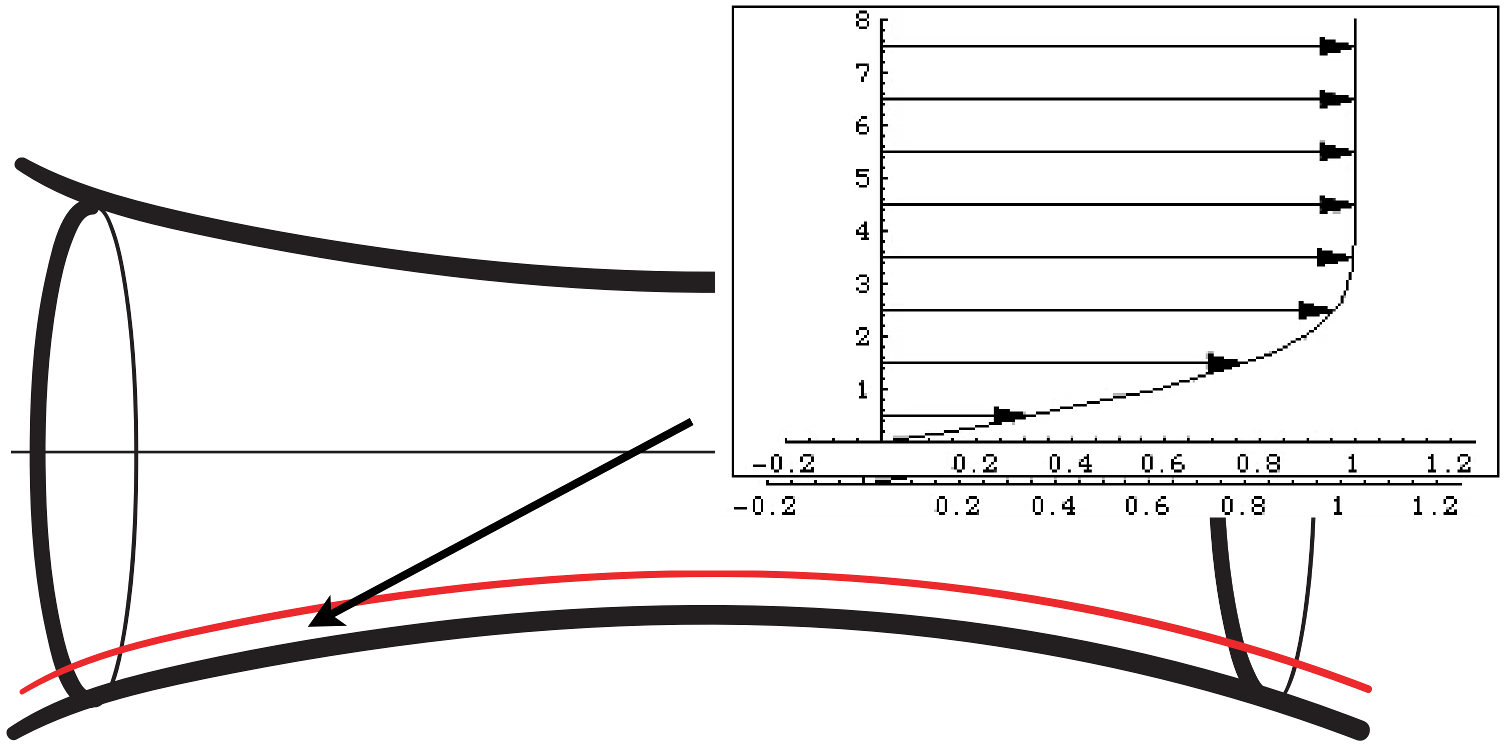
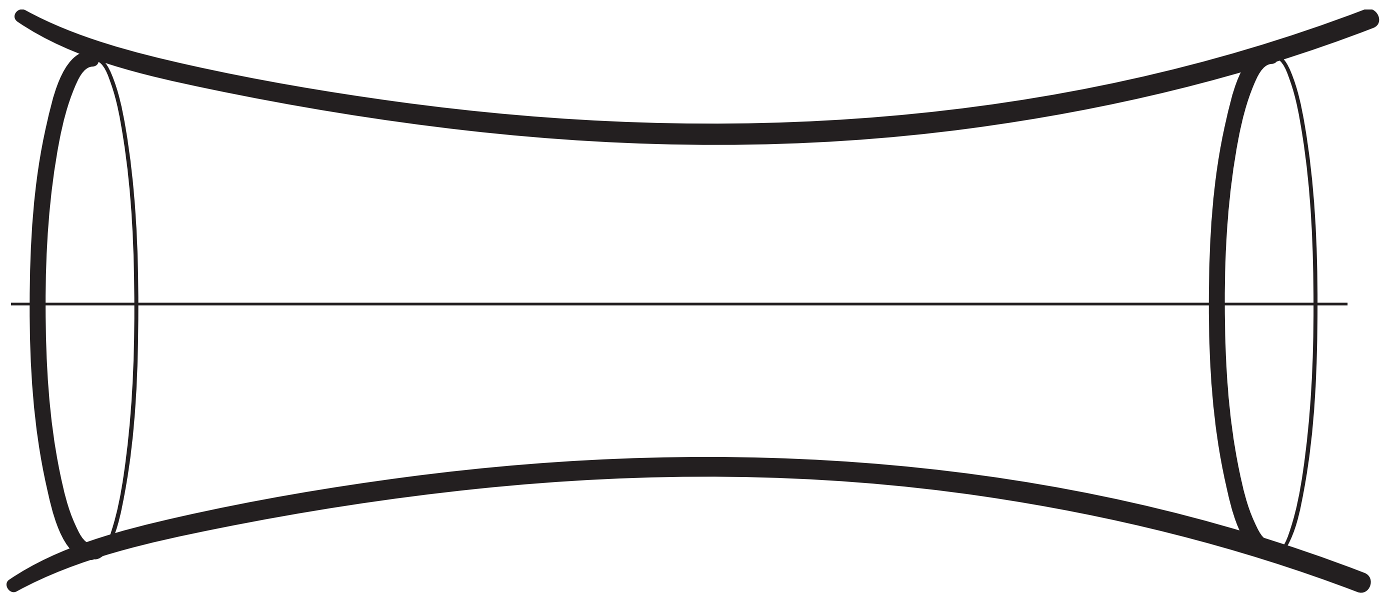


Choice of the family of simple profiles

In a steady flow it is natural to use Falkner Skan

Integral resolution



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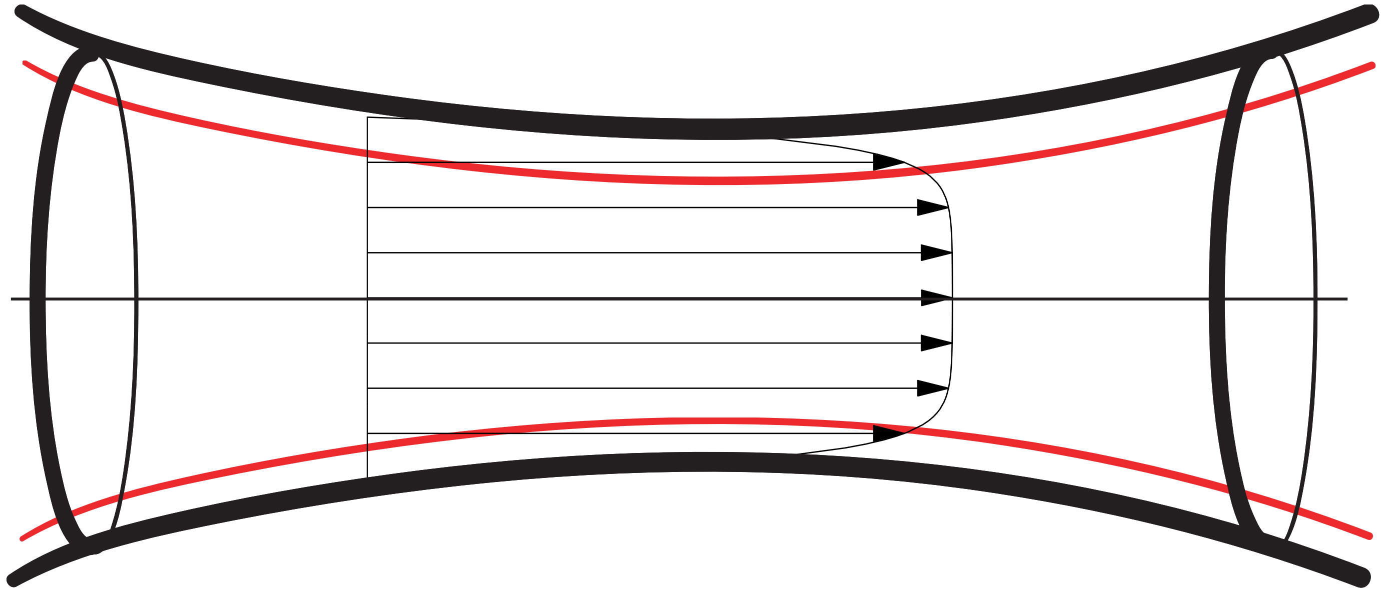
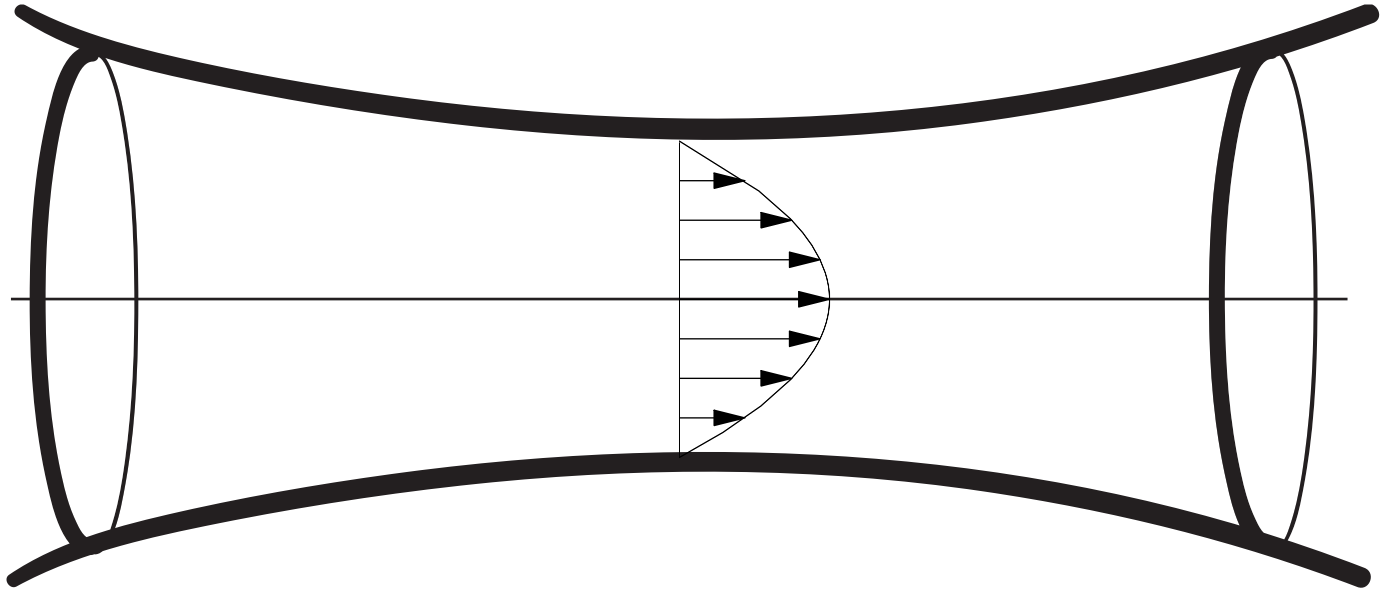
Integral resolution

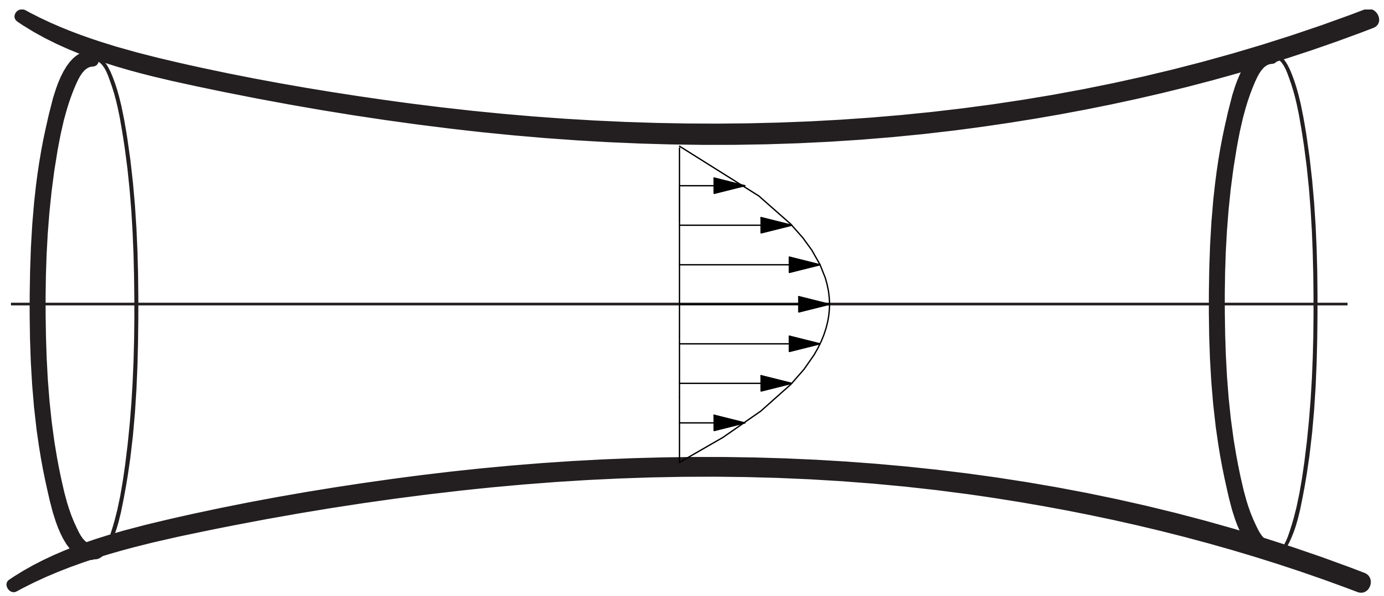


Numerical resolution:
finite differences

Interactive Boundary Layer

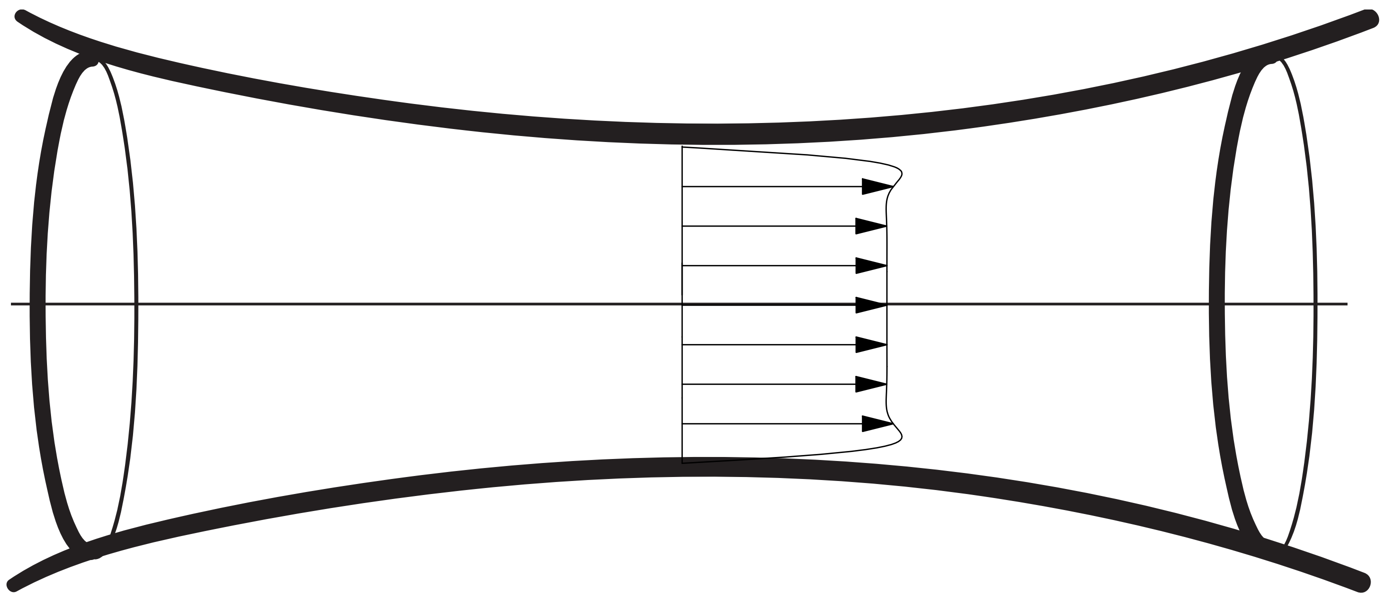


Interactive Boundary Layer

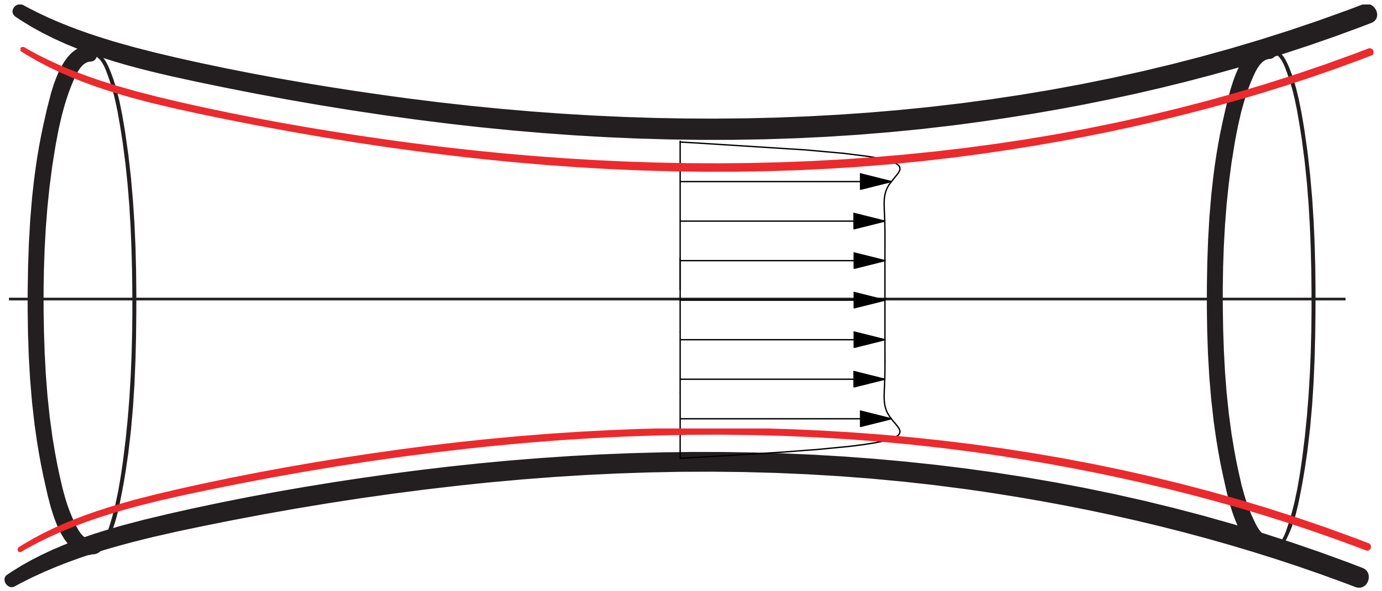


IBL is included in RNSP

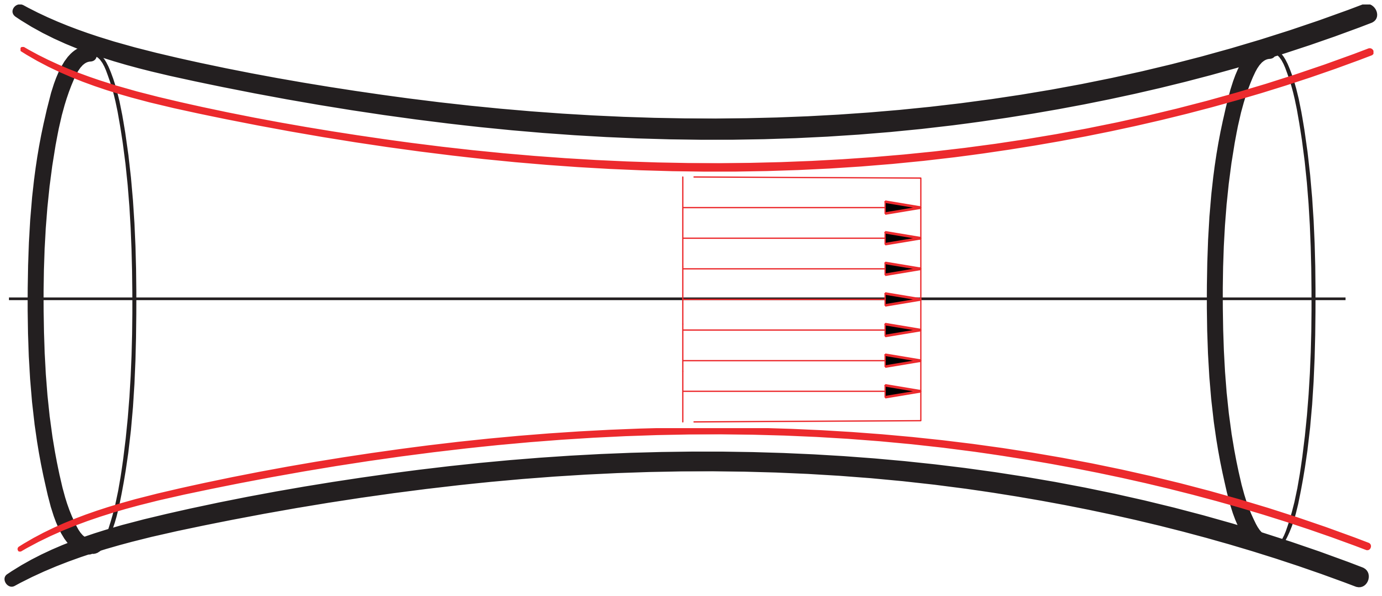
Interactive Boundary Layer



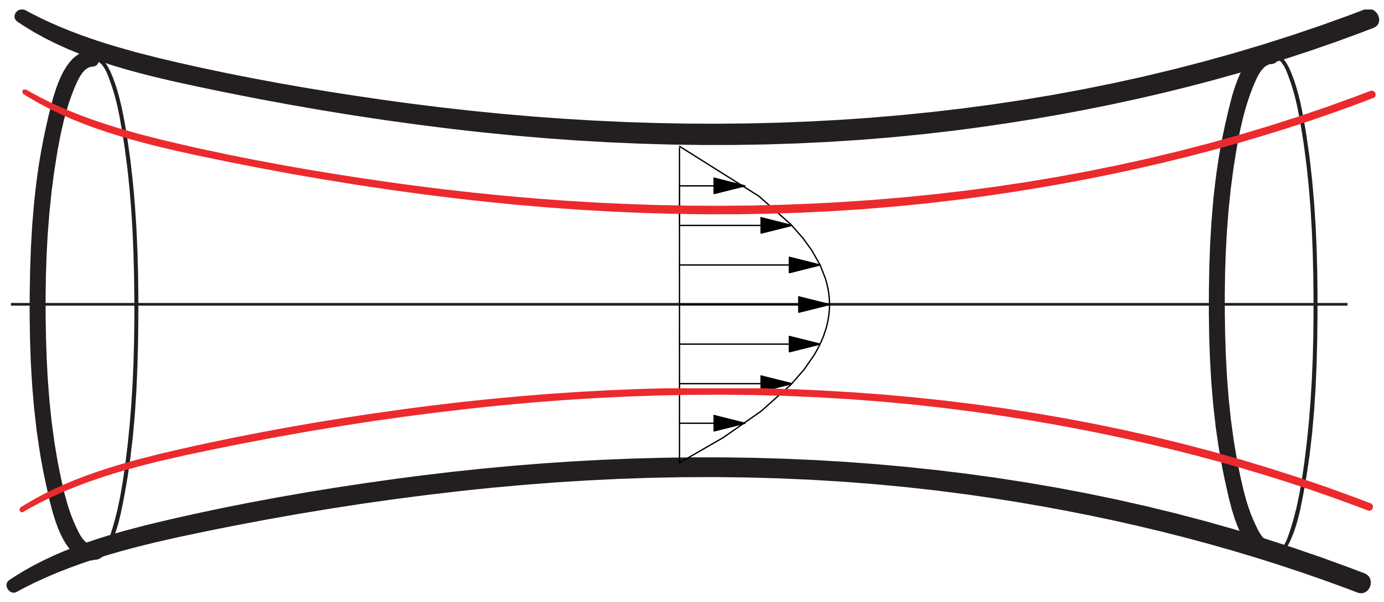
Interactive Boundary Layer



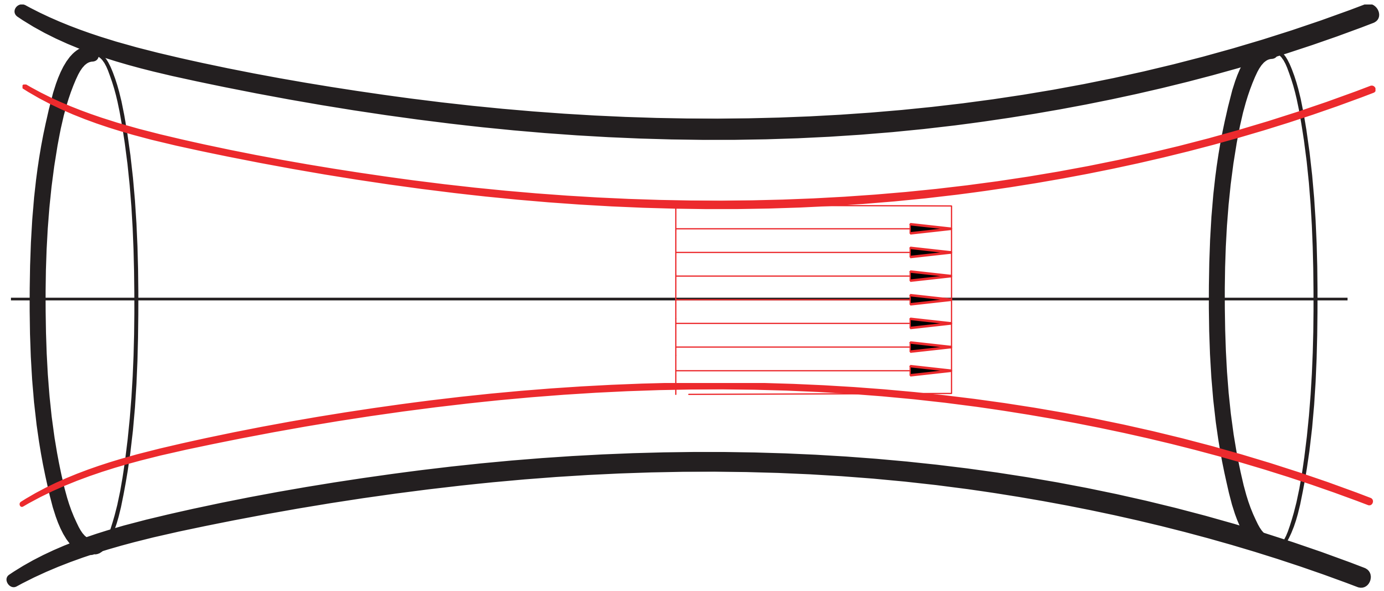
Interactive Boundary Layer



Interactive Boundary Layer



Interactive Boundary Layer



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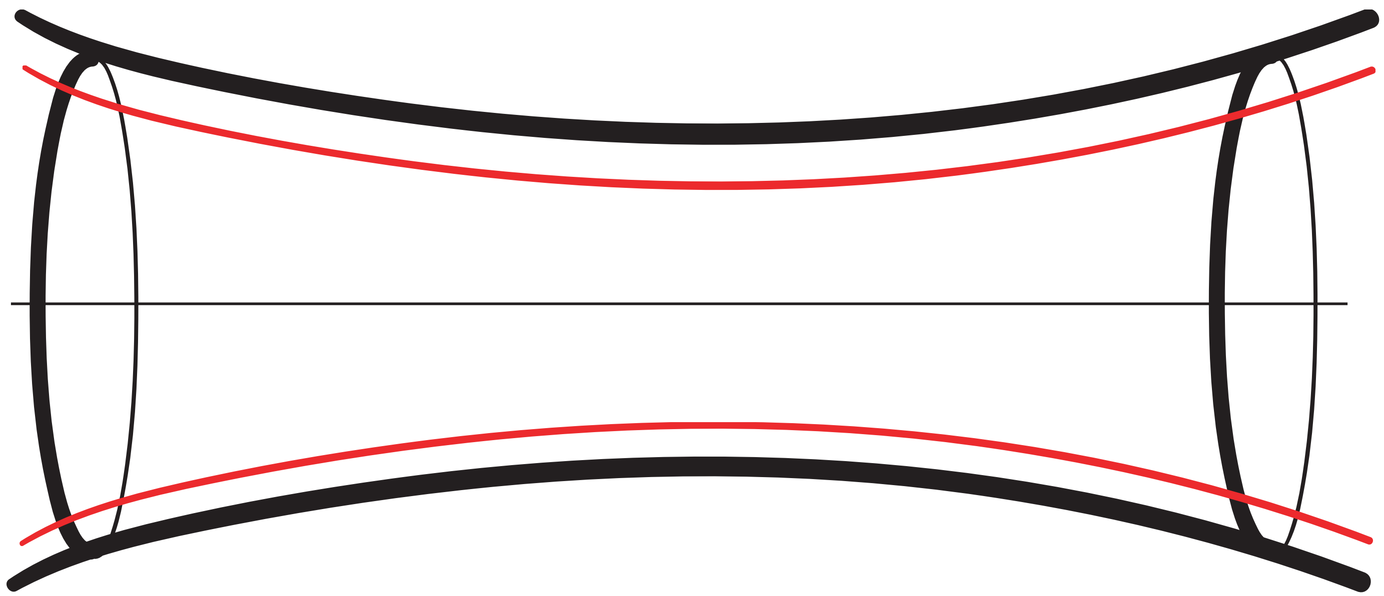
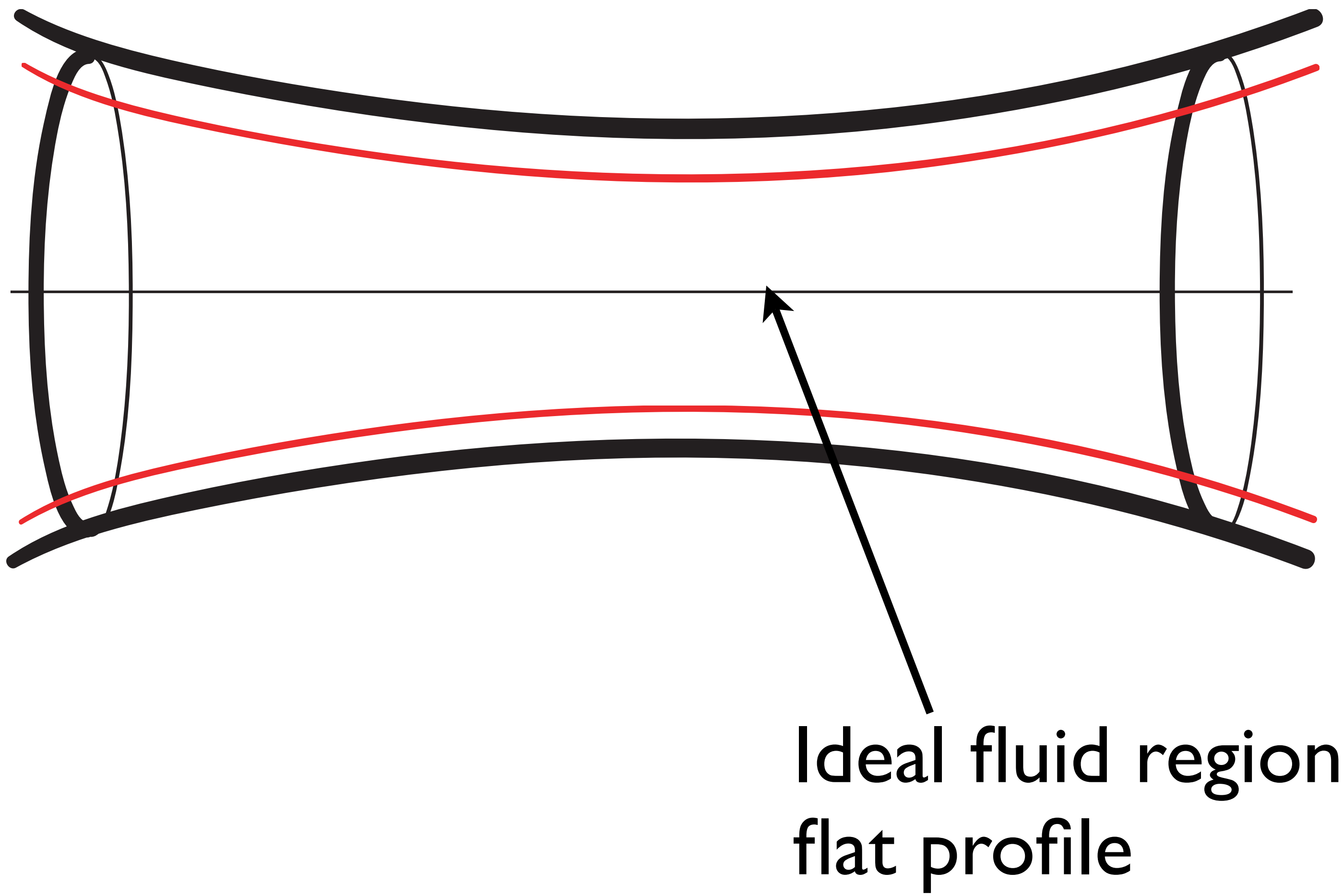
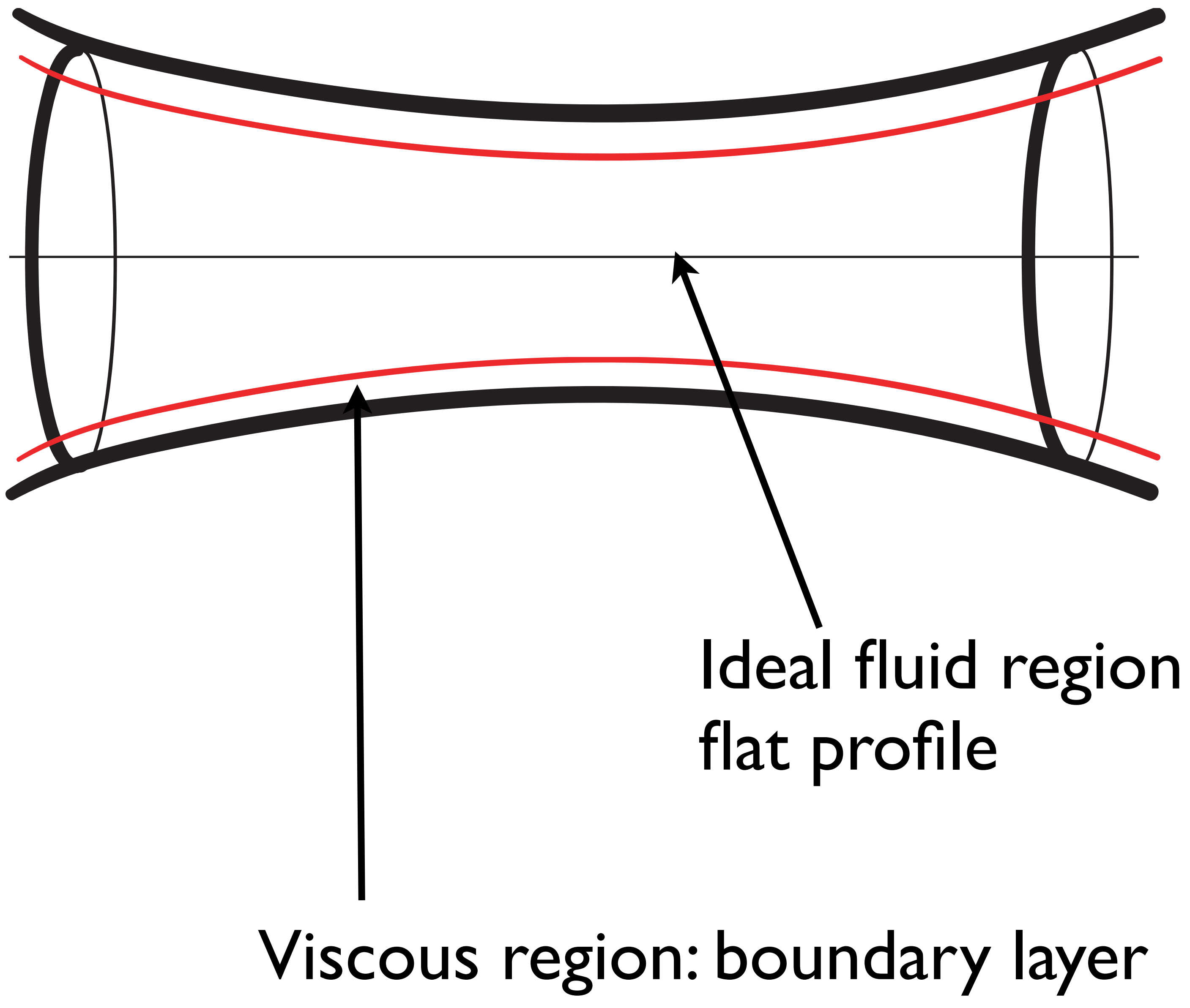


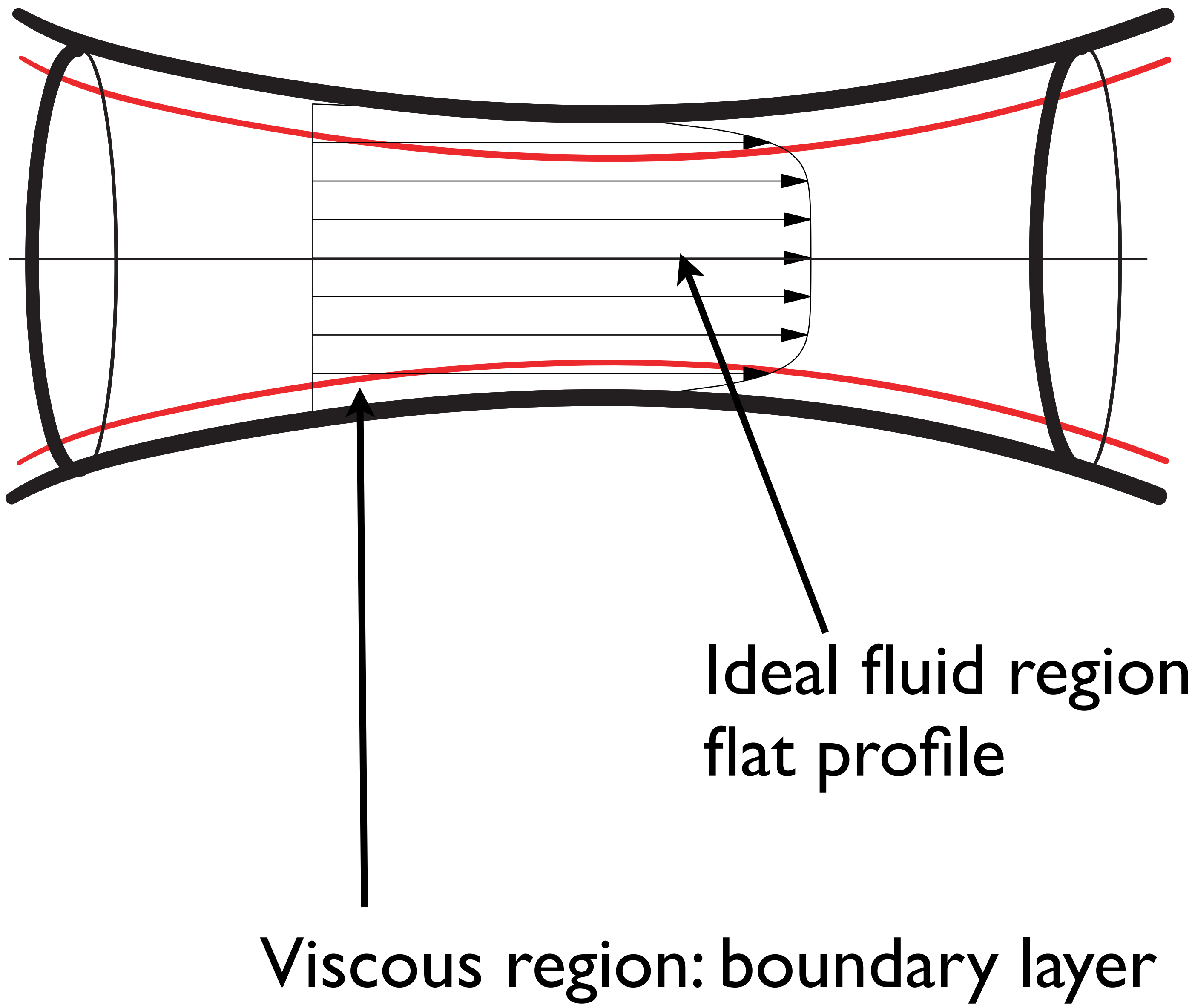
Interactive Boundary Layer



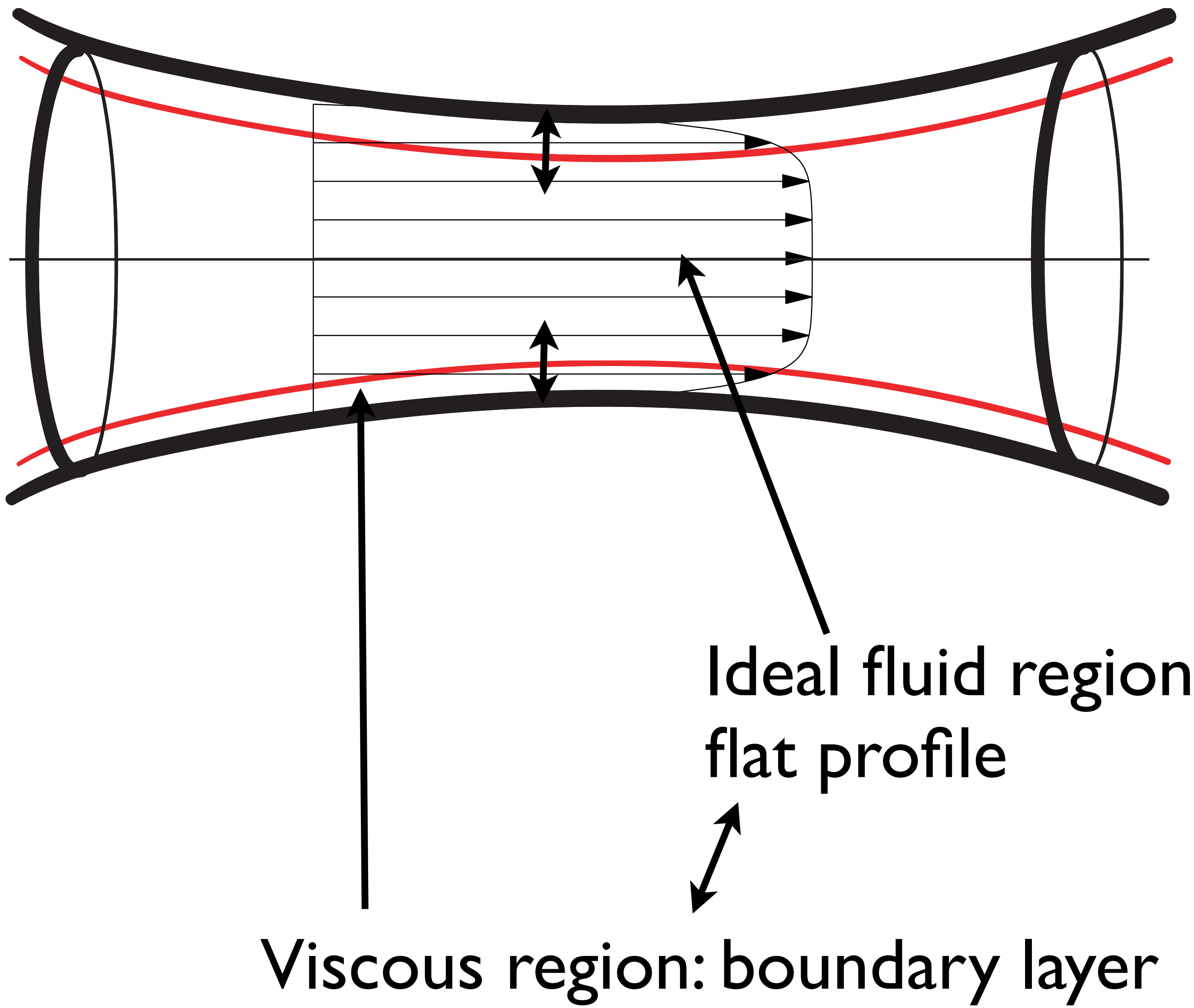
Interactive Boundary Layer



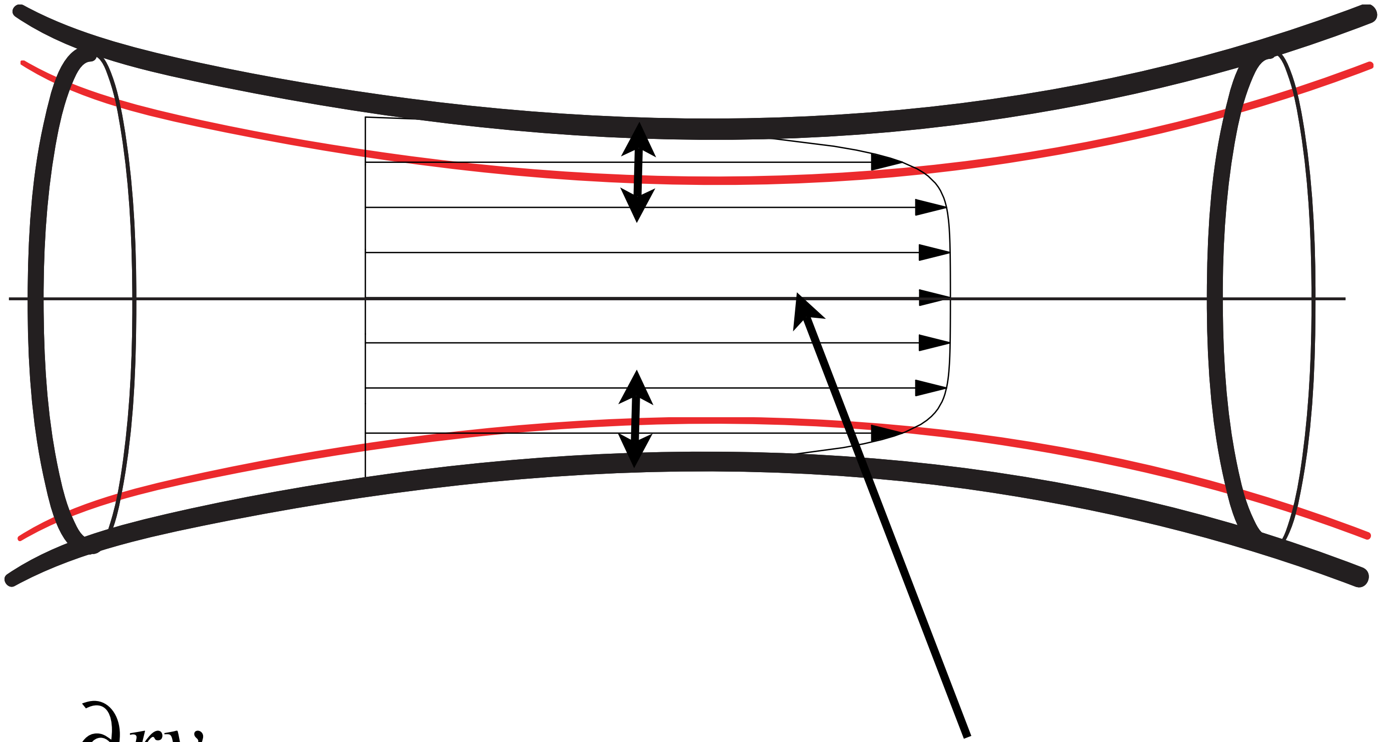
Interactive Boundary Layer



Integral resolution



Interactive Boundary Layer

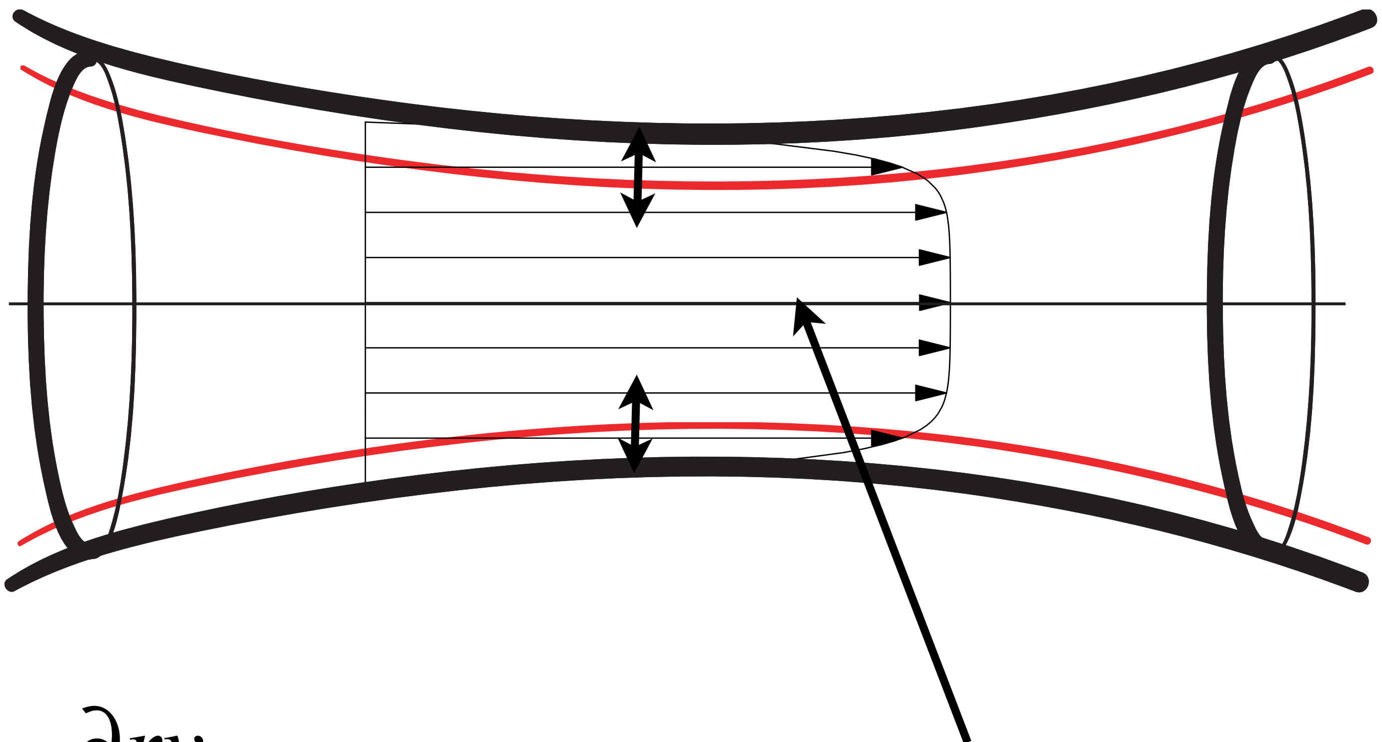


Ideal fluid region
flat profile

$$\frac{\partial u}{\partial x} + \frac{\partial rv}{r\partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho\partial x} + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r} \qquad 0 = -\frac{\partial p}{\rho\partial r}$$

