Équations de Navier Stokes Réduites pour les écoulements biomécaniques: échelles caractéristiques et conditions aux limites.

Reduced Navier Stokes equations in Biomechanics: scales / boundary conditions

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Introduction

Aim: find out simplier equations than Navier Stokes

Well adapted for "real time simulations" / image processing

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Starting from Navier Stokes (Axi)

- we simplify NS to a Reduced set of equations
  - which contains the physical scales,
  - the most important phenomena
- much more simple set of equations: Integral equations (1D)
- cross comparisons in some cases of NS/ RNSP/ Integral

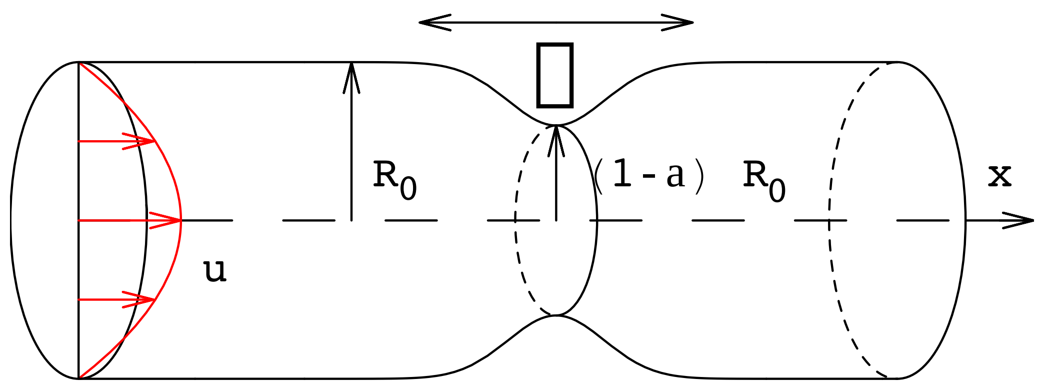
3 Applications

- Application 1/3: steady flow in a rigid axi symmetrical arterial stenosis evaluation of the maximum value of Wall Shear Stress

- Application 2/3: steady flow in a rigid 2D symmetrical glottis evaluation of the pressure drop

- Application 3/3: unsteady flow in an elastic axi symmetrical artery comparing Reduced equations and Integral equations

RNSP Scales



Using:

$x^* = xR_0Re$, $r^* = rR_0$, $u^* = U_0u$, $v^* = \frac{U_0}{Re}v$,
$p^* = p_0^* + \rho_0 U_0^2 p$ and $\tau^* = \frac{\rho U_0^2}{Re}\tau$

the following partial differential system is obtained from Navier Stokes as $Re \to \infty$:

RNSP: Reduced Navier Stokes/ Prandtl System

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$$\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv = 0,$$

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\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0, \\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u),
\end{aligned}
$$

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$+$ The boundary conditions.

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- axial symmetry $\big(\partial_r u = 0$ and $v = 0$ at $r = 0\big)$,

- no slip condition at the wall $\big(u = v = 0$ at $r = 1 - f(x)\big)$,

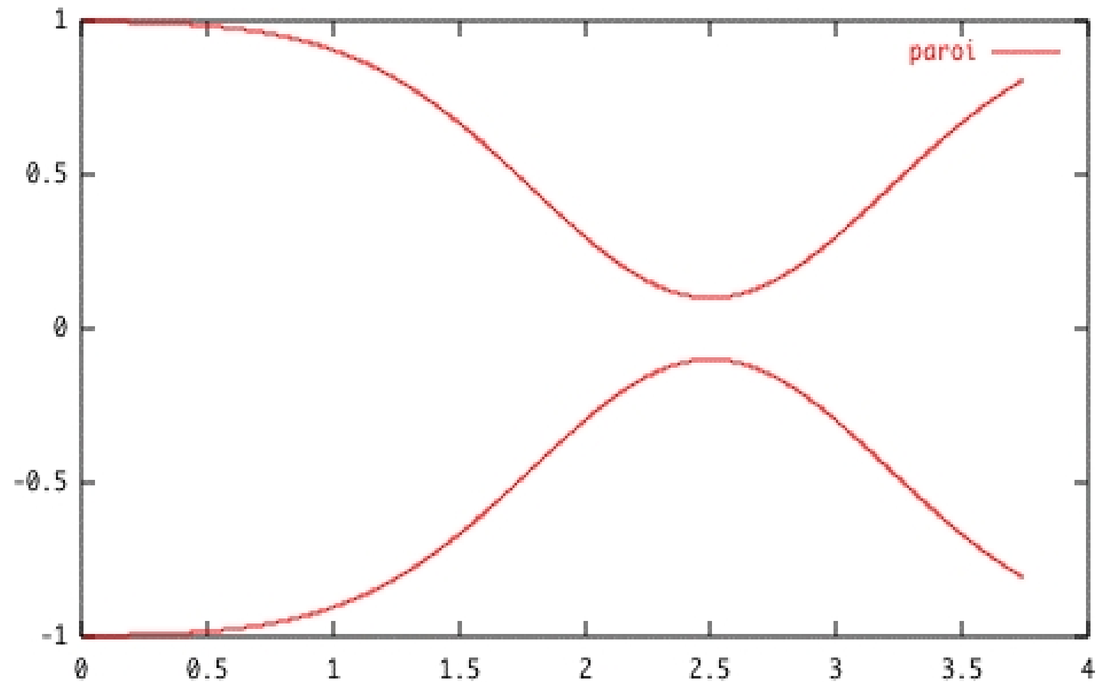
- the entry velocity profiles $\big(u(0, r)$ and $v(0, r)\big)$ are given