Interactive Boundary Layer

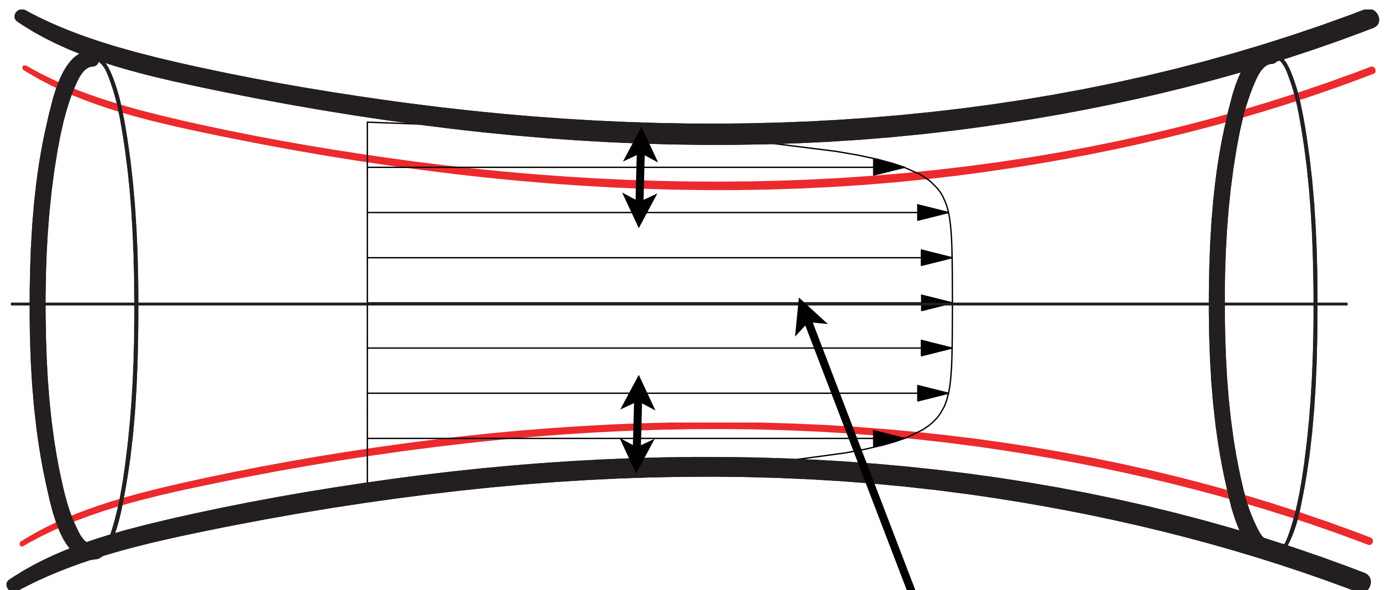


Ideal fluid region
flat profile

$$\frac{\partial u}{\partial x}+\frac{\partial rv}{r\partial r}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial r}u=-\frac{\partial p}{\rho\partial x}+\nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r} \qquad 0=-\frac{\partial p}{\rho\partial r}$$

Interactive Boundary Layer



Ideal fluid region
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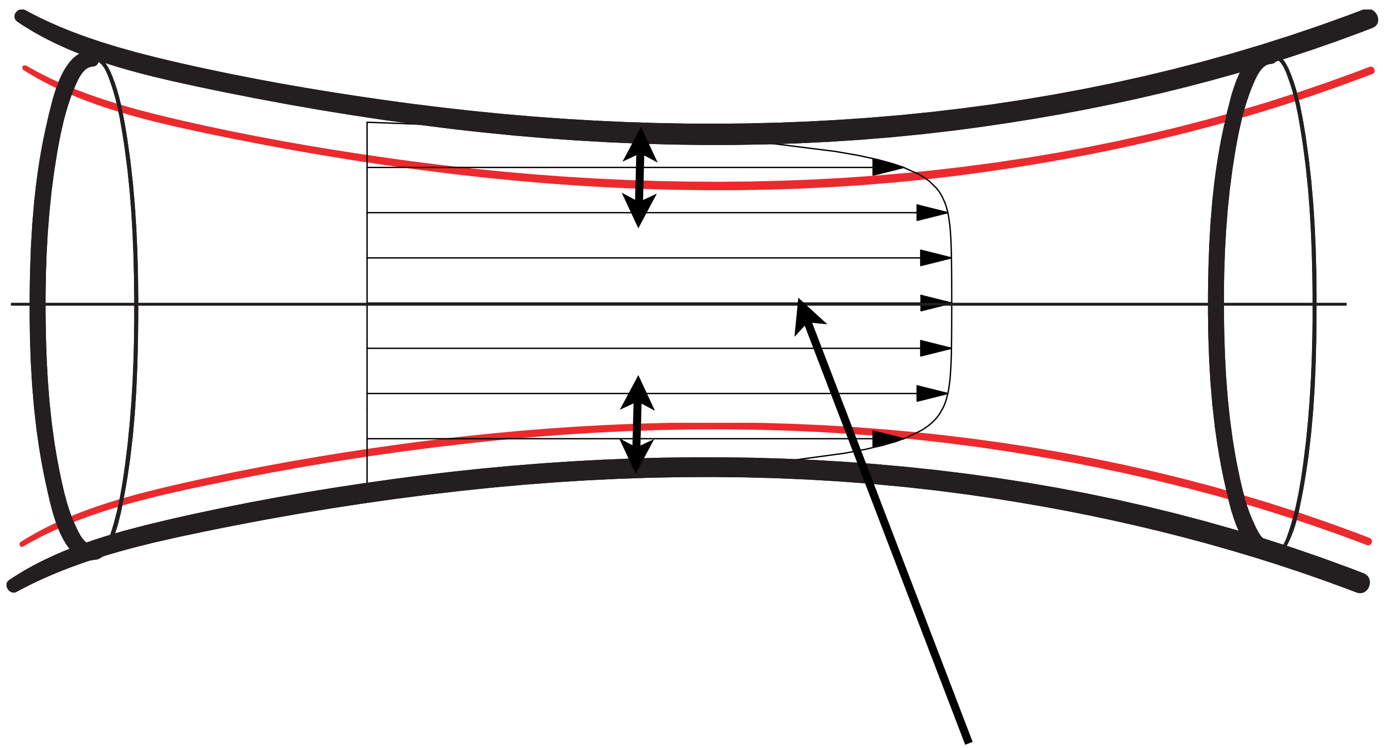
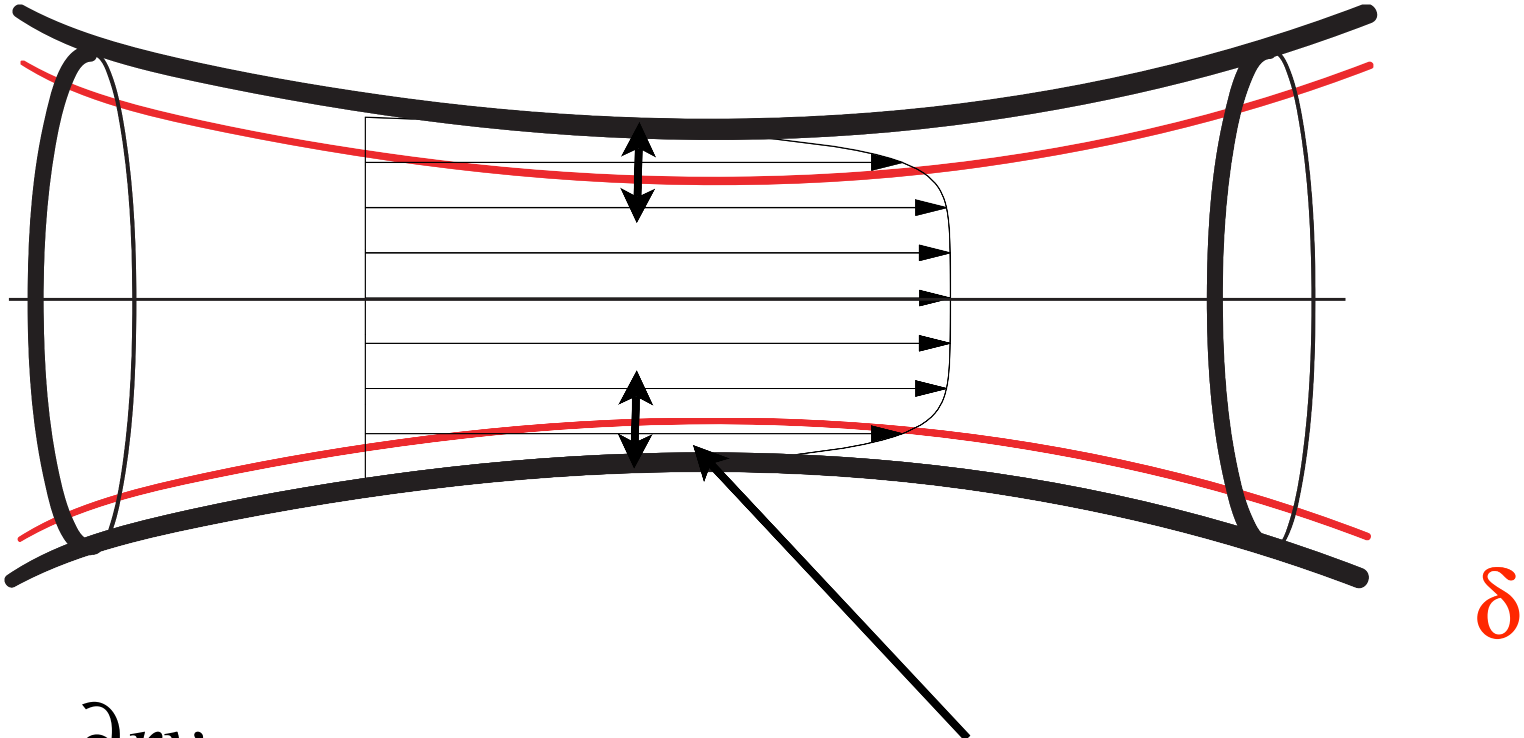
steady/ or large convective acceleration

Interactive Boundary Layer



$$U_e S = cst$$

Interactive Boundary Layer



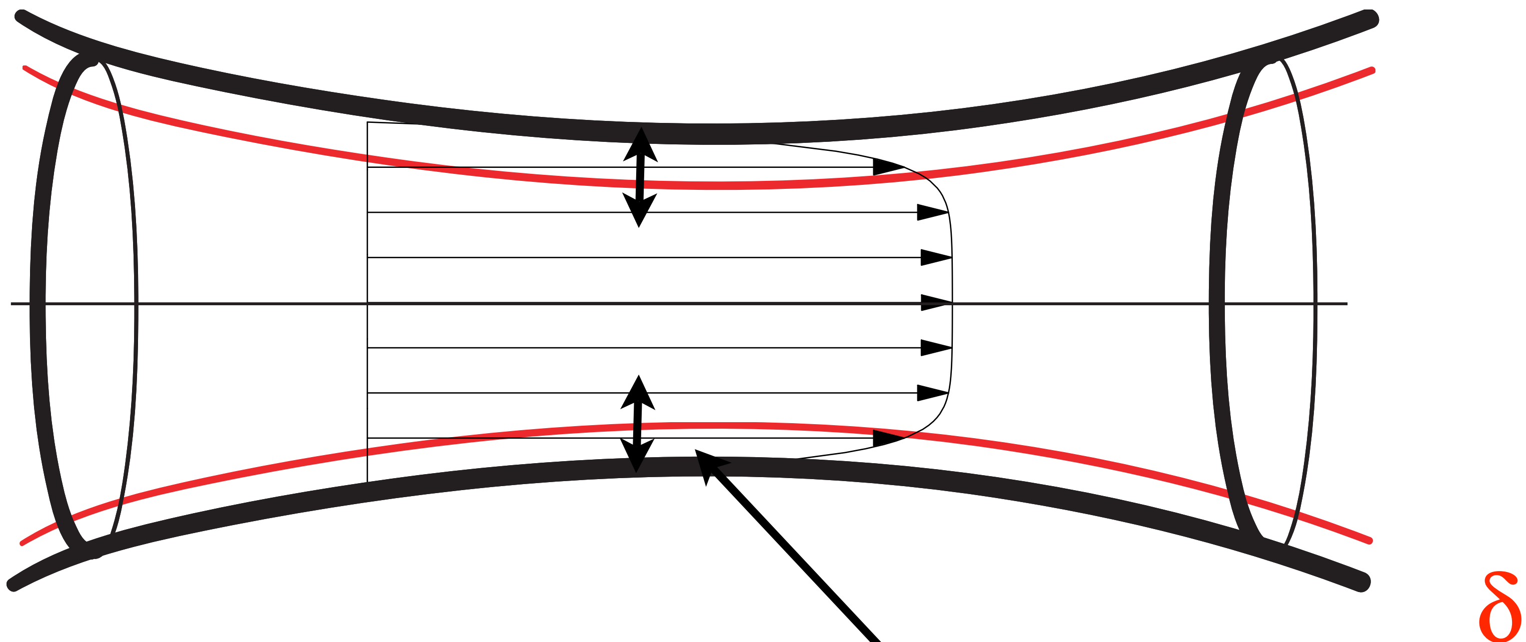
Viscous region: boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r} \qquad 0 = -\frac{\partial p}{\rho \partial r}$$

steady/ or large convective acceleration

Interactive Boundary Layer

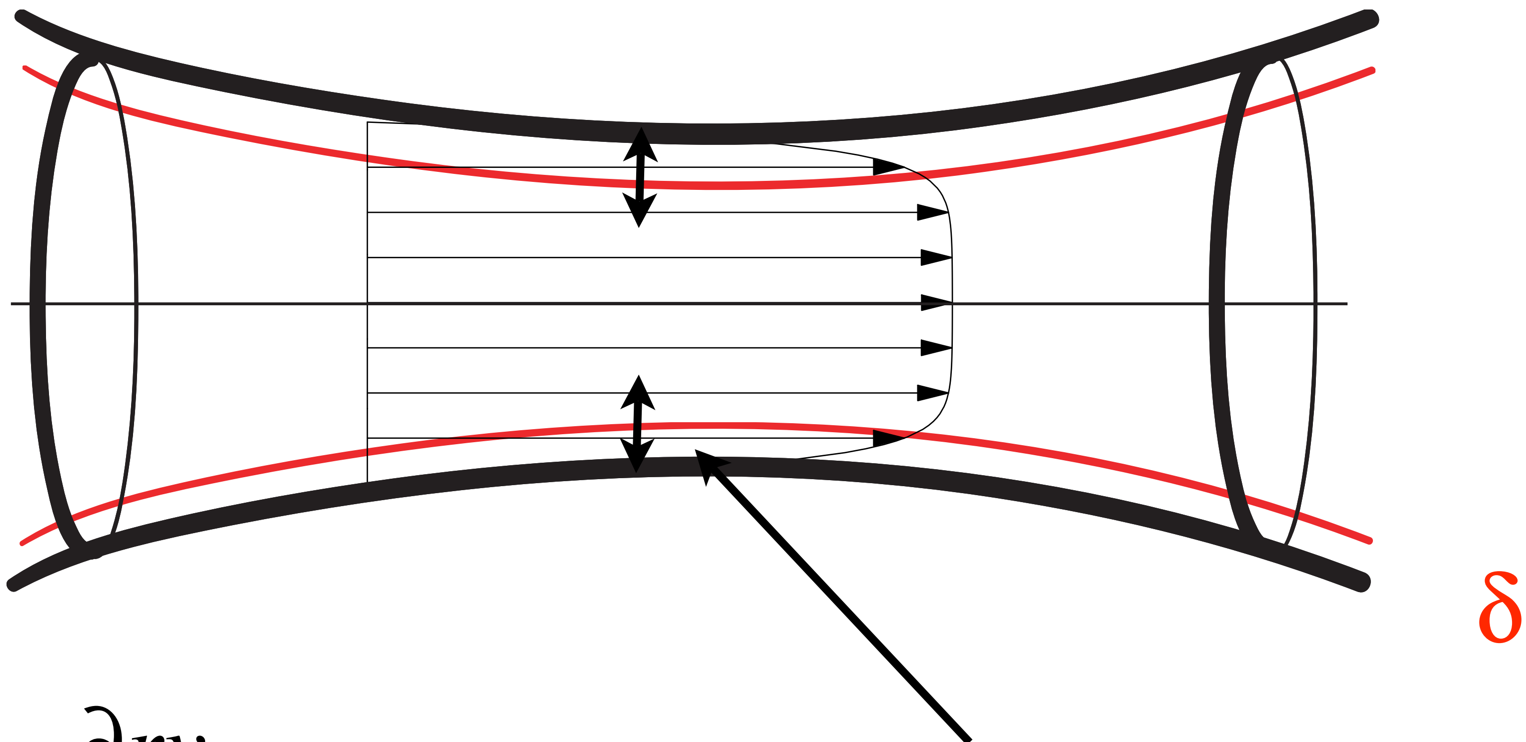


$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{U_0^2}{\lambda} = -\frac{\partial p}{\rho \partial x} + \frac{\nu}{U_0 \lambda} \frac{\lambda^2}{\delta^2} \frac{U_0^2}{\lambda} \qquad 0 = -\frac{\partial p}{\rho \partial r}$$

steady/ or large convective acceleration

Interactive Boundary Layer



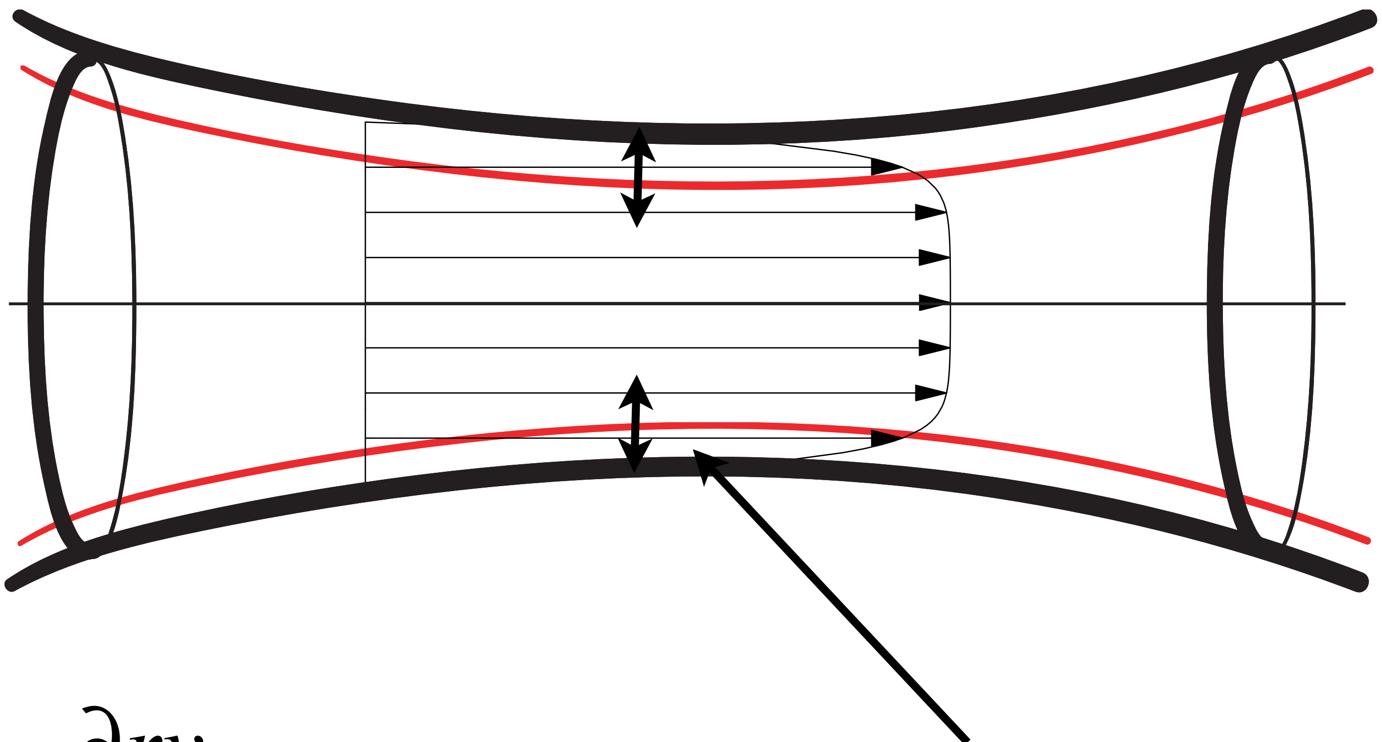
Viscous region: boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\boxed{\frac{U_0^2}{\lambda}} = -\frac{\partial p}{\rho \partial x} + \boxed{\frac{1}{Re}\frac{\lambda^2}{\delta^2}\frac{U_0^2}{\lambda}} \quad 0 = -\frac{\partial p}{\rho \partial r}$$

steady/ or large convective acceleration

Interactive Boundary Layer



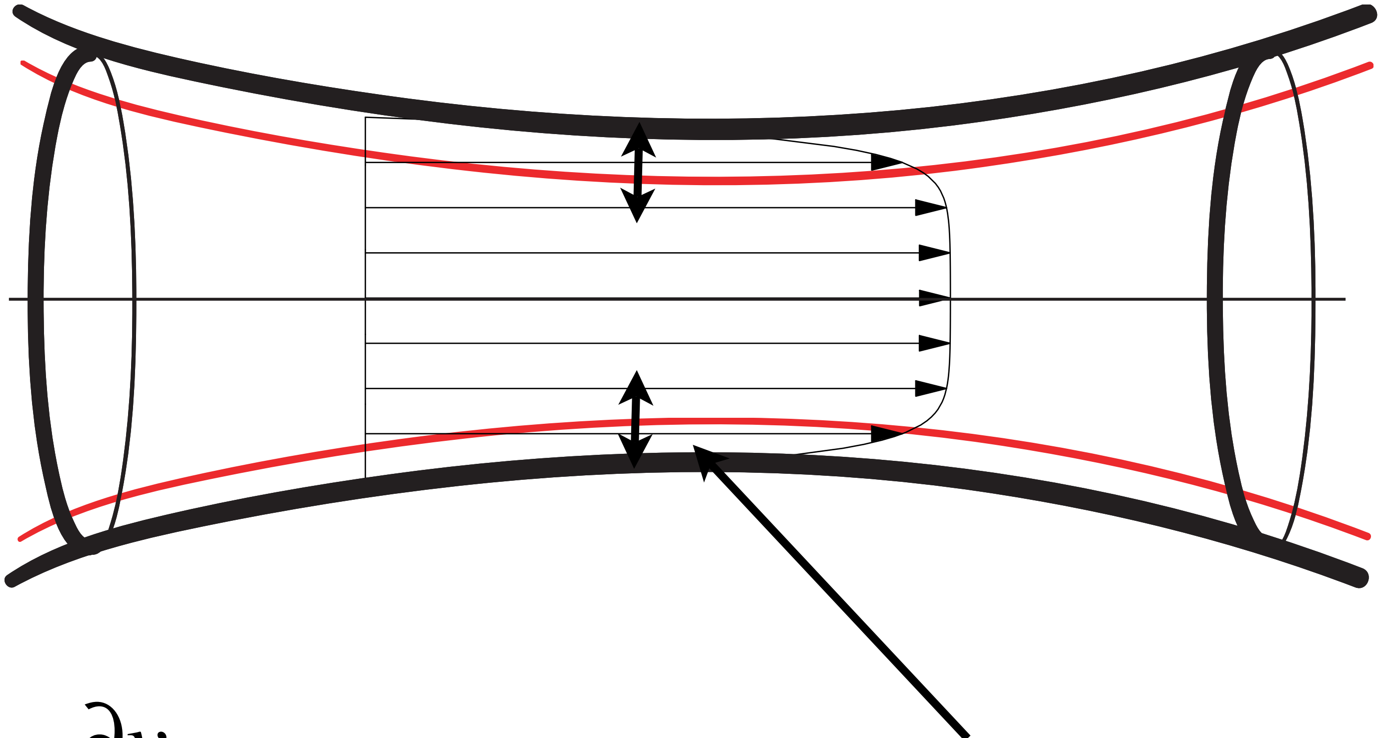
$$\delta \sim \frac{\lambda}{Re^{1/2}}$$

Viscous region: boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{U_0^2}{\lambda} = -\frac{\partial p}{\rho \partial x} + \frac{1}{Re}\frac{\lambda^2}{\delta^2}\frac{U_0^2}{\lambda} \qquad 0 = -\frac{\partial p}{\rho \partial r}$$

Interactive Boundary Layer



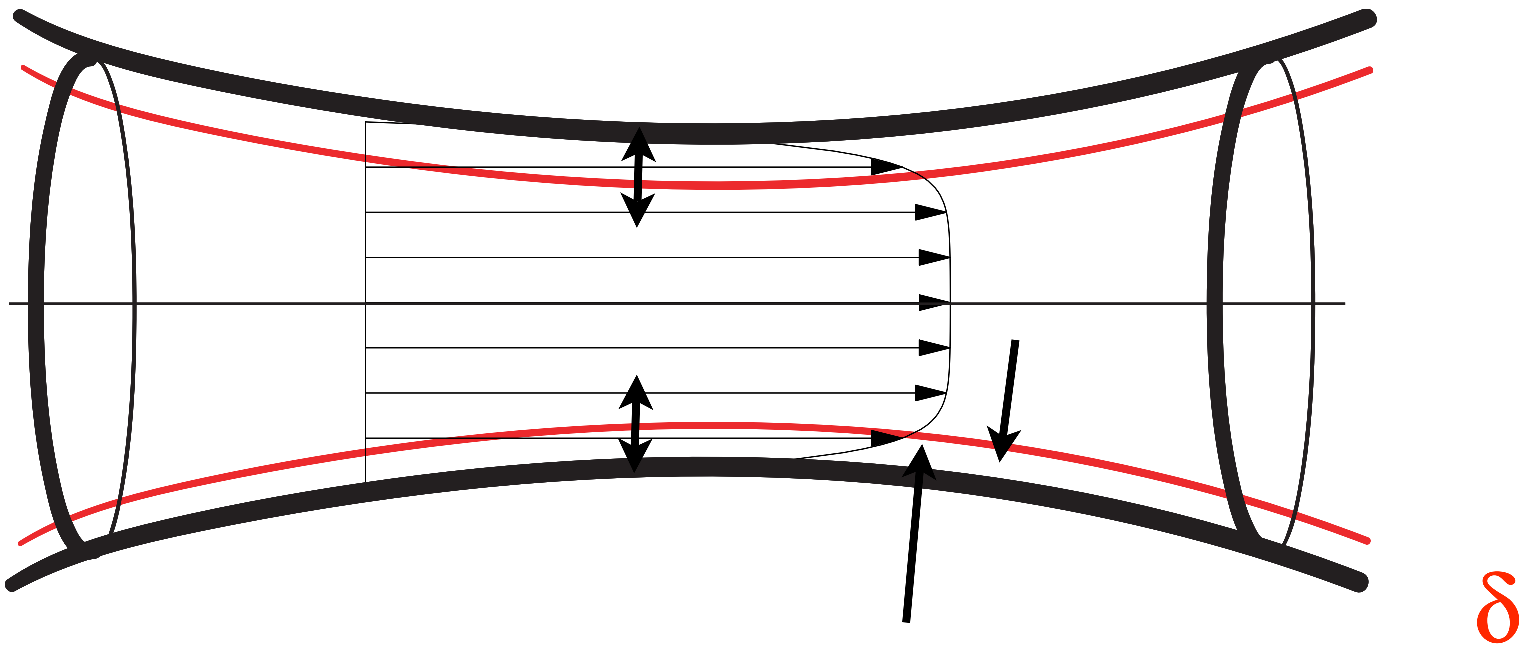
$$\delta \sim \frac{\lambda}{Re^{1/2}}$$

Viscous region: boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = -\frac{\partial p}{\partial x} + \frac{\partial^2}{\partial n^2}u \qquad 0 = -\frac{\partial p}{\partial n}$$

Interactive Boundary Layer

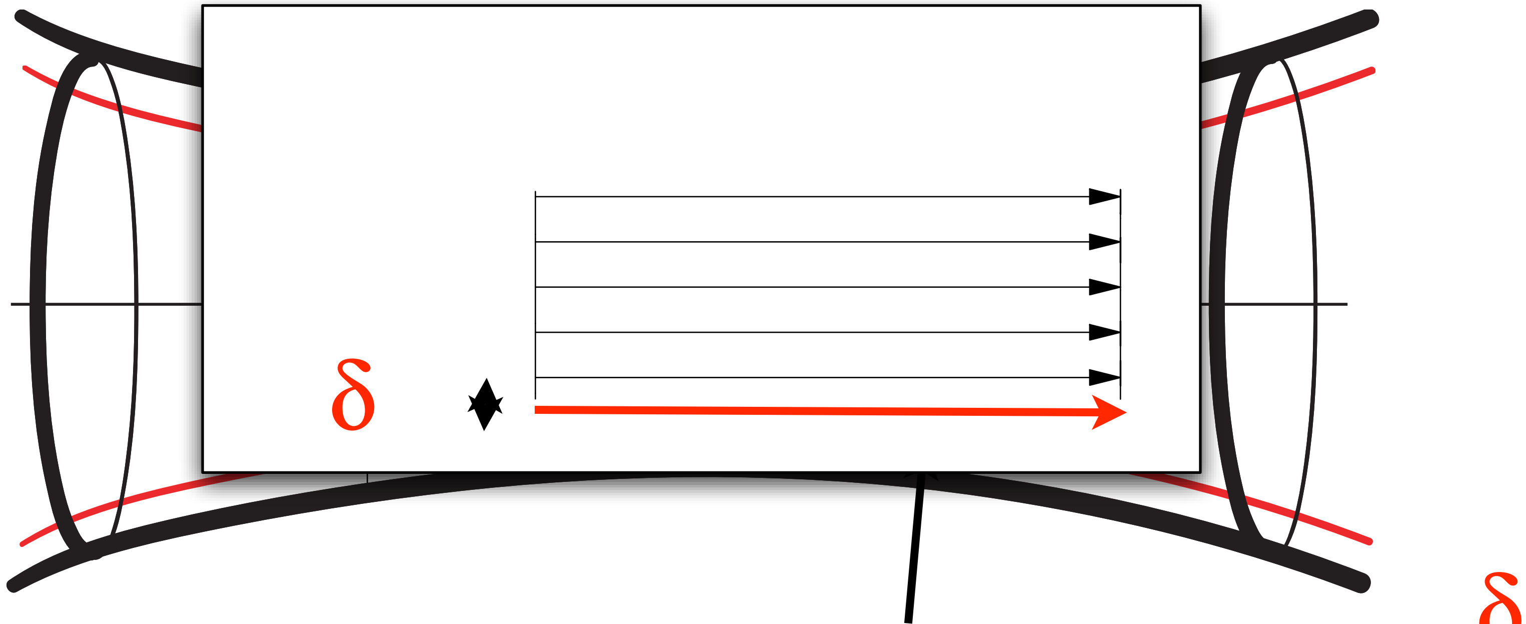


Matching of velocity
from invicid/ viscous

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = -\frac{\partial p}{\partial x} + \frac{\partial^2}{\partial n^2}u \qquad 0 = -\frac{\partial p}{\partial n}$$

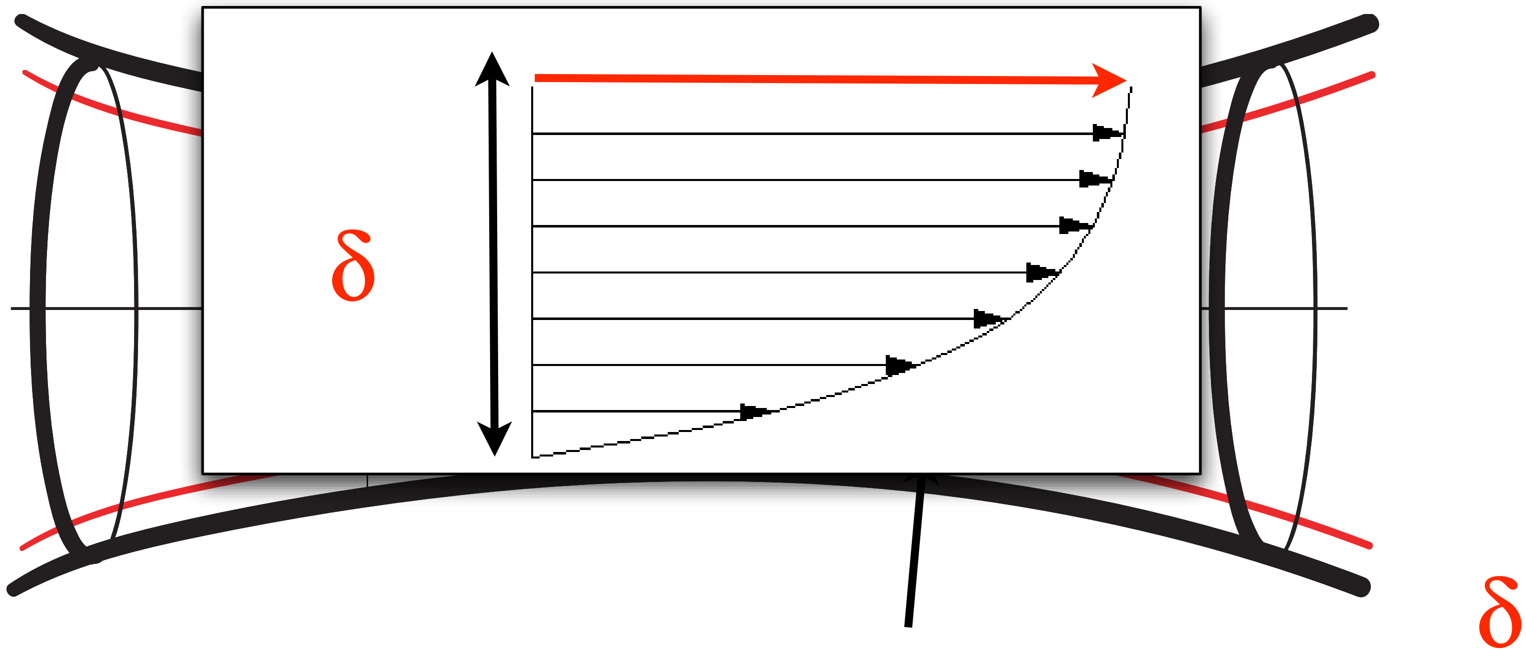
Interactive Boundary Layer



Matching of velocity from invicid/ viscous

U_e at the wall

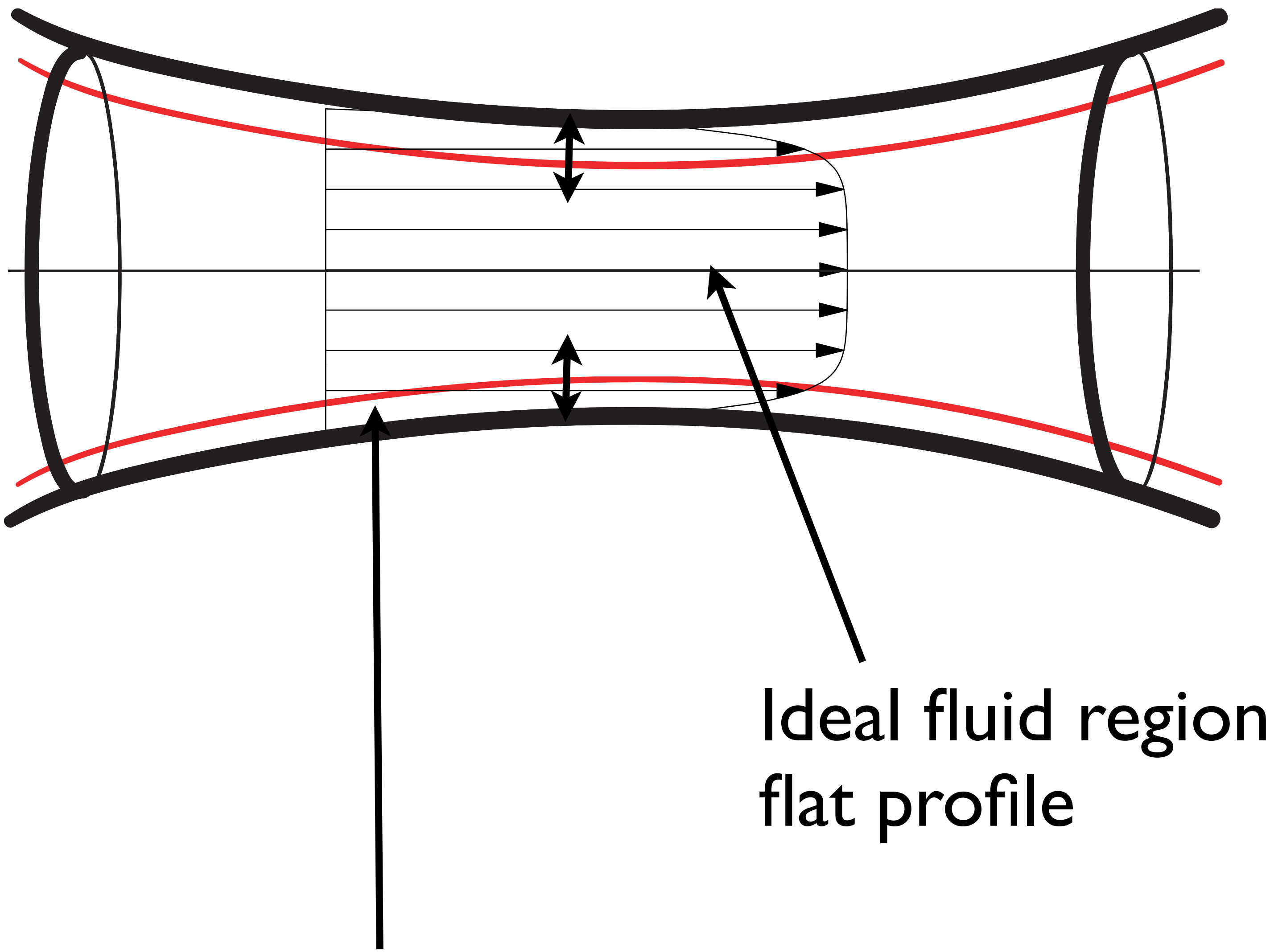
Interactive Boundary Layer



U_e at the wall

is the velocity at the edge of the boundary layer $u(x,\infty)$
at "infinity"