- *no* output condition in $x_{out} = \frac{x^*_{out}}{R_0 Re}$

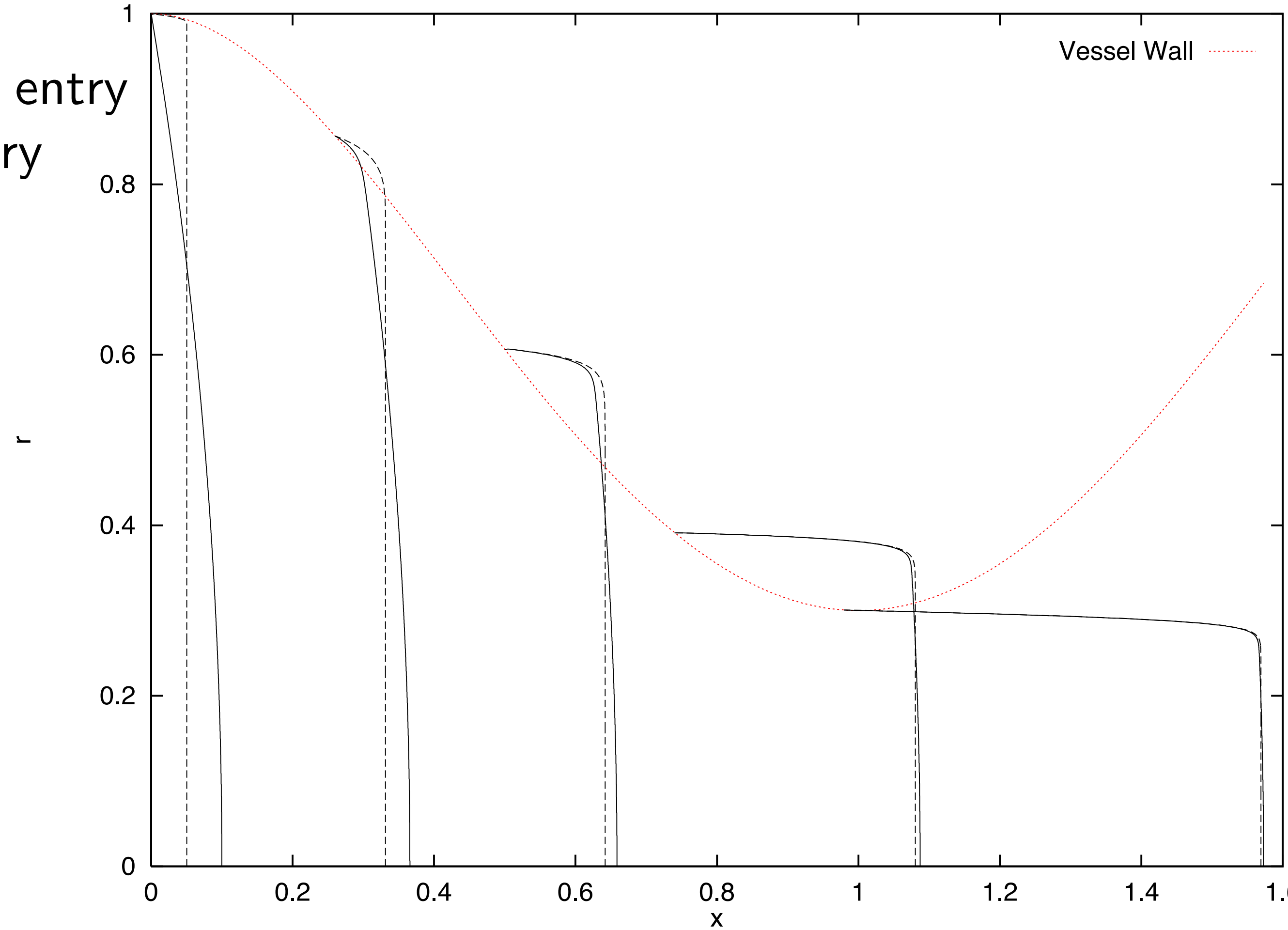
- streamwise marching, even when flow separation.

Application 1/3: Flow in an arterial stenosis

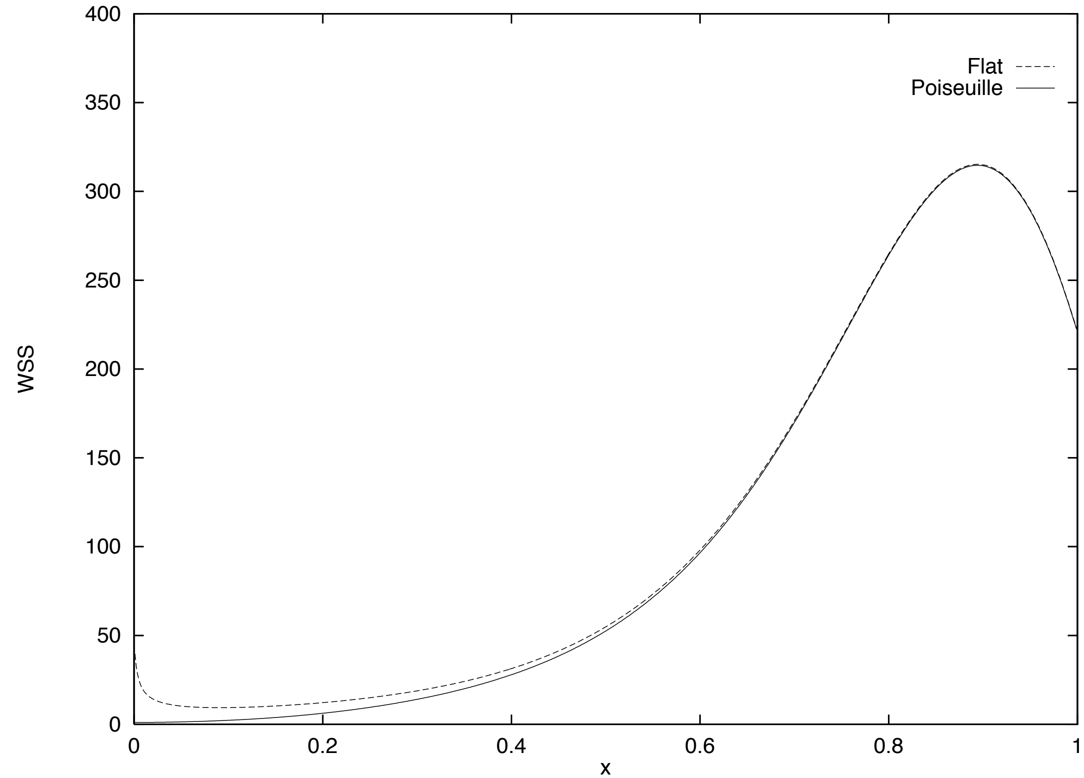
collaboration with S. Lorthois IMFT
(F. Cassot & M.-P. Vergnes, INSERM, + B. de Bruin RuG)



Evolution of the velocity profile along the convergent part of a 70% stenosis ($Re = 500$) ;
solid line: Poiseuille entry
broken line: flat entry



Wall Shear Stress

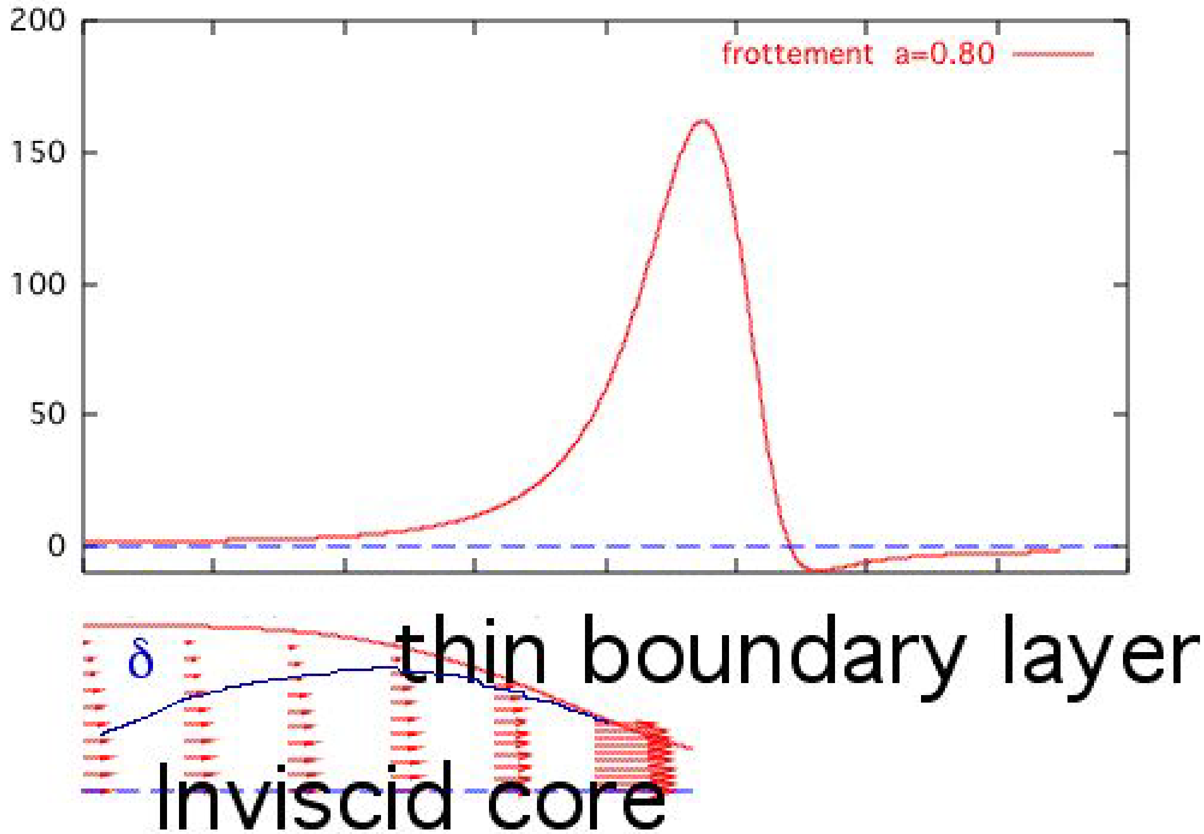


Evolution of the WSS distribution along the convergent part of a 70% stenosis ($Re = 500$) ; solid line: Poiseuille entry profile ; broken line: flat entry profile.