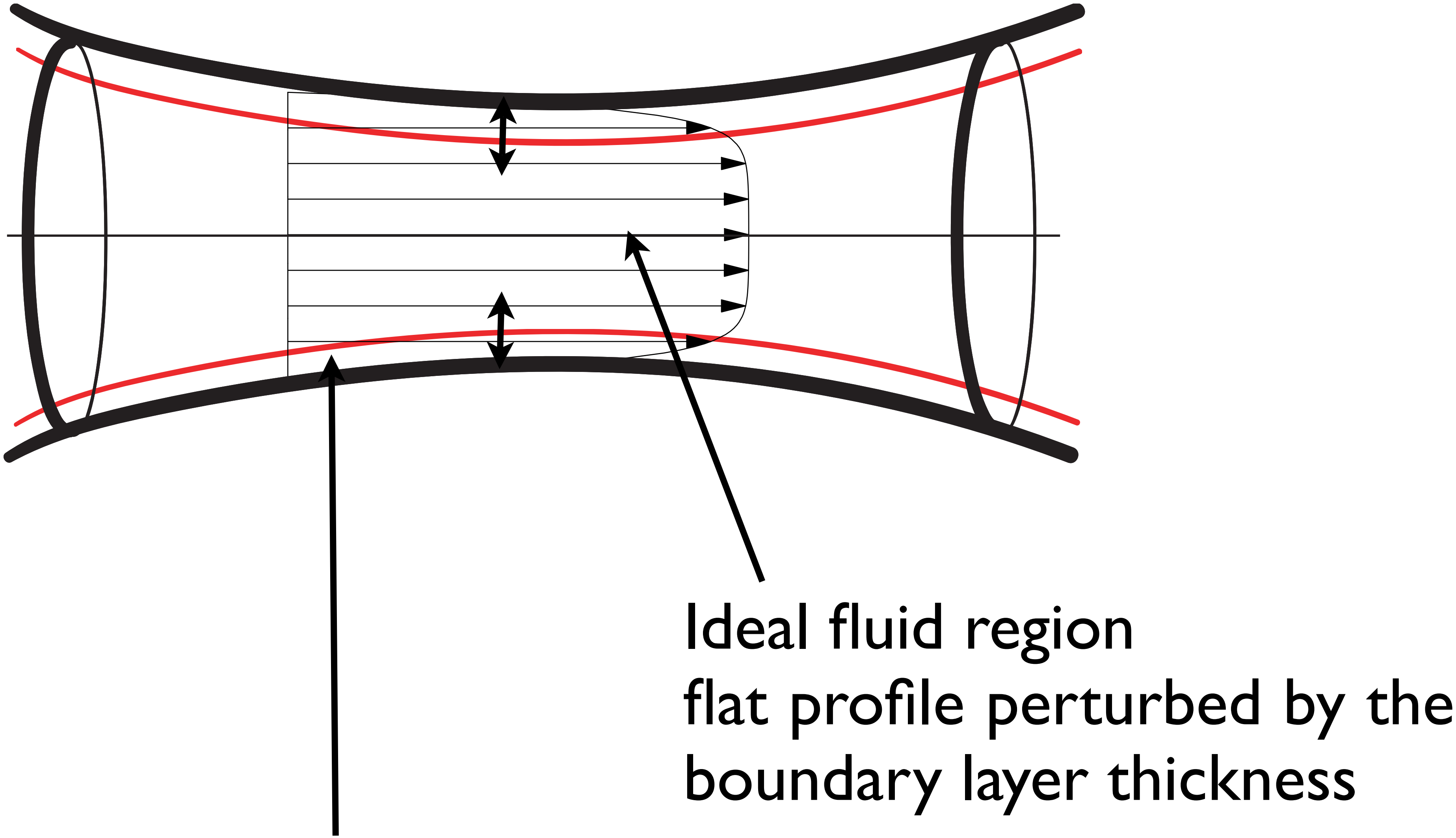
Interactive Boundary Layer



$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn$$

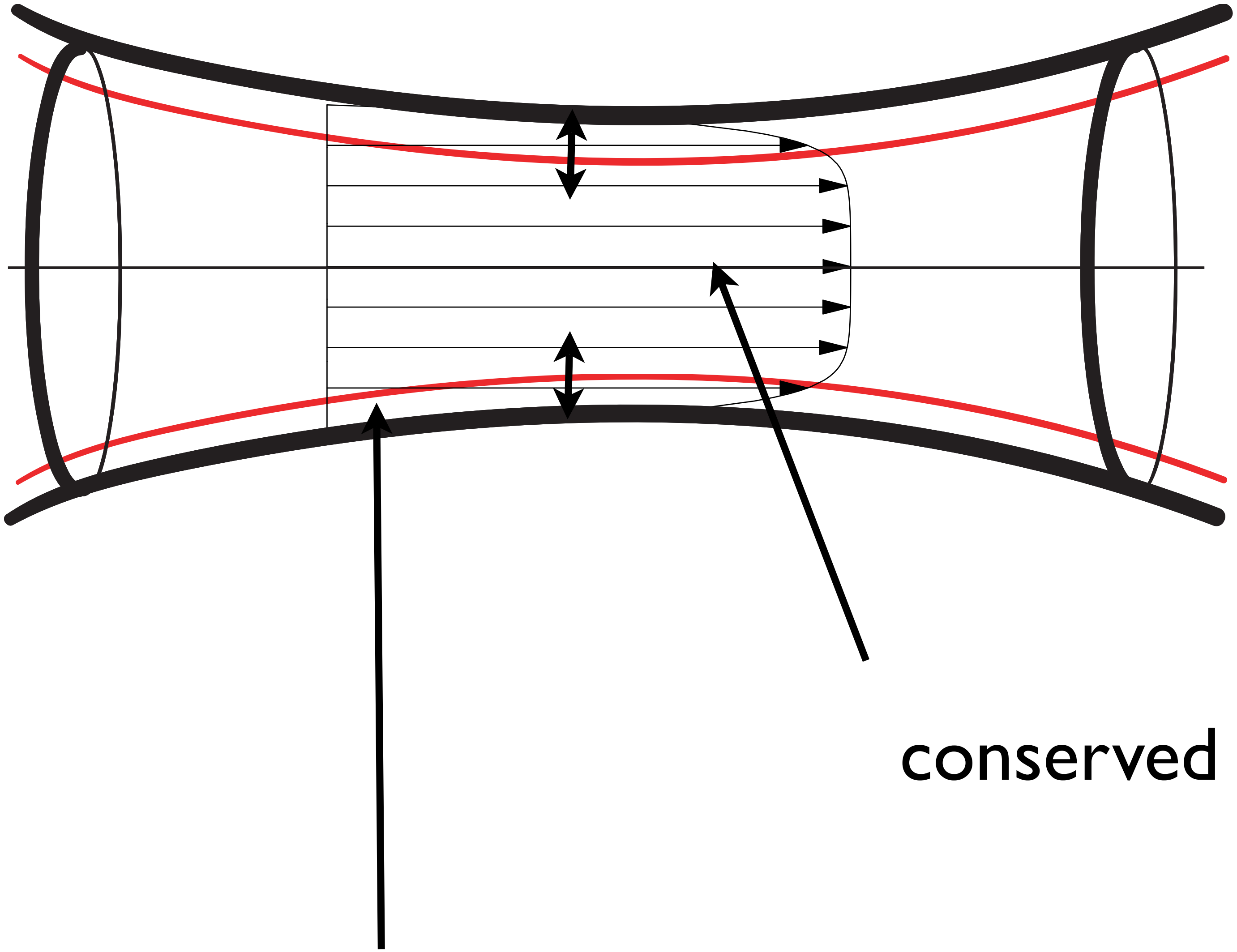
displacement of stream lines

Interactive Boundary Layer



$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U_e}\right) dn$$

Interactive Boundary Layer



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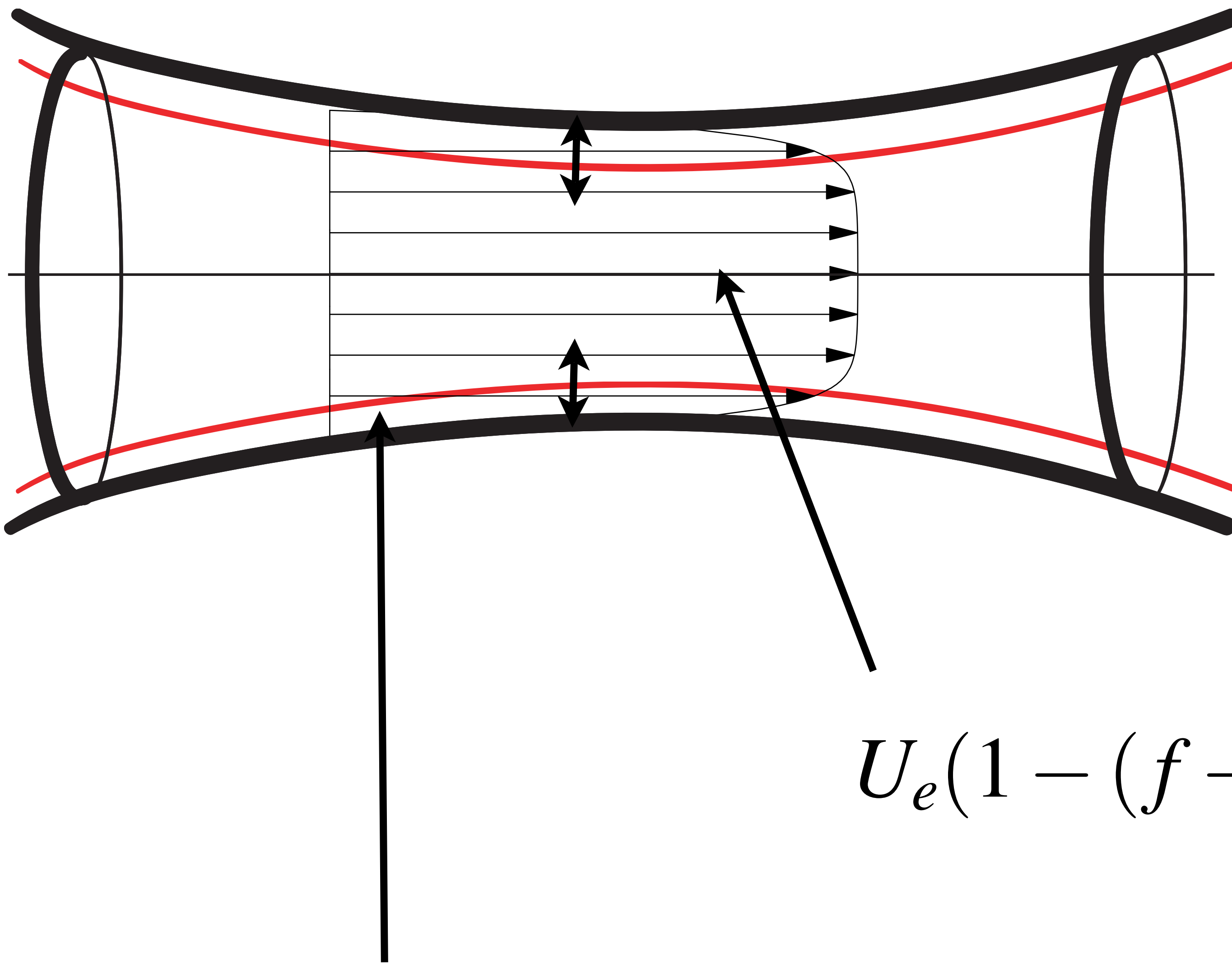
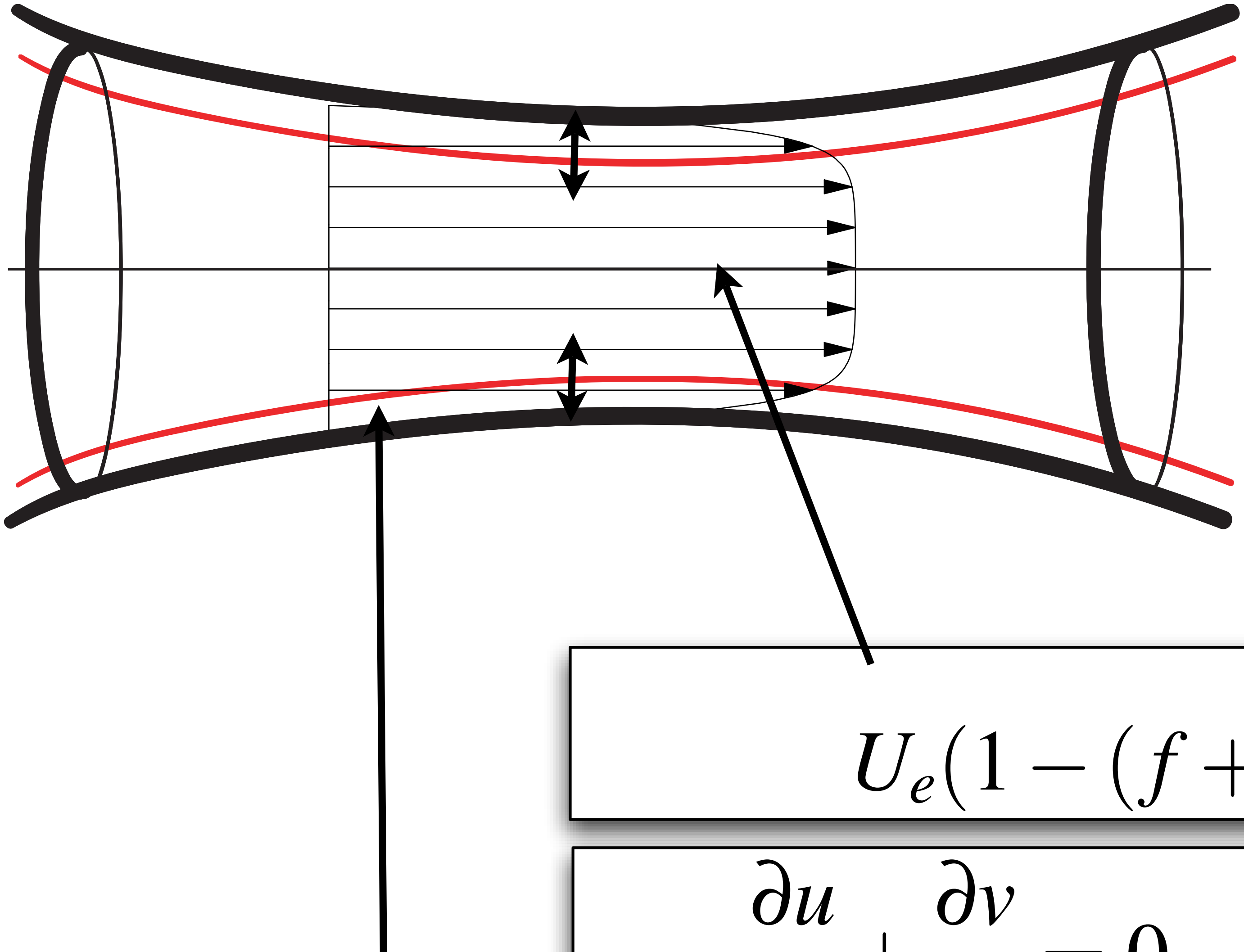
Interactive Boundary Layer



$$U_e(1-(f+\delta_1))^2 = 1$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn$$

Interactive Boundary Layer



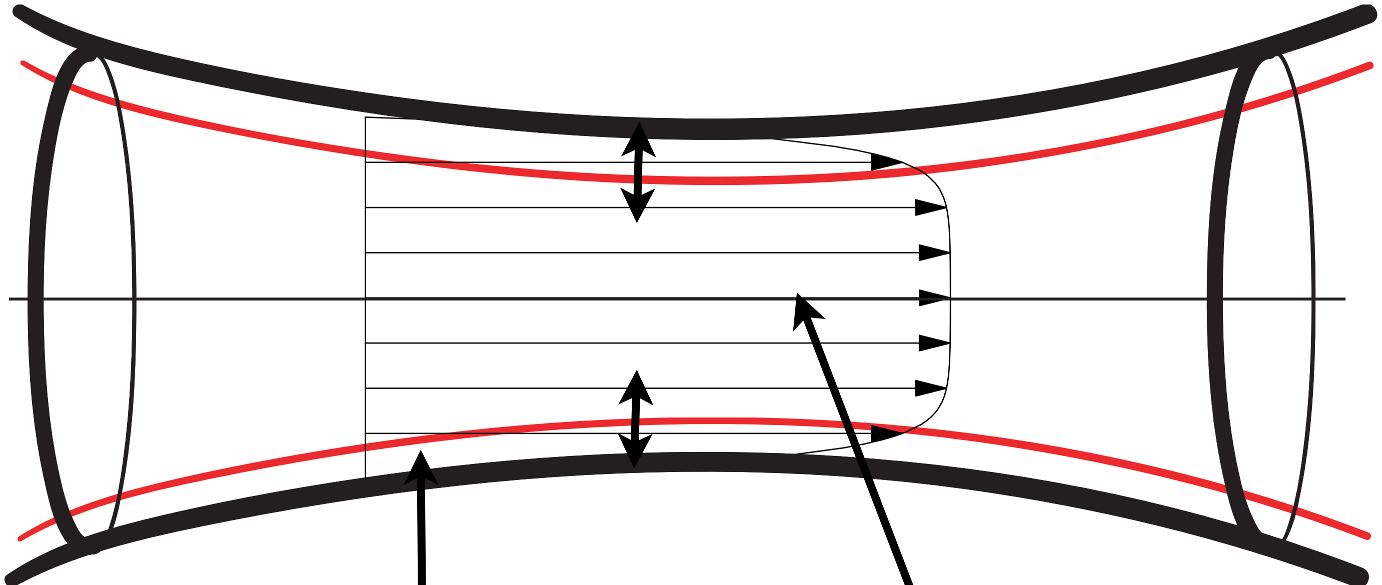
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e})dn$$

Interactive Boundary Layer



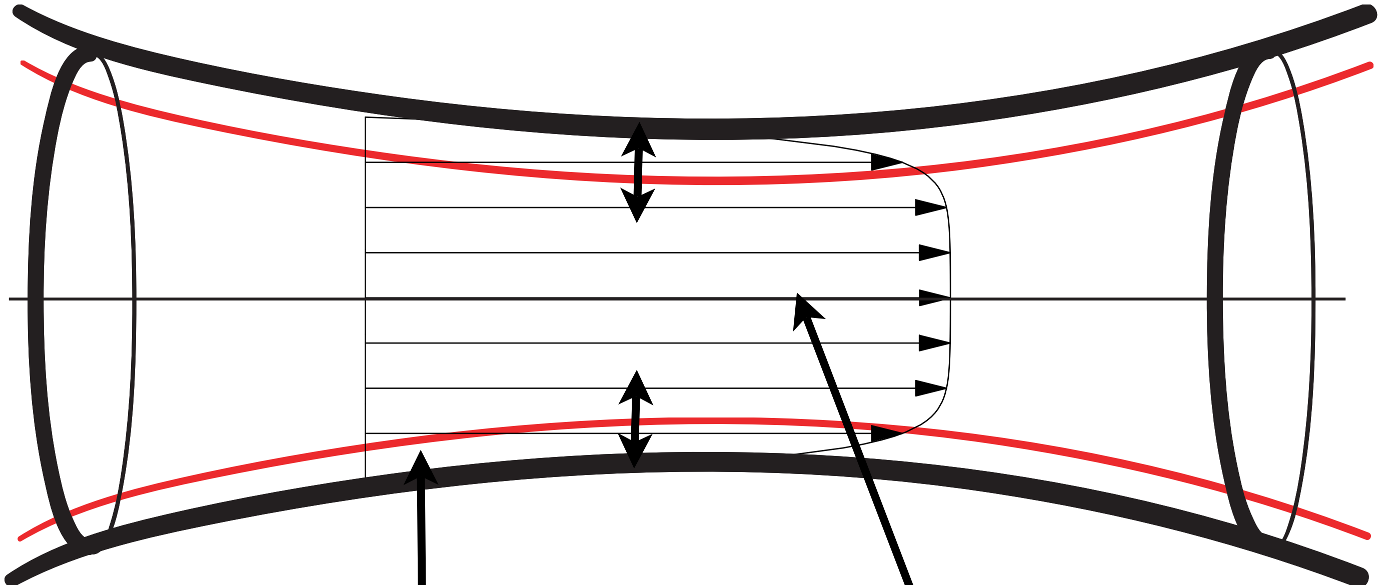
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \qquad u(x,\infty) = U_e$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e})dn$$

Interactive Boundary Layer



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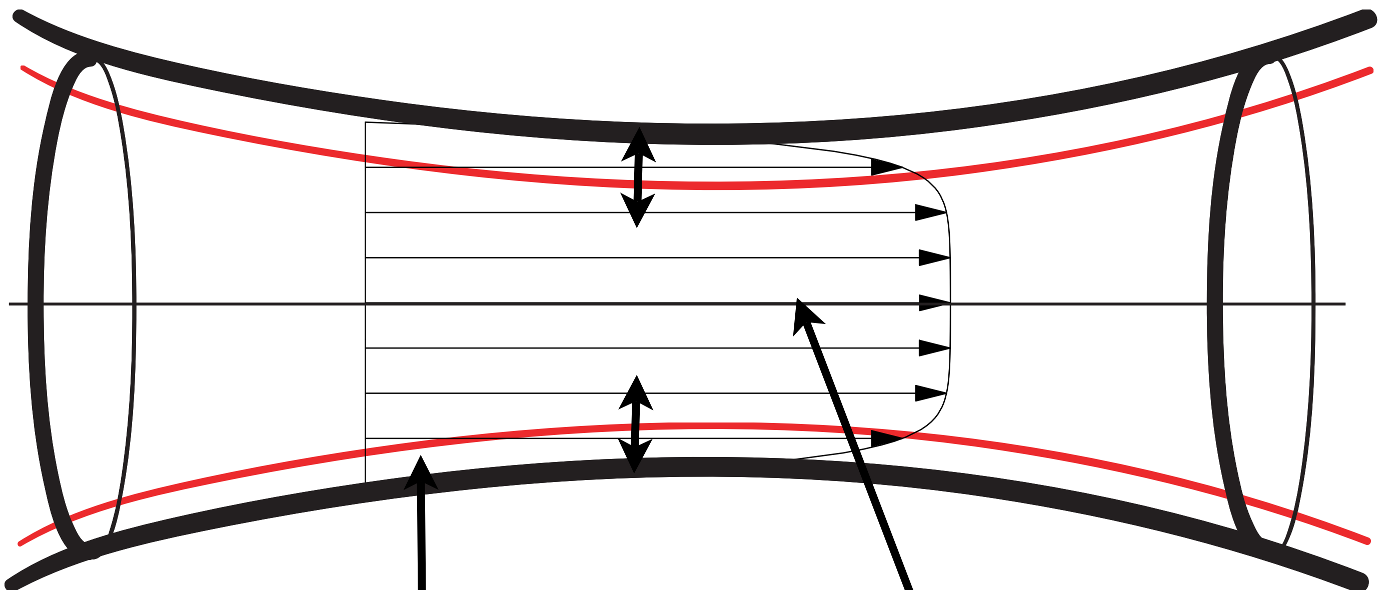
$$\delta_1 = \int_0^\infty (1-\frac{u}{U_e})dn$$

Interactive Boundary Layer



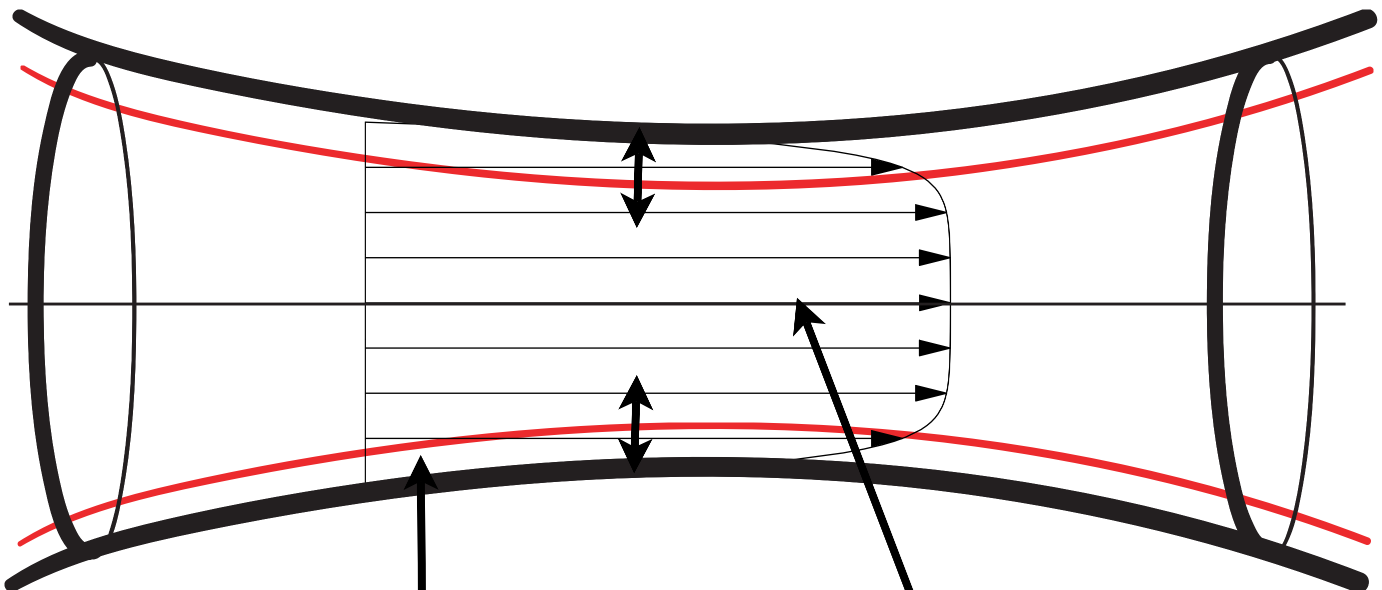
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial n}=0 \qquad u(x,\infty)=U_e$$

$$u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial n}u=\frac{dU_e}{dx}+\frac{\partial^2}{\partial n^2}u$$

$$\delta_1=\int_0^\infty\left(1-\frac{u}{U_e}\right)dn$$

Interactive Boundary Layer



Coupled System to solve

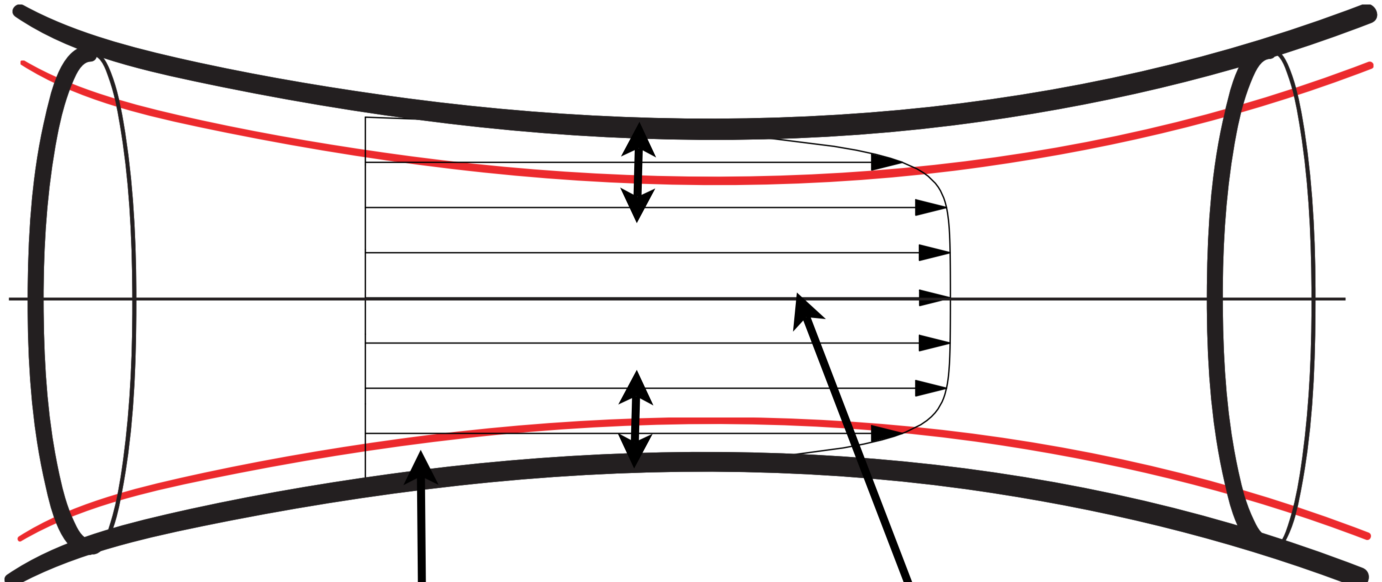
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \qquad u(x,\infty) = U_e$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e})dn$$

Interactive Boundary Layer



Coupled System to solve

$$U_e(1-(f+\delta_1))^2 = 1$$

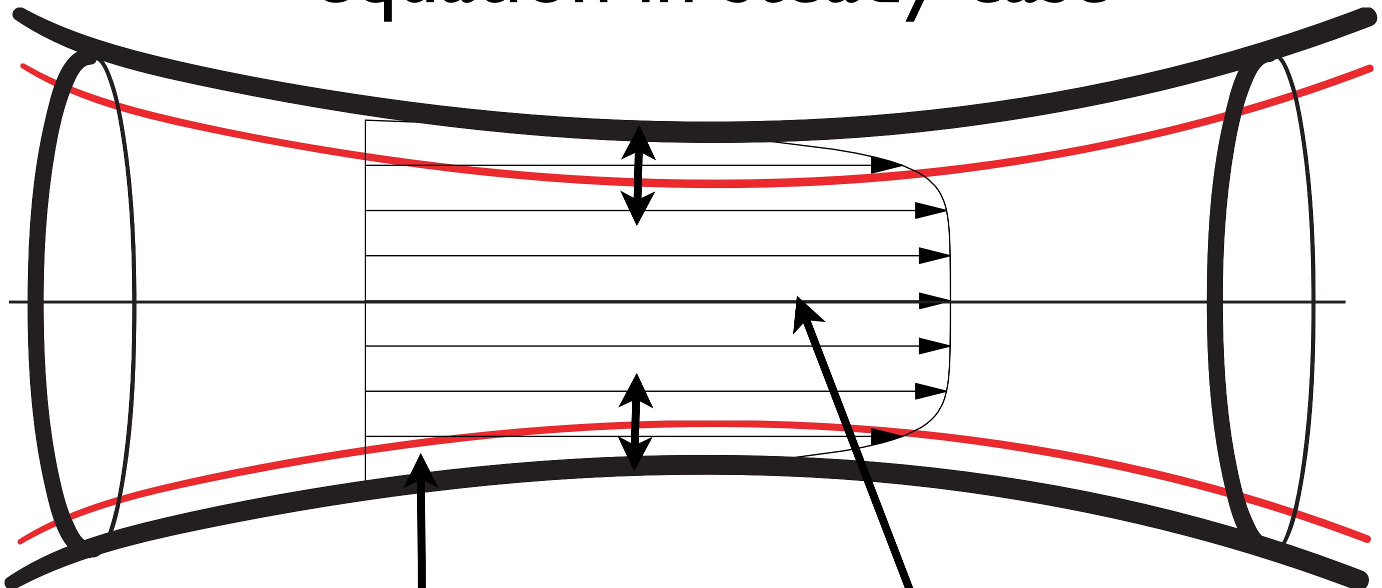
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \qquad u(x,\infty) = U_e$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\int_0^\infty dn$$

$$\delta_1 = \int_0^\infty (1-\frac{u}{U_e})dn$$

Integral resolution equation in steady case



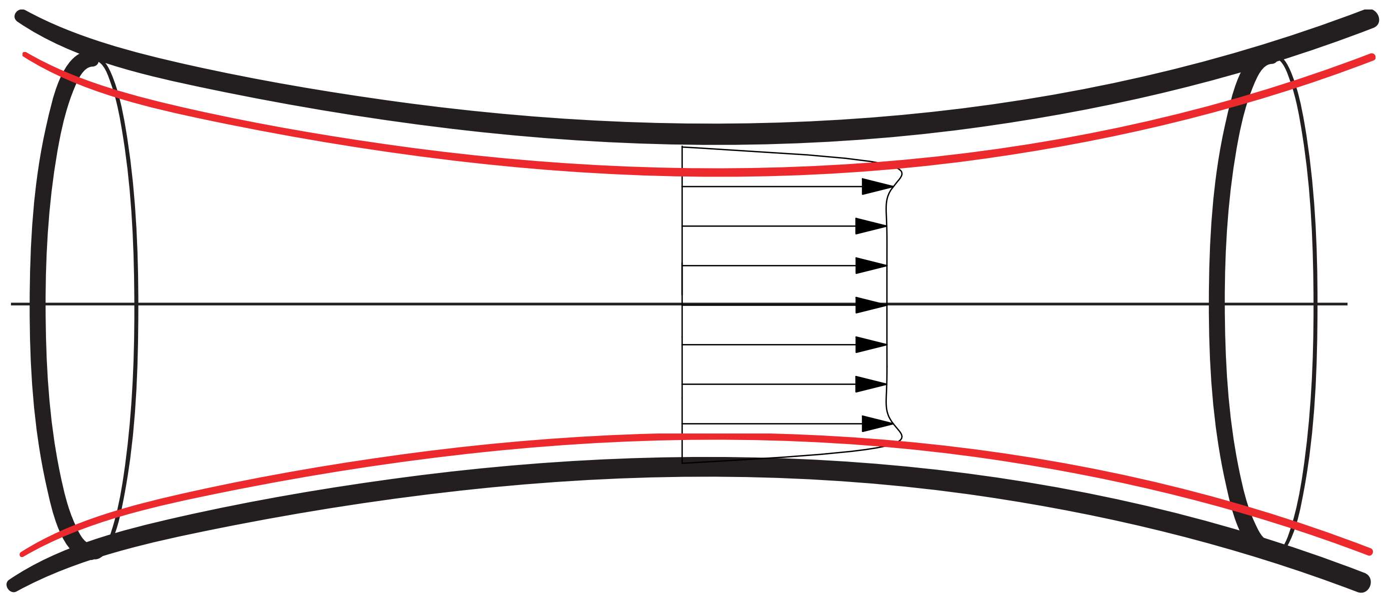
Coupled System to solve

$$U_e(1-(f+\delta_1))^2 = 1$$

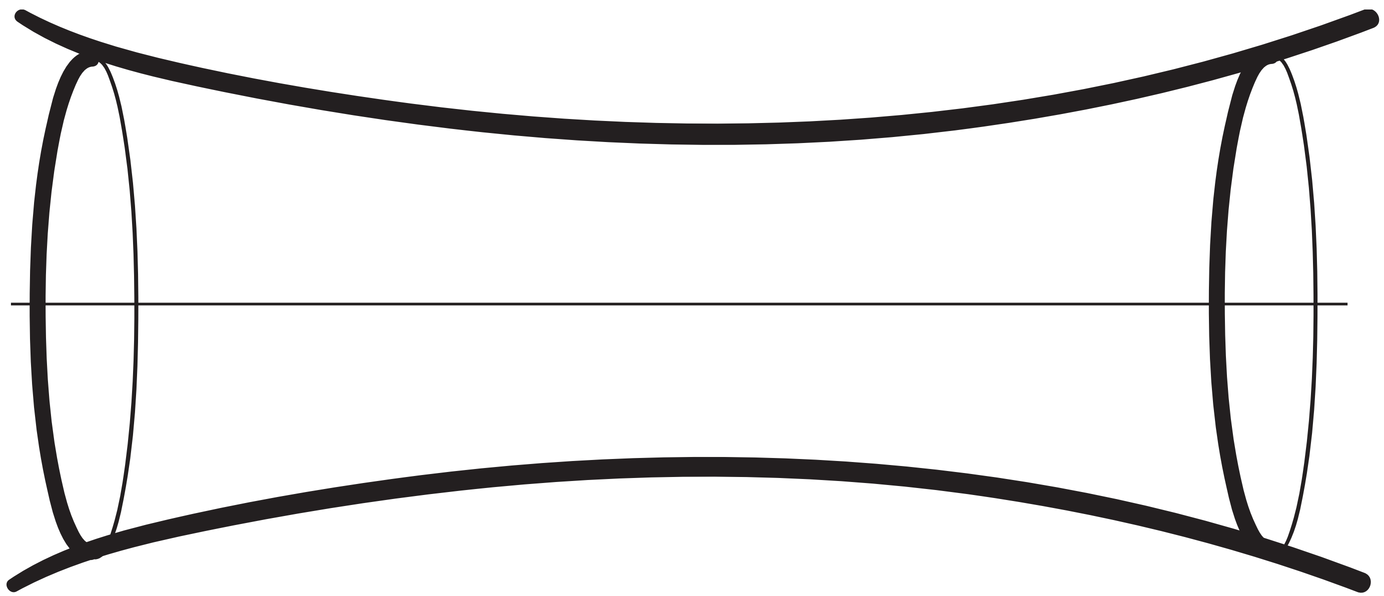
$$\frac{d}{dx}\left(\frac{\delta_1}{H}\right) + \frac{\delta_1}{U_e}\left(1+\frac{2}{H}\right)\frac{dU_e}{dx} = \frac{f_2 H}{\delta_1 U_e}$$

$$\delta_1 = \int_0^\infty \left(1-\frac{u}{U_e}\right)dn$$

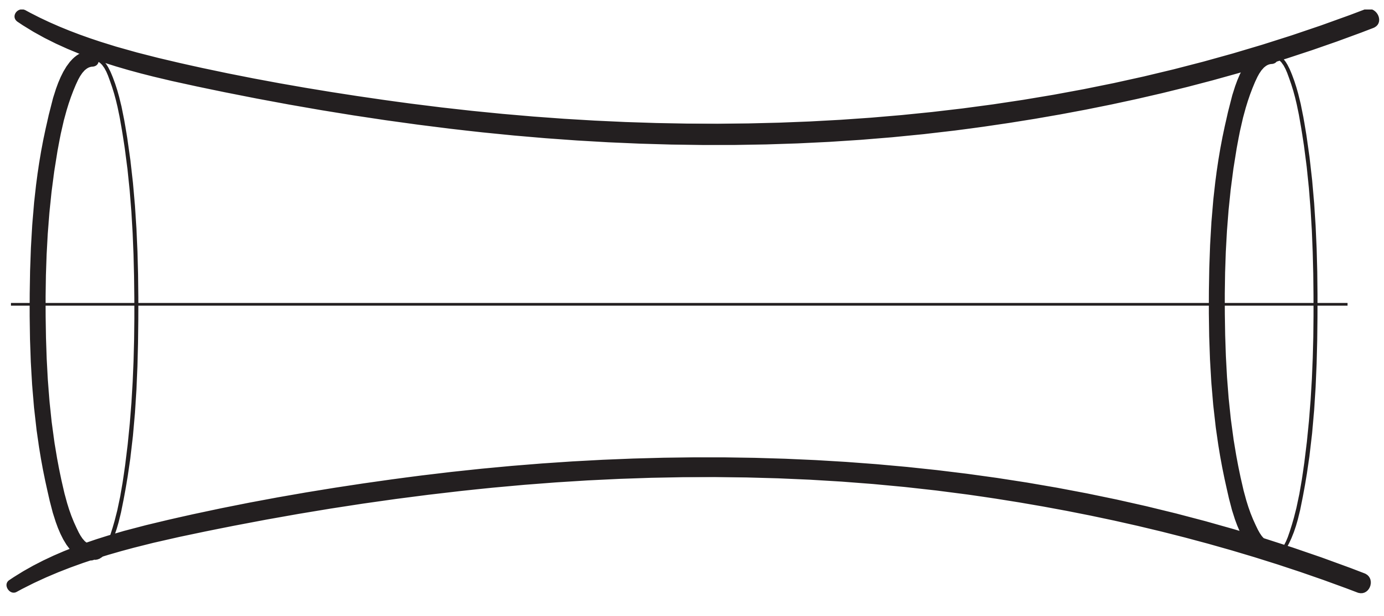
Interactive Boundary Layer



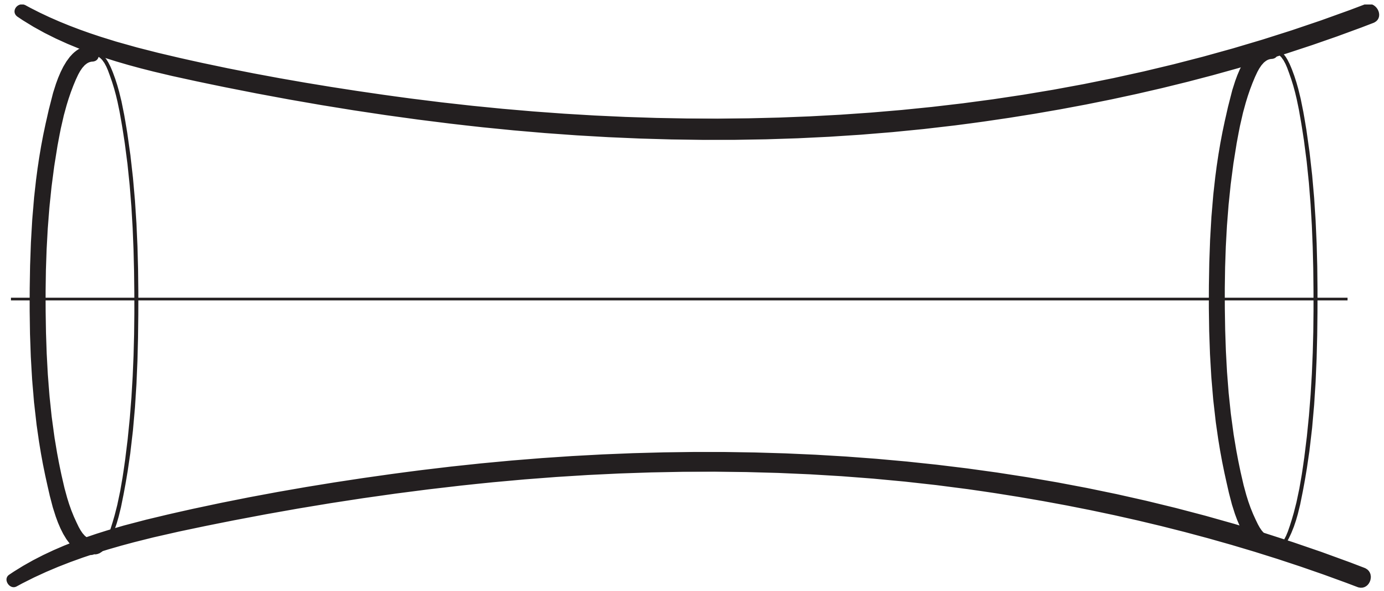
IBL is included in RNSP



RNSP includes usual 1D equations
RNSP includes Womersley profiles
RNSP includes Boundary Layer Theories (IBL)



Comparisons

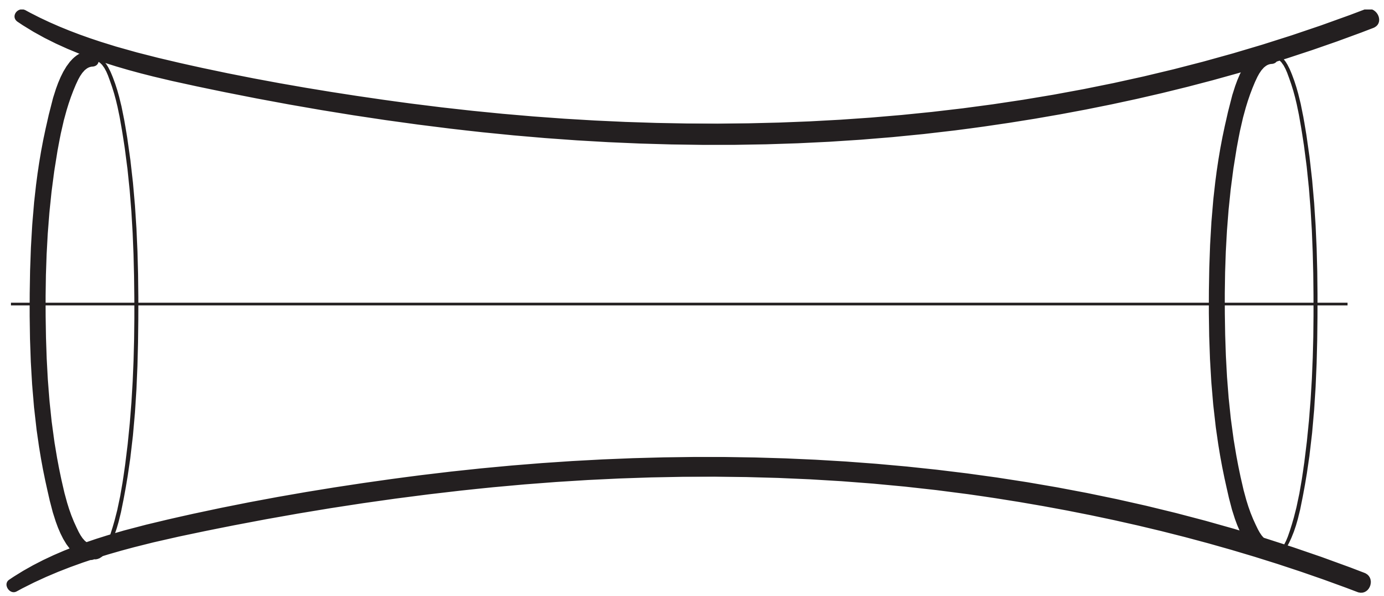


Comparisons

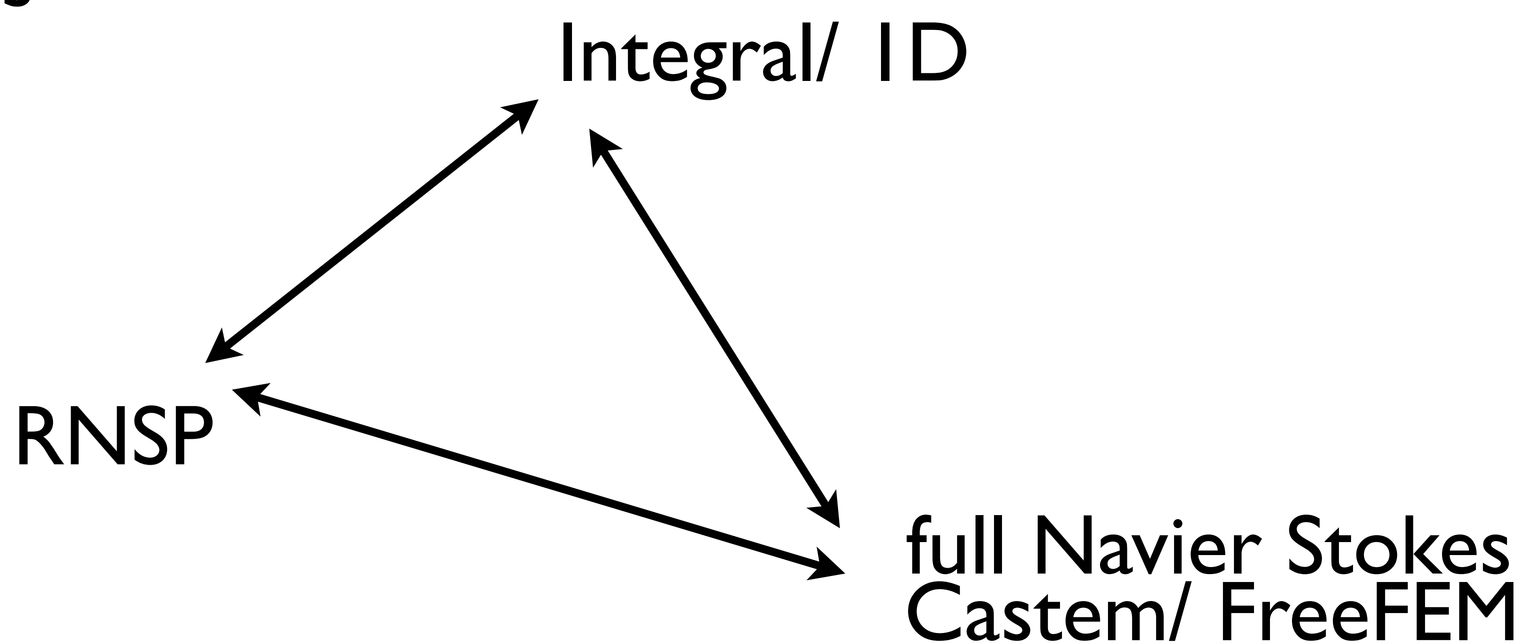
Integral/ 1D

↕

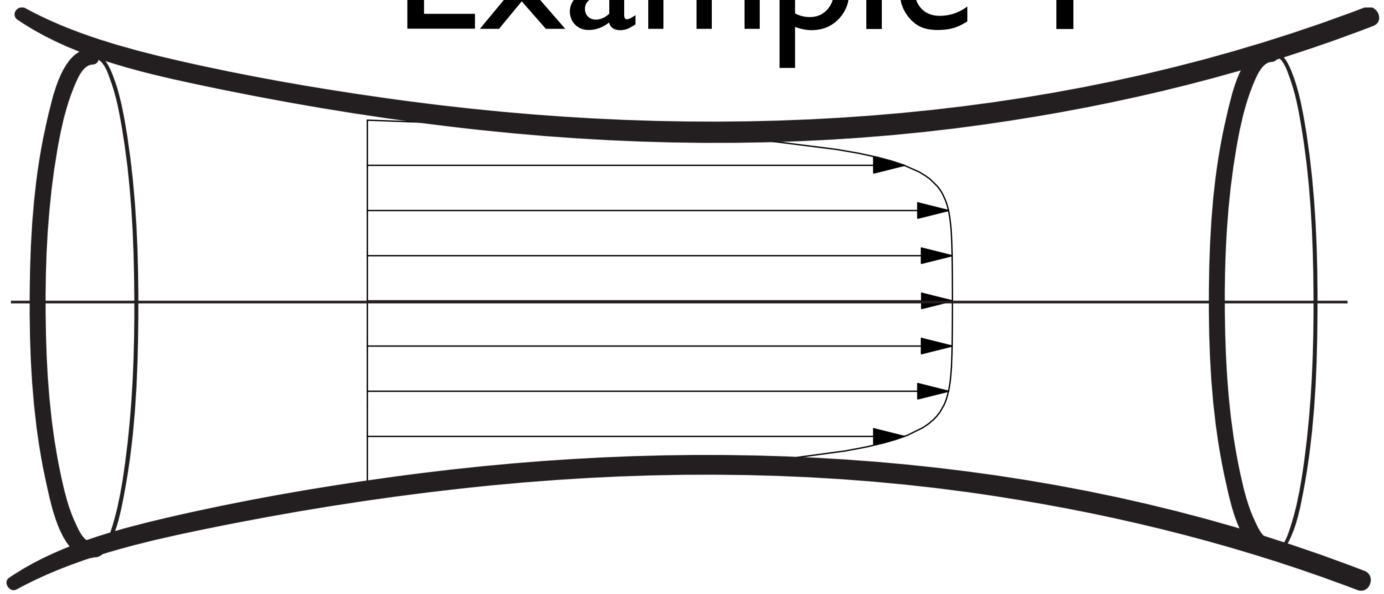
full Navier Stokes
Castem/ FreeFEM



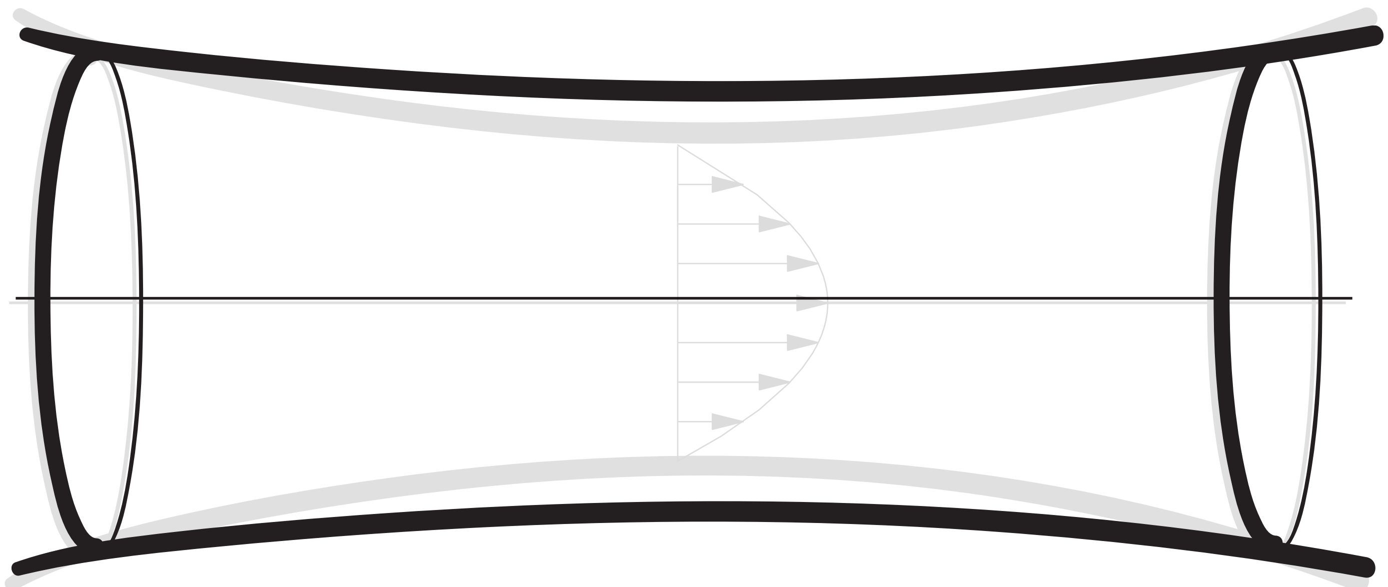
Comparisons

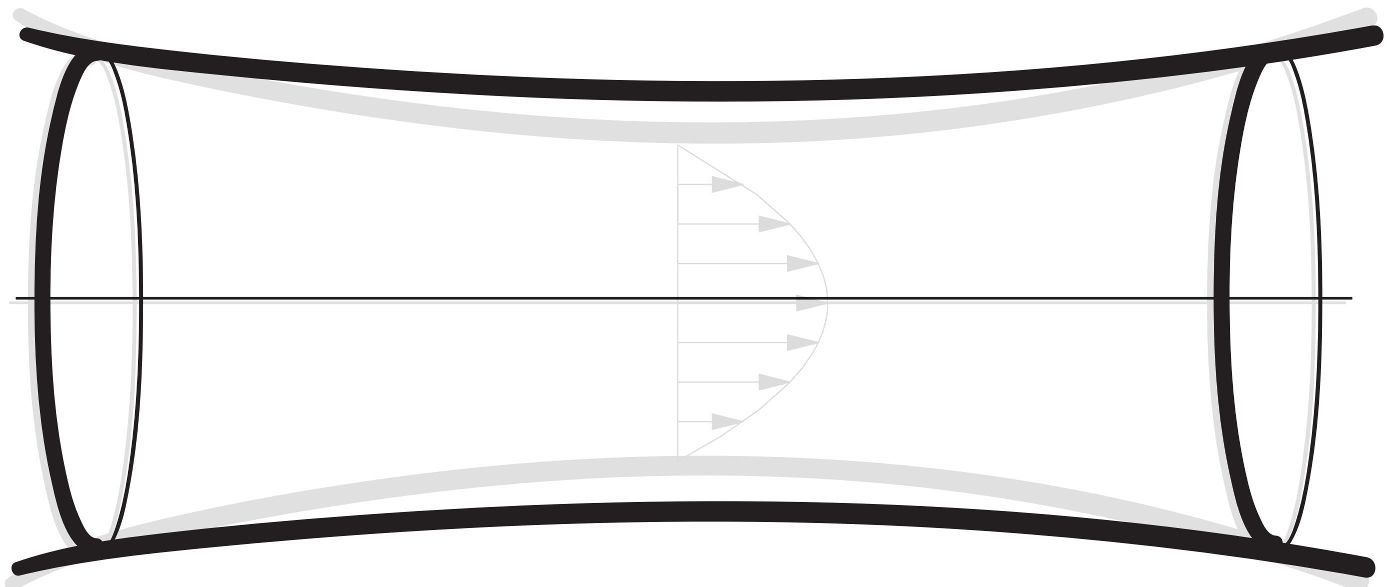


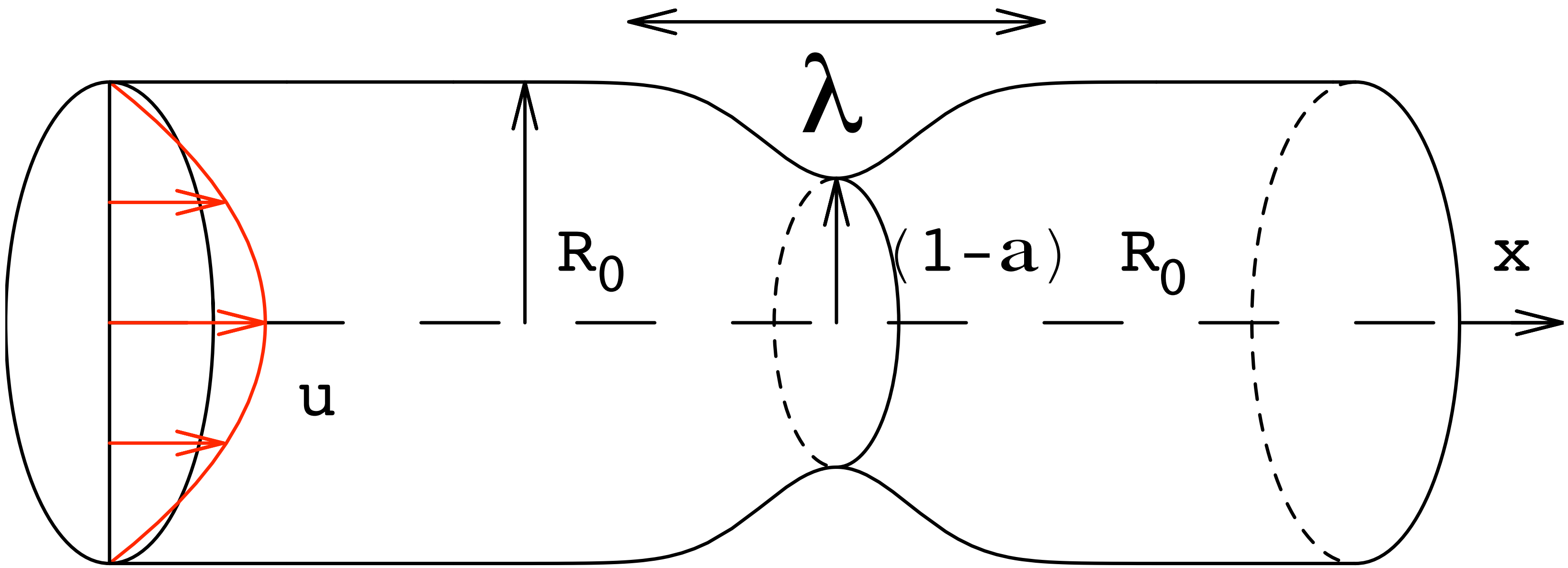
Example 1



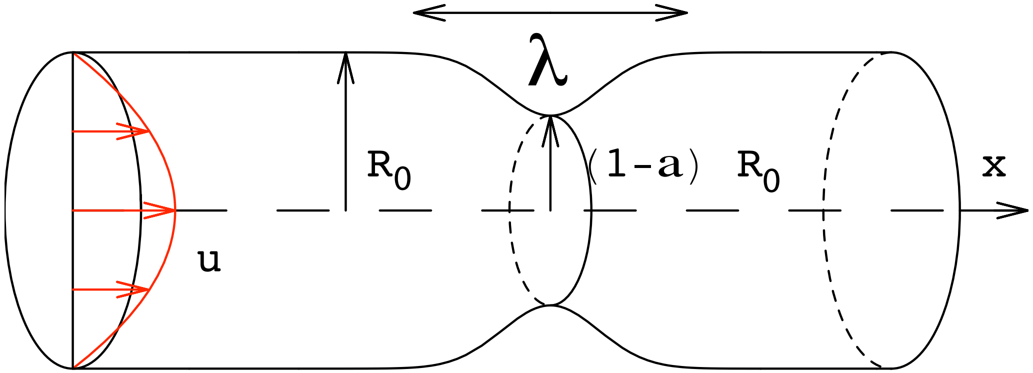
- Flow in a stenozed vessel
- steady, rigid wall






λ
R_0
(1-a) R_0
x
u

RNSP Scales

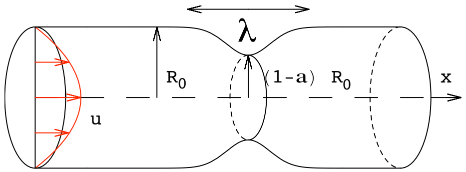


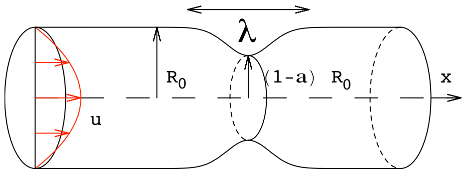
Using:

$x^* = xR_0Re$, $r^* = rR_0$, $u^* = U_0 u$, $v^* = \frac{U_0}{Re}v$,
$p^* = p_0^* + \rho_0 U_0^2 p$ and $\tau^* = \frac{\rho U_0^2}{Re}\tau$

the following partial differential system is obtained from Navier Stokes as $Re \to \infty$:

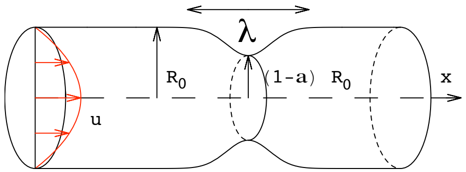
RNSP: Reduced Navier Stokes/ Prandtl System





RNSP: Reduced Navier Stokes/ Prandtl System

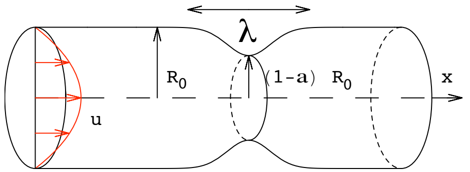
$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0, \\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), \\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$



RNSP: Reduced Navier Stokes/ Prandtl System

$$\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0, \\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), \\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}$$

+ The boundary conditions.



RNSP: Reduced Navier Stokes/ Prandtl System

$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0, \\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), \\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$

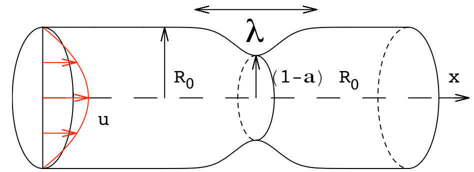
- axial symmetry ($\partial_r u = 0$ and $v = 0$ at $r = 0$),

- no slip condition at the wall ($u = v = 0$ at $r = 1 - f(x)$),

- the entry velocity profiles ($u(0, r)$ and $v(0, r)$) are given

- *no* output condition in $x_{out} = \frac{x^*_{out}}{R_0 Re}$

- streamwise marching, even when flow separation.



RNSP: Reduced Navier Stokes/ Prandtl System

$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0, \\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), \\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$

Parabolic Problem - Marching Problem

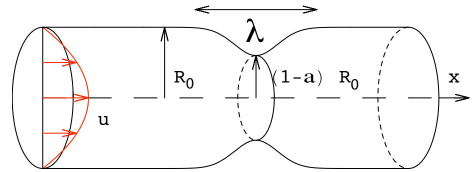
- axial symmetry ($\partial_r u = 0$ and $v = 0$ at $r = 0$),

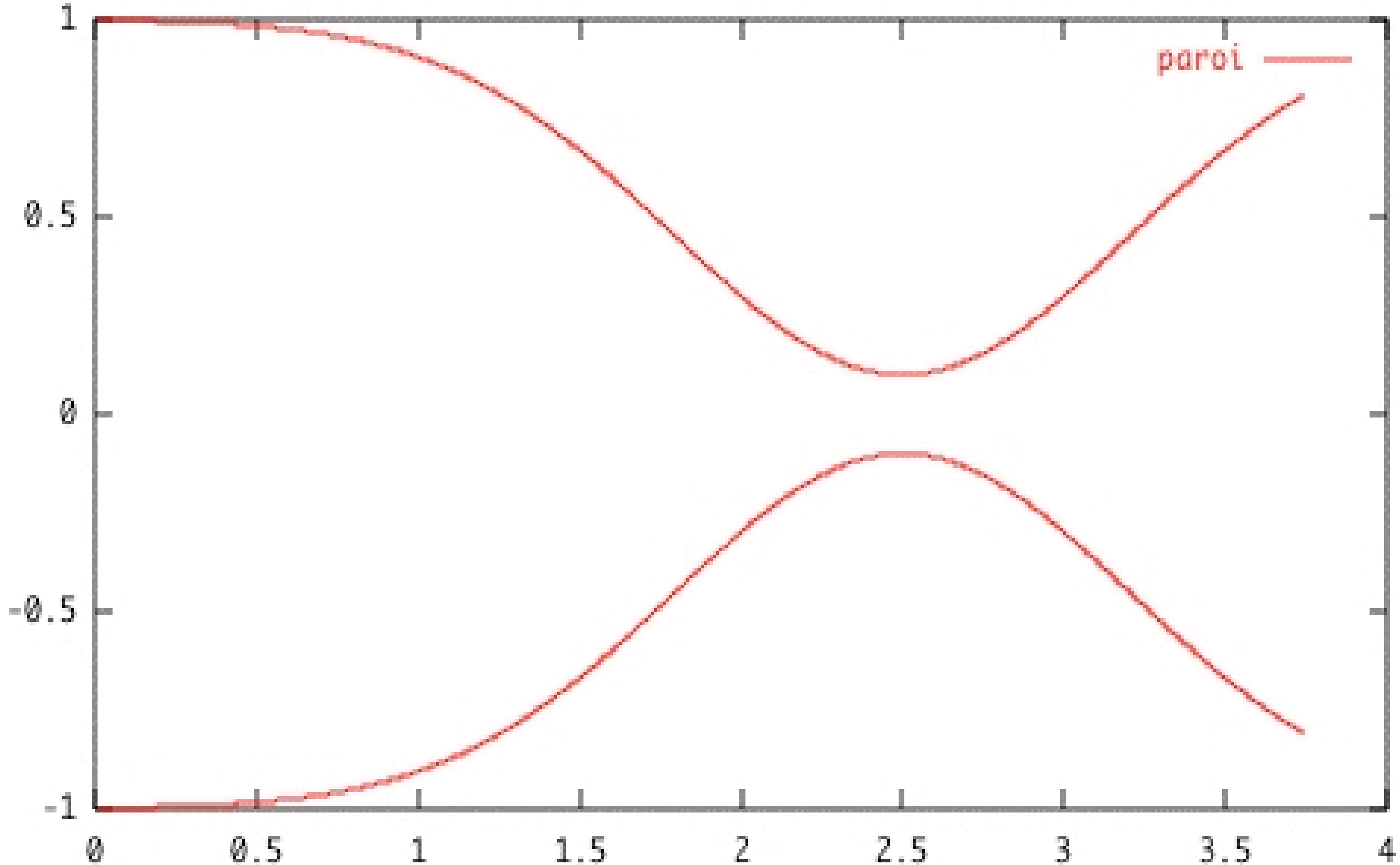
- no slip condition at the wall ($u = v = 0$ at $r = 1 - f(x)$),

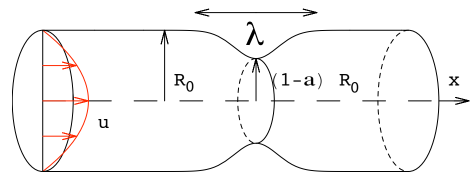
- the entry velocity profiles ($u(0, r)$ and $v(0, r)$) are given

- *no* output condition in $x_{out} = \frac{x^*_{out}}{R_0 Re}$

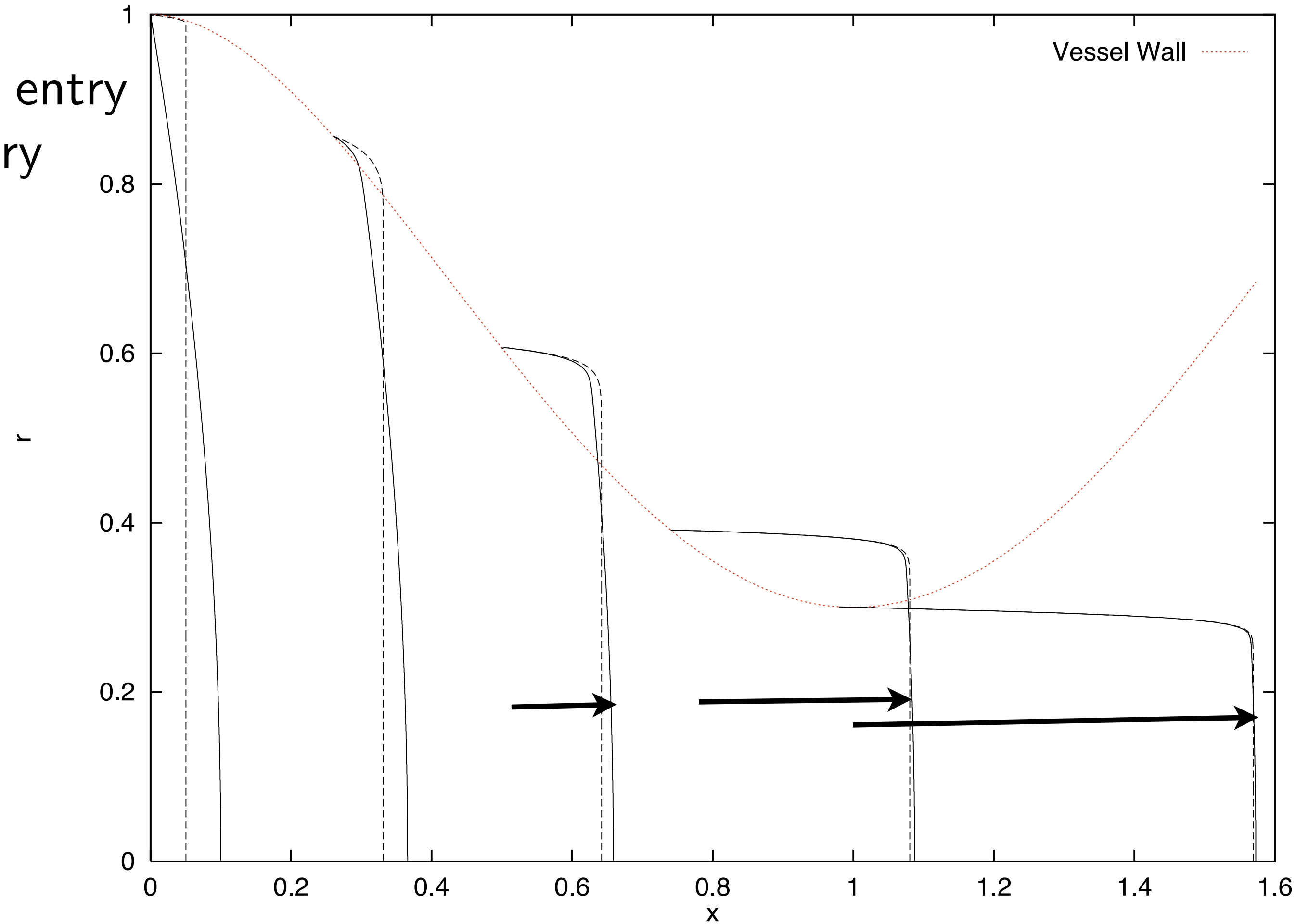
- streamwise marching, even when flow separation.

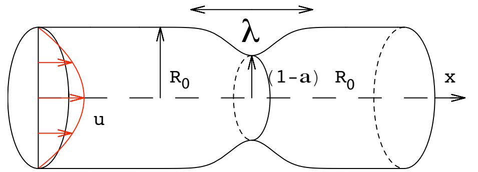

λ
R_0
(1-a) R_0
x
u


paroi
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
3
3.5
4

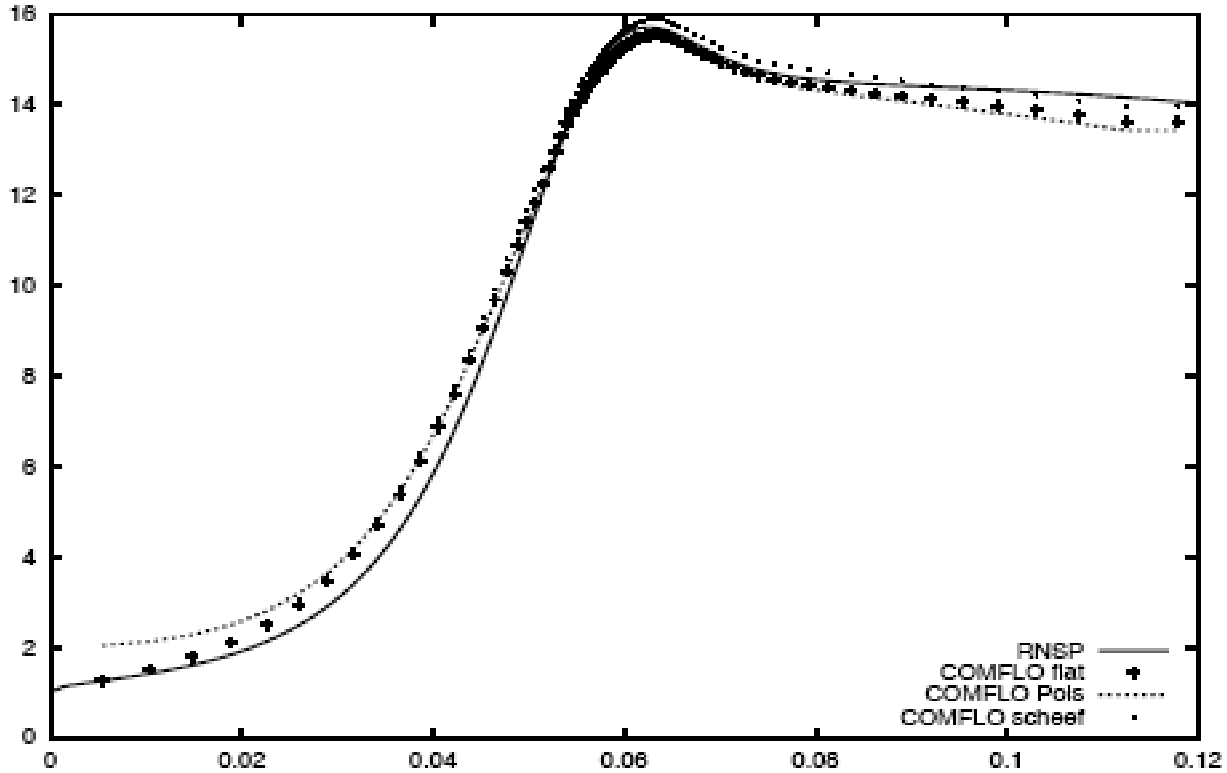


Evolution of the velocity profile along the convergent part of a 70% stenosis $(Re = 500)$;
solid line: Poiseuille entry
broken line: flat entry

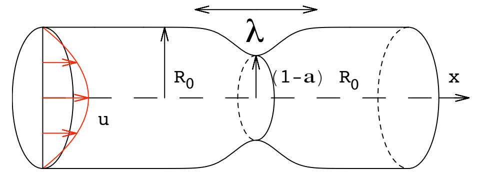




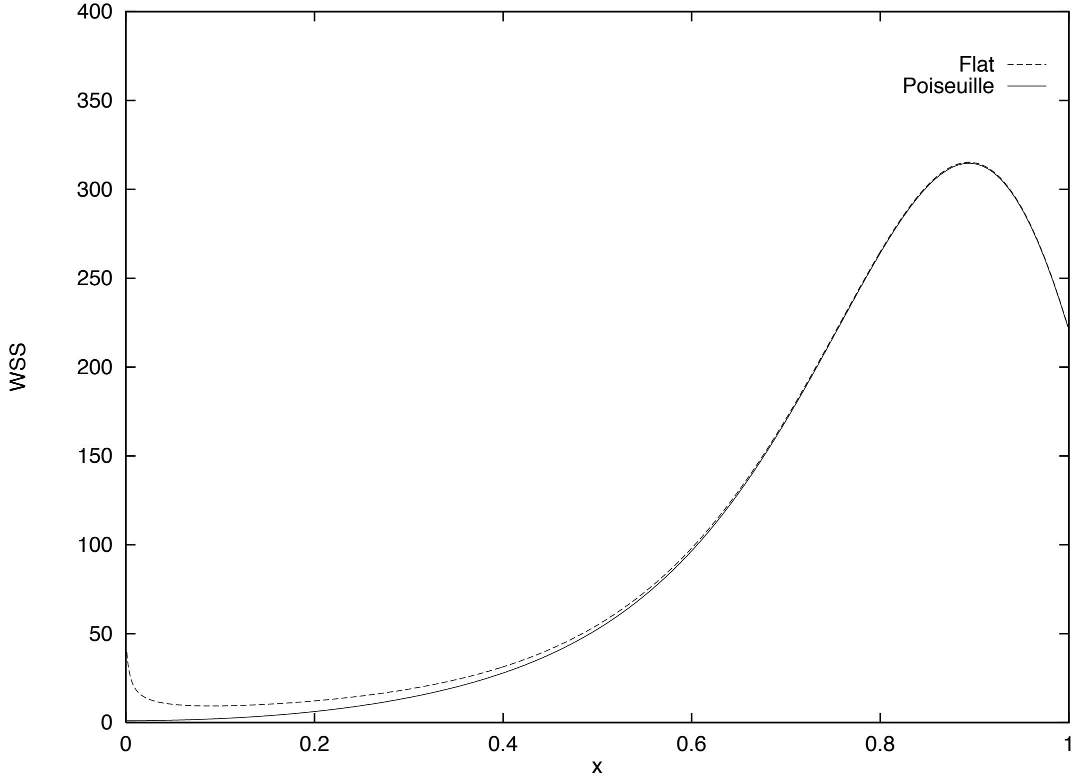
Testing asymmetry in the entry profile



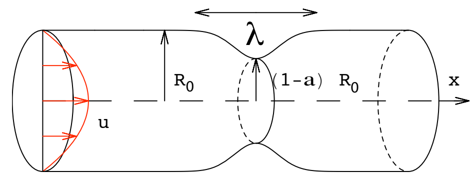
The velocities in the middle for Comflo and RNS.
Comflo uses here 50X50X100 points. Dimensionless scales!

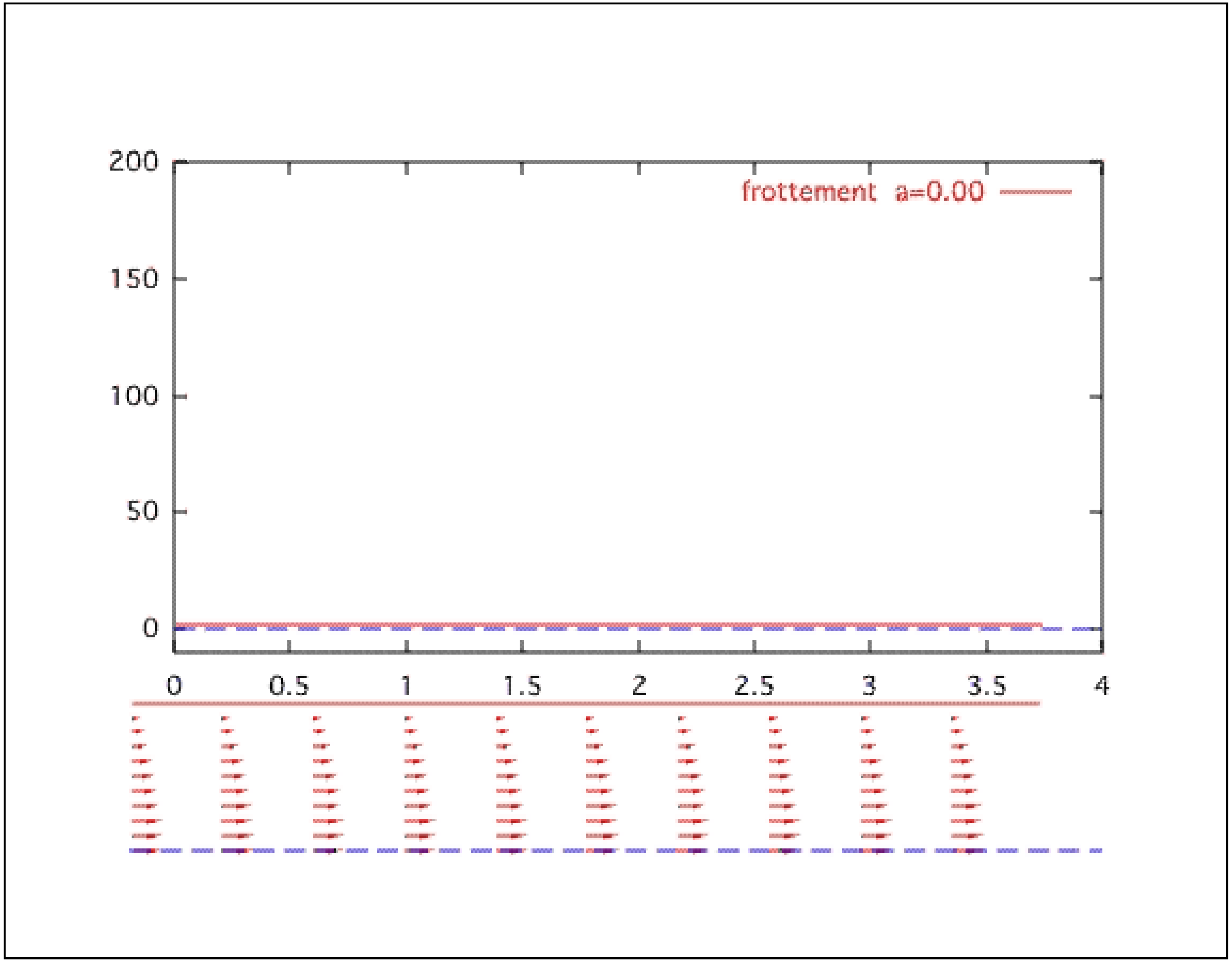


Wall Shear Stress

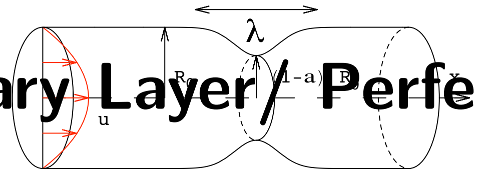


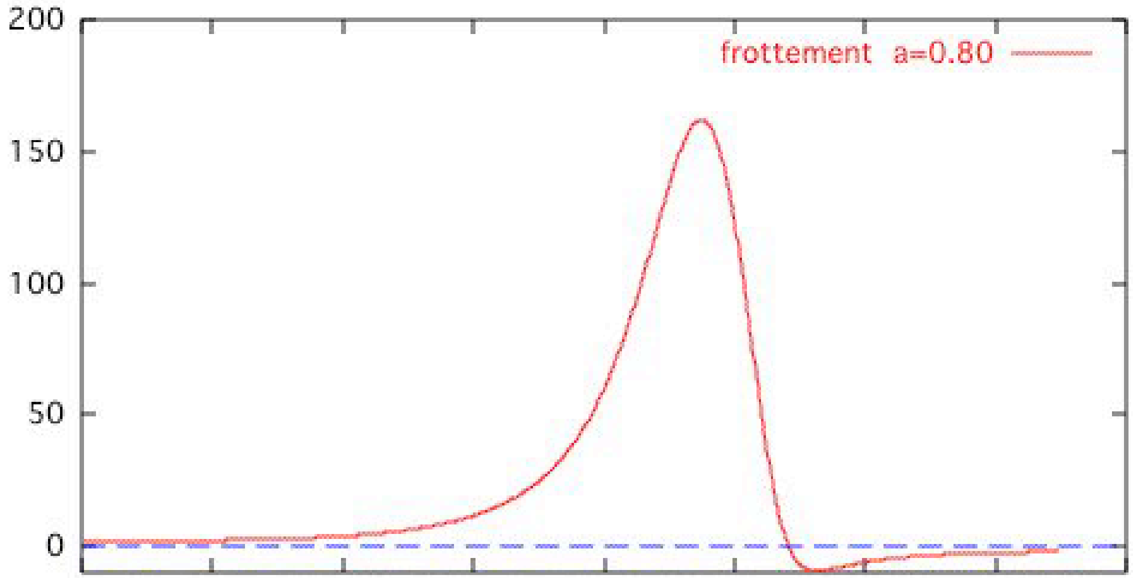
Evolution of the WSS distribution along the convergent part of a 70% stenosis ($Re = 500$) ; solid line: Poiseuille entry profile ; broken line: flat entry profile.

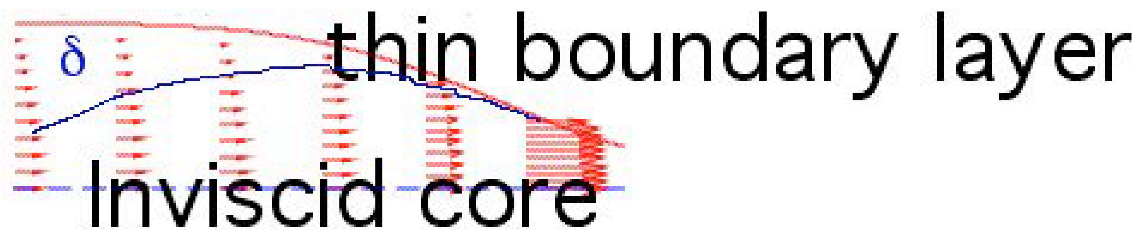

λ
R0
(1-a) R0
x
u

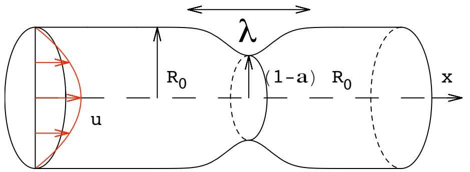

frottement a=0.00
200
150
100
50
0
0
0.5
1
1.5
2
2.5
3
3.5
4

Boundary Layer / Perfect Fluid

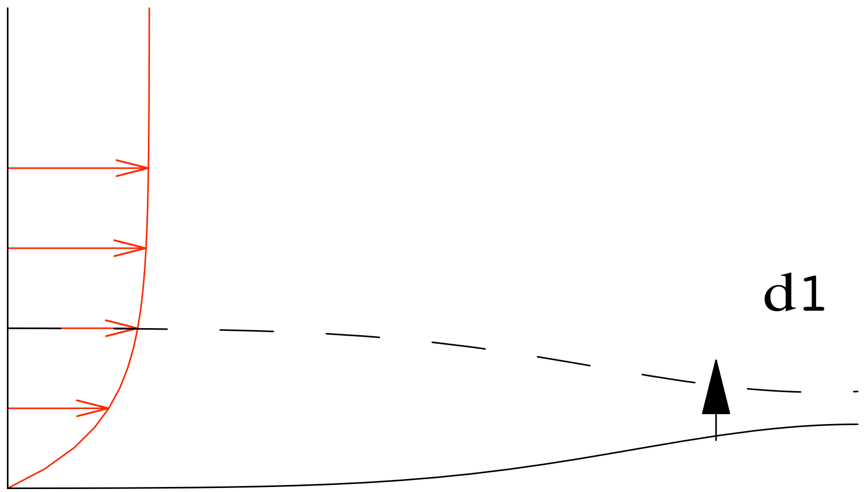




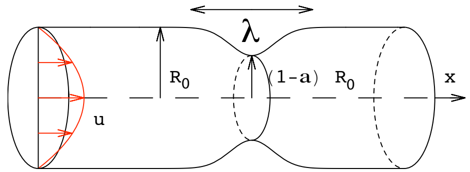




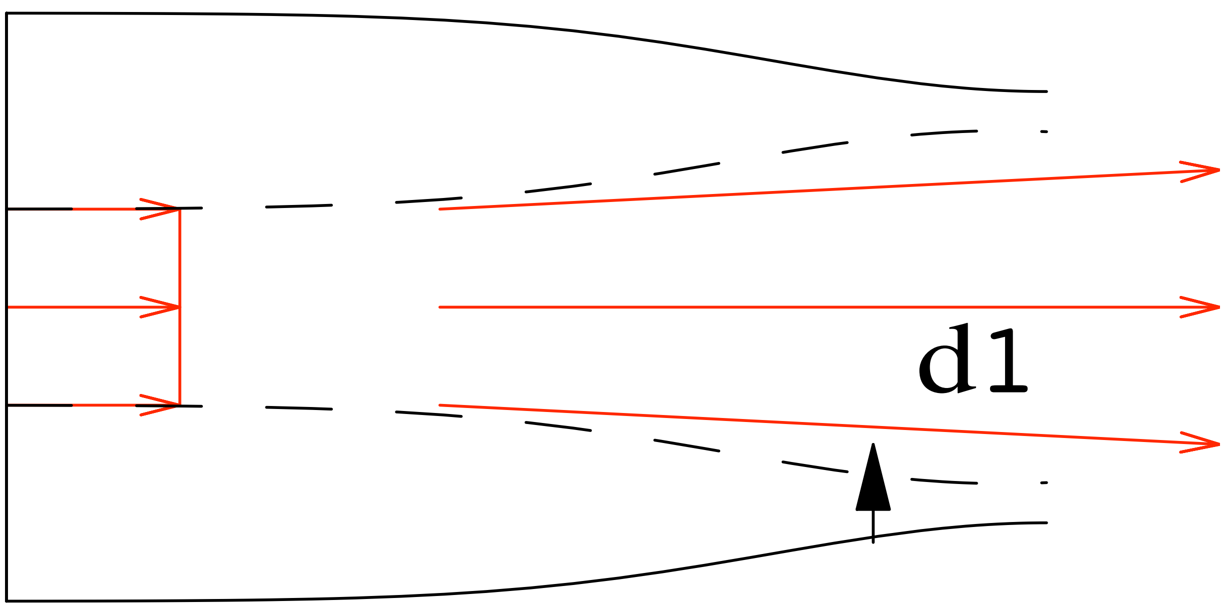
Boundary Layer/ Perfect Fluid



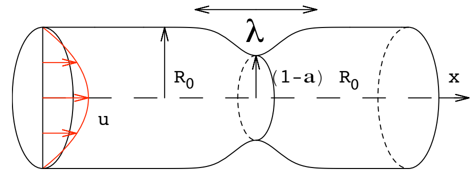
The boundary layer is generated near the wall
δ_1 is the displacement thickness.



Boundary Layer/ Perfect Fluid



The displacement thickness acts as a "new" wall!
→Interacting Boundary Layer (IBL)



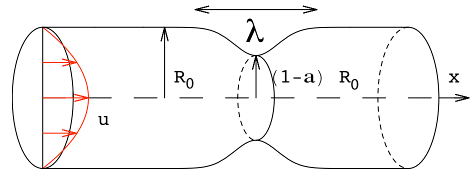
RNSP/ IBL

After rescalling:
$r = R(\bar{x}) - (\lambda/Re)^{-1/2}\bar{y}$, $u = \bar{u}$, $v = (\lambda/Re)^{1/2}\bar{v}$ and $x - x_b = (\lambda/Re)\bar{x}$, $p = \bar{p}$, where x_b is the position of the bump, the RNSP(x) set gives the final IBL (interacting Boundary Layer) problem as follows:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{n}} = 0$$
$$(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{n}}) = \bar{u}_e\frac{d\bar{u}_e}{d\bar{s}} + \frac{\partial}{\partial \bar{n}}\frac{\partial \bar{u}}{\partial \bar{n}}$$

with: $\bar{u}(\bar{x},0) = 0$, $\bar{v}(\bar{x},0) = 0$ $\bar{u}(\bar{x},\infty) = u_e$, where $\bar{\delta}_1 = \int_0^\infty (1 - \frac{\bar{u}}{\bar{u}_e})d\bar{n}$, and

$$\bar{u}_e = \frac{1}{(R^2 - 2((\lambda/Re)^{-1/2})\bar{\delta}_1)}.$$



IBL integral: 1D equation

$$\frac{d}{d\bar{x}}(\frac{\bar{\delta}_1}{H}) = \bar{\delta}_1(1+\frac{2}{H})\frac{d\bar{u}_e}{d\bar{x}} + \frac{f_2 H}{\bar{\delta}_1 \bar{u}_e},$$

$$\bar{u}_e = \frac{1}{(R^2 - 2(\lambda/Re)^{-1/2}\bar{\delta}_1)}.$$

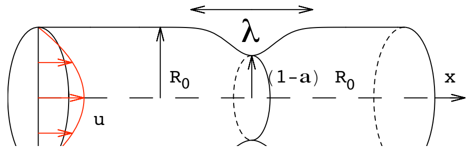
To solve this system, a closure relationship linking H and f_2 to the velocity and the displacement thickness is needed.

Defining $\Lambda_1 = \bar{\delta}_1^2 \frac{d\bar{u}_e}{d\bar{x}}$,

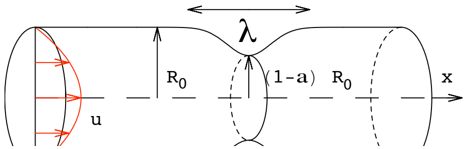
the system is closed from the resolution of the Falkner Skan system as follows:

if $\Lambda_1 < 0.6$ then $H = 2.5905exp(-0.37098\Lambda_1)$, else $H = 2.074$.

From H,f_2 is computed as $f_2 = 1.05(-H^{-1} + 4H^{-2})$.

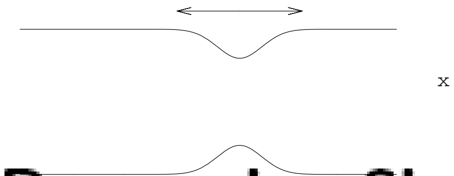


IBL integral: 1D equation Simplified Shear Stress



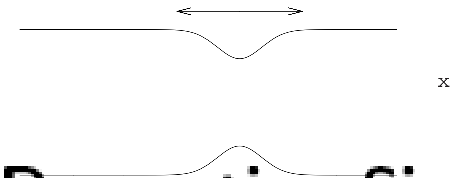
IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation)



IBL integral: 1D equation Simplified Shear Stress

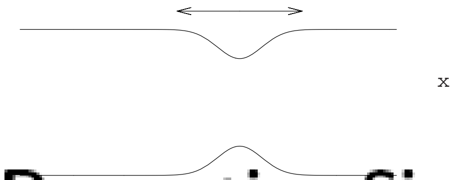
- variation of velocity (flux conservation) $U_0 \to U_0/(1-\alpha-\delta_1)^2$



IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation) $U_0 \to U_0/(1-\alpha-\delta_1)^2$

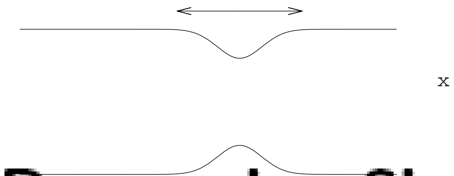
- acceleration: boundary layer $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$,



IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation) $U_0 \to U_0/(1-\alpha-\delta_1)^2$

- acceleration: boundary layer $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$, with $Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2 \nu} = \frac{Re \lambda}{(1-\alpha)^2}$

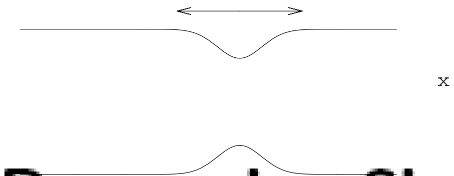


IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation) $U_0 \to U_0/(1-\alpha-\delta_1)^2$

- acceleration: boundary layer $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$, with $Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2\nu} = \frac{Re\lambda}{(1-\alpha)^2}$

- WSS $=$ (variation of velocity)/(boundary layer thickness)

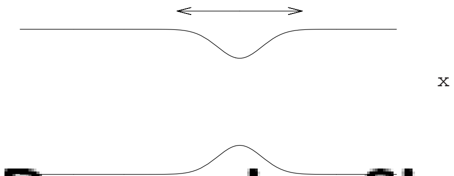


IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation) $U_0 \to U_0/(1-\alpha-\delta_1)^2$

- acceleration: boundary layer $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$, with $Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2\nu} = \frac{Re\lambda}{(1-\alpha)^2}$

- WSS = (variation of velocity)/(boundary layer thickness) $=\frac{(Re/\lambda)^{1/2}}{(1-\alpha)^3}$



IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation) $U_0 \to U_0/(1-\alpha-\delta_1)^2$

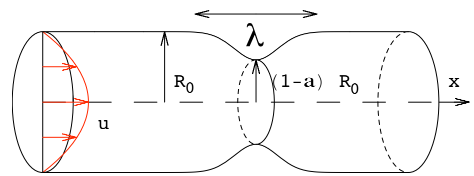
- acceleration: boundary layer $\delta_1 \simeq \frac{\lambda}{\sqrt{Re_\lambda}}$, with $Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2\nu} = \frac{Re\lambda}{(1-\alpha)^2}$

- WSS = (variation of velocity)/(boundary layer thickness) $=\frac{(Re/\lambda)^{1/2}}{(1-\alpha)^3}$

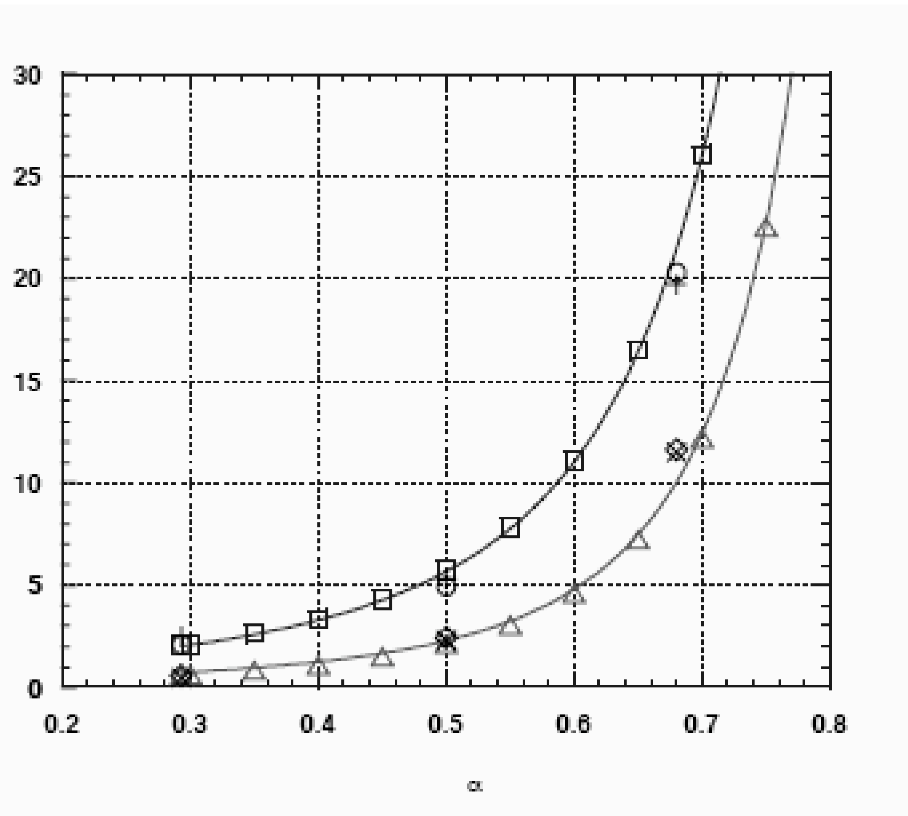
A simple formula as been settled:

$$WSS = (\mu\frac{\partial u^*}{\partial y^*})/((\mu\frac{4U_0}{R})) \sim .22\frac{((Re/\lambda)^{1/2}+3)}{(1-\alpha)^3}$$

Reynolds number is no longer Re but $Re\lambda$ and $(Re/\lambda)^{1/2}$ is the inverse of the relative boundary layer thickness.



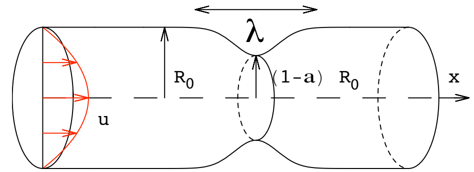
IBL integral: Comparison with Navier Stokes (Siegel et al. 1994)

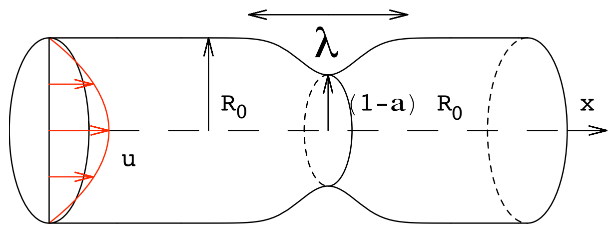


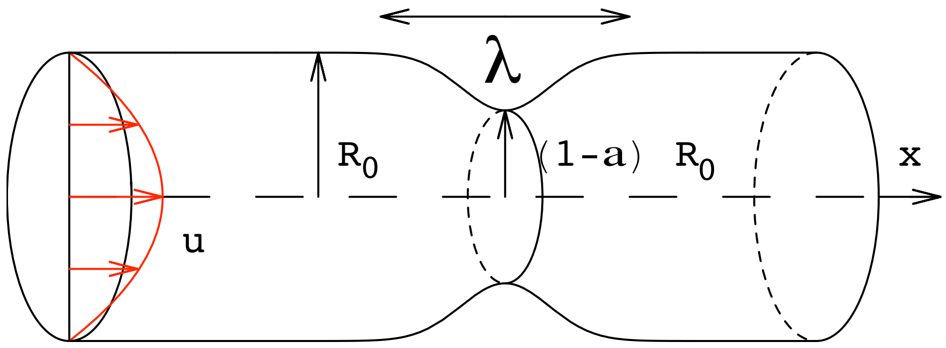
$$WSS = aRe^{1/2} + b$$

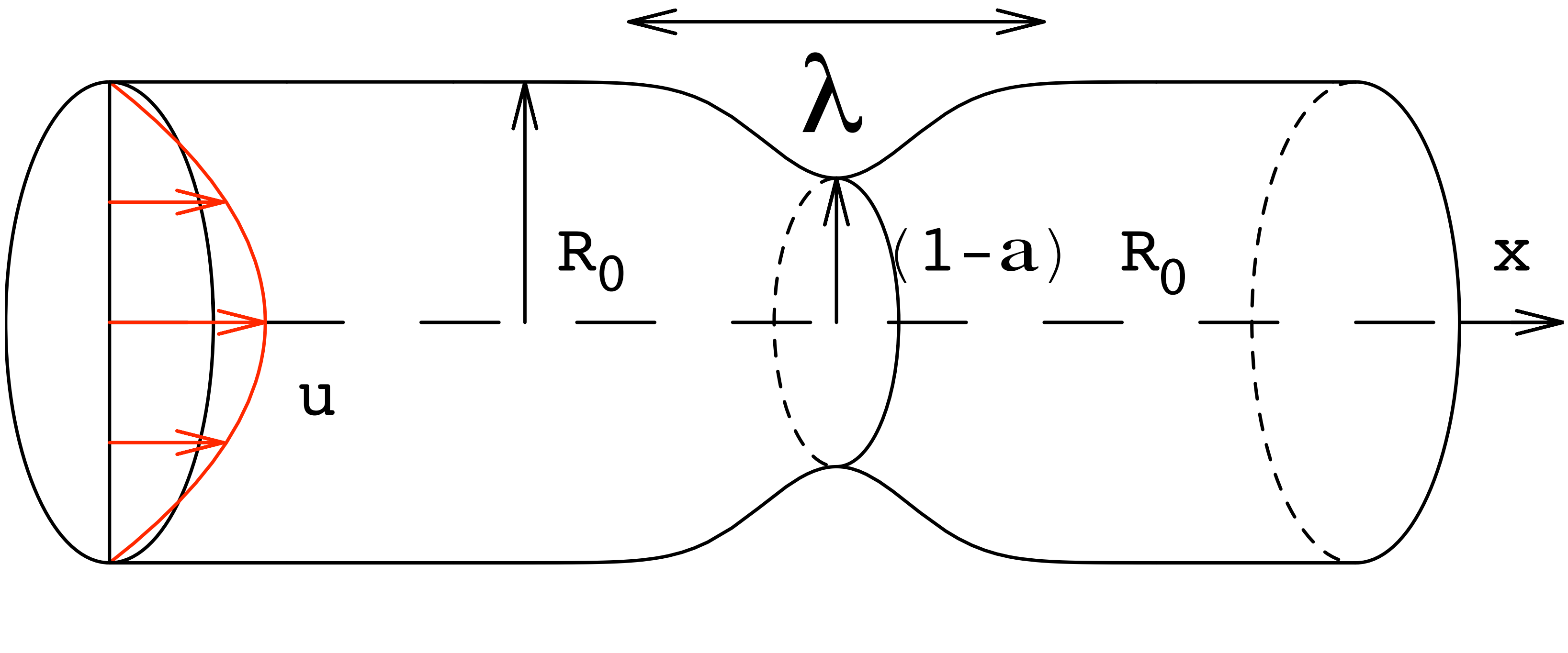
Coefficient a and b for the maximum WSS.
solid lines with △ and "square" : coefficient a and b obtained using the IBL integral method ;

◇ : coefficient a derived from Siegel for $\lambda = 3$;
× : coefficient a derived from Siegel for $\lambda = 6$;
◯ : coefficient b derived from Siegel for $\lambda = 3$;
+ : coefficient b derived from Siegel for $\lambda = 6$.