Example of numerical resolution

Various values of the stenosis degree: $\underline{animation}$

Boundary Layer/ Perfect Fluid



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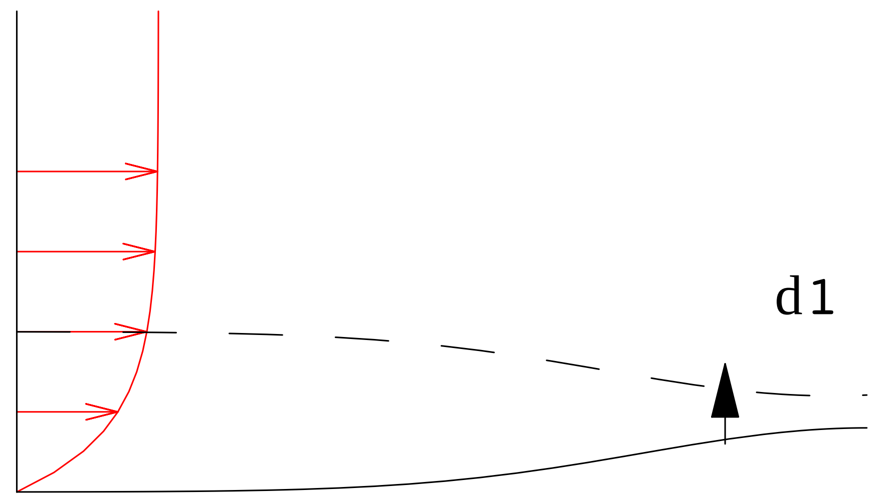
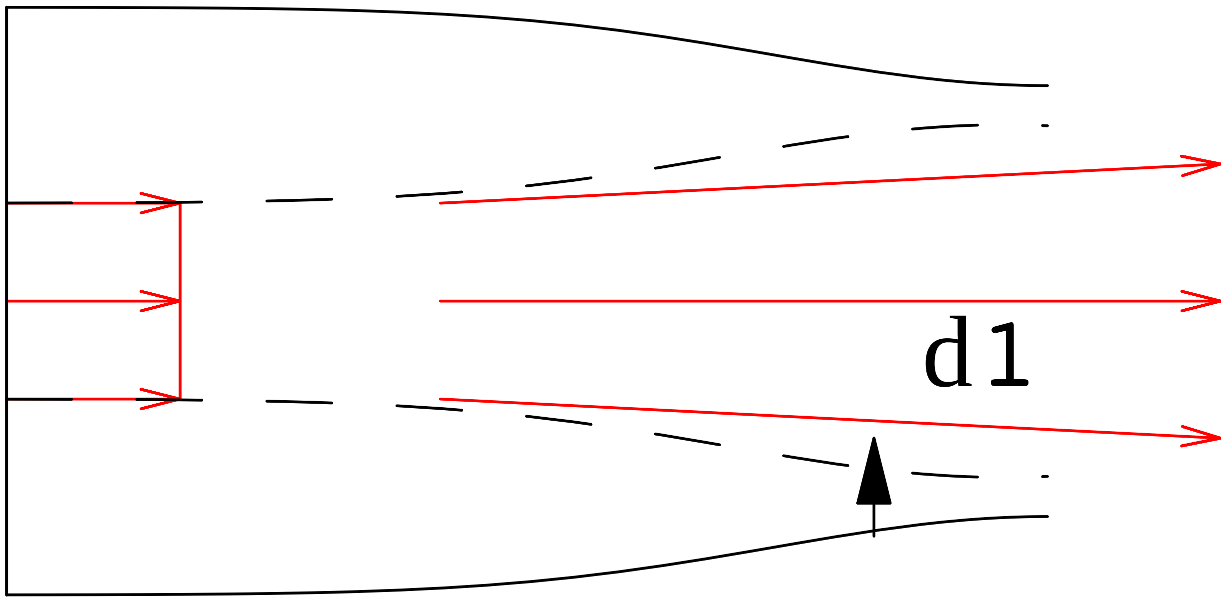


The boundary layer is generated near the wall
δ_1 is the displacement thickness.

Boundary Layer/ Perfect Fluid



The displacement thickness acts as a "new" wall!
→Interacting Boundary Layer (IBL)

RNSP/ IBL

After rescalling:
$r = R(\bar{x}) - (\lambda/Re)^{-1/2}\bar{y}$, $u = \bar{u}$, $v = (\lambda/Re)^{1/2}\bar{v}$ and $x - x_b = (\lambda/Re)\bar{x}$, $p = \bar{p}$, where x_b is the position of the bump, the RNSP(x) set gives the final IBL (interacting Boundary Layer) problem as follows:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{n}} = 0$$

$$(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{n}}) = \bar{u}_e \frac{d\bar{u}_e}{d\bar{s}} + \frac{\partial}{\partial \bar{n}}\frac{\partial \bar{u}}{\partial \bar{n}}$$

with: $\bar{u}(\bar{x}, 0) = 0$, $\bar{v}(\bar{x}, 0) = 0$ $\bar{u}(\bar{x}, \infty) = u_e$, where $\bar{\delta}_1 = \int_0^\infty (1 - \frac{\bar{u}}{\bar{u}_e}) d\bar{n}$, and

$$\bar{u}_e = \frac{1}{(R^2 - 2((\lambda/Re)^{-1/2})\bar{\delta}_1)}.$$

IBL integral: 1D equation

$$\frac{d}{d\bar{x}}(\frac{\bar{\delta}_1}{H}) = \bar{\delta}_1(1+\frac{2}{H})\frac{d\bar{u}_e}{d\bar{x}} + \frac{f_2 H}{\bar{\delta}_1 \bar{u}_e},$$
$$\bar{u}_e = \frac{1}{(R^2 - 2(\lambda/Re)^{-1/2}\bar{\delta}_1)}.$$

To solve this system, a closure relationship linking H and f_2 to the velocity and the displacement thickness is needed.

Defining $\Lambda_1 = \bar{\delta}_1^2 \frac{d\bar{u}_e}{d\bar{x}}$,

the system is closed from the resolution of the Falkner Skan system as follows:

if $\Lambda_1 < 0.6$ then $H = 2.5905 exp(-0.37098\Lambda_1)$, else $H = 2.074$.

From H,f_2 is computed as $f_2 = 1.05(-H^{-1} + 4H^{-2})$.