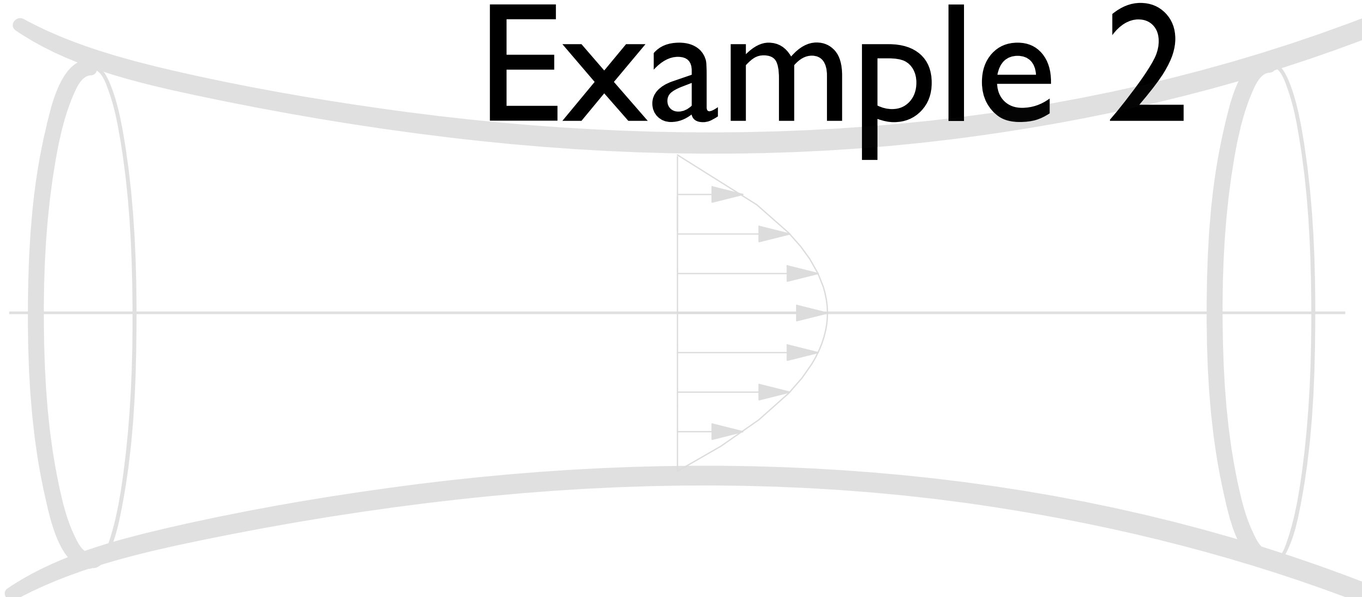

λ
R_0
(1-a) R_0
x
u


λ
R_0
(1-a) R_0
x
u

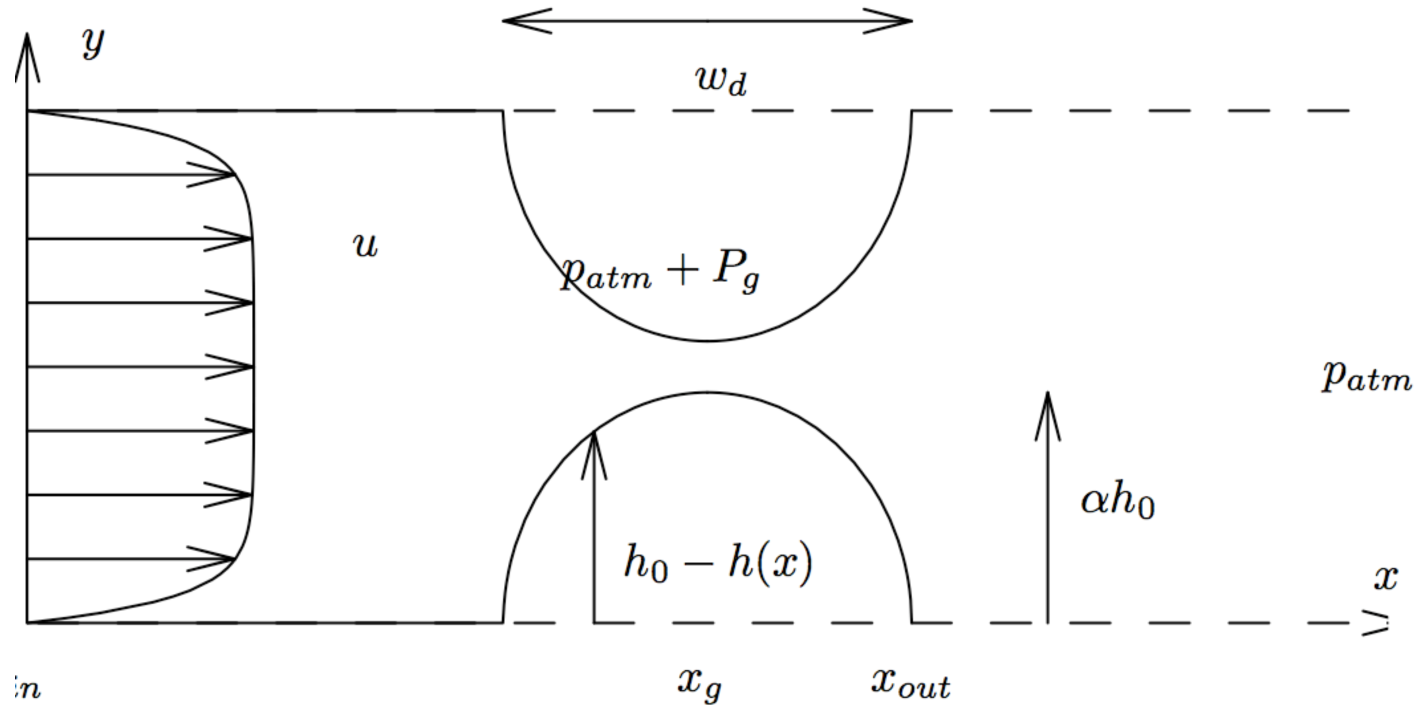

λ
R0
(1-a) R0
x
u


λ
u
R0
(1-a) R0
x

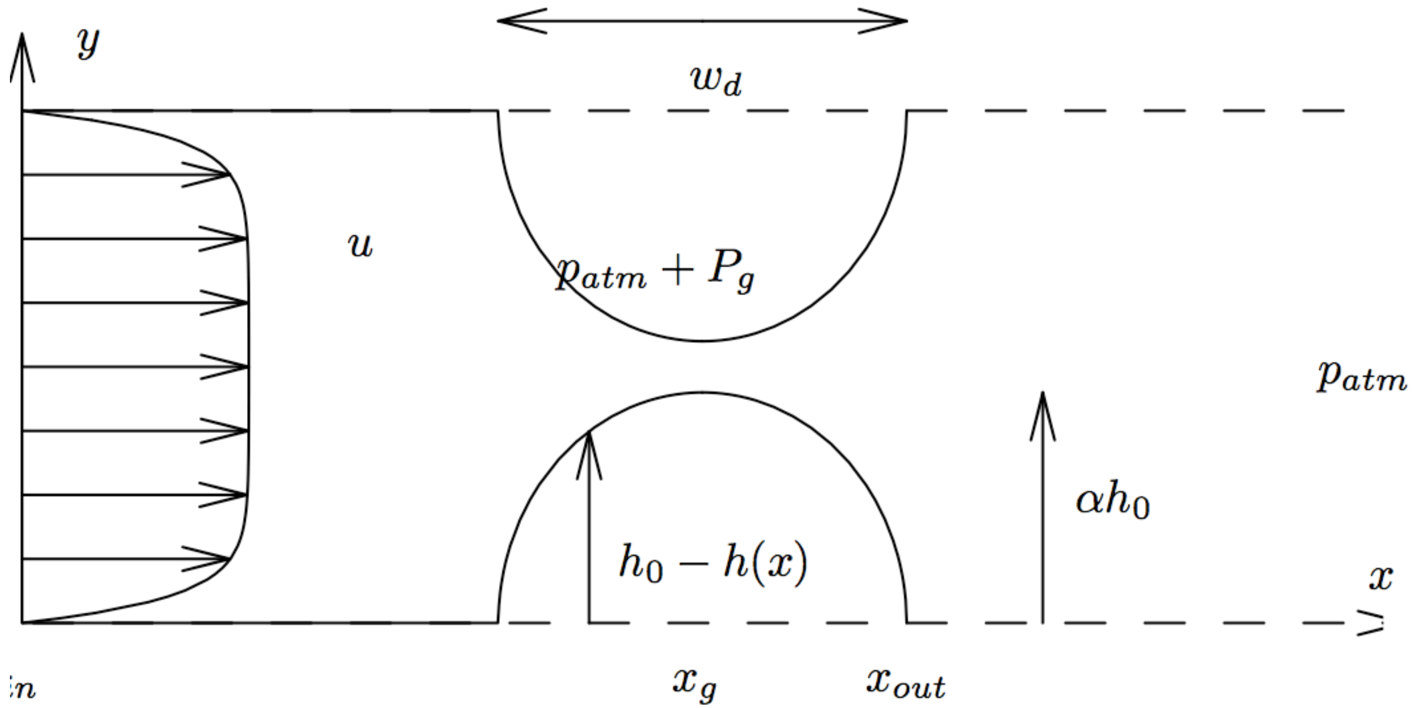
Example 2



- Flow in a 2D stenozed vessel
- steady, rigid wall



- Flow in a stenozed vessel
- steady, rigid wall

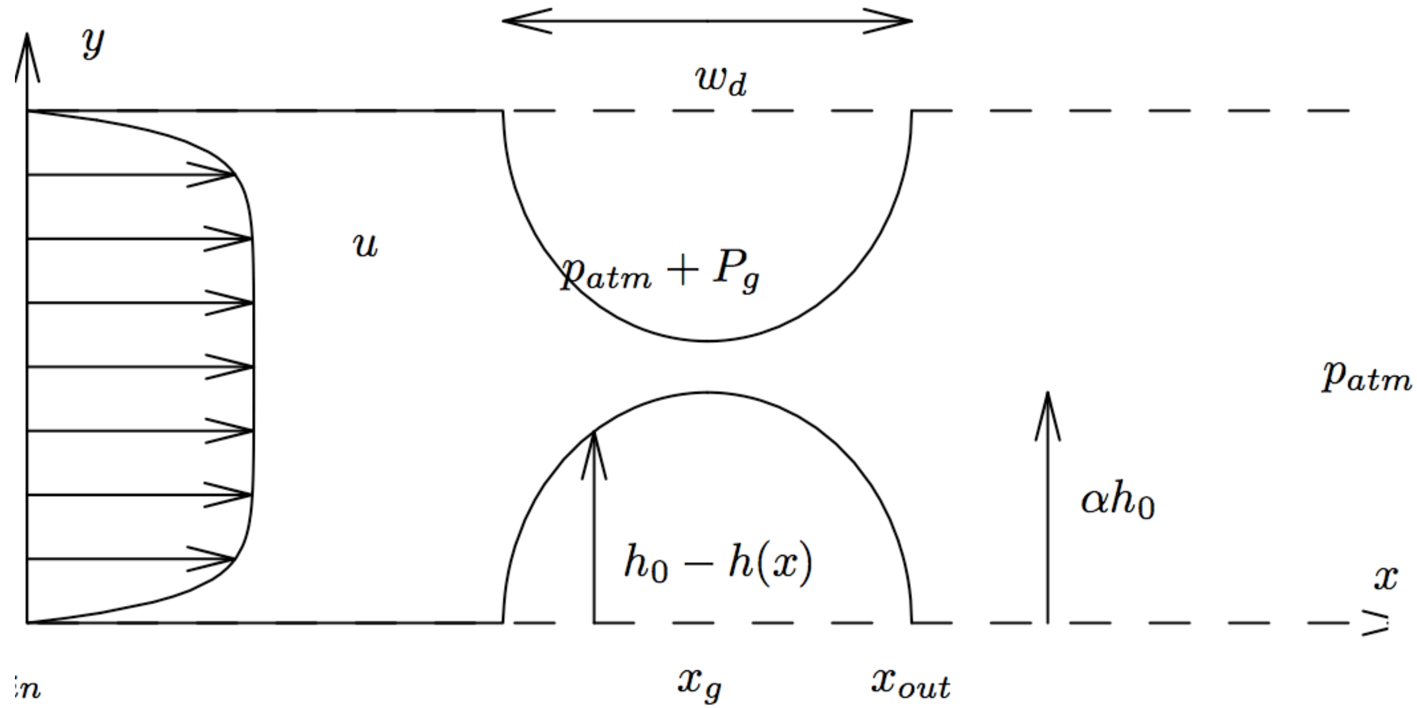


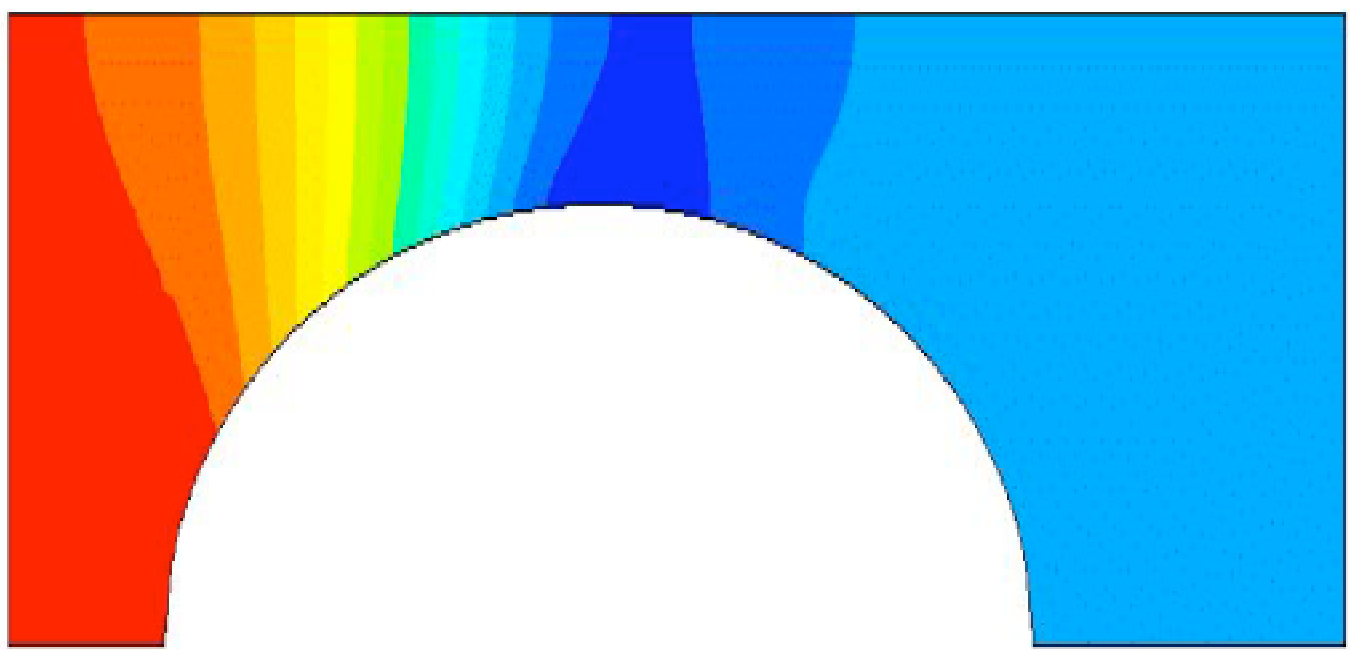
$$\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v=0$$

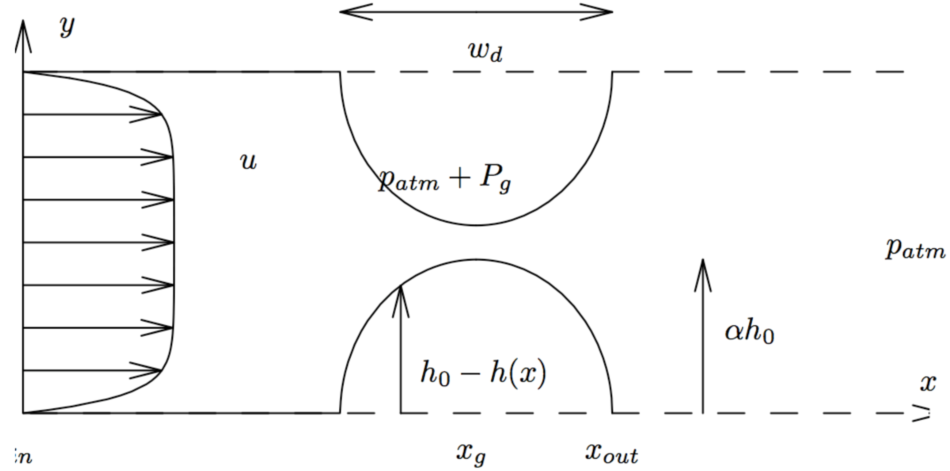
$$u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial y}u=-\frac{\partial}{\partial x}p+\frac{\partial^2}{\partial y^2}u$$

$$0=-\frac{\partial}{\partial y}p$$

RNSP non dimensional


y
w_d
u
$p_{atm} + P_g$
p_{atm}
αh_0
$h_0 - h(x)$
x
x_g
x_{out}





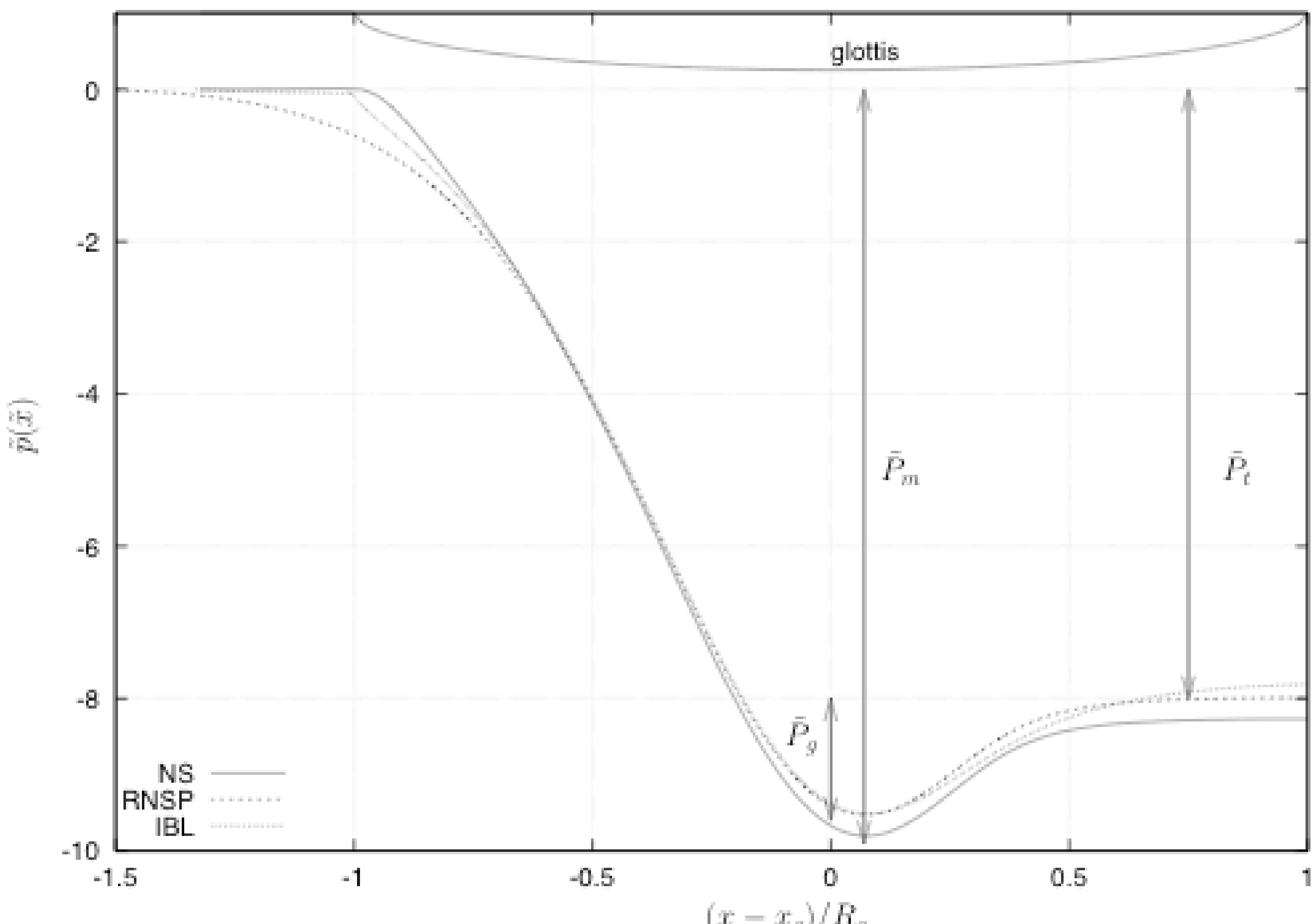
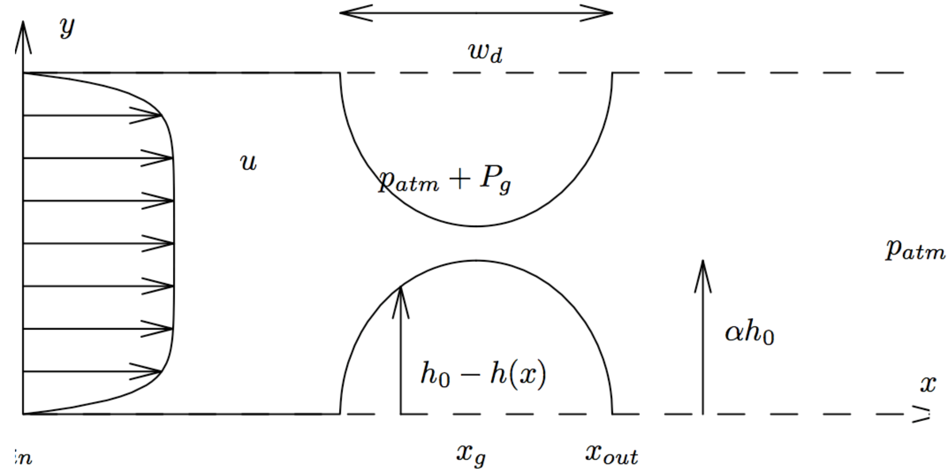


Fig. 2 A comparison between computed non-dimensional pressure for the three models (NS, IBL and RNSP, in this last case the wall has



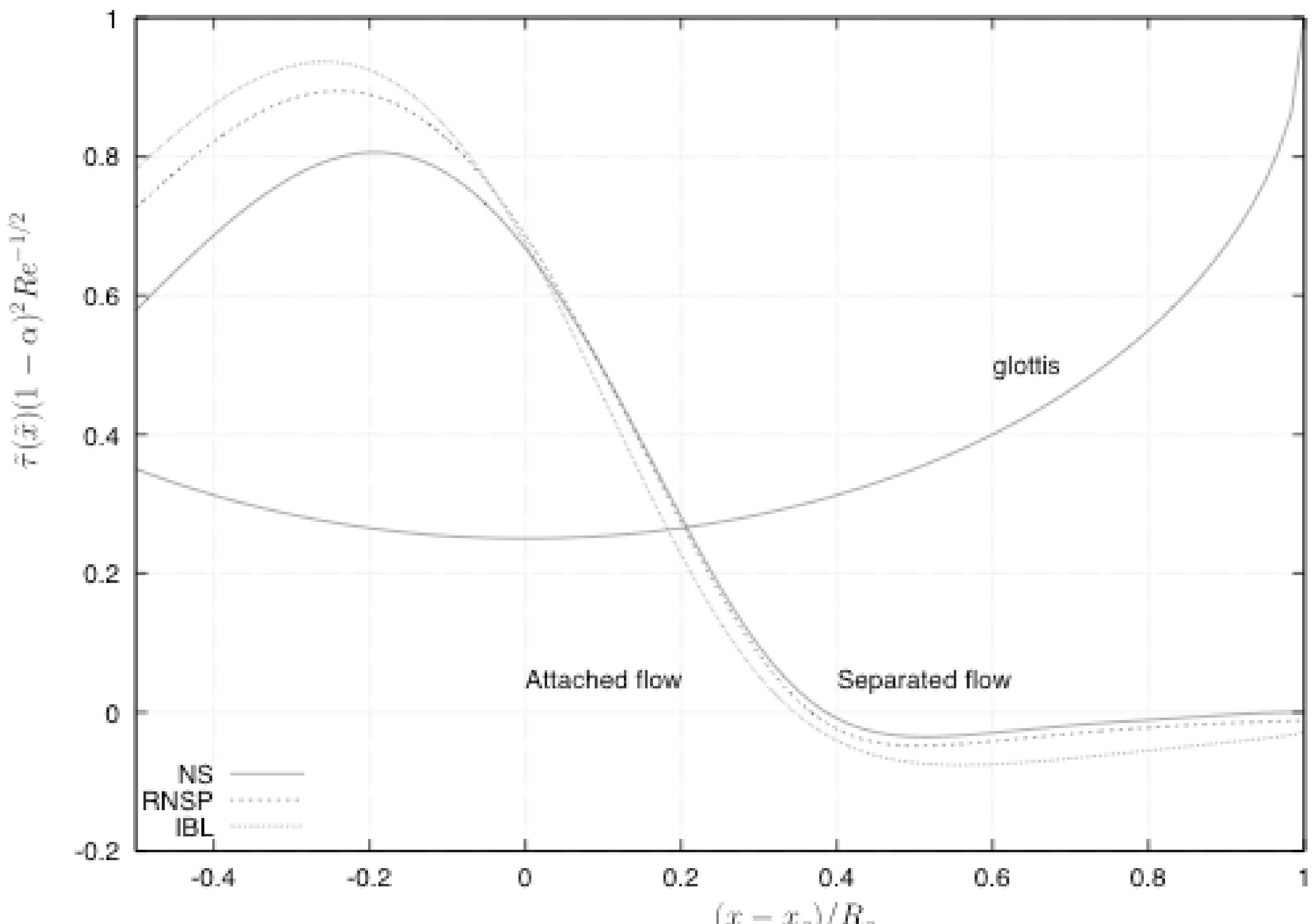
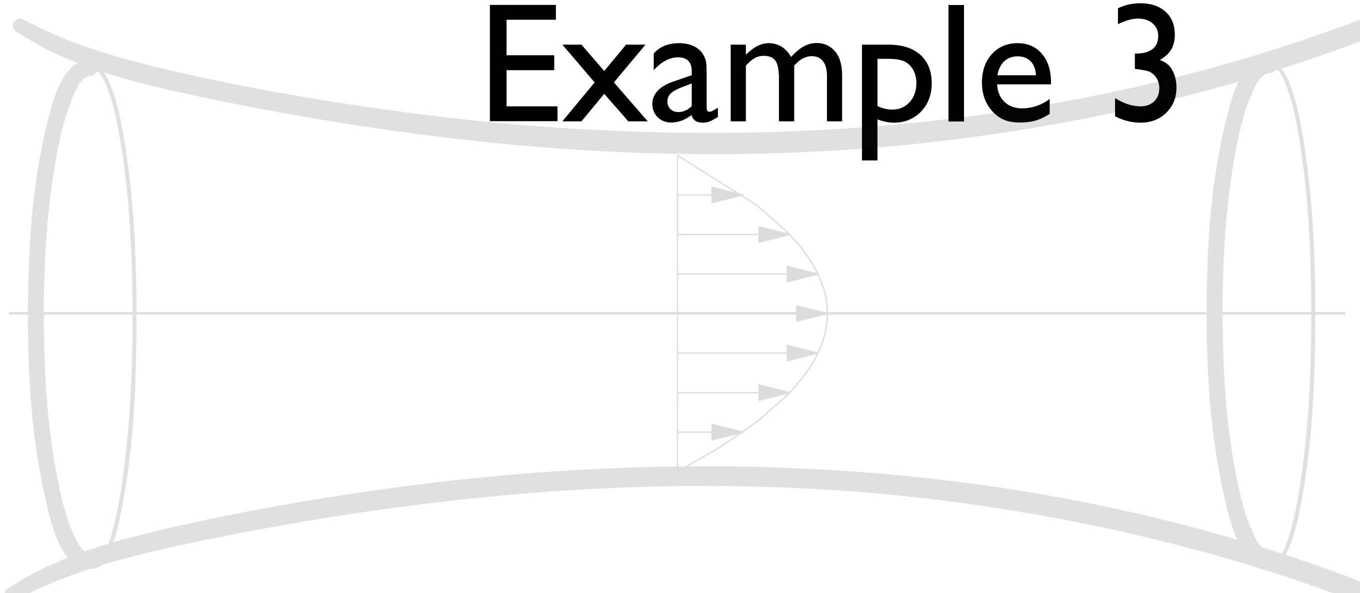


Fig. 4. A comparison between computed skin friction divided by $(0.47 + 2.07)(1-\alpha)^{-1}/\tilde{\delta}_{...} \simeq (1-\alpha)^{-2} Re^{1/2}$ for the three models

Example 3



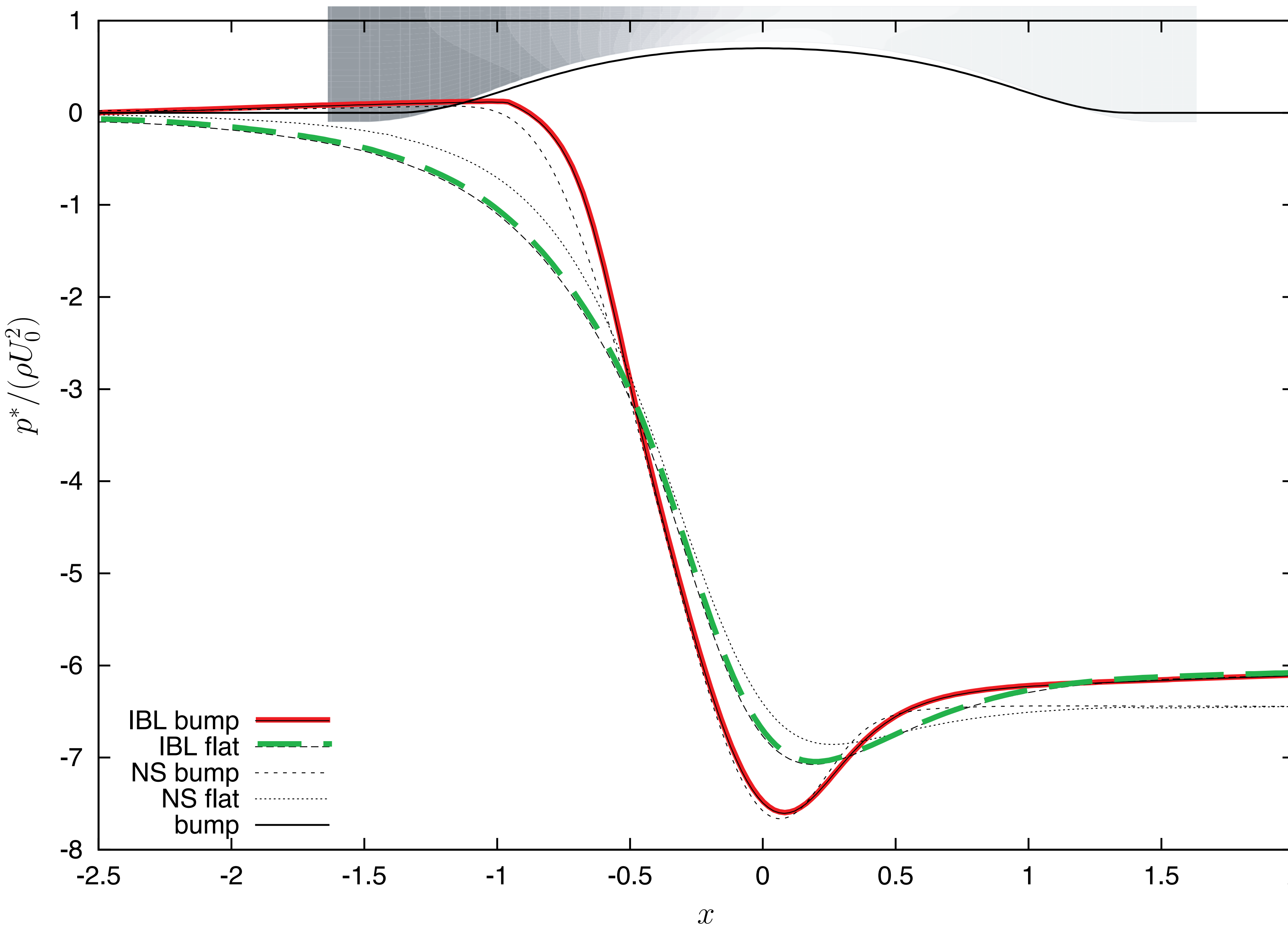
- Flow in a stenozed vessel
- steady, rigid wall
- non symetrical case

non symmetrical case

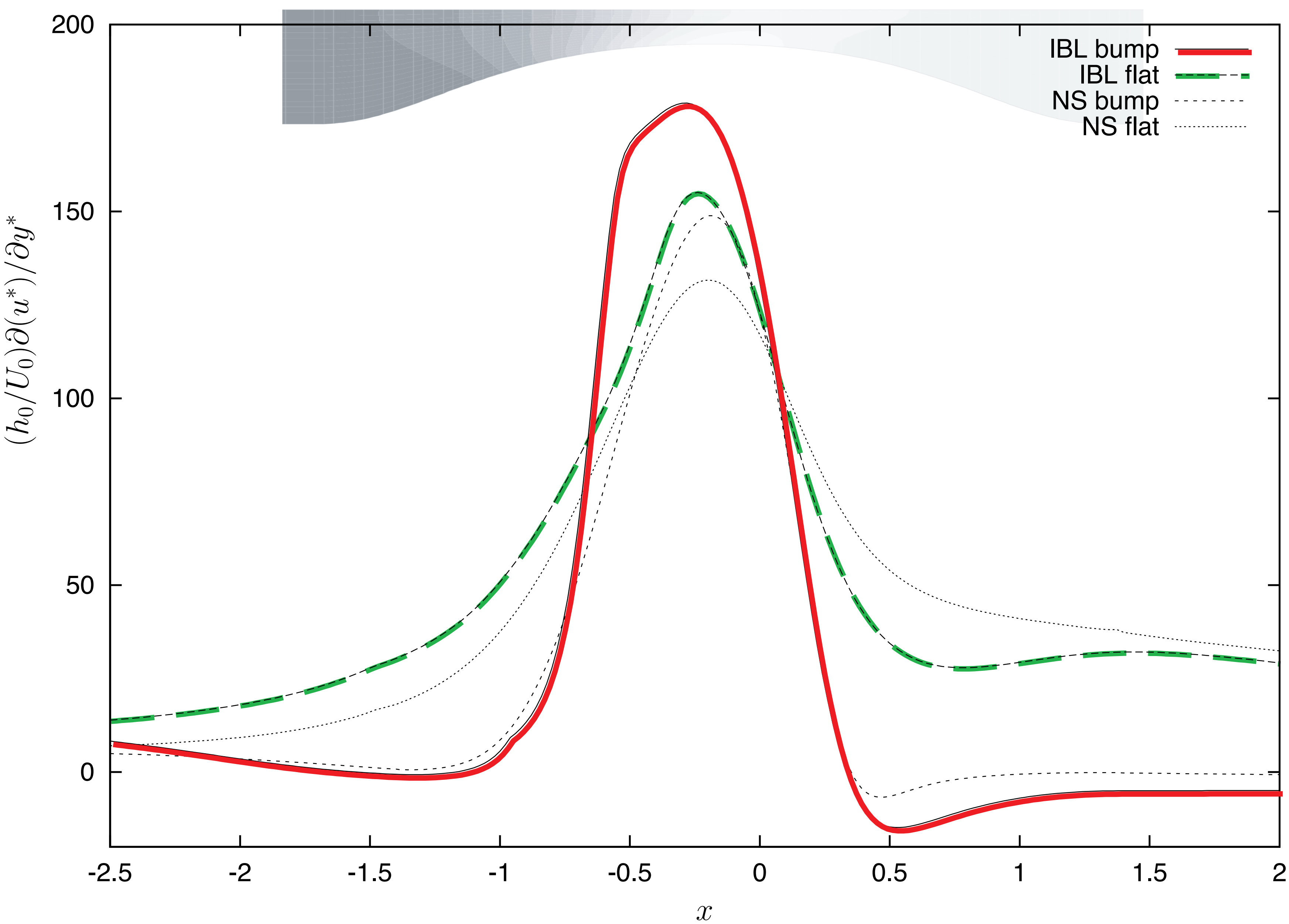
- RNSP
- modified integral method to take into account the transverse variation of pressure
- NS

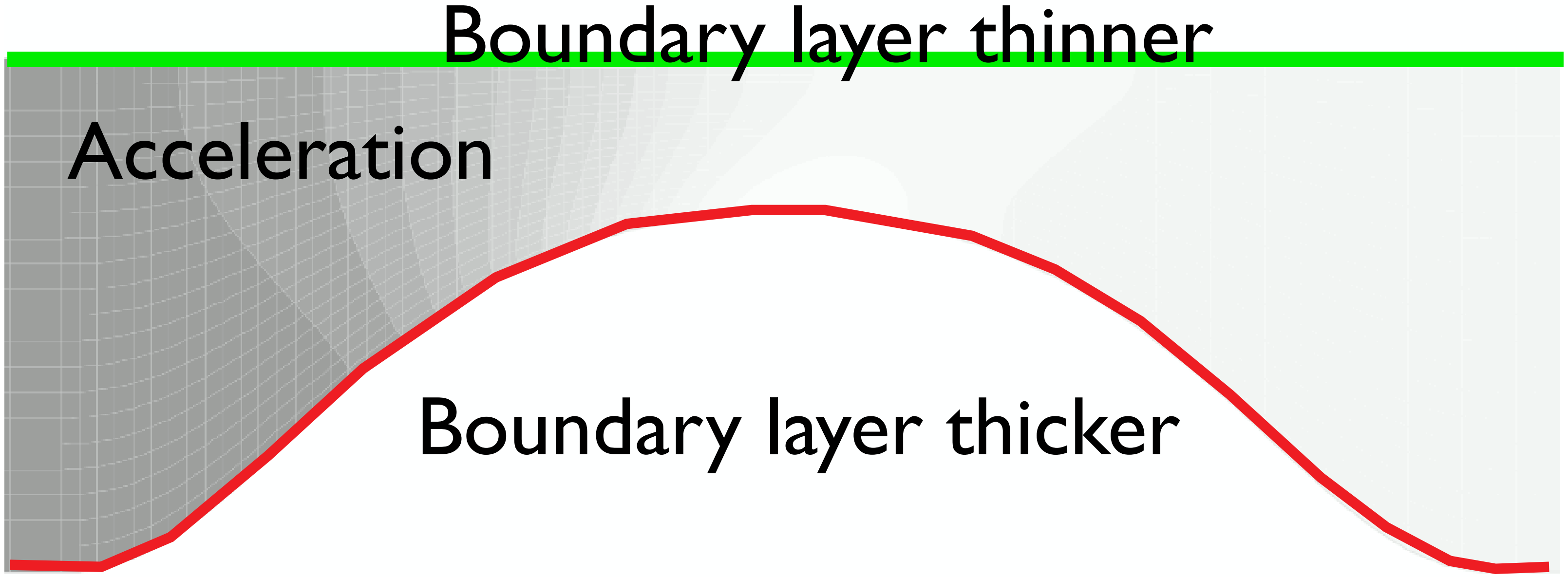
non symmetrical case

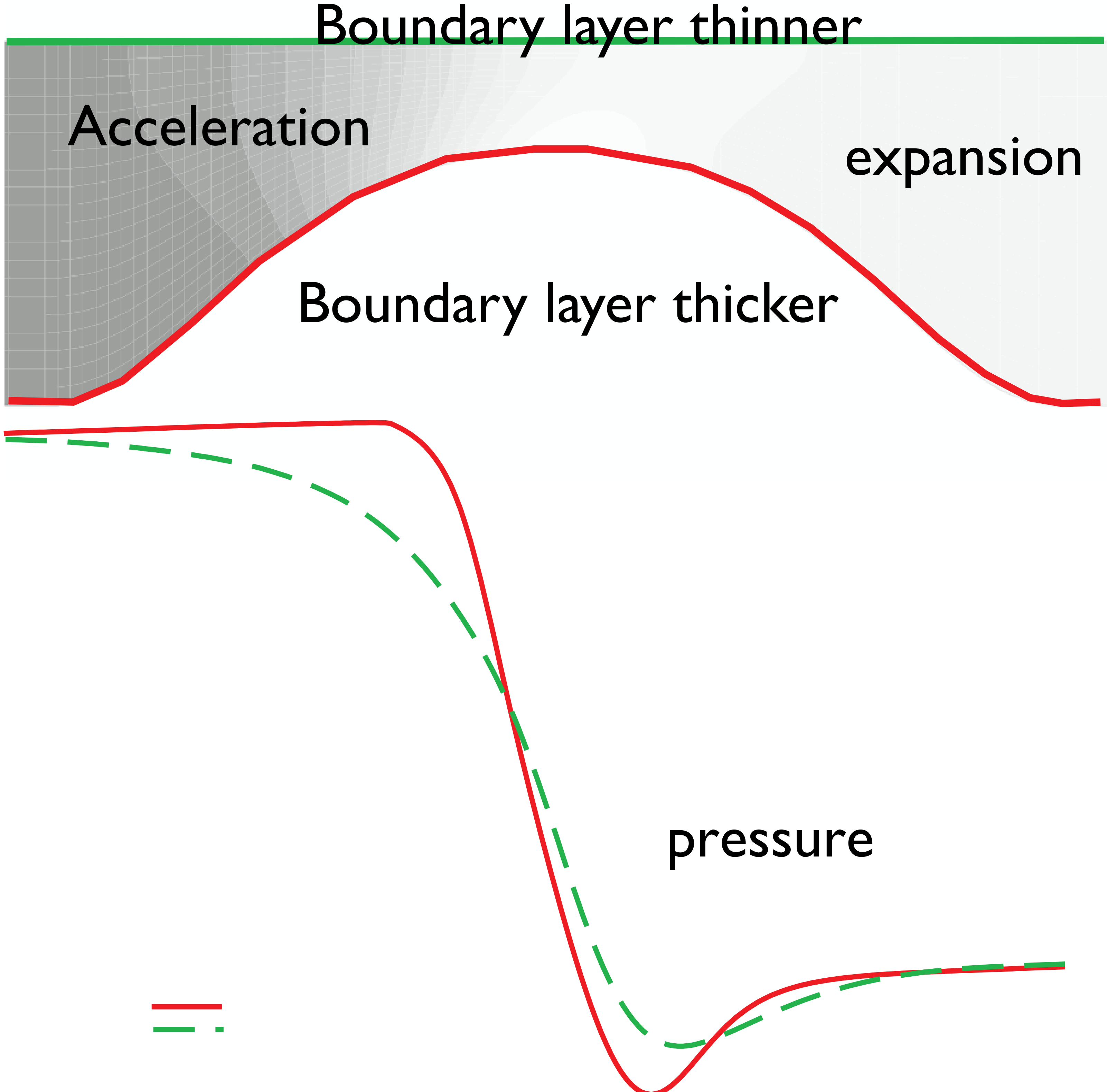
- RNSP
- modified integral method to take into account the transverse variation of pressure
- NS

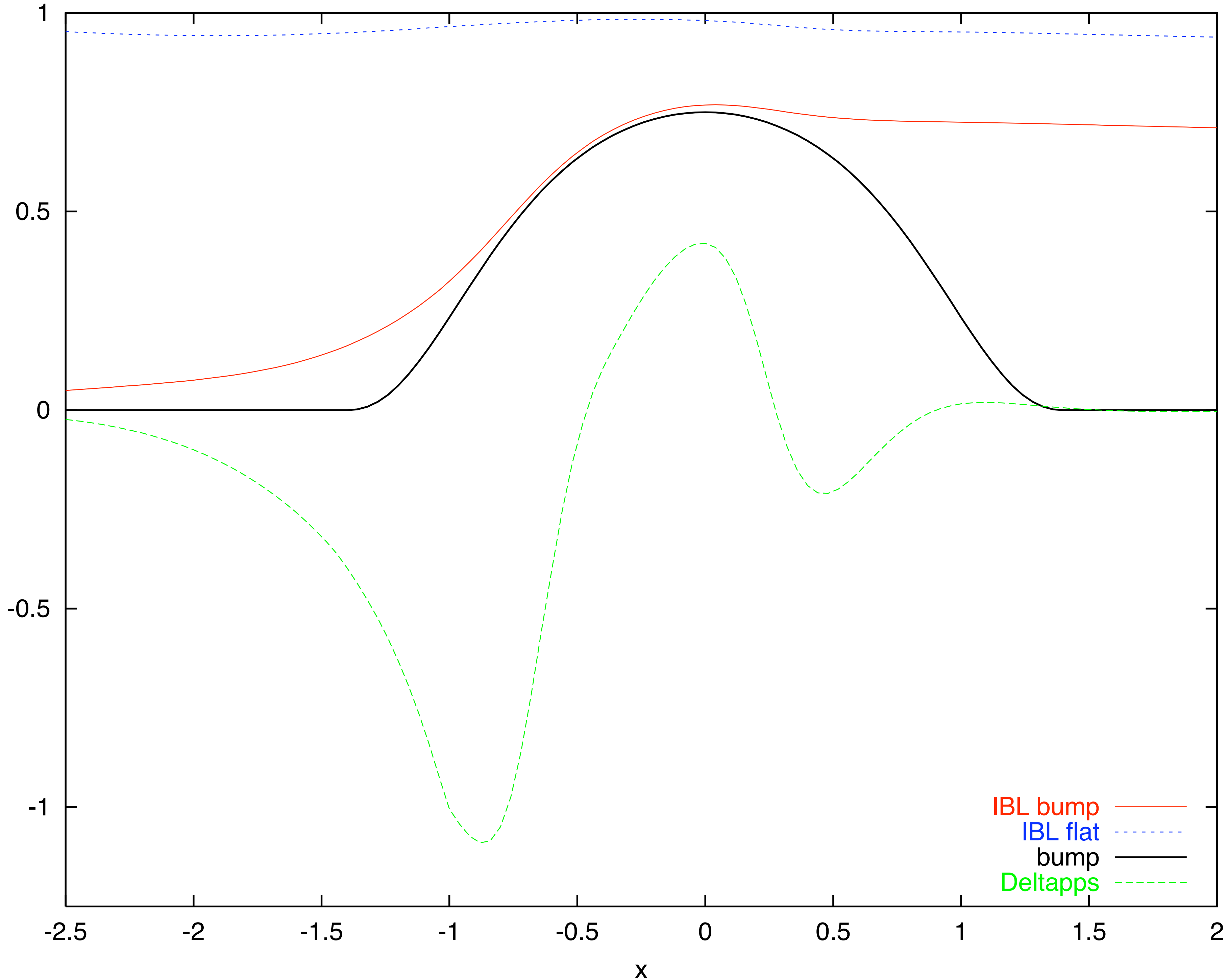


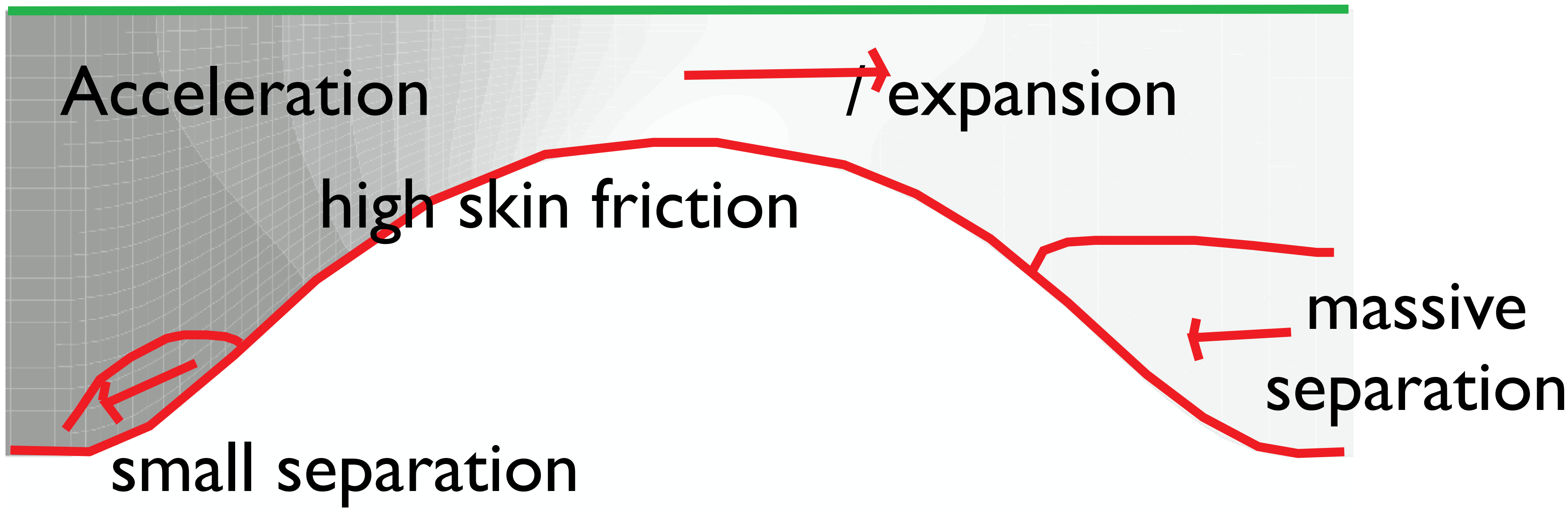
1
0
-1
-2
-3
-4
-5
-6
-7
-8
$p^*/(\rho U_0^2)$
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
IBL bump
IBL flat
NS bump
NS flat
bump