IBL integral: 1D equation Simplified Shear Stress

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- WSS $=$ (variation of velocity)/(boundary layer thickness)

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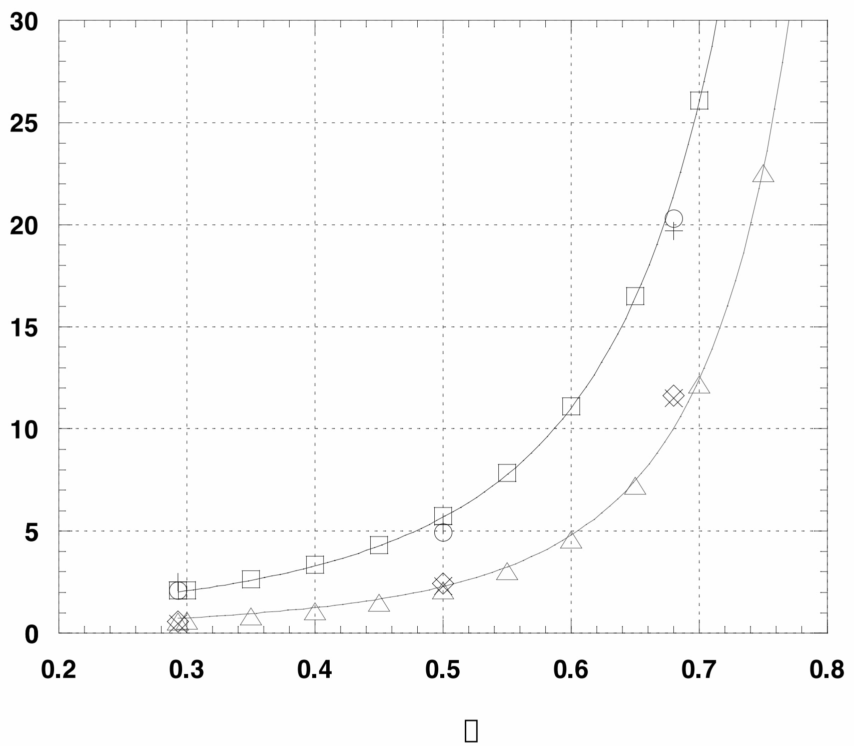
A simple formula as been settled:

$$WSS = (\mu\frac{\partial u^*}{\partial y^*})/((\mu\frac{4U_0}{R})) \sim .22\frac{((Re/\lambda)^{1/2}+3)}{(1-\alpha)^3}$$

Reynolds number is no longer Re but $Re\lambda$ and $(Re/\lambda)^{1/2}$ is the inverse of the relative boundary layer thickness.

IBL integral: Comparison with Navier Stokes (Siegel et al. 1994)

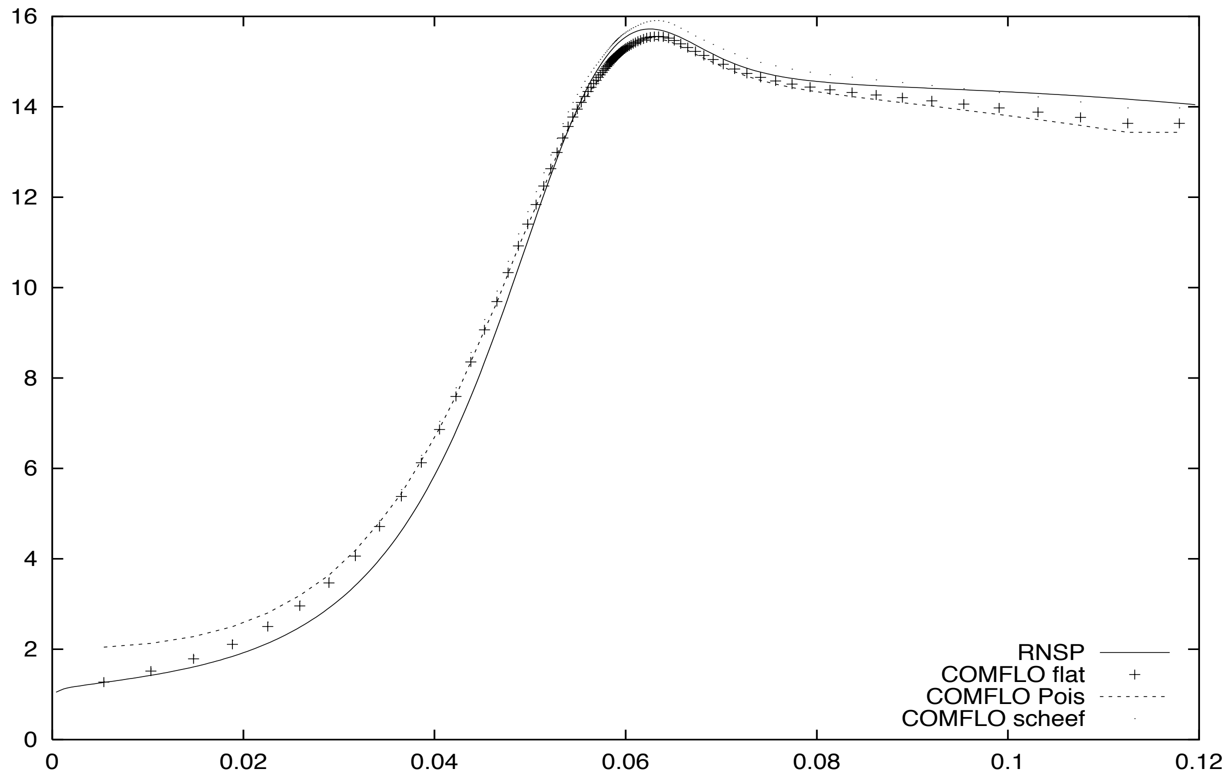


$$WSS = aRe^{1/2} + b$$

Coefficient a and b for the maximum WSS.
solid lines with $\triangle$ and "square" : coefficient a and b obtained using the IBL integral method ;

$\diamond$: coefficient a derived from Siegel for $\lambda = 3$;
$\times$: coefficient a derived from Siegel for $\lambda = 6$;
$\bigcirc$: coefficient b derived from Siegel for $\lambda = 3$;
$+$: coefficient b derived from Siegel for $\lambda = 6$.

Testing asymmetry in the entry profile



The velocities in the middle for Comflo and RNS.
Comflo uses here 50X50X100 points. Dimensionless scales!