IBL bump
IBL flat
NS bump
NS flat
200
150
100
50
0
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
$(h_0/U_0)\partial(u^*)/\partial y^*$
x


Boundary layer thinner
Acceleration
Boundary layer thicker


Boundary layer thinner
Acceleration
expansion
Boundary layer thicker
pressure

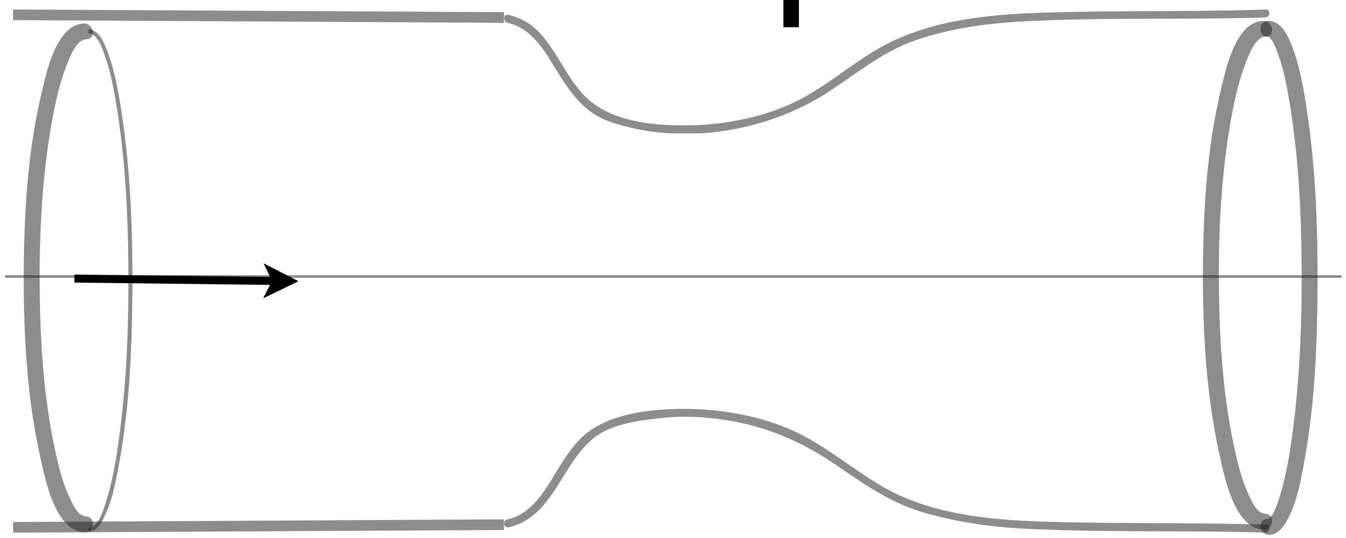

1
0.5
0
-0.5
-1
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
IBL bump
IBL flat
bump
Deltapps