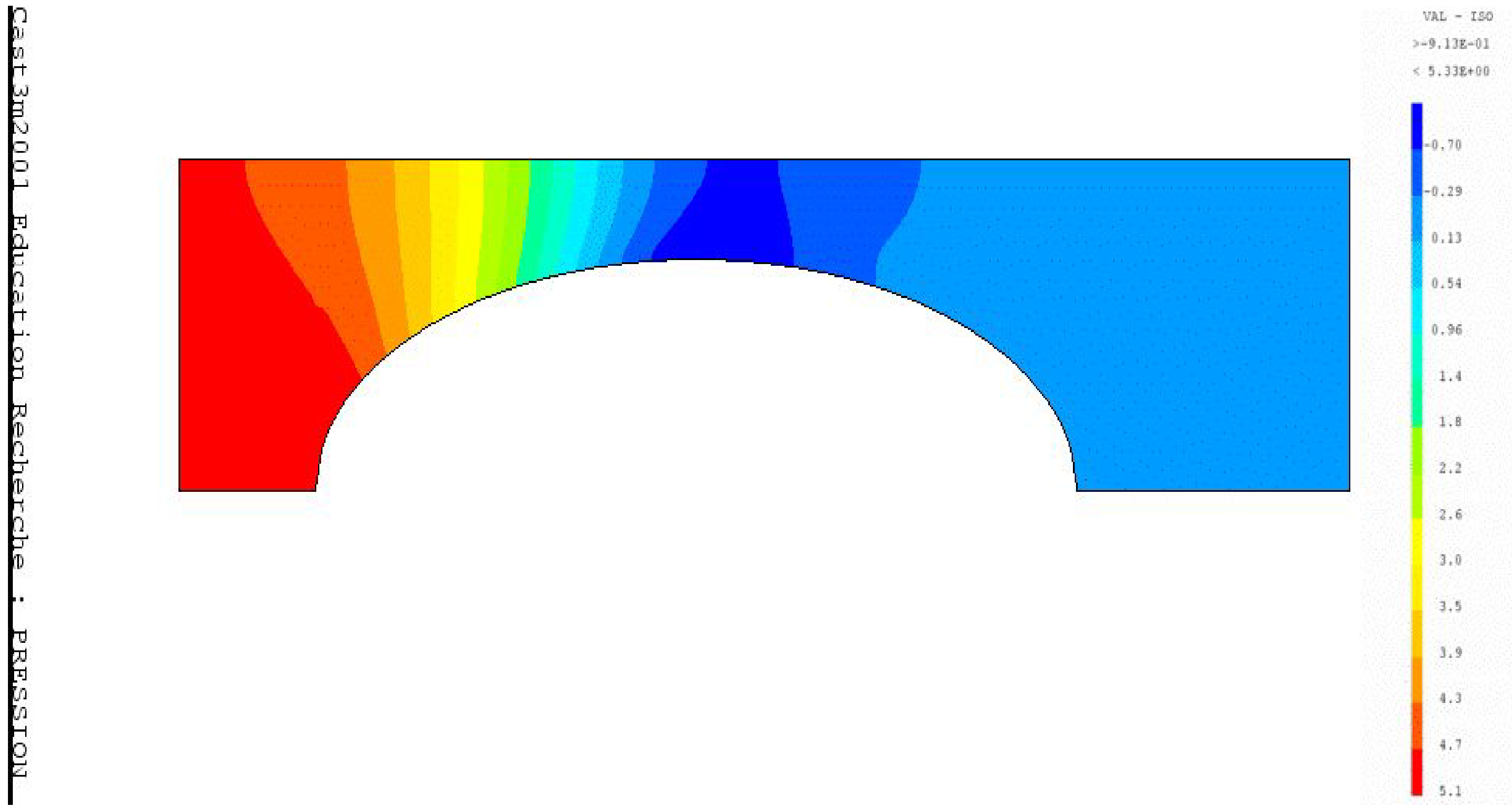
Application 2/3: Flow in the glottis

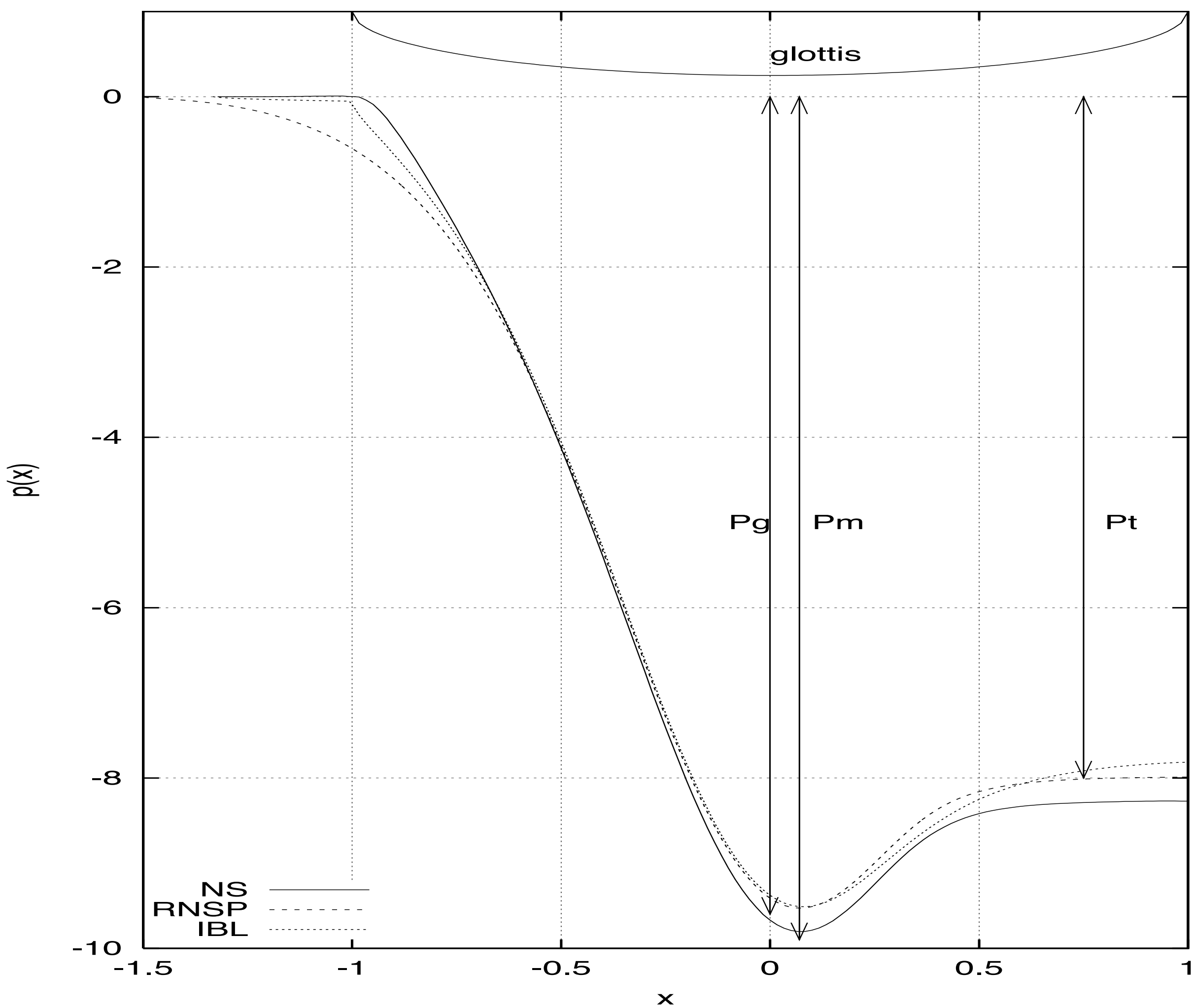
collaboration E. Berger (LMM), M. Deverge (TUE), C. Vilain (ICP)
& A. Hirschberg (TUE) + B. de Bruin (RuG)

idem in 2D !!!

$$\begin{aligned} \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v &= 0, && (1) \\ u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u &= -\frac{\partial}{\partial x}p + \frac{\partial^2}{\partial y^2}u, \qquad 0 = -\frac{\partial}{\partial y}p. && (2) \end{aligned}$$

Application 2/3: Flow in the glottis





glottis
0
-2
-4
-6
-8
-10
p(x)
Pg
Pm
Pt
NS
RNSP
IBL
-1.5
-1
-0.5
0
0.5
1
x

Application 2/3: Flow in the glottis

- nearly constant "shape":
  $K_e = P_t/P_m$ is nearly constant $K_e \simeq 0.82$
  $K_g = P_g/P_m$ is nearly constant $K_g \simeq 0.97$.

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$$P_m \simeq -\frac{1}{2}\left(\frac{1}{(1-\alpha-\delta_{1c})^2} - 1\right).$$

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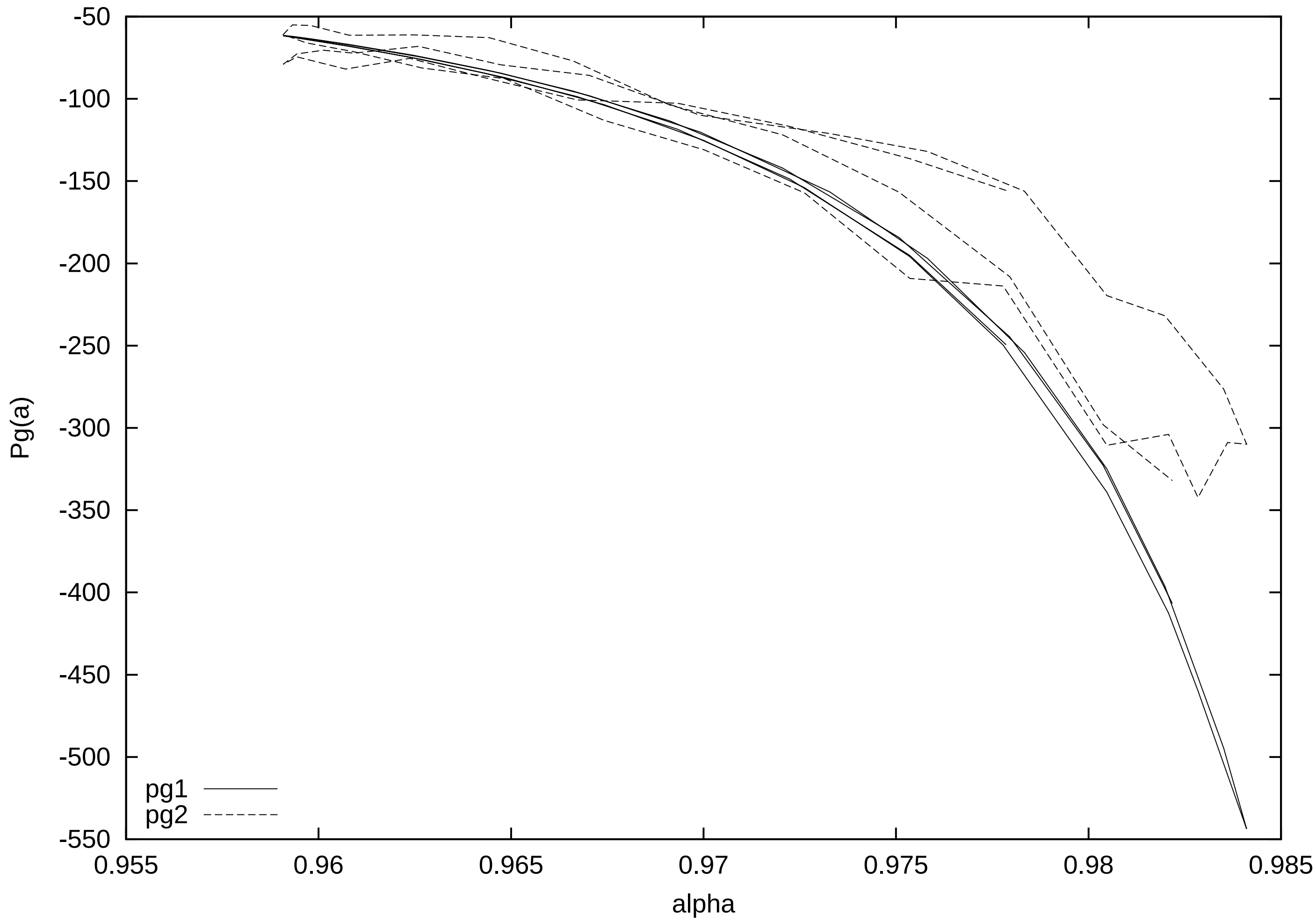
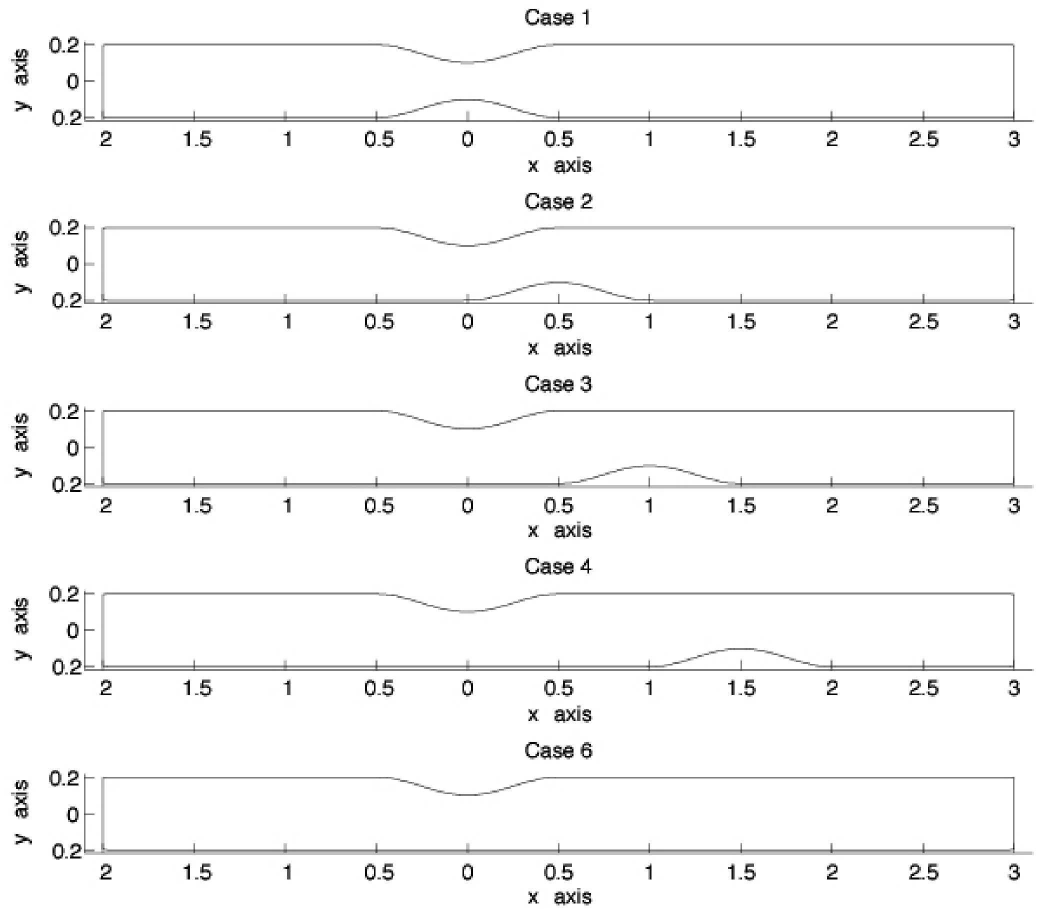
- with $\delta_{1c} \simeq (1-\alpha)(Re)^{-1/2}$

Experiments



"pg1" : $(-0.17 P_t^* / \rho U_0^2)$, "pg2"$(P_g^* / \rho U_0^2)$ (P_t^* and P_g^* experimental pressures)

Testing upstream influence (Re=200)