Acceleration / expansion
high skin friction
massive
separation
small separation

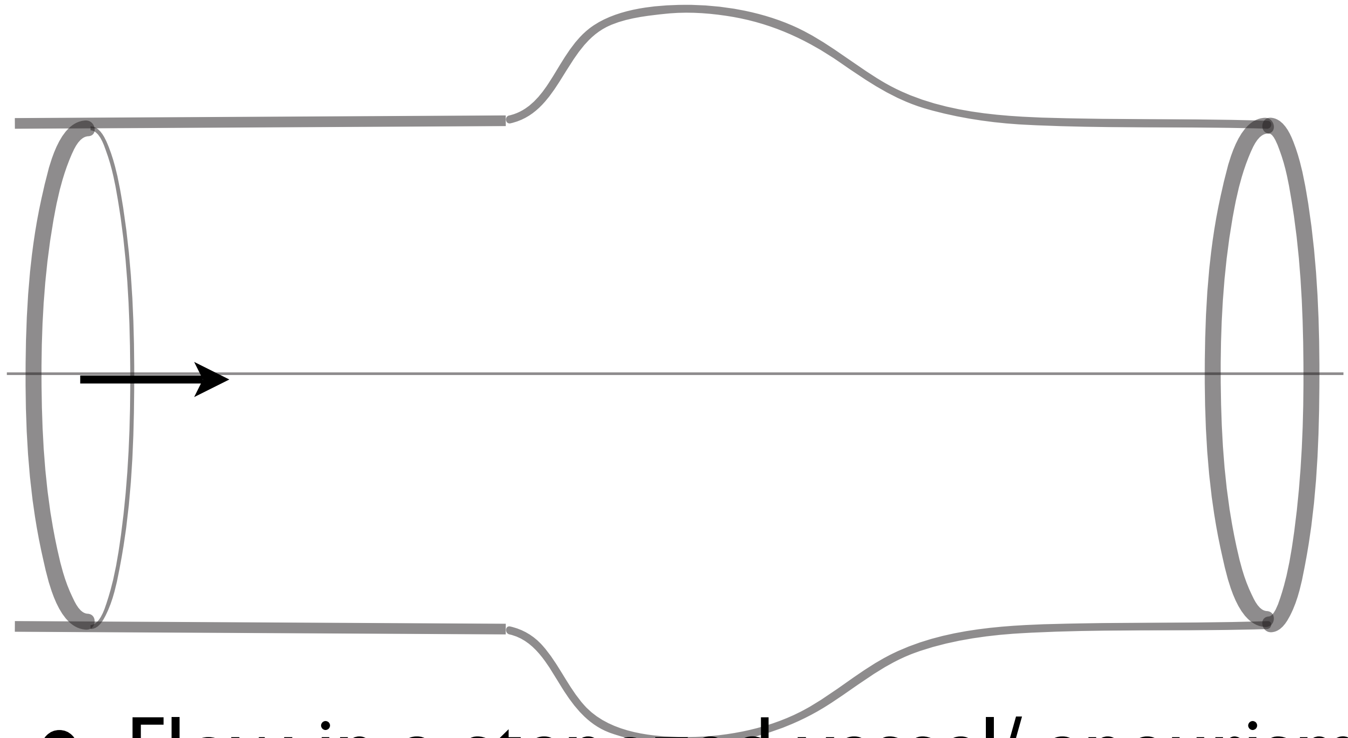
Example 4

- Flow in a stenozed vessel/ aneurism
- unsteady, rigid wall

Example 2

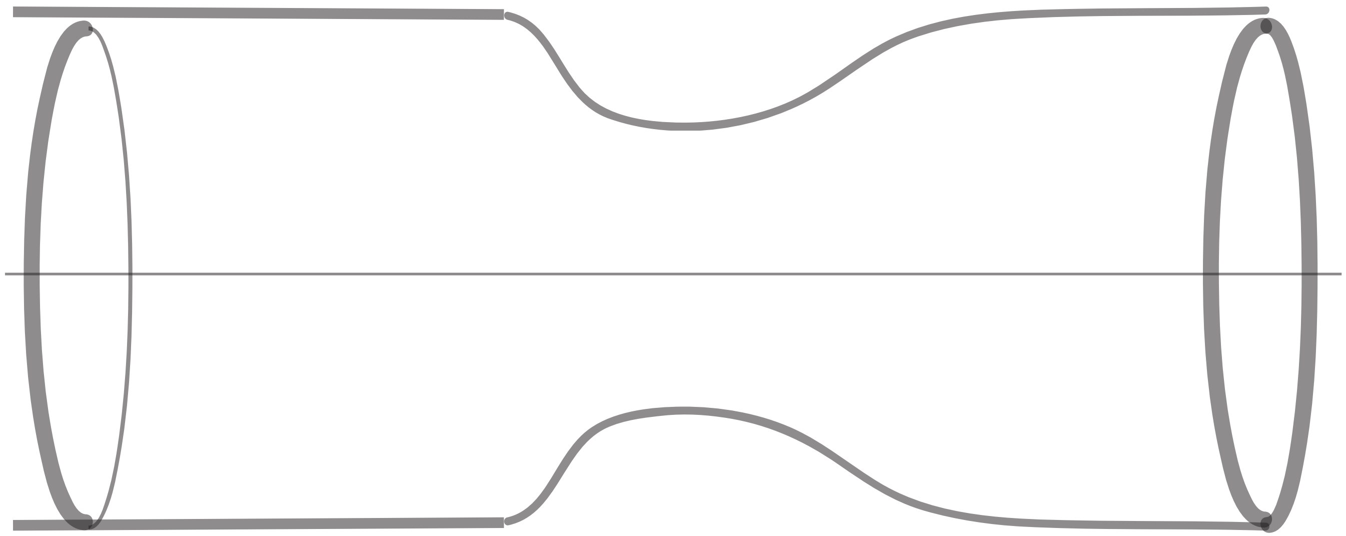


- Flow in a stenozed vessel/
- unsteady, rigid wall

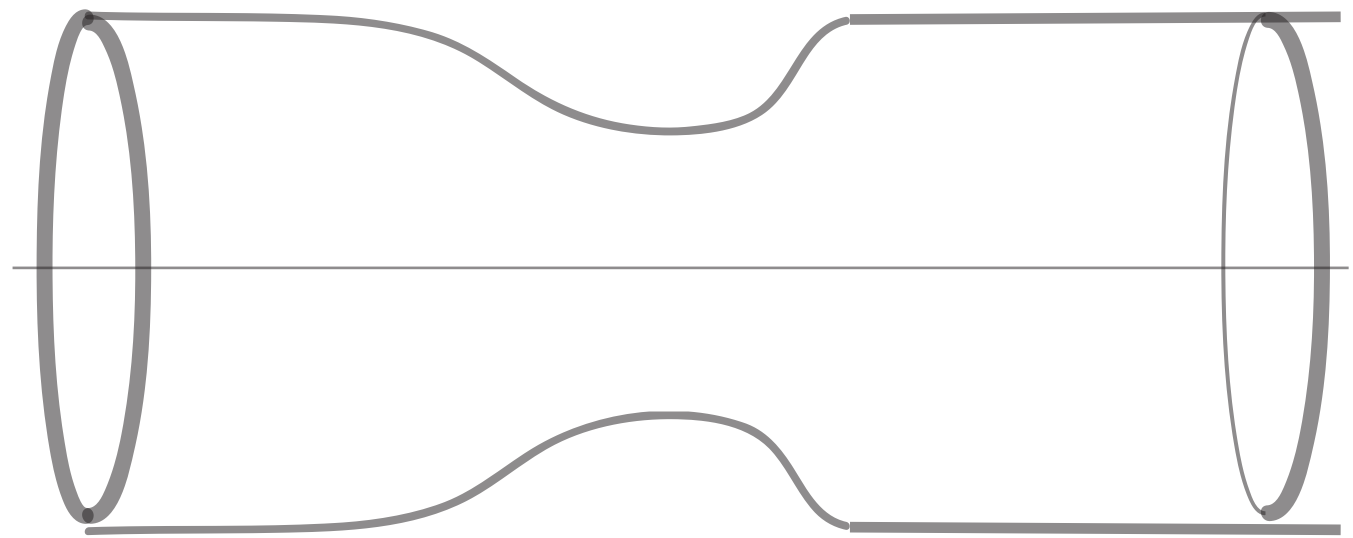


- Flow in a stenozed vessel/ aneurism
- unsteady, rigid wall

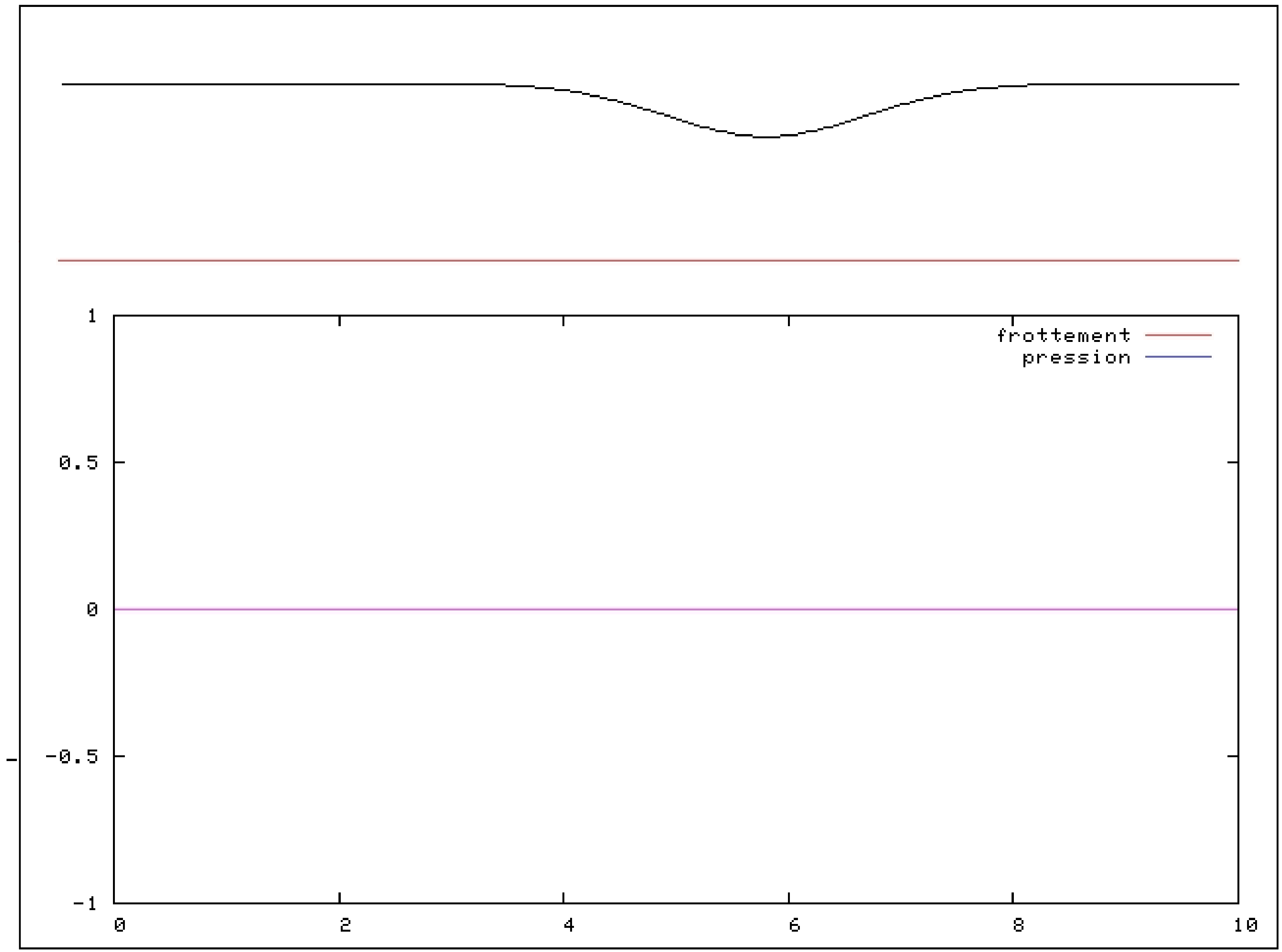
- Stenosis



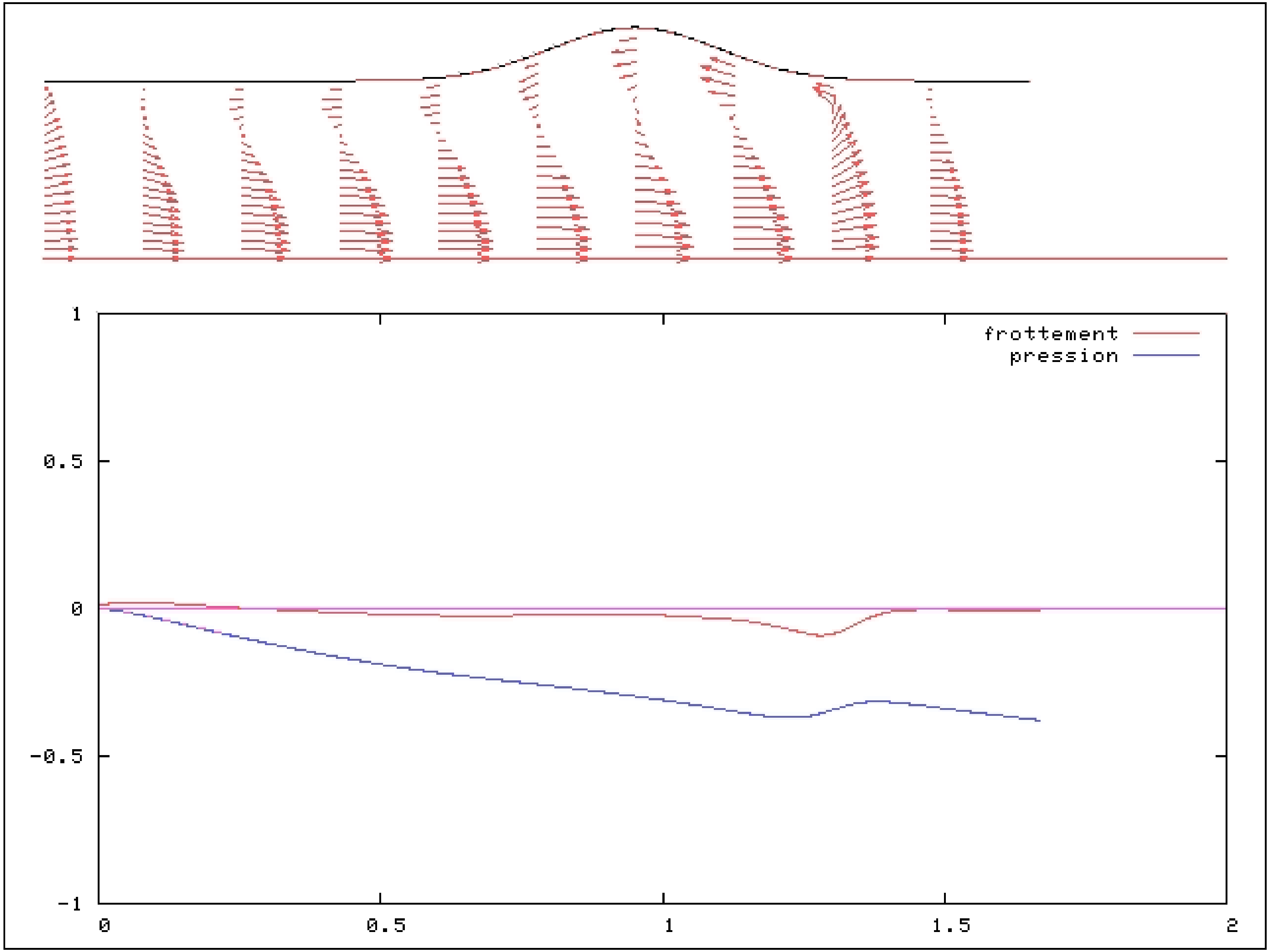
- Stenosis

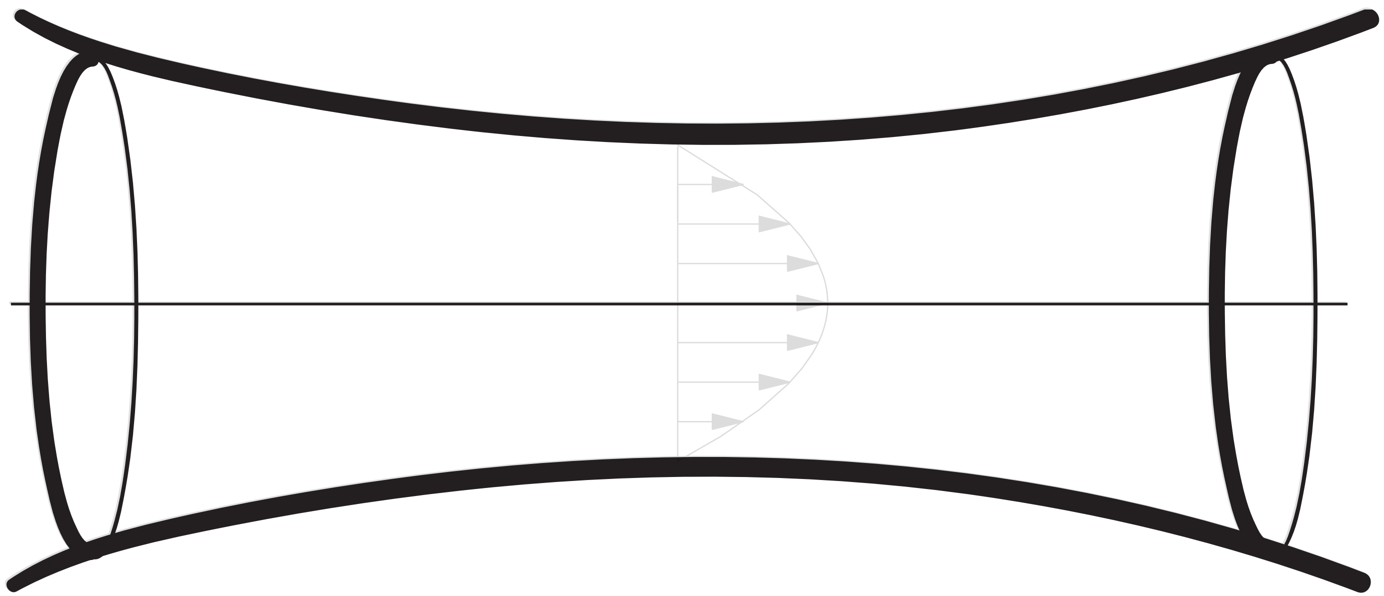


- Stenosis

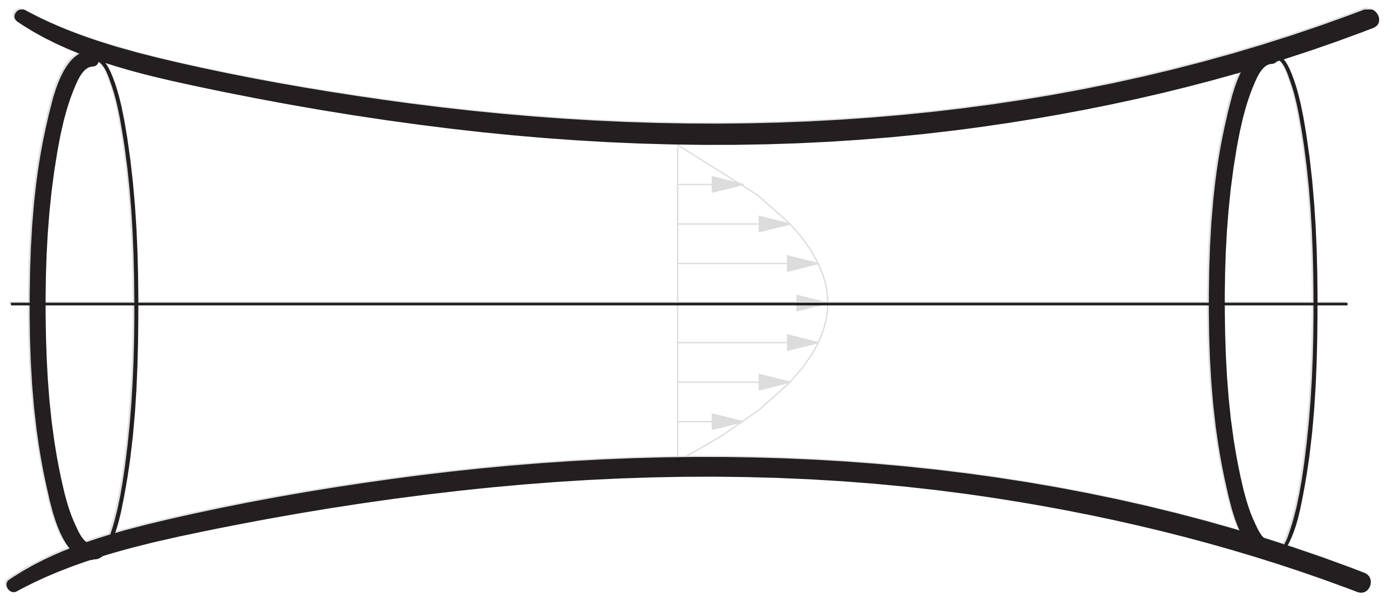


- Aneurism

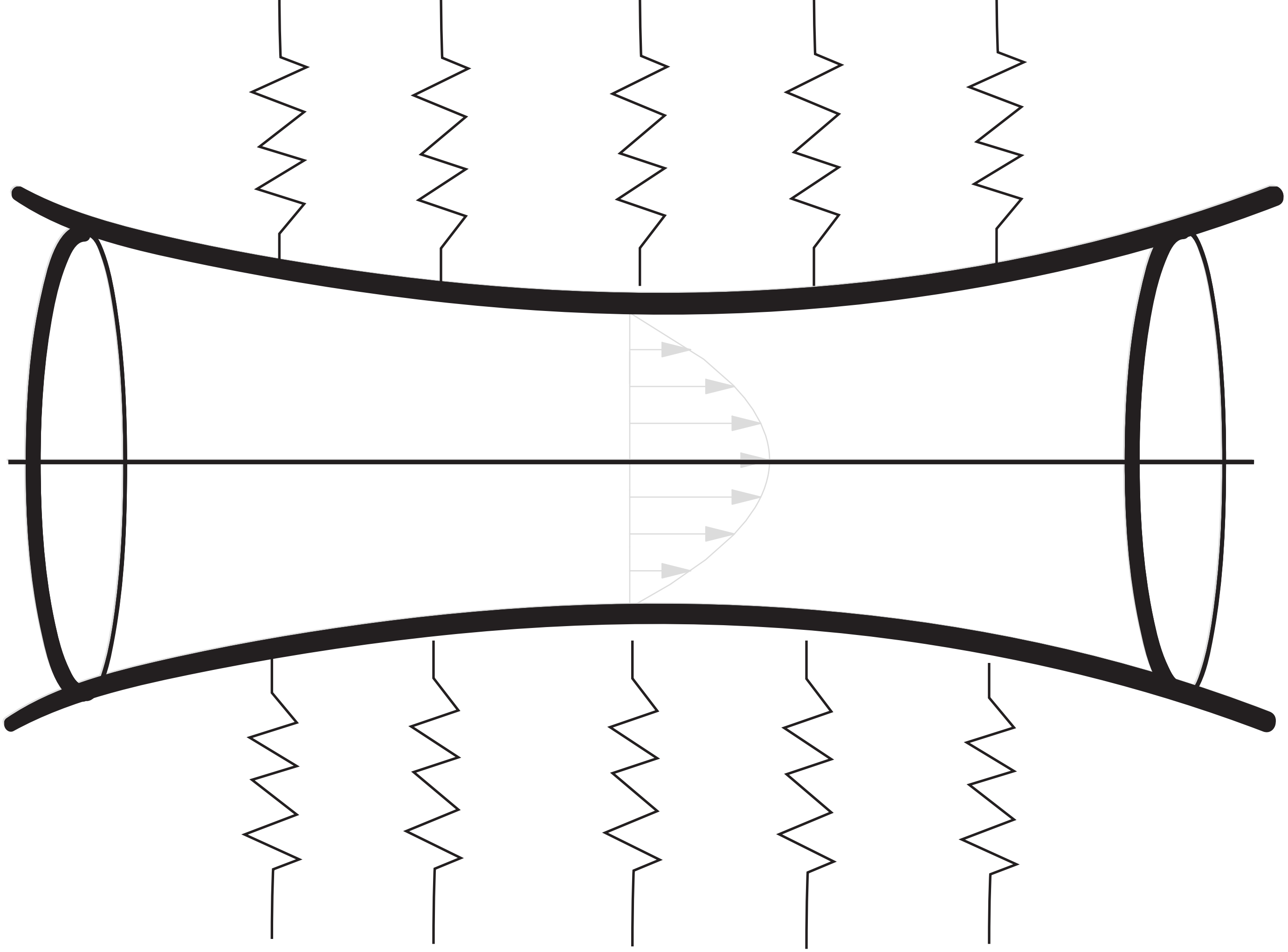




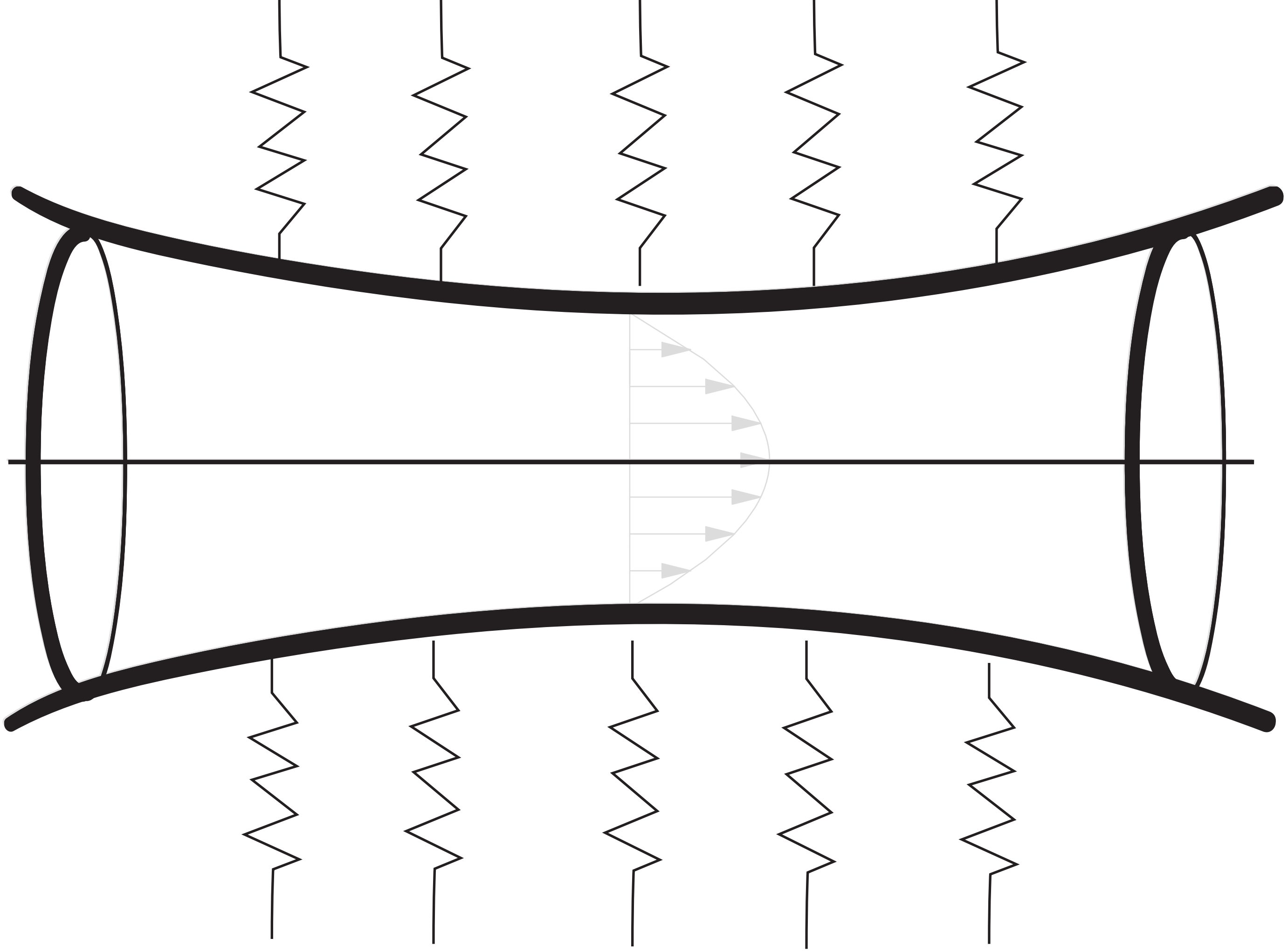
Up to now, the wall was rigid

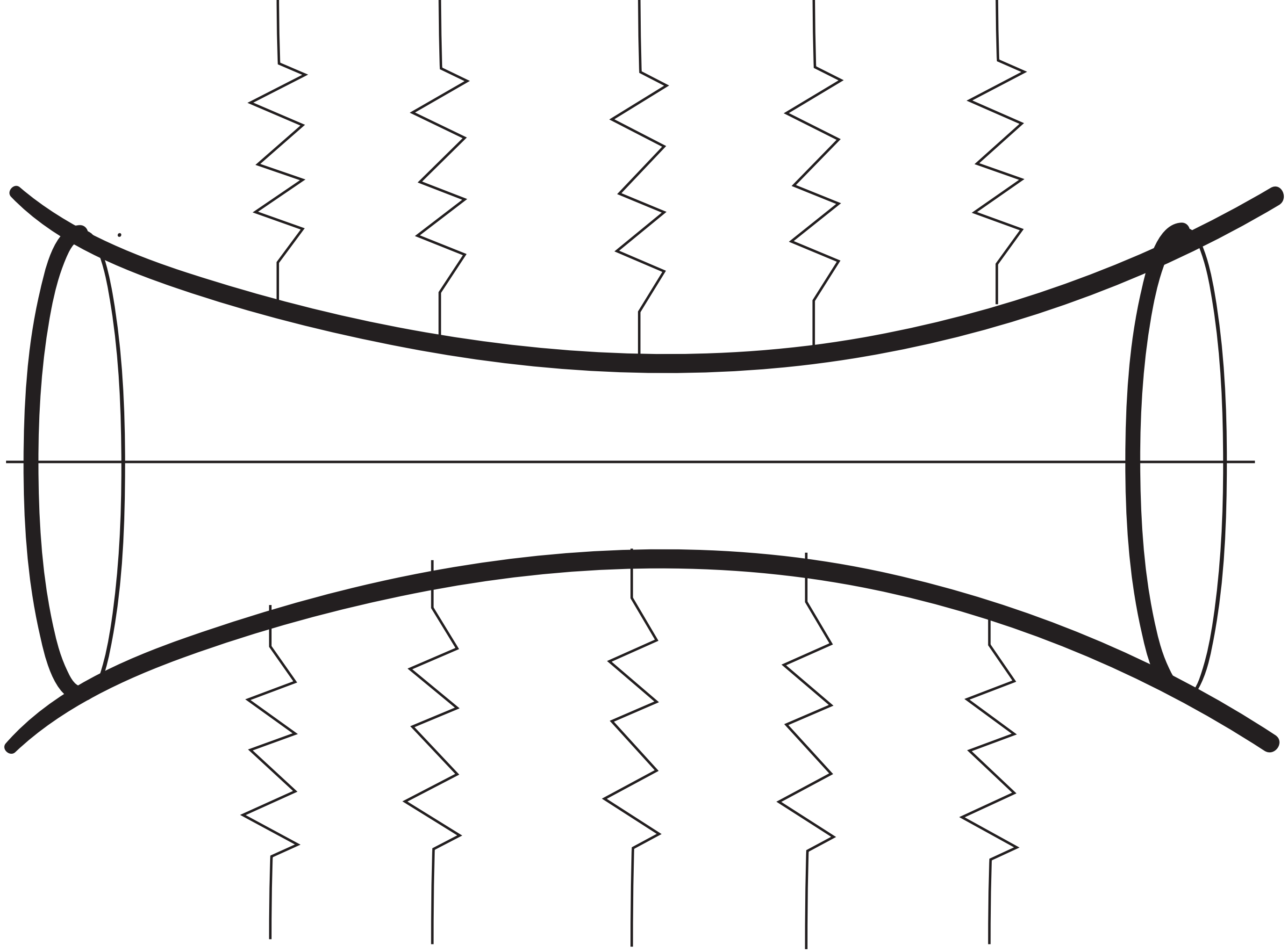


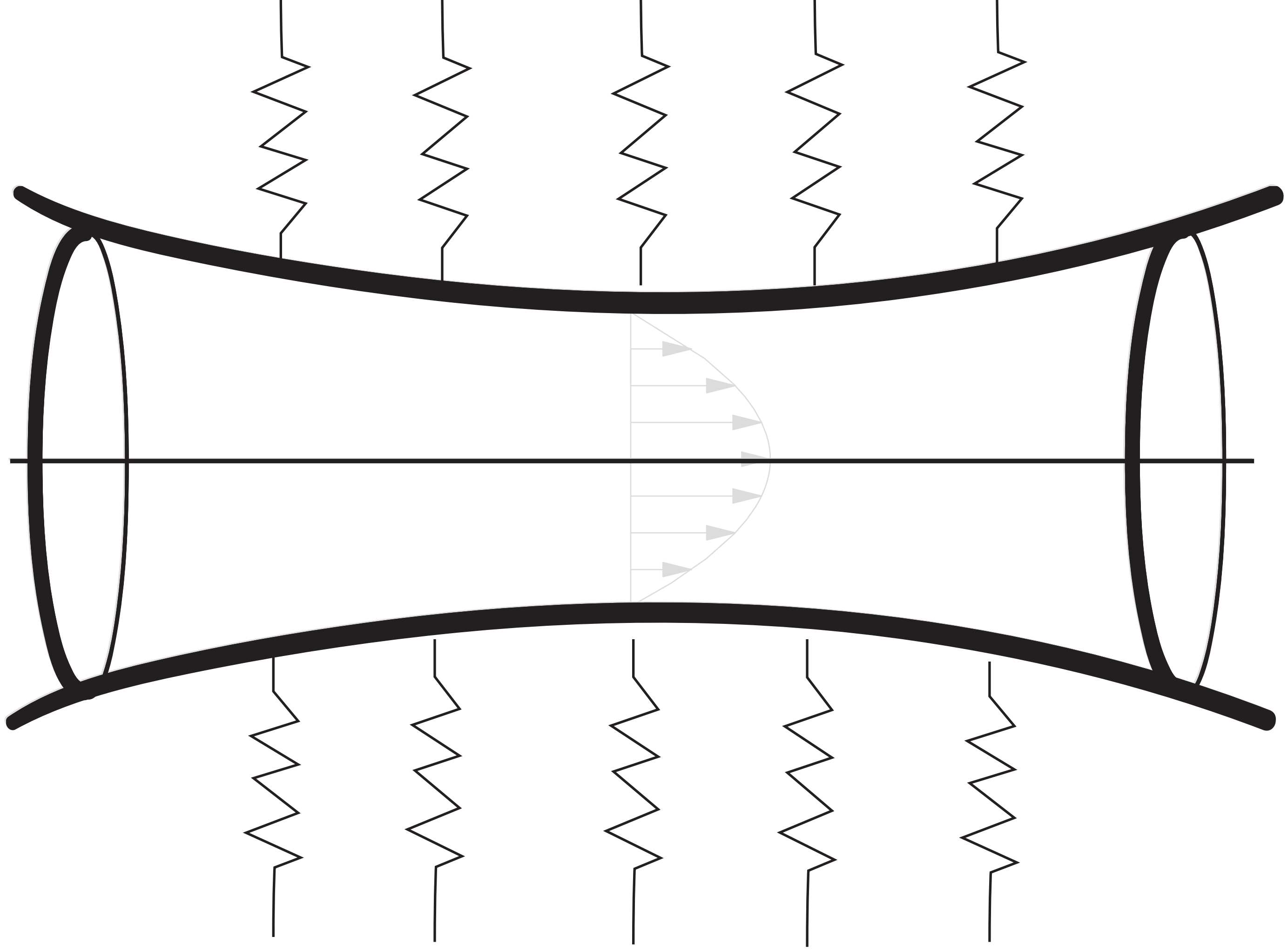
we use a simple elastic model

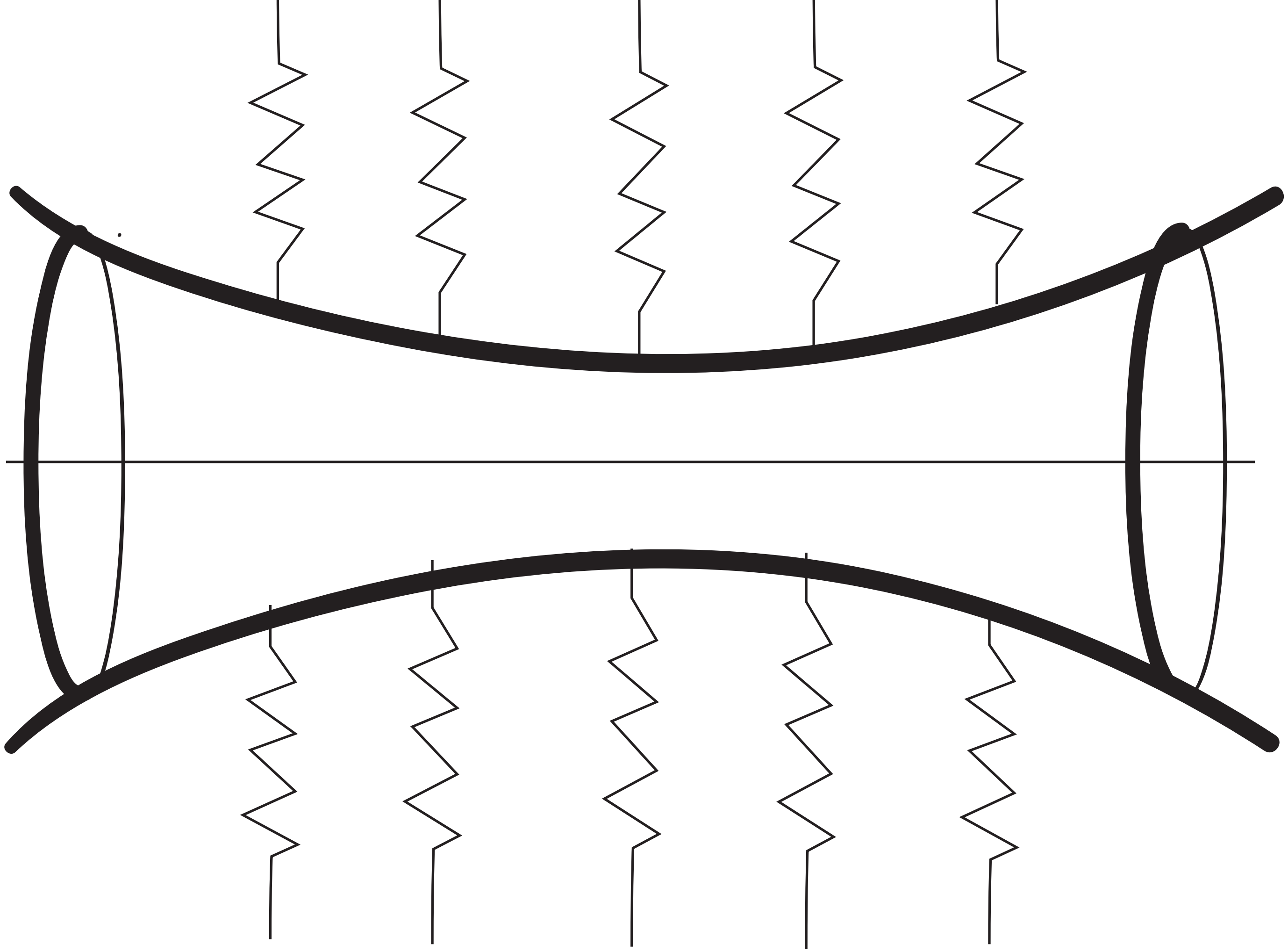


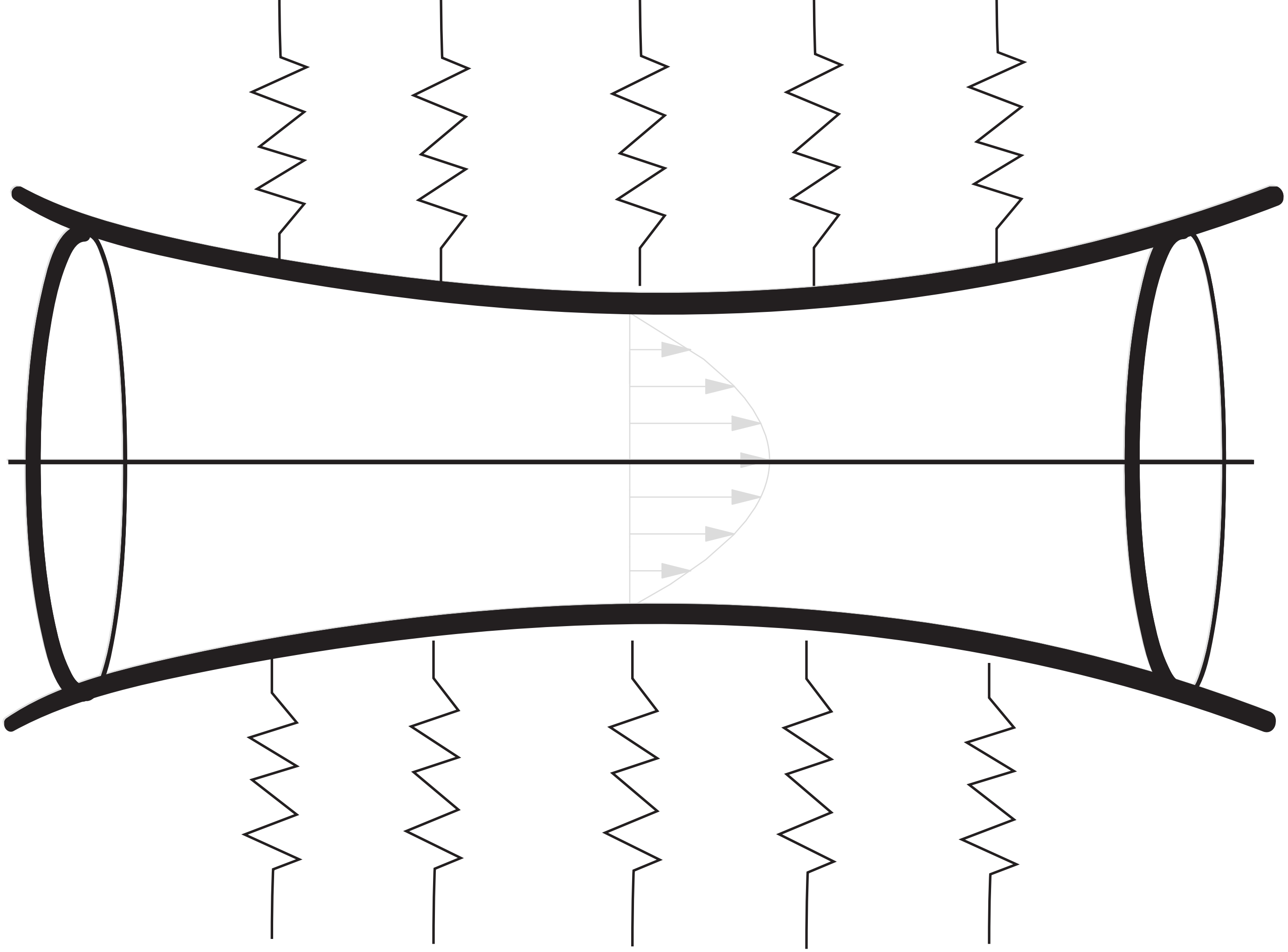
we use a simple elastic model



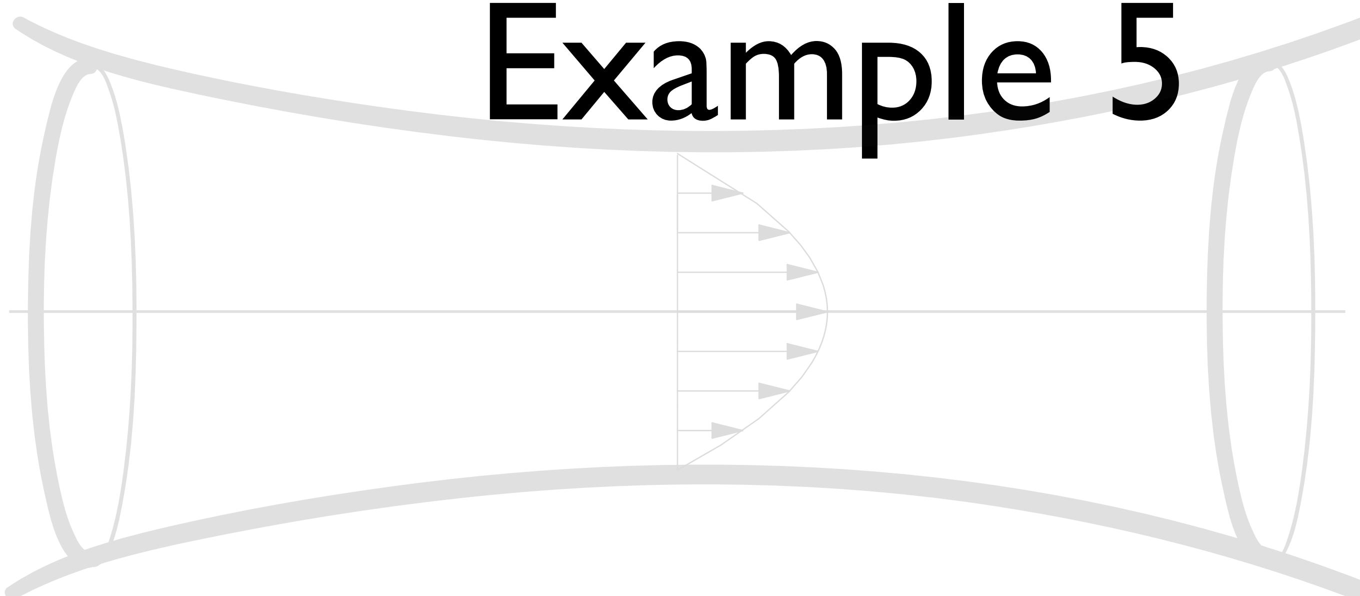




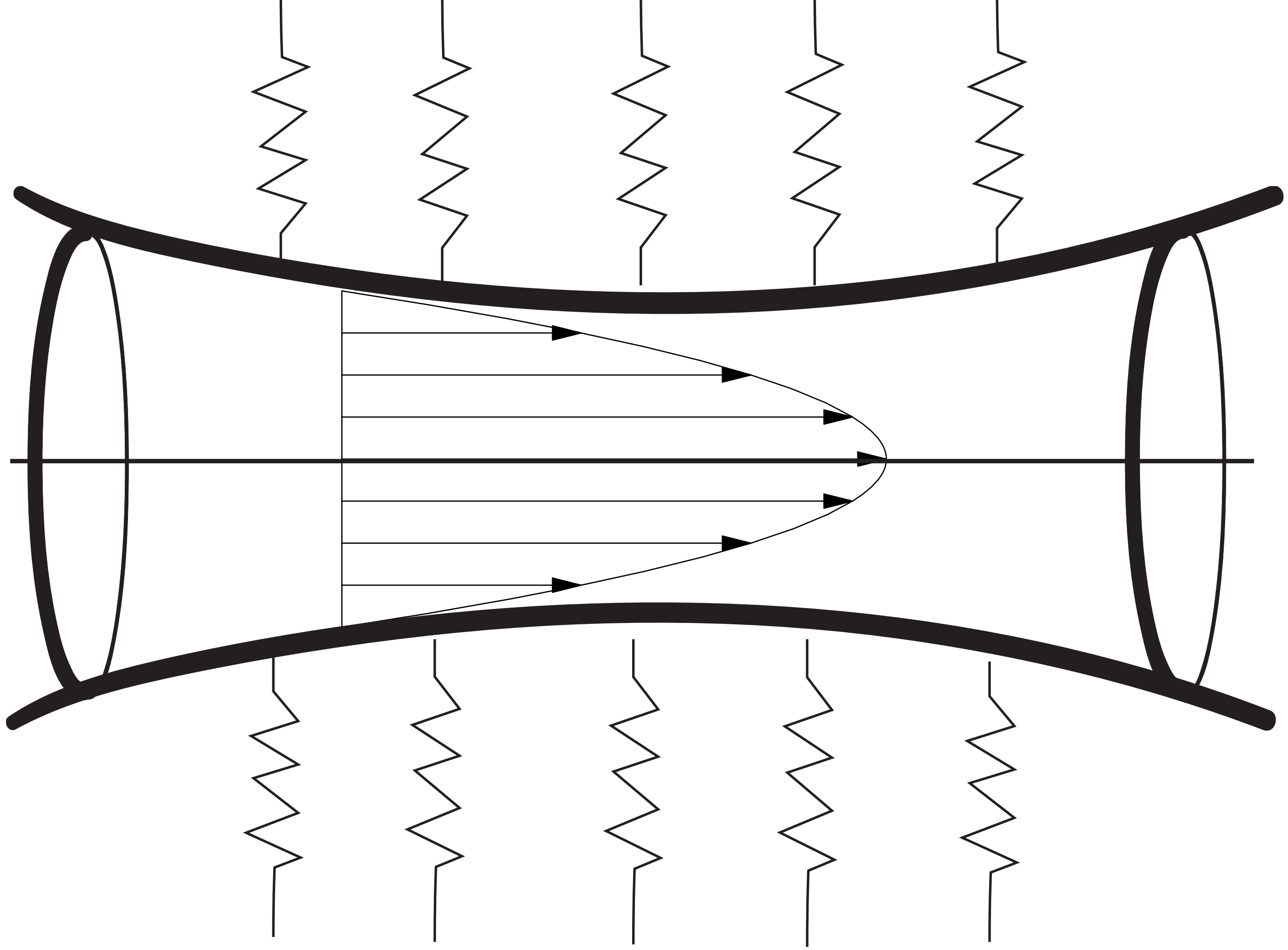




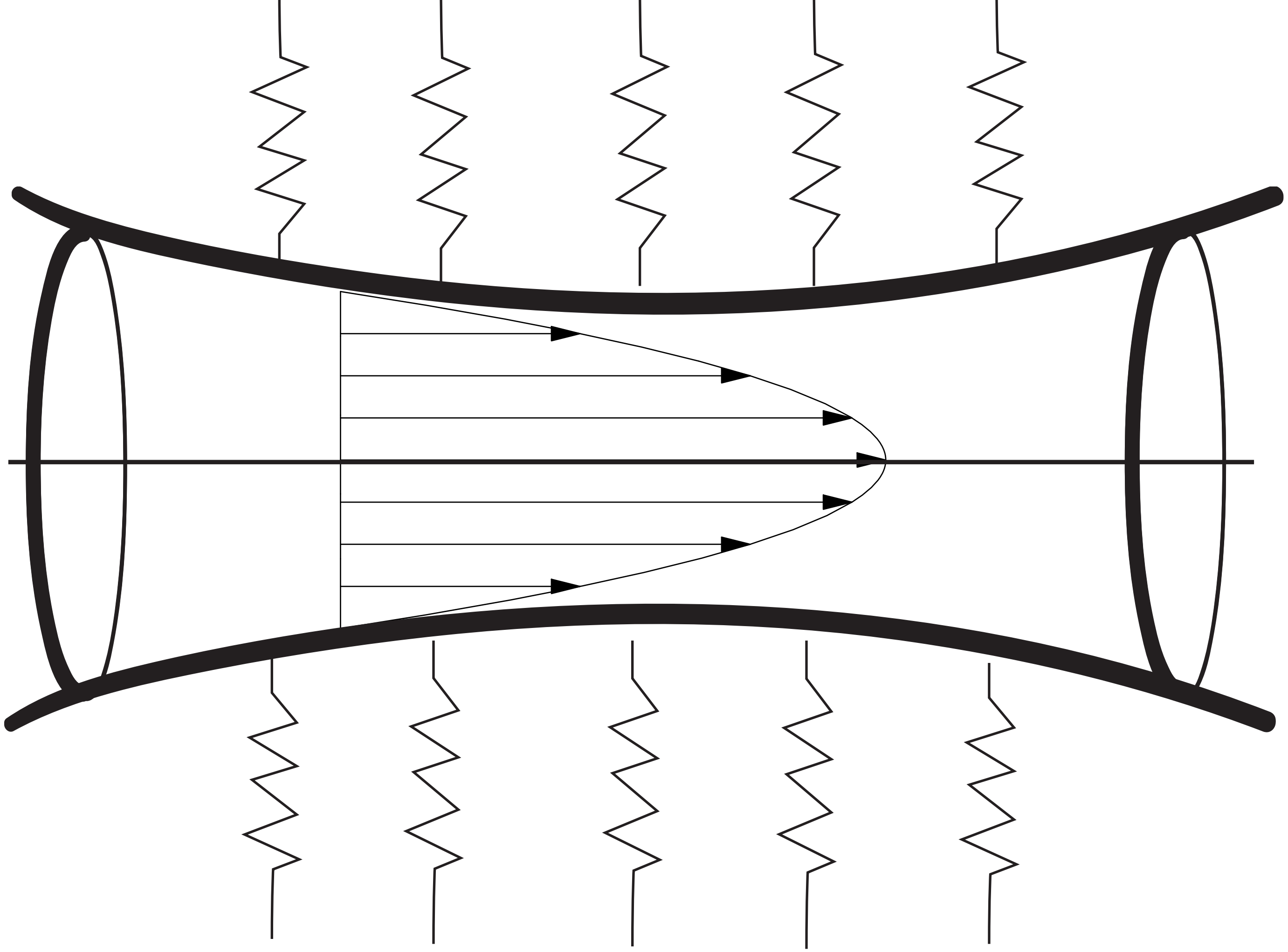
Example 5



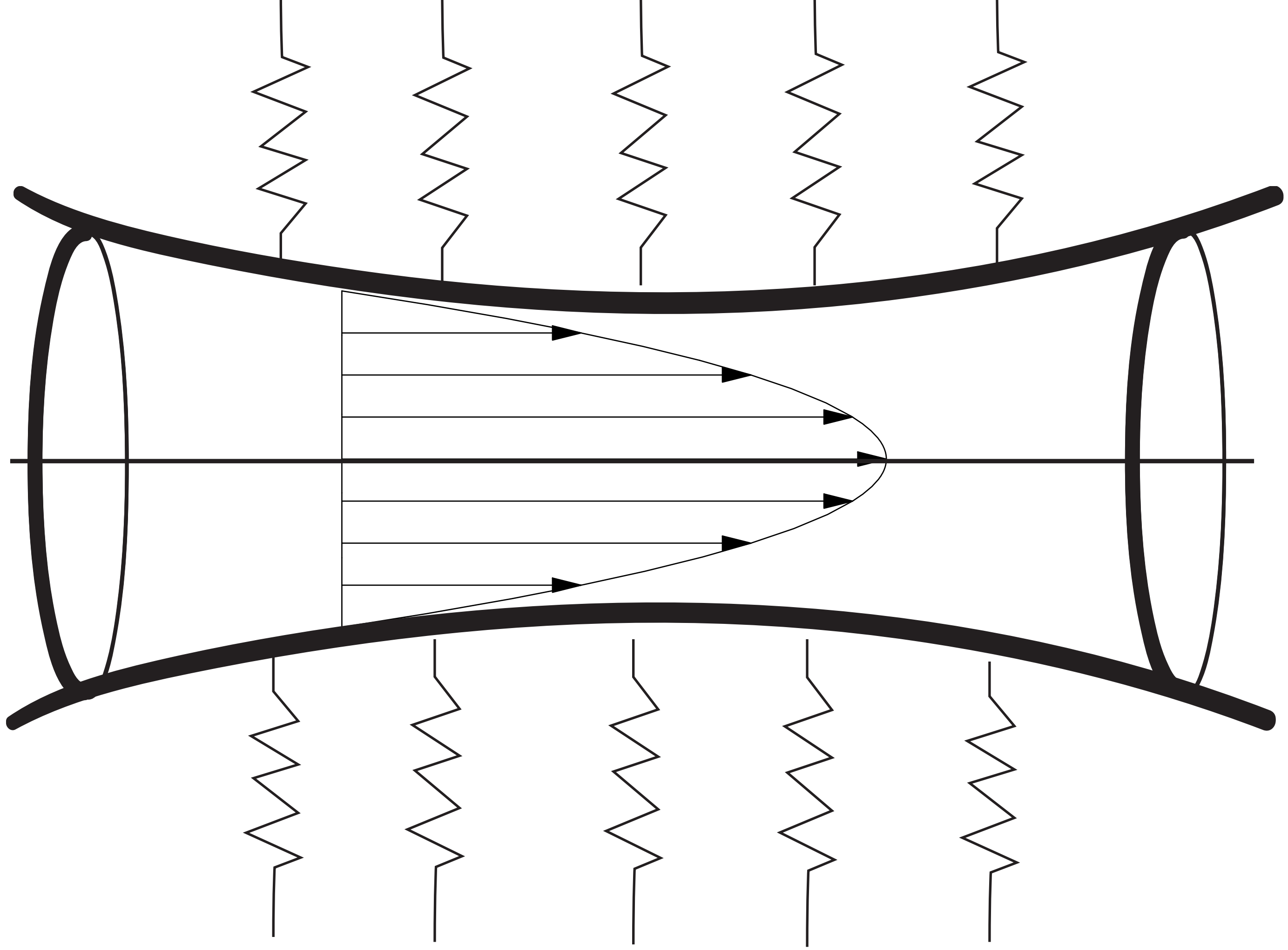
- Flow in a collapsible tube
- unsteady, elastic wall, no inertia



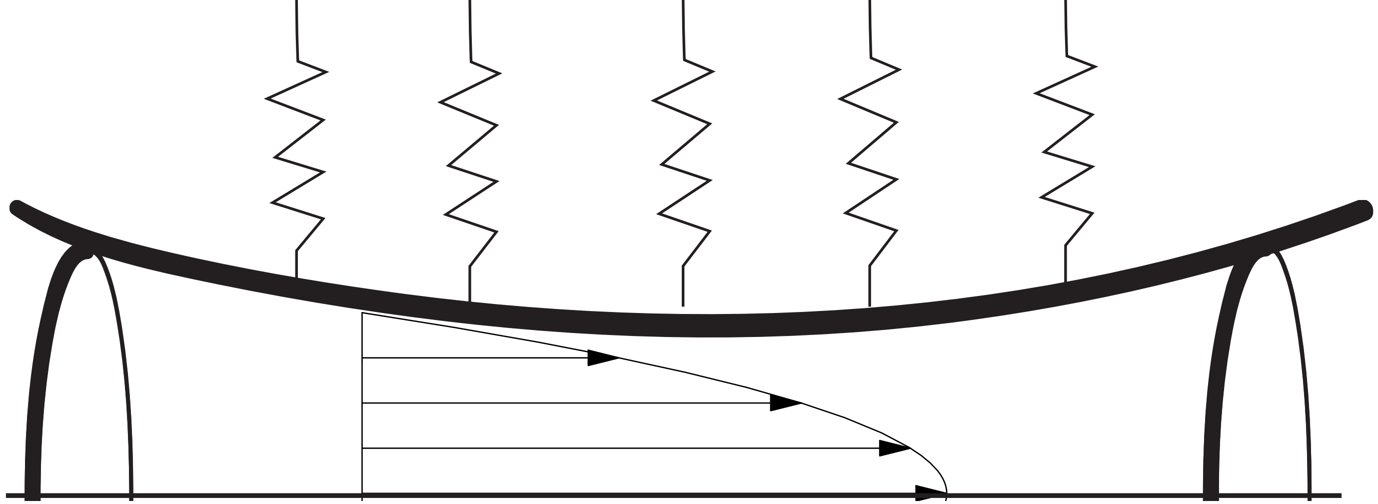
Collapsible tube

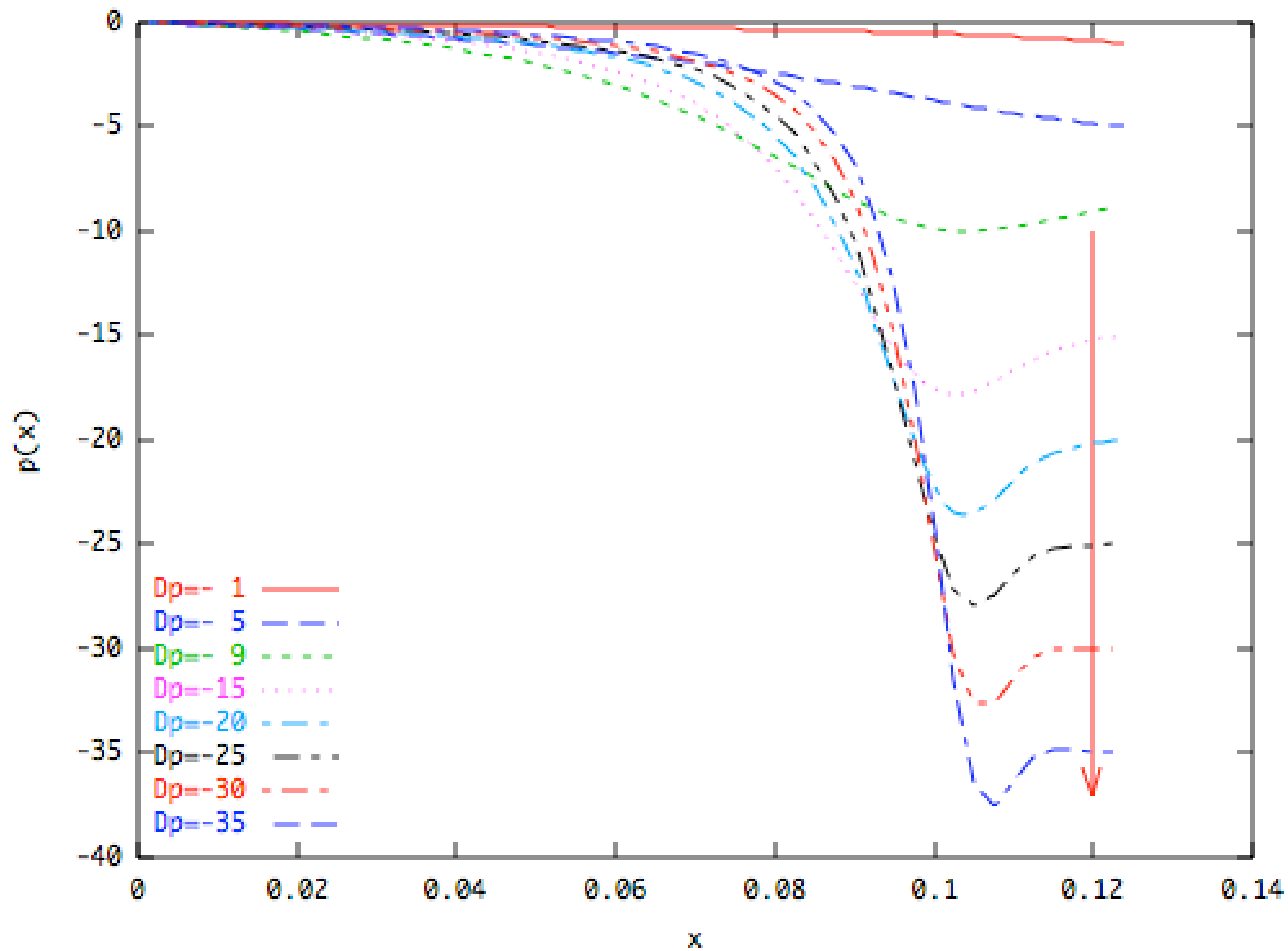


R^n gives p^{n+1}

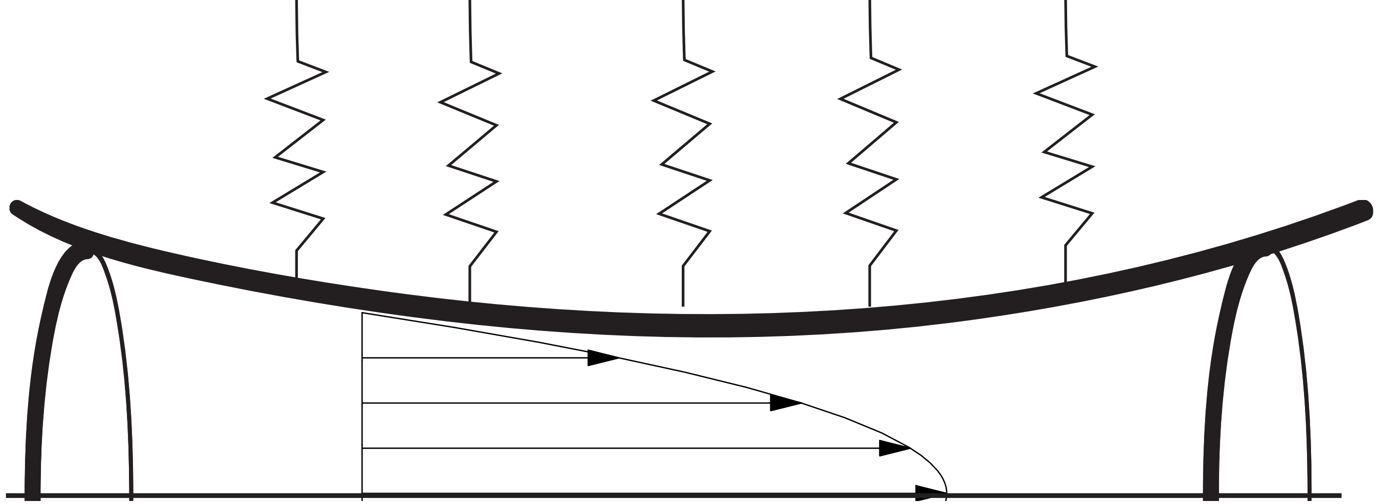


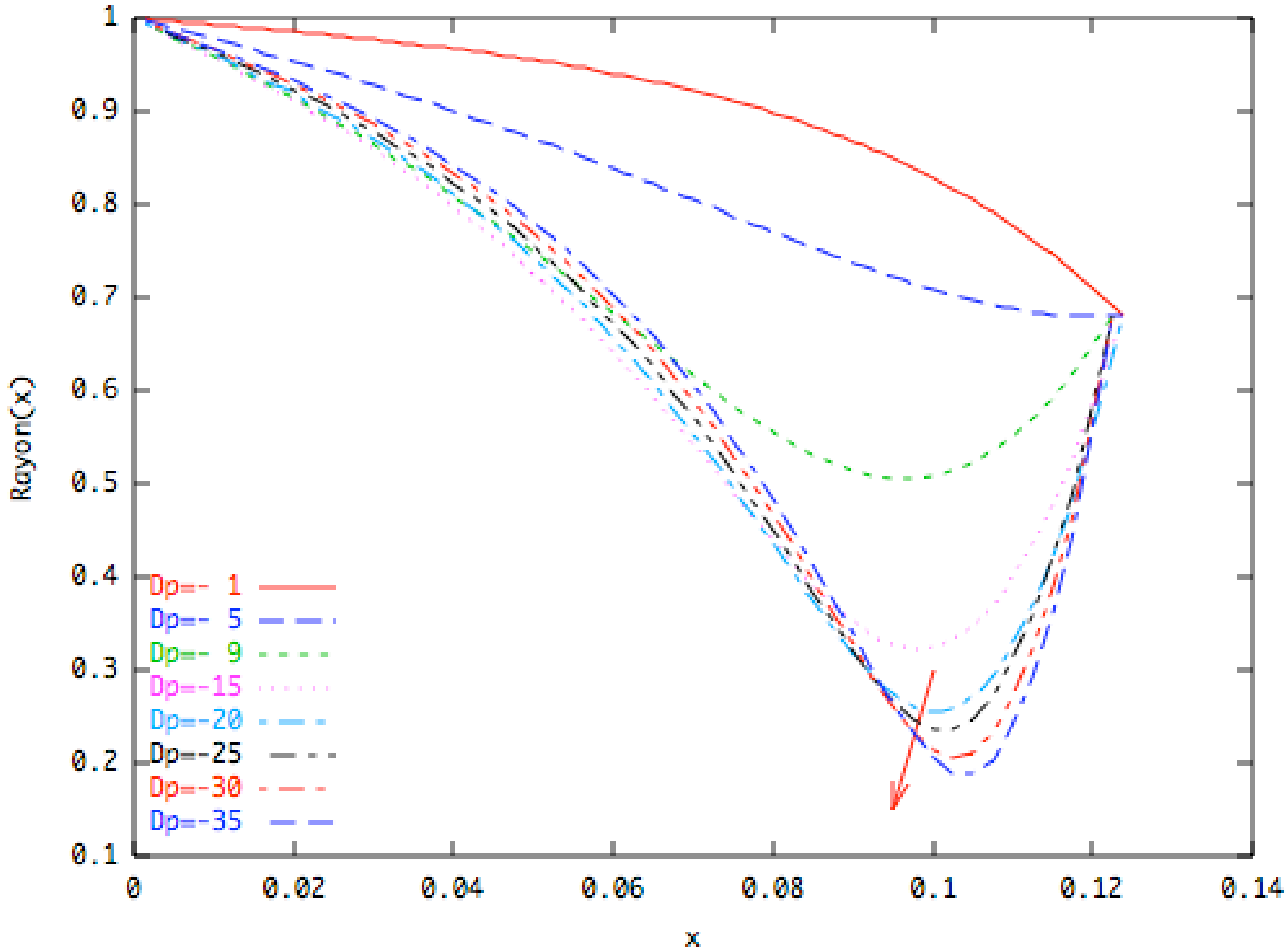
R^n gives p^{n+1} $\longrightarrow$ $p^{n+1} = k(R^{n+1} - 1)$



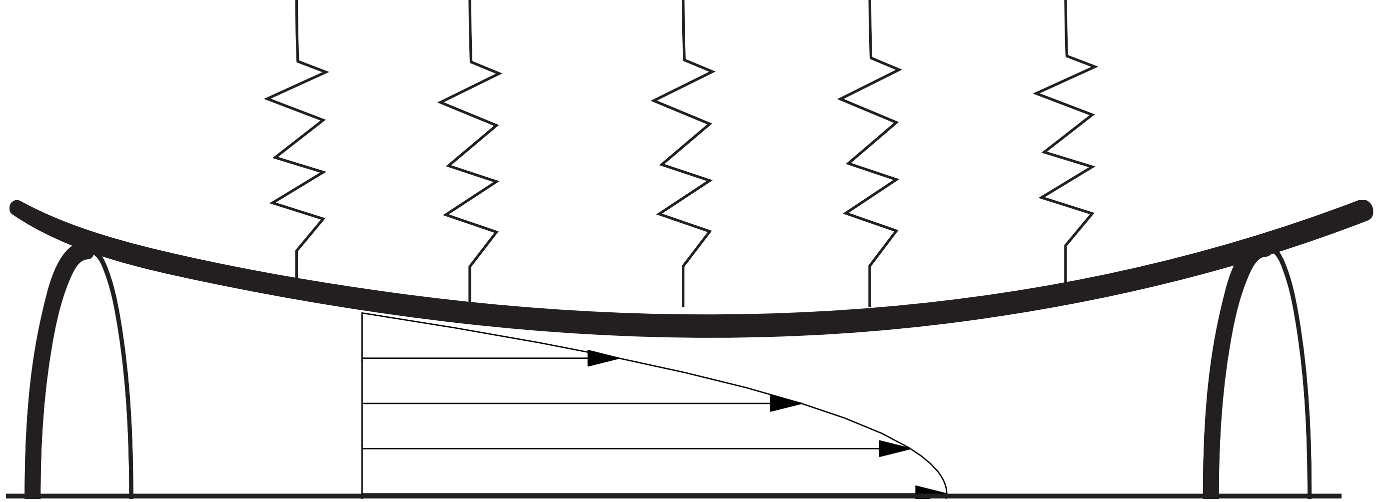


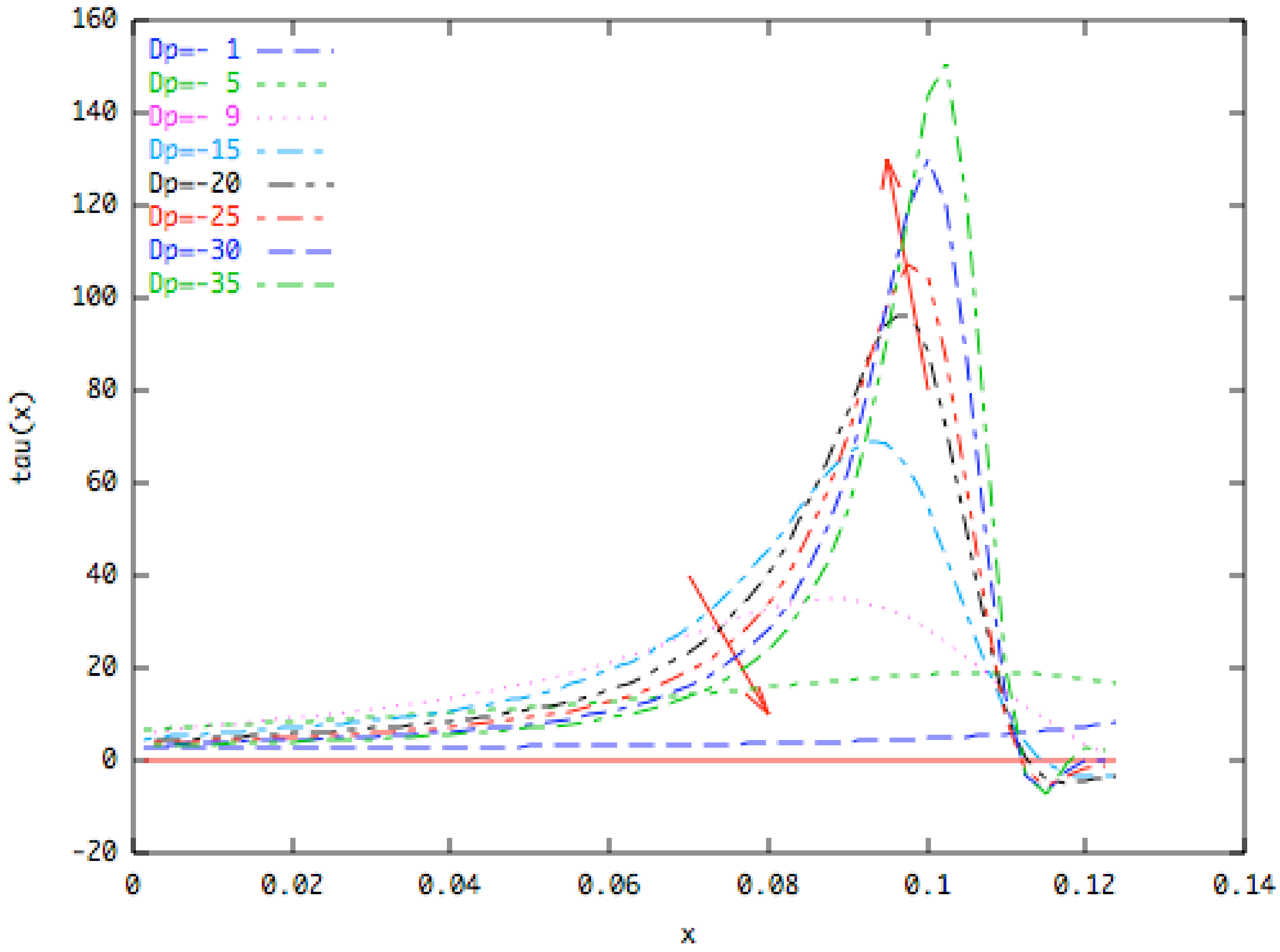
0
-5
-10
-15
-20
-25
-30
-35
-40
p(x)
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
x
Dp=- 1
Dp=- 5
Dp=- 9
Dp=-15
Dp=-20
Dp=-25
Dp=-30
Dp=-35



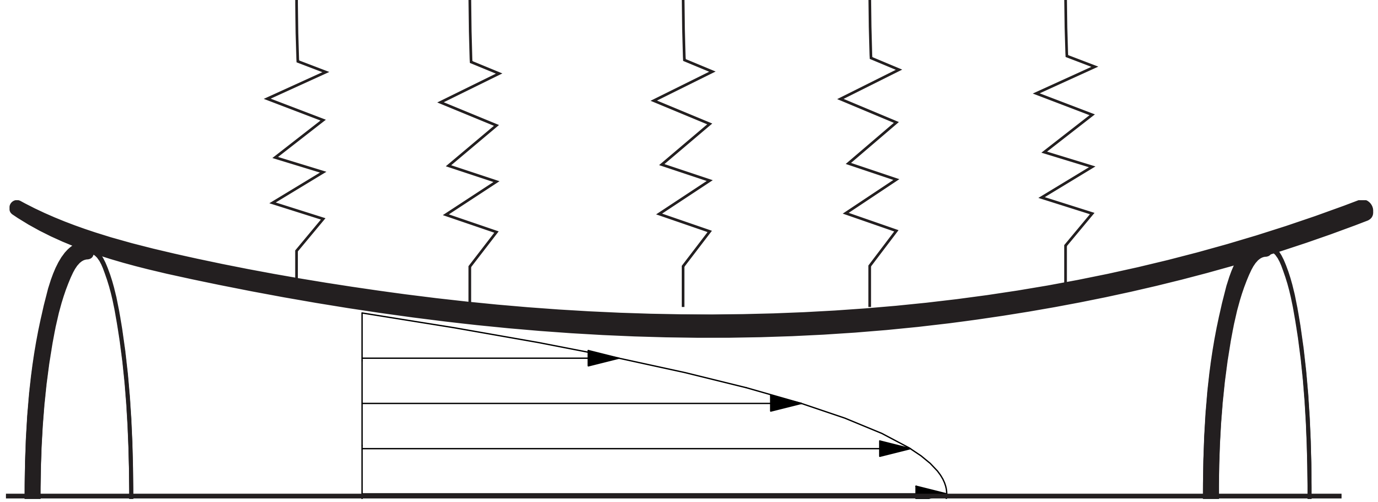


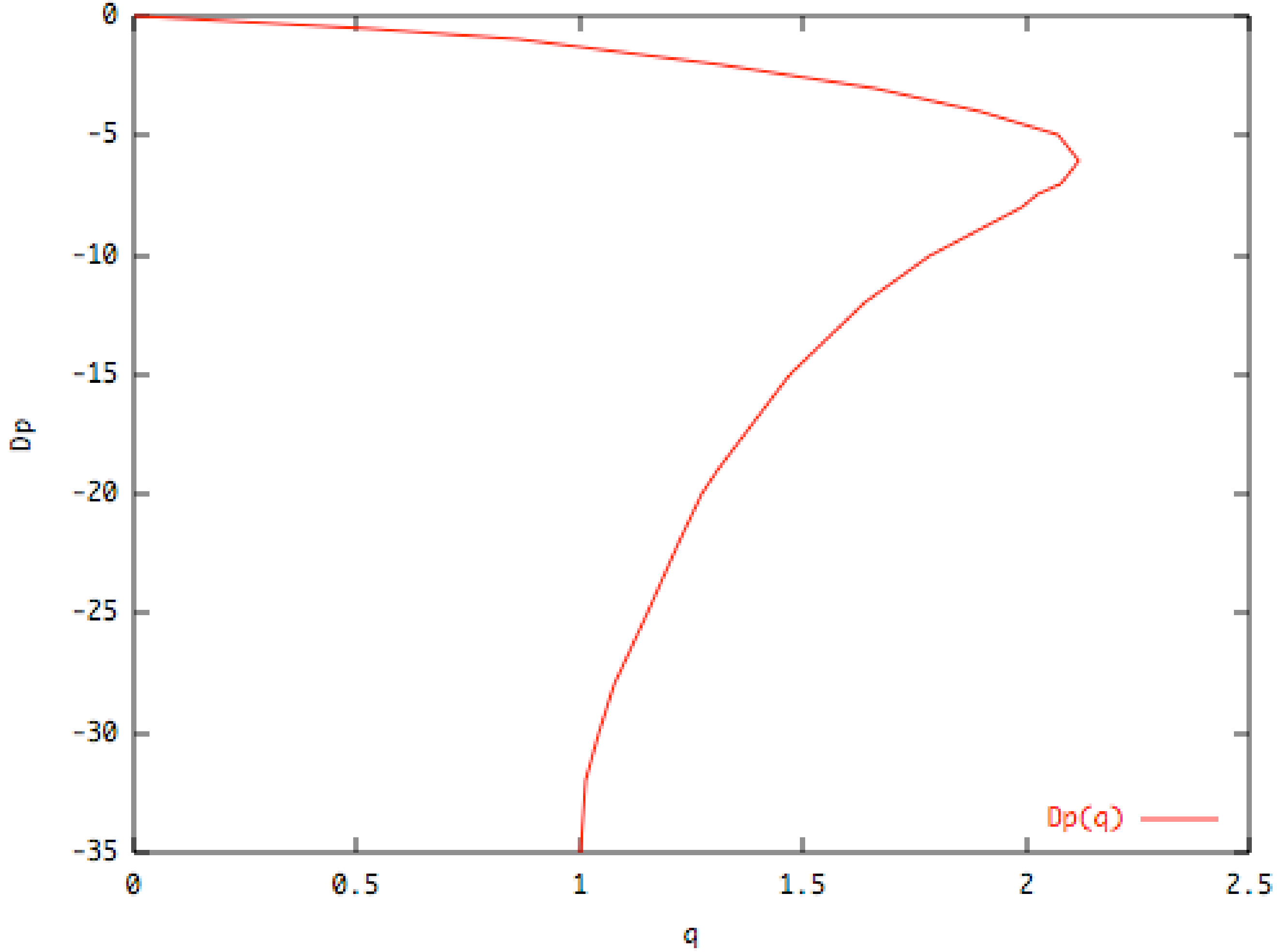
Rayon(x)
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
Dp=- 1
Dp=- 5
Dp=- 9
Dp=-15
Dp=-20
Dp=-25
Dp=-30
Dp=-35



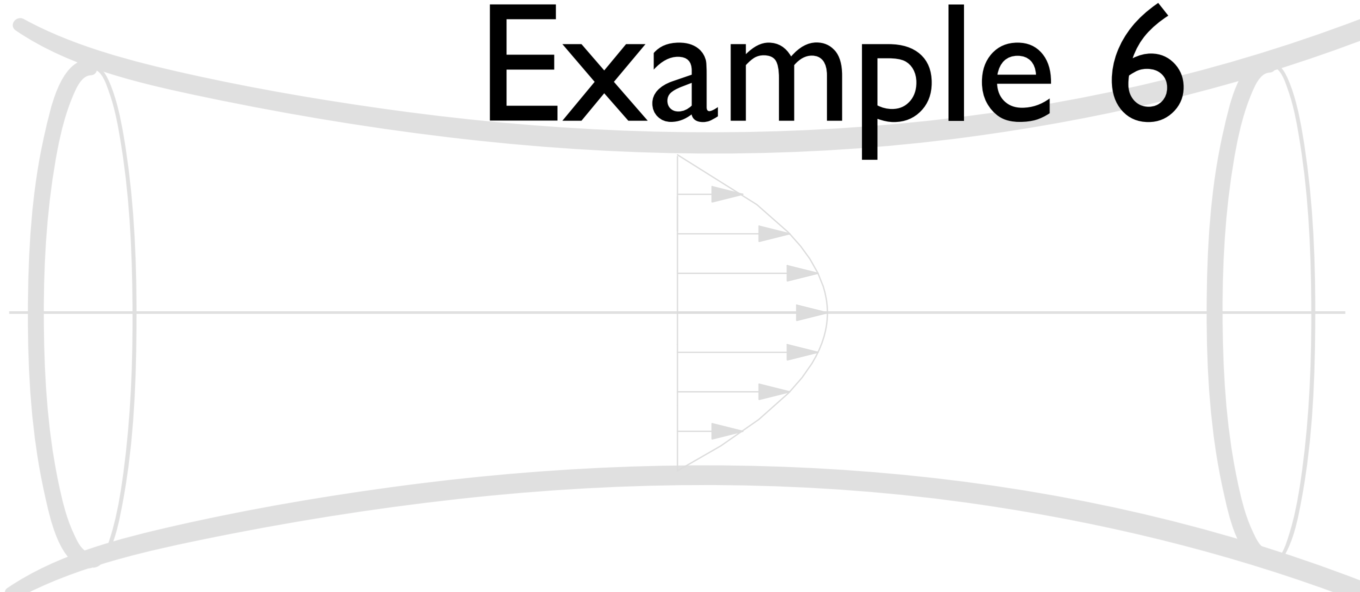


Dp=- 1
Dp=- 5
Dp=- 9
Dp=-15
Dp=-20
Dp=-25
Dp=-30
Dp=-35
tau(x)
x
160
140
120
100
80
60
40
20
0
-20
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14

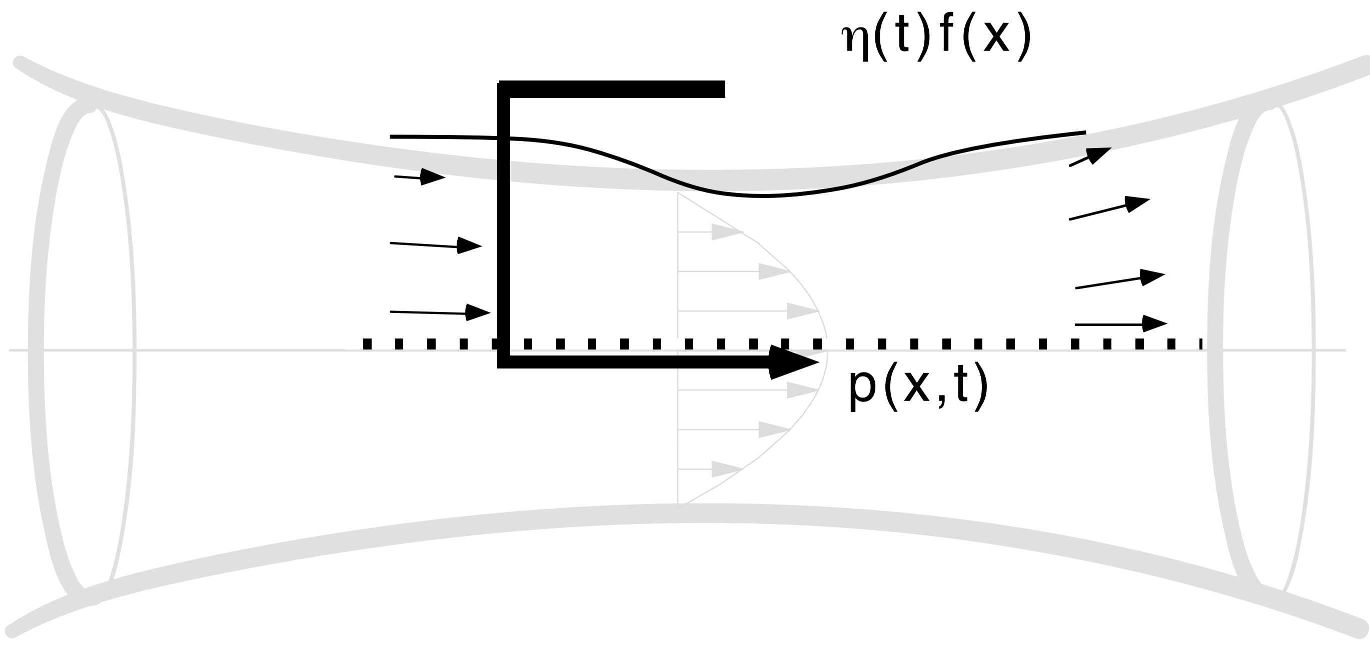



0
-5
-10
-15
-20
-25
-30
-35
Dp
0
0.5
1
1.5
2
2.5
q
Dp(q)

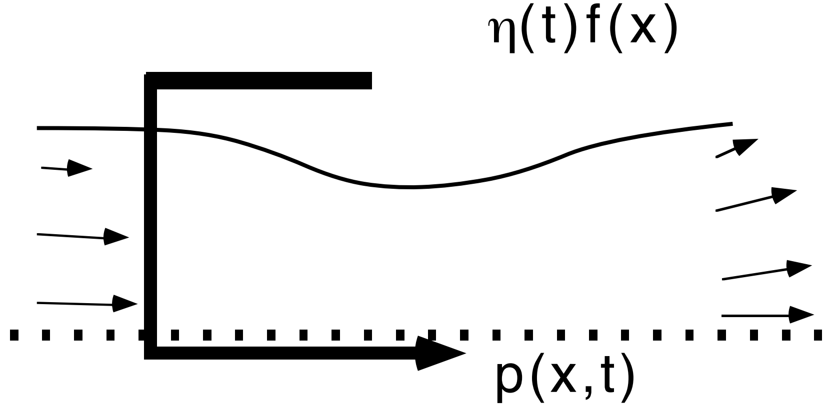
Example 6

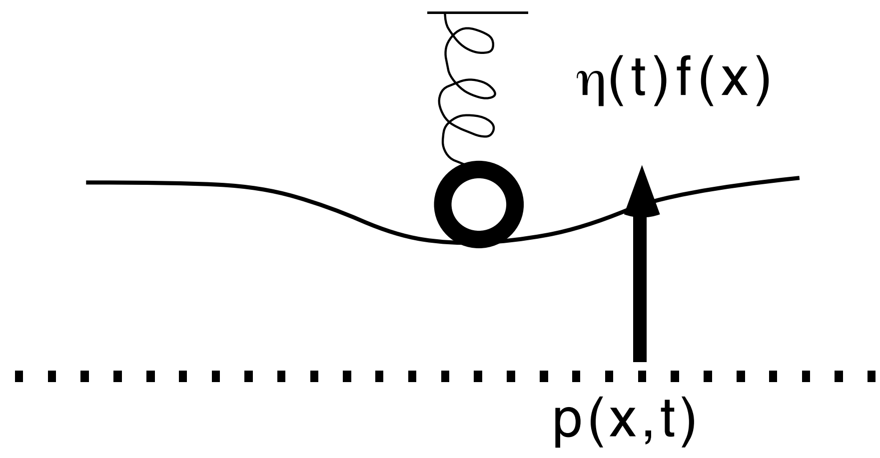


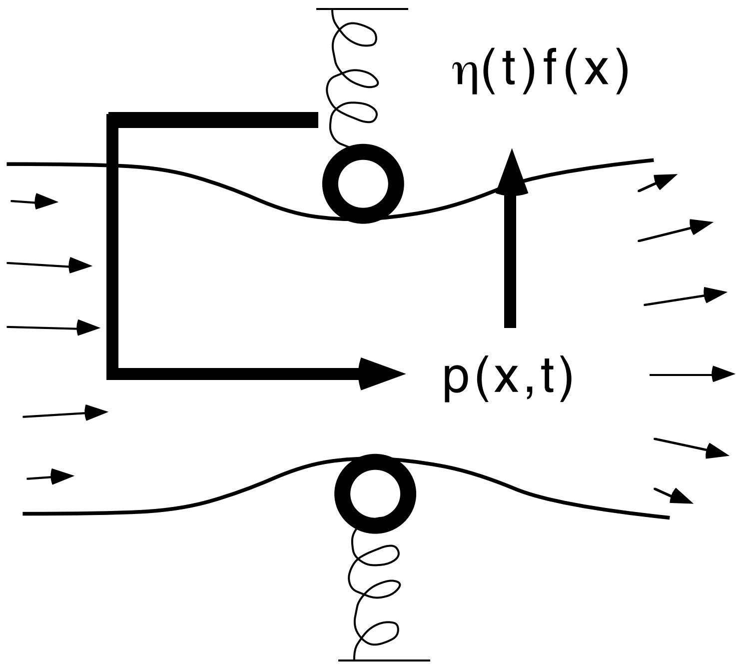
- flow with elastic wall with mass (glottis)