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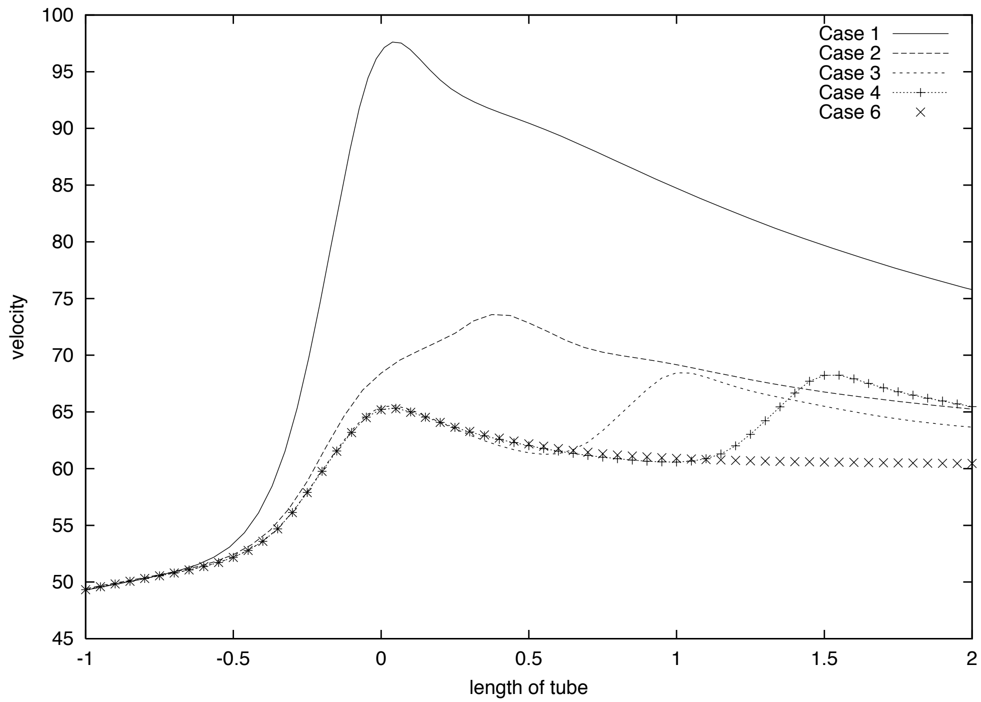

The velocity in the middle of the tube for stenosis 1, 2, 3, 4 and 6. Dimensions in cm

Application 3/3: Flow in an elastic artery

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$$\varepsilon_2 = \frac{\delta R}{R_0}, \qquad \alpha = R_0\sqrt{\frac{2\pi/T}{\nu}}$$

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$+$ The boundary conditions: here hyperbolical ($R(x_{in},t)$ and $R(x_{out},t)$) given.

Flow in an elastic artery: integral relations

- new integral equations: adapting Von Kármán integral methods

The key is to integrate the equations with respect to the variable $\eta = r/R$ from the centre of the pipe to the wall $(0 \leq \eta \leq 1)$.

- U_0, the velocity along the axis of symmetry,

- q a kind of loss of flux (δ_1),

- Γ a kind of loss of momentum flux (δ_2):

$$U_0(x,t) = u(x, \eta = 0, t), \quad q = R^2(U_0 - 2\int_0^1 u\eta d\eta) \;\; \& \;\; \Gamma = R^2(U_0^2 - 2\int_0^1 u^2\eta d\eta).$$

Flow in an elastic artery: integral relations

$$\frac{\partial R^2}{\partial t} + \varepsilon_2 \frac{\partial}{\partial x}(R^2 U_0 - q) = 0, \quad R = 1 + \varepsilon_2 h.$$

Integrating RNSP, with the help of the boundary conditions, we obtain the equation for $q(x,t)$:

$$\frac{\partial q}{\partial t} + \varepsilon_2 (\frac{\partial}{\partial x}\Gamma - U_0 \frac{\partial}{\partial x} q) = -2\frac{2\pi}{\alpha^2}\tau, \qquad \tau = (\frac{\partial u}{\partial \eta})|_{\eta=1} - (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$

From the same equation evaluated on the axis of symmetry (in $\eta = 0$), we obtain an equation for the velocity along the axis $U_0(x,t)$:

$$\frac{\partial U_0}{\partial t} + \varepsilon_2 U_0 \frac{\partial U_0}{\partial x} = -\frac{\partial p}{\partial x} + 2\frac{2\pi}{\alpha^2}\frac{\tau_0}{R^2}, \qquad \tau_0 = (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$

Boundary conditions ($h(x_{in}, t)$ and $h(x_{out}, t)$) given

Closure

The two previous relations introduced the values of the friction in $\eta = 0$, the axis of symmetry: $((\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0})$ and the skin friction in $\eta = 1$, at the wall: $((\frac{\partial u}{\partial \eta})|_{\eta=1})$.

- Information has been lost here, so we need a closure relation between (Γ, τ, τ_0) and (q, R, U_0).

- we have to imagine a velocity profile and deduce from it relations linking Γ, τ and τ_0 and q, U_0 et R.

Closure: Womersley

• the most simple idea is to use the profiles from the analytical linearized solution given by Womersley (1955) for

$$(j_r + i j_i) = \left(\frac{1 - \frac{J_0(i^{3/2}\alpha\eta)}{J_0(i^{3/2}\alpha)}}{1 - \frac{1}{J_0(i^{3/2}\alpha)}} \right).$$

• assume that the velocity distribution in the following has the same dependence on η. It means that we suppose that the fundamental mode imposes the radial structure of the flow.

The coefficients of closure

- by integration/ derivation, we obtain:

$$\Gamma = \gamma_{qq}\frac{q^2}{R^2} + \gamma_{qu} q U_0 + \gamma_{uu} R^2 U_0^2, \quad \tau = \tau_q \frac{q}{R^2} + \tau_u U_0 \quad \tau_0 = \tau_{0q}\frac{q}{R^2} + \tau_{0u} U_0.$$

The coefficients $((\gamma_{qq}, \gamma_{qu}, \gamma_{uu}), (\tau_q, \tau_u), (\tau_{0q}, \tau_{0u}))$ are only functions of α.

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$$\begin{aligned}
\gamma_{uu} &= 1 - \int j_i^2/(\int j_i)^2 - (2\int j_r j_i)/\int j_i - \int j_r^2 + \\
&\quad +(2\int j_i^2 \int j_r)/(\int j_i)^2 + (2\int j_i j_r \int j_r)/\int j_i - \\
&\quad -(\int j_i^2 (\int j_r)^2)/(\int j_i), \\
\tau_{0u} &= \partial_\eta^2 j_{r\eta=0} + \partial_\eta^2 j_{i\eta=0}/\int j_i - (\partial_\eta^2 j_{i\eta=0}\int j_r)/\int j_i.
\end{aligned}$$

Remarks

- The main difference from other integral methods in our approach is the introduction of an auxillary partial differential relation obtained from an aeronautical analogy. Instead of q, Γ and U_0 authors mainly use Q, Q_2 and U_0:

$$Q = \int_0^R 2\pi u r dr \qquad Q/\pi = U_0 R^2 - q$$

$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad Q_2/\pi = U_0^2 R^2 - \Gamma$$

Remarks

Substracting our third from our second equation we obtain the classical system:

$$2\pi R\frac{\partial R}{\partial t}+\varepsilon_2\frac{\partial}{\partial x}(Q)=0,$$

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The effect of the skin friction ($\tau_1 = \frac{2\pi}{\alpha^2}(\frac{\partial u}{\partial \eta})|_{\eta=1}$) is often estimated by $\tau_1 = -\frac{8\pi}{\alpha^2}\frac{Q}{\pi R^3}$, true for a Poiseuille flow only. It may be replaced by an unsteady relation such as:

$$T_\tau \frac{\partial \tau_1}{\partial t} + \tau_1 = -\frac{8}{\alpha^2}(Q + T_Q \frac{\partial Q}{\partial t} + ...)$$

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$$\frac{\partial Q}{\partial t} + \varepsilon_2\frac{\partial}{\partial x}(Q_2) = \ -\pi R^2\frac{\partial p}{\partial x} + \pi\frac{2\pi}{\alpha^2}(\frac{\partial u}{\partial \eta})|_{\eta=1}$$

The effect of the skin friction $(\tau_1 = \frac{2\pi}{\alpha^2}(\frac{\partial u}{\partial \eta})|_{\eta=1})$ is often estimated by $\tau_1 = -\frac{8\pi}{\alpha^2}\frac{Q}{\pi R^3}$, true for a Poiseuille flow only. It may be replaced by an unsteady relation such as:

$$T_\tau\frac{\partial \tau_1}{\partial t} + \tau_1 = -\frac{8}{\alpha^2}(Q + T_Q\frac{\partial Q}{\partial t} + ...)$$

- We do not claim that our description is better, but for a sinusoidal input we find again (at any frequency) the Womersley linear solution. Our profiles are realistic in the sense that they present overshoots in the core and back flow near the wall.