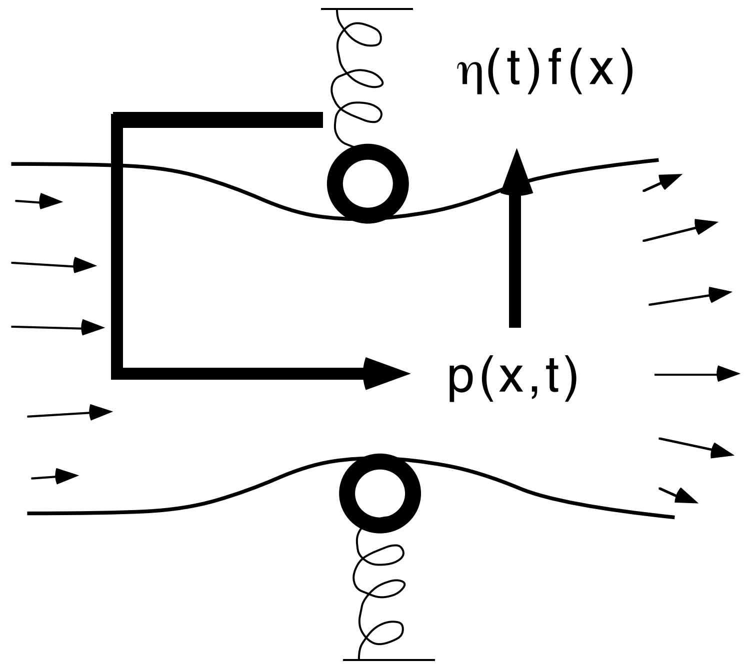


η(t)f(x)
p(x,t)

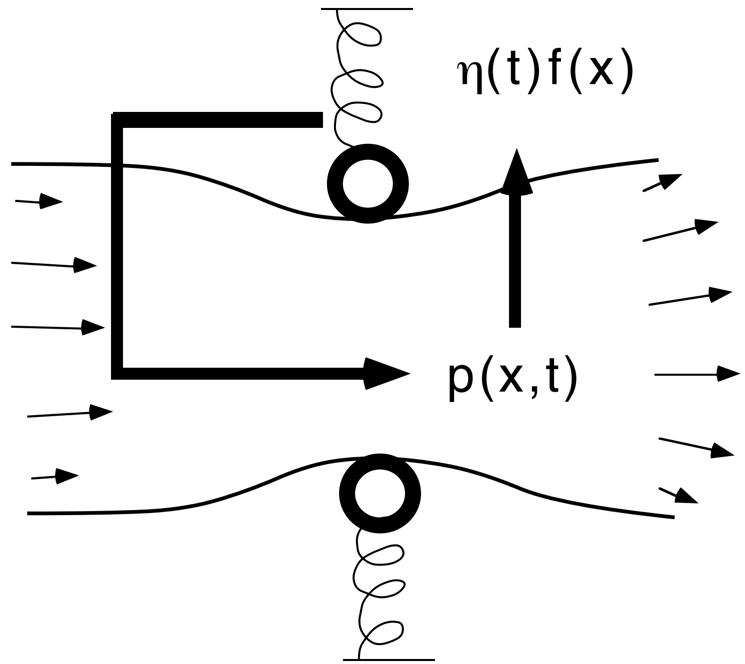

η(t)f(x)
p(x,t)


η(t)f(x)
p(x,t)

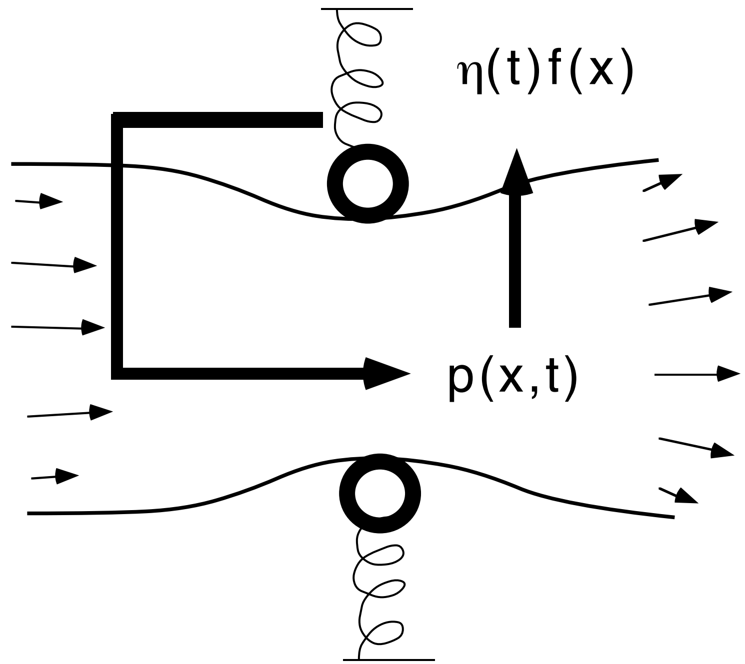

η(t) f(x)
p(x,t)



$$\mu\frac{\partial^2\eta}{\partial t^2}+k\eta=-p$$

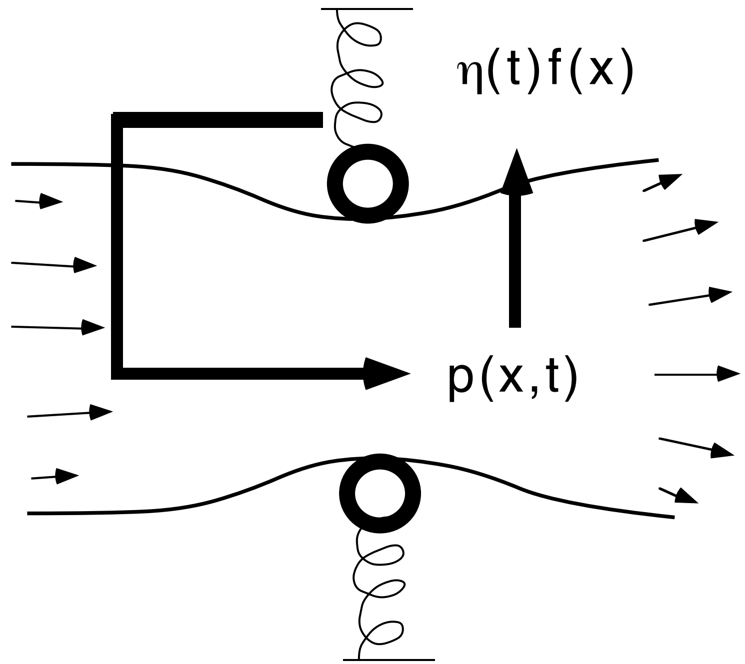


Newmark method for the spring:
prediction/ correction



Newmark method for the spring:
prediction/ correction

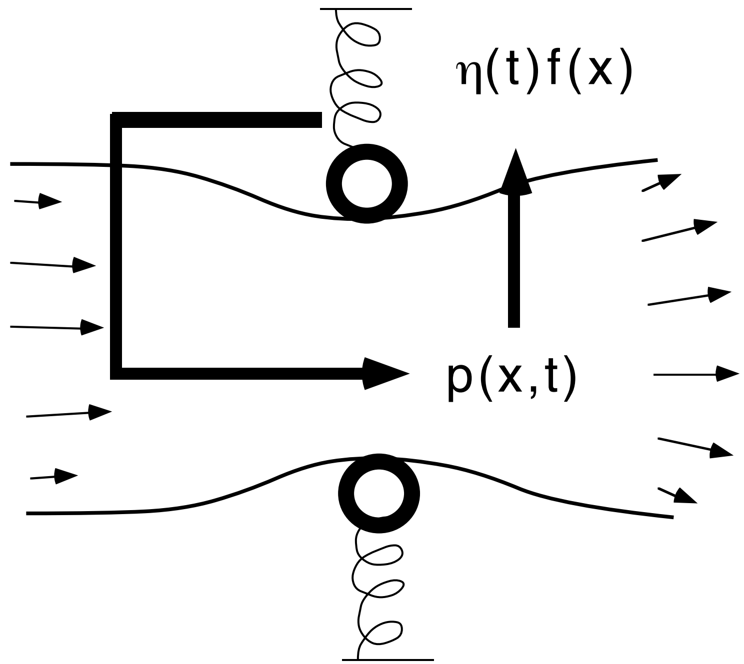
$$\eta^n, \frac{\partial \eta^n}{\partial t} \xrightarrow{\text{fluid}} p$$



Newmark method for the spring:
prediction/ correction

$$\eta^n, \frac{\partial \eta^n}{\partial t} \xrightarrow{\text{fluid}} p \qquad \eta^e, \frac{\partial \eta^e}{\partial t}$$

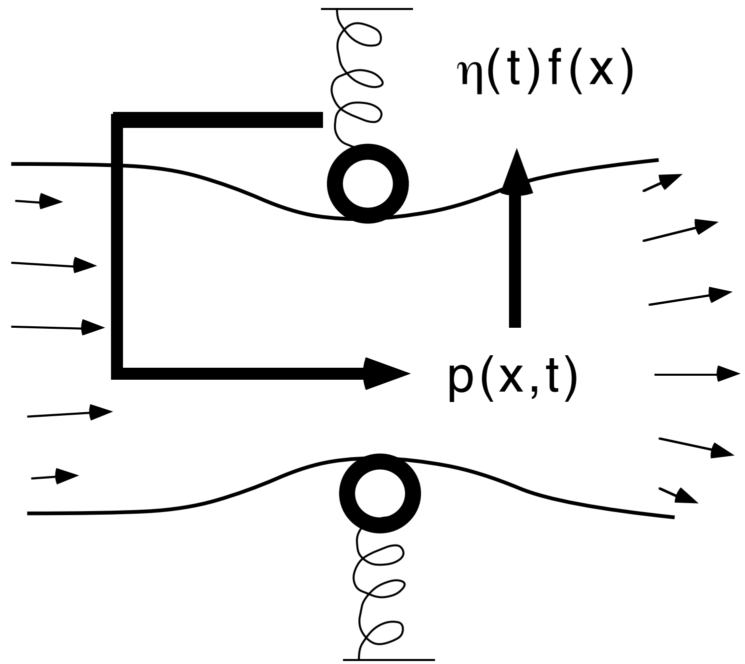
spring-prediction



Newmark method for the spring:
prediction/ correction

$$\eta^n, \frac{\partial \eta^n}{\partial t} \xrightarrow{\text{fluid}} p \qquad \eta^e, \frac{\partial \eta^e}{\partial t} \xrightarrow{\text{fluid}} p^e$$

spring-prediction

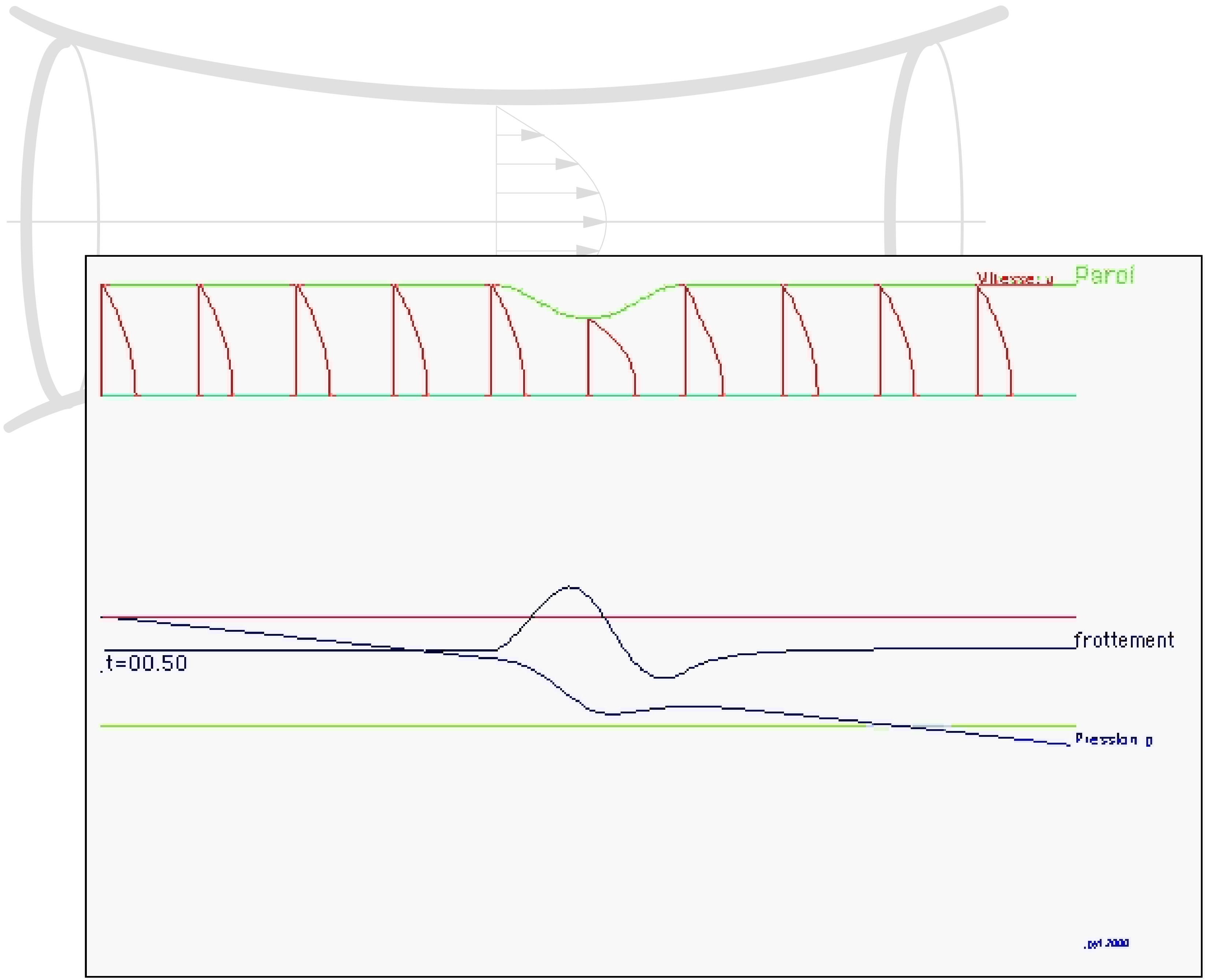


Newmark method for the spring: prediction/ correction

$$\eta^n, \frac{\partial \eta^n}{\partial t} \xrightarrow{\text{fluid}} p \qquad \eta^e, \frac{\partial \eta^e}{\partial t} \xrightarrow{\text{fluid}} p^e \qquad \eta^{n+1}, \frac{\partial \eta^{n+1}}{\partial t}$$

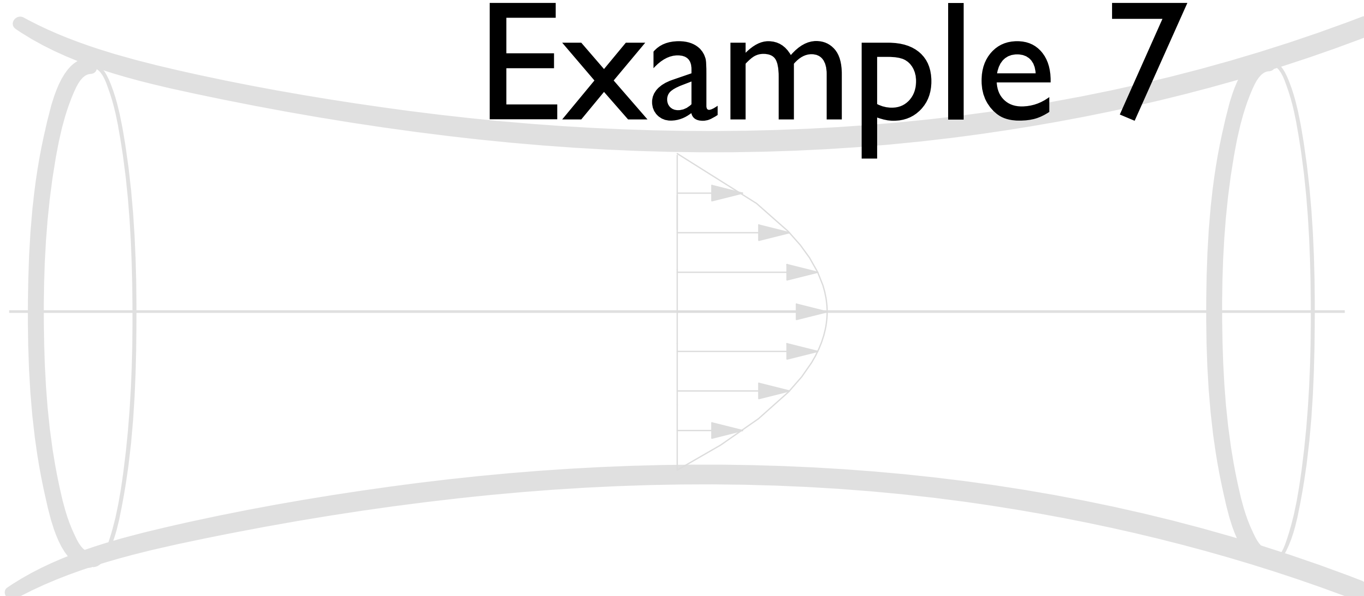
spring-prediction

spring- correction

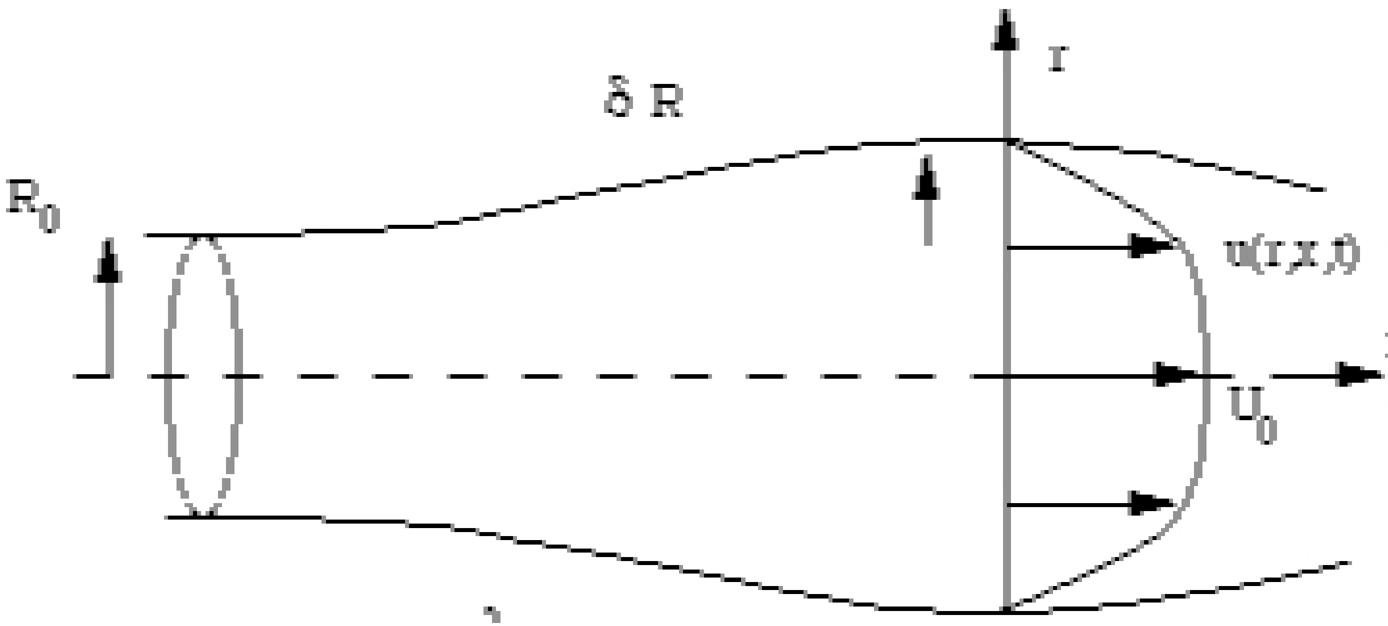


Vitesse: u
Paroi
frottement
t=00.50
Pression p

Example 7



flow in arteries



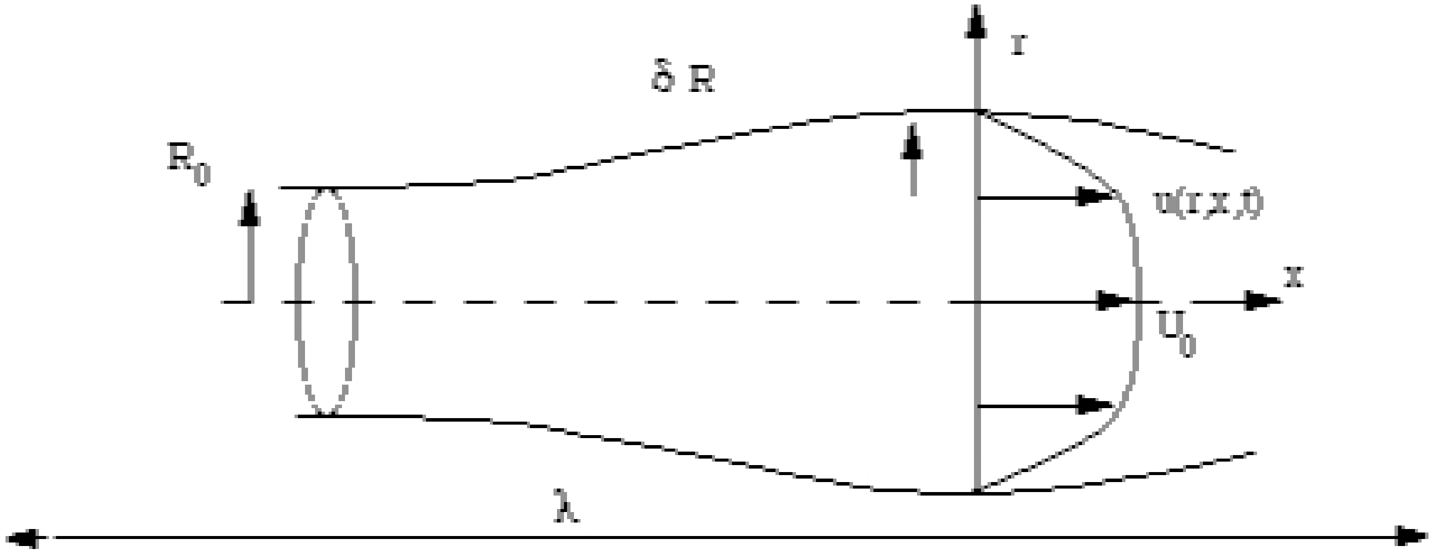
$$\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv = 0,$$

$$\frac{\partial u}{\partial t} + \varepsilon_2(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) = -\frac{\partial p}{\partial x} + \frac{2\pi}{\alpha^2}\frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), 0 = -\frac{\partial p}{\partial r}.$$

$$\varepsilon_2 = \frac{\delta R}{R_0}, \qquad \alpha = R_0\sqrt{\frac{2\pi/T}{\nu}}$$

introducing wall elasticity: $p(x,t) = k(R(x,t) - R_0)$

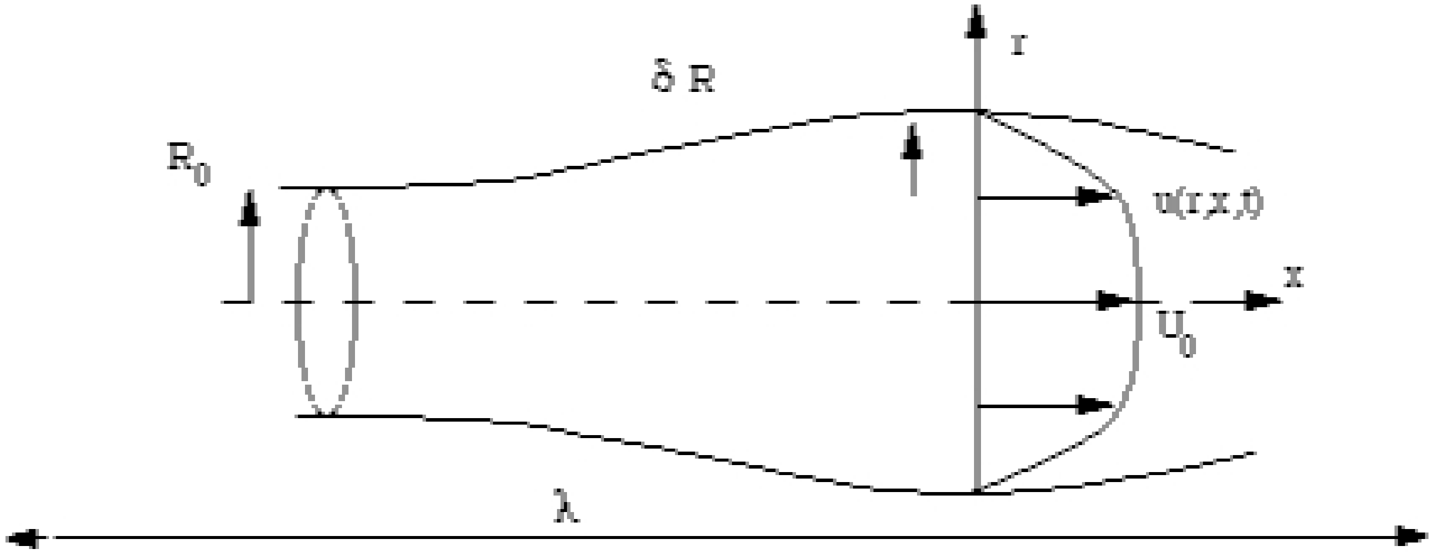
+ The boundary conditions: here hyperbolical ($R(x_{in}, t)$ and $R(x_{out}, t)$) given



week coupling

$$\frac{u^{n+1}-u^n}{\Delta t}+NL(u^n)=-\frac{\partial p^n}{\rho\partial x}+\nu\frac{\partial}{r\partial r}r\frac{\partial u^{n+1}}{\partial r}$$

$$v^{n+1}(R^n)=-\int_0^{R^n}\frac{r}{R^n}\frac{\partial u^{n+1}}{\partial x}dr$$



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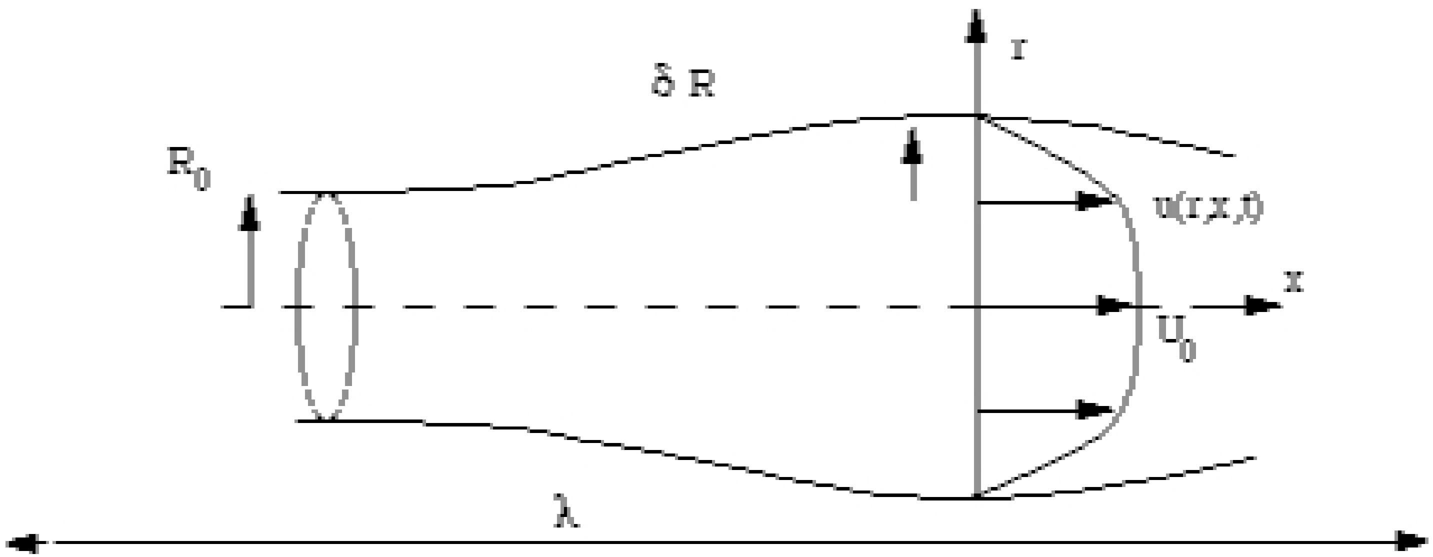

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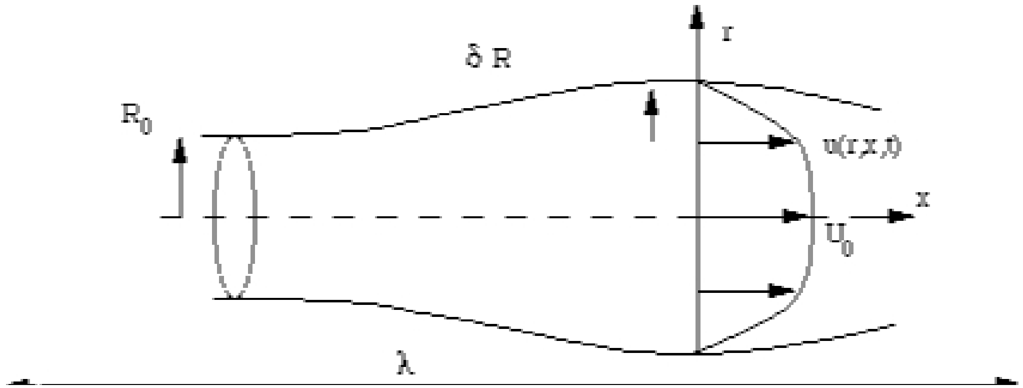
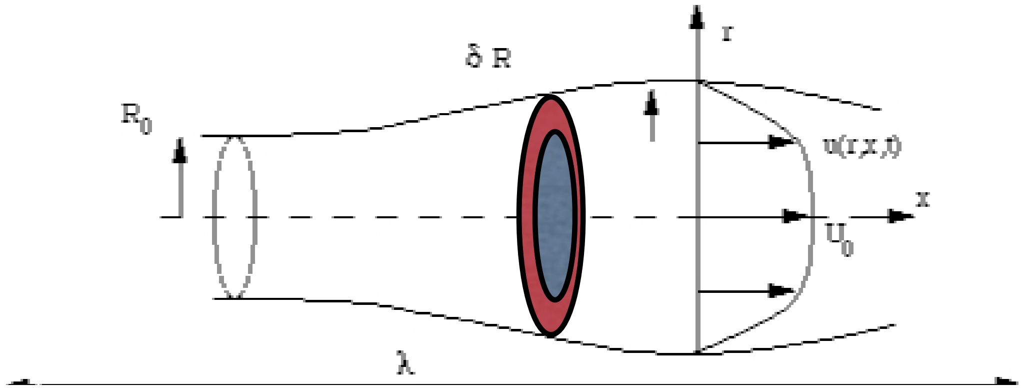
$$R^{n+1}=R^n+v^{n+1}(R^n)\Delta t \qquad p^{n+1}=k(R^{n+1}-R_0)$$



Flow in an elastic artery: integral relations

- new integral equations: adapting Von Kármán integral methods

The key is to integrate the equations with respect to the variable $\eta = r/R$ from the centre of the pipe to the wall ($0 \leq \eta \leq 1$).



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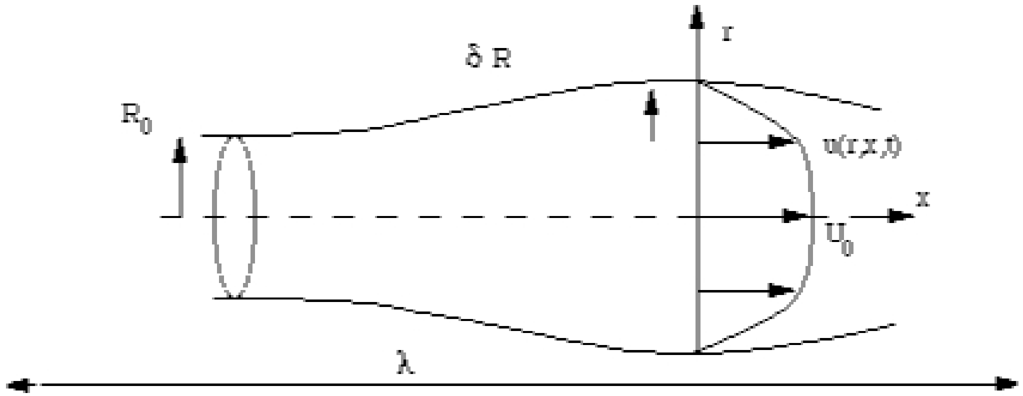
- U_0, the velocity along the axis of symmetry,

- q a kind of loss of flux (δ_1),

- Γ a kind of loss of momentum flux (δ_2):

$$U_0(x,t) = u(x, \eta = 0, t), \quad q = R^2(U_0 - 2\int_0^1 u\eta d\eta) \;\; \& \;\; \Gamma = R^2(U_0^2 - 2\int_0^1 u^2 \eta d\eta).$$



Flow in an elastic artery: integral relations

$$\frac{\partial R^2}{\partial t} + \varepsilon_2 \frac{\partial}{\partial x}(R^2 U_0 - q) = 0, \quad R = 1 + \varepsilon_2 h.$$

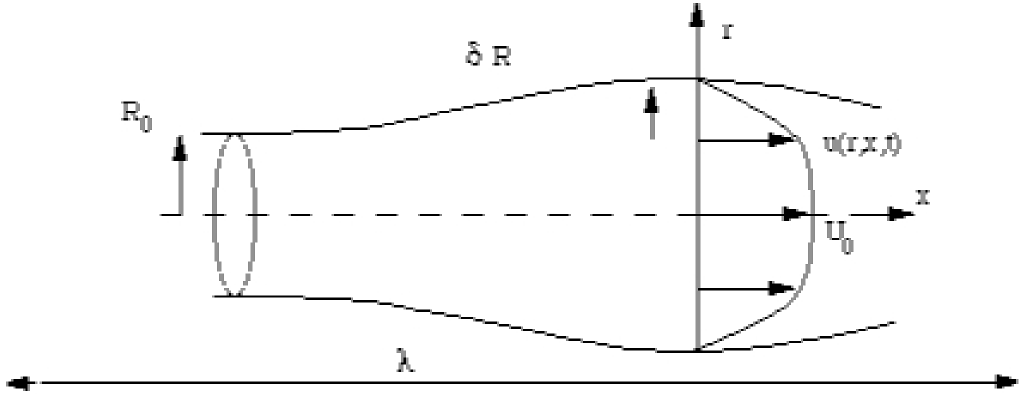
Integrating RNSP, with the help of the boundary conditions, we obtain the equation for $q(x,t)$:

$$\frac{\partial q}{\partial t} + \varepsilon_2 (\frac{\partial}{\partial x}\Gamma - U_0 \frac{\partial}{\partial x} q) = -2\frac{2\pi}{\alpha^2}\tau, \qquad \tau = (\frac{\partial u}{\partial \eta})|_{\eta=1} - (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$

From the same equation evaluated on the axis of symmetry (in $\eta = 0$), we obtain an equation for the velocity along the axis $U_0(x,t)$:

$$\frac{\partial U_0}{\partial t} + \varepsilon_2 U_0 \frac{\partial U_0}{\partial x} = -\frac{\partial p}{\partial x} + 2\frac{2\pi}{\alpha^2}\frac{\tau_0}{R^2}, \qquad \tau_0 = (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$

Boundary conditions ($h(x_{in}, t)$ and $h(x_{out}, t)$) given

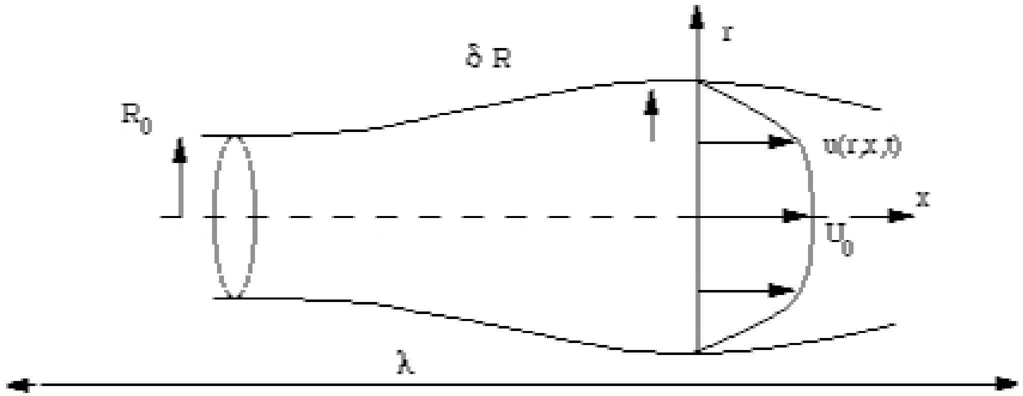


Closure

The two previous relations introduced the values of the friction in $\eta = 0$, the axis of symmetry: $((\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0})$ and the skin friction in $\eta = 1$, at the wall: $((\frac{\partial u}{\partial \eta})|_{\eta=1})$.

- Information has been lost here, so we need a closure relation between (Γ, τ, τ_0) and (q, R, U_0).

- we have to imagine a velocity profile and deduce from it relations linking Γ, τ and τ_0 and q, U_0 et R.

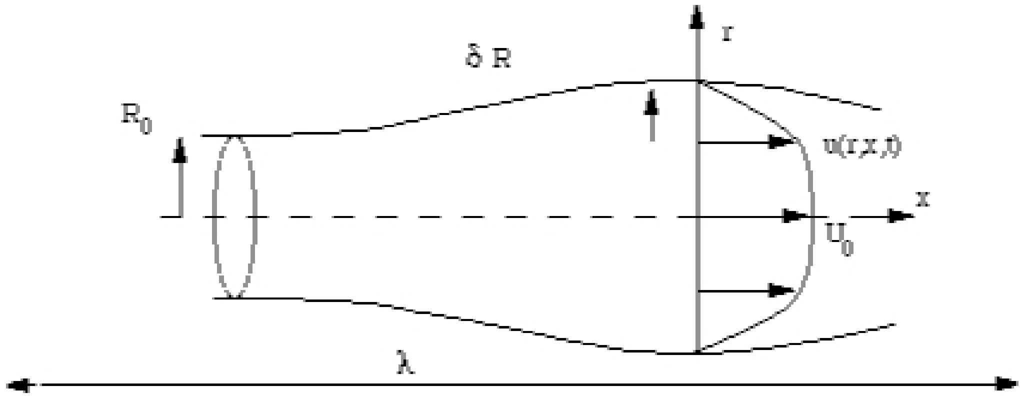


Closure: Womersley

• the most simple idea is to use the profiles from the analytical linearized solution given by Womersley (1955) for

$$(j_r + i j_i) = \left(\frac{1 - \frac{J_0(i^{3/2}\alpha\eta)}{J_0(i^{3/2}\alpha)}}{1 - \frac{1}{J_0(i^{3/2}\alpha)}} \right).$$

• assume that the velocity distribution in the following has the same dependence on η. It means that we suppose that the fundamental mode imposes the radial structure of the flow.

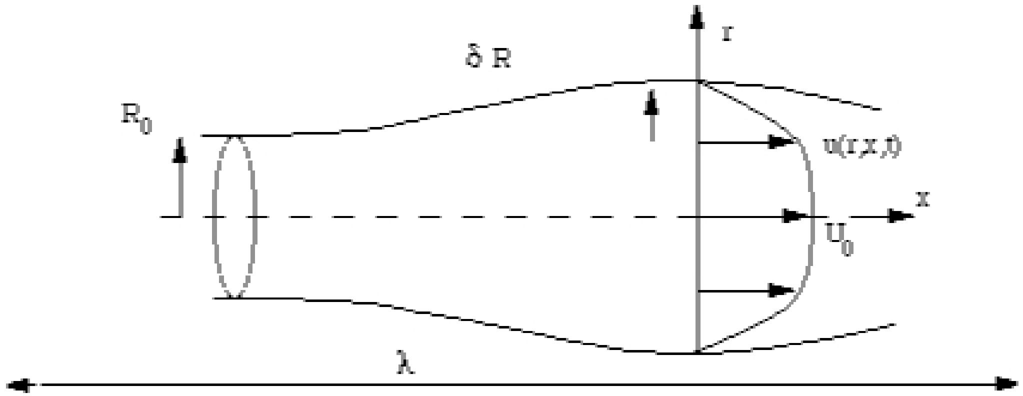


The coefficients of closure

- by integration/ derivation, we obtain:

$$\Gamma = \gamma_{qq}\frac{q^2}{R^2} + \gamma_{qu} q U_0 + \gamma_{uu} R^2 U_0^2, \quad \tau = \tau_q \frac{q}{R^2} + \tau_u U_0 \quad \tau_0 = \tau_{0q}\frac{q}{R^2} + \tau_{0u} U_0.$$

The coefficients $((\gamma_{qq}, \gamma_{qu}, \gamma_{uu}), (\tau_q, \tau_u), (\tau_{0q}, \tau_{0u}))$ are only functions of α.



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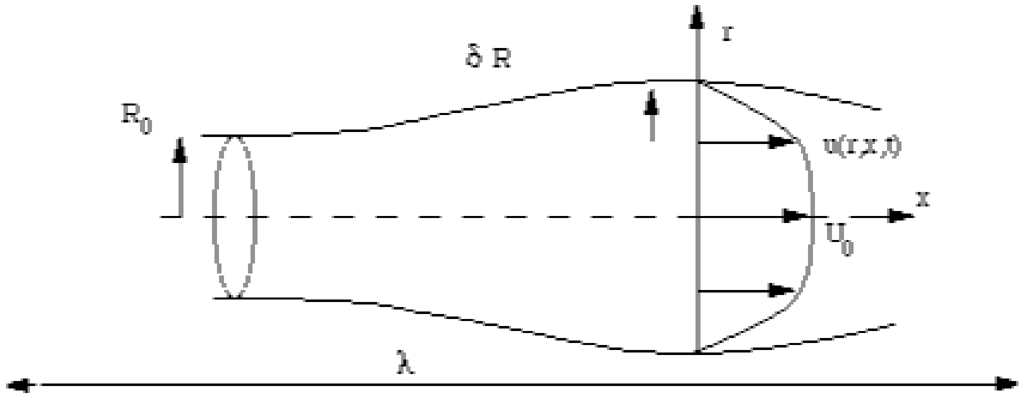
$$\begin{aligned}
\gamma_{uu} &= 1 - \int j_i^2/(\int j_i)^2 - (2\int j_r j_i)/\int j_i - \int j_r^2 + \\
&+(2\int j_i^2 \int j_r)/(\int j_i)^2 + (2\int j_i j_r \int j_r)/\int j_i - \\
&-(\int j_i^2(\int j_r)^2)/(\int j_i), \\
\tau_{0u} &= \partial_\eta^2 j_{r\eta=0} + \partial_\eta^2 j_{i\eta=0}/\int j_i - (\partial_\eta^2 j_{i\eta=0}\int j_r)/\int j_i.
\end{aligned}$$



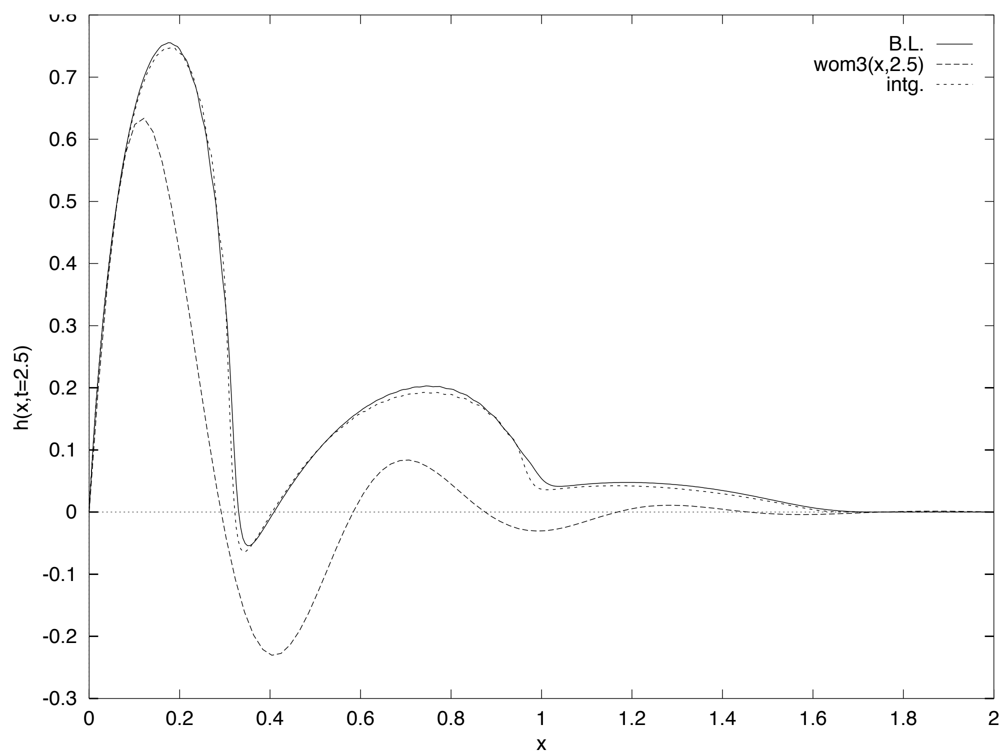


Figure 1: The displacement of the wall ($h(x, t = 2.5)$) as a function of x is plotted here at time $t = 2.5$. The dashed line (wom3(x,2.5)) is the Womersley solution (reference), the solid line (B.L.) is the result of the Boundary Layer code and the dots (intg) are the results of the integral method ($\alpha = 3$, $k_1 = 1$, $k_2 = 0$ and $\varepsilon_2 = 0.2$).

Conclusion

- starting from Navier Stokes
- set of simple equations RNSP
- set of more simple equations Integral
-
-
-
-

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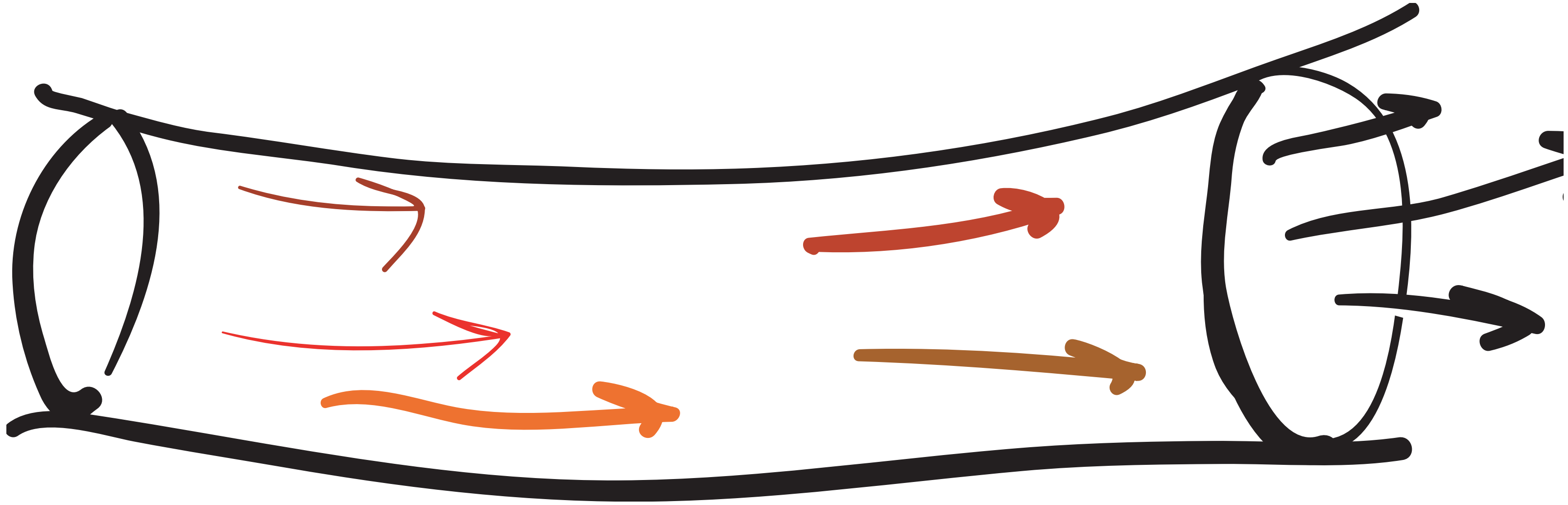
Conclusion

- starting from Navier Stokes
- set of simple equations RNSP
- set of more simple equations Integral
- Good agreement with full Navier Stokes
- “explain” the features of the flow
- boundary conditions for full NS
- real time simulation

- Use Acrobat Reader 7.05 to see animations
- Updated version may be found here.



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