Comparison RNSP/ Integral 1D/ pure Womersley

- RNSP

$$\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv = 0,$$
$$\frac{\partial u}{\partial t} + \varepsilon_2(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) = -\frac{\partial p}{\partial x} + \frac{2\pi}{\alpha^2}\frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), 0 = -\frac{\partial p}{\partial r}.$$
$$p(x,t) = k(R(x,t) - R_0)$$

- Integral 1D

$$\frac{\partial R^2}{\partial t} + \varepsilon_2\frac{\partial}{\partial x}(R^2U_0 - q) = 0, \quad R = 1 + \varepsilon_2 h.$$
$$\frac{\partial q}{\partial t} + \varepsilon_2(\frac{\partial}{\partial x}\Gamma - U_0\frac{\partial}{\partial x}q) = -2\frac{2\pi}{\alpha^2}\tau, \qquad \tau = (\frac{\partial u}{\partial \eta})|_{\eta=1} - (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$
$$\frac{\partial U_0}{\partial t} + \varepsilon_2 U_0\frac{\partial U_0}{\partial x} = -\frac{\partial p}{\partial x} + 2\frac{2\pi}{\alpha^2}\frac{\tau_0}{R^2}, \qquad \tau_0 = (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$
$$p(x,t) = k(h(x,t))$$

- Womersley

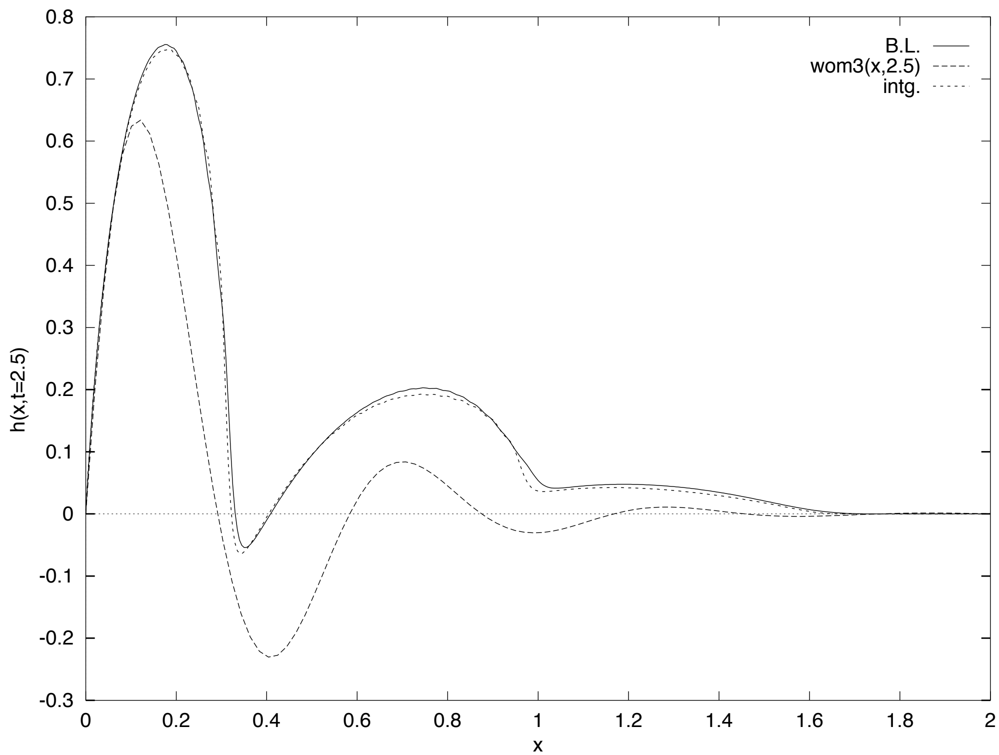
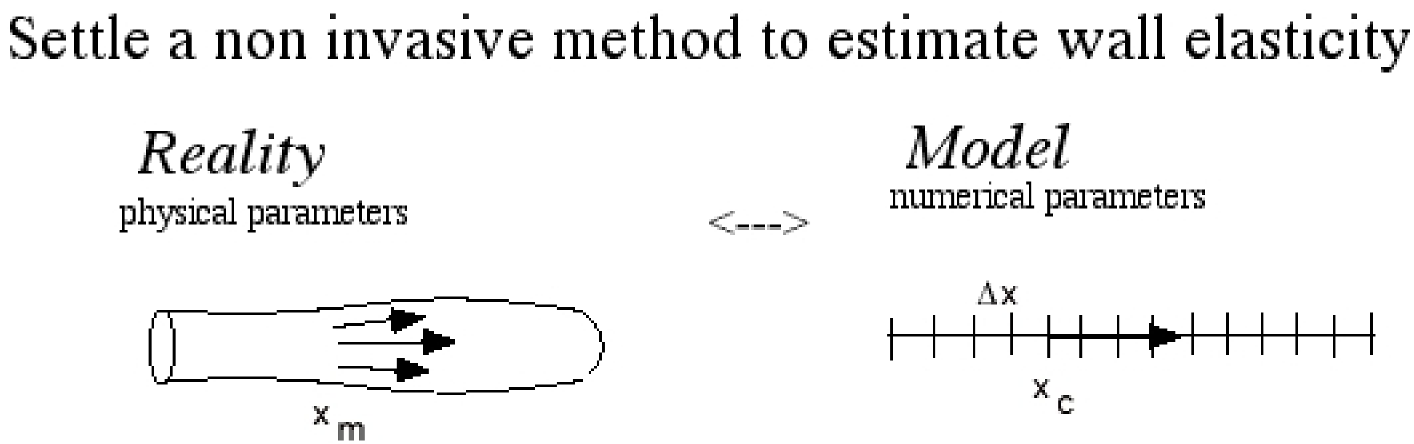


Figure 1: The displacement of the wall ($h(x, t = 2.5)$) as a function of x is plotted here at time $t = 2.5$. The dashed line (wom3(x,2.5)) is the Womersley solution (reference), the solid line (B.L.) is the result of the Boundary Layer code and the dots (intg) are the results of the integral method ($\alpha = 3$, $k_1 = 1$, $k_2 = 0$ and $\varepsilon_2 = 0.2$).

inverse method

using RNSP equ. as synthetic datas, inverse method (retropropagation...)



Minimisation between "mesure" and 1D computation at one point.

Conclusion

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Starting from Navier Stokes

- a simple set of equations: RNSP

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evaluation of skin friction
evaluation of pressure drop
Boundary conditions...
Well adapted for "real time computations/ simulations/ visualisations"...