Upstream influence in mixed convection at small Richardson number on triple, double and single deck scales

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Abstract: We investigate the boundary layer flow with thermal effects on a flate plate perturbed by a bump or a thermal spot. We introduce the buoyancy effects in the framework of "triple deck" theory. This is a particular case of the mixed thermal convection problem which is known to lead to a breakdown when the plate is cooled, and when the governing parameter, the "Richardson number", is of order one. Here we try to study the onset of the phenomenon since we work at small Richardson number. For this simplified problem, the usual "triple deck" structure reduces to a "double deck" one when this number increases, and a new term in the pressure displacement relation, accounting for the hydrostatic pressure change across the main deck, appears from the analysis. The linearized and non linearized solutions of this new set of equation show upstream influence of a thermal spot or a bump on the boundary layer.

1. Introduction, state of the art

We investigate the flow régime which occurs when a laminar, uniform, stationary and two- dimensional flow of a Newtonian fluid meets a horizontal flat plate which is at a different temperature from the free stream. The plate is cooled (or heated) by natural convection by Archimedes force and by forced convection. This problem is known as "mixed convection" because both effect compete; the magnitude of this competition is the Richardson number defined here by $J = \frac{g\alpha\Delta T_0}{U_\infty^2}\frac{L}{\sqrt{Re}}$ (in fact our study deals with small J, and we will show that $\tilde{J} = JRe^{1/8}$ is more convenient). This number gauges the ratio of thermal coupling effect, modelled by Boussinesq approximation, versus dynamic effects. We suppose that viscosity does not change with temperature, see Méndez & Treviño (1992) for this influence, but without buoyancy.

The Reynolds number ($Re = \frac{U_\infty L}{\nu}$) is large, so, viscous and conductive effects in the fluid (Prandtl number of order one, but this is not restrictive) are localized in a small neighbourhood of the wall. This leads to the system (see next references) deduced from Prandtl boundary layer theory, with no slip condition, outer flow matching, and imposed temperature:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})u=-\frac{\partial}{\partial x}p+\frac{\partial^2}{\partial y^2}u, \qquad (1a,1b)$$

$$\frac{\partial}{\partial y}p=J\theta, \quad (u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})T=\frac{\partial^2}{\partial y^2}T. \qquad (1c,1d)$$

Relation (1c) defines J. Unfortunately, it has been shown by Schneider & Wasel (1985), Daniels & Gargaro (1993) and Daniels (1992) (for example) that boundary layer equations, when coupled with energy equation by buoyancy become abruptly singular at a (relatively small) fixed abscissa.

A very simple way to understand this, is to over simplify Daniels's work (1992). Suppose that there is an algebraic departure from the non buoyant solution for the displacement thickness $\delta \sim x^a$; then, the pressure ($\partial p/\partial y = J\theta$), after integration through boundary layer behaves like $p \sim \delta \sim x^a$; but the speed ($u\partial u/\partial x \sim -\partial p/\partial x$) is now $u \sim x^{(a/2)}$; next we obtain the boundary layer thickness ($u\partial u/\partial x \sim \partial^2 u/\partial y^2$) as $\delta \sim x^{((2-a)/4)}$. This gives $a = 2/5$, a more cumbersome calculation gives $a = 0.43$... Another over simplified point of view will be explained in the last paragraph.

Thus, we propose to revisit this problem of "mixed convection" on the basis of "triple deck" (Neiland (1970) and Brown *et al.* (1975)), we will not smooth the singularity but we will show that this approach captivates some features like: strong self induced upstream influence (in case of cold wall), no upstream influence and Tollmien Schlichting (Smith (1979)) waves (in case of hot wall). Linearized and non linear computed (by a finite difference scheme) solutions will be presented in the case of bump or thermal spot.

Our problem is different from the problem studied by Sykes (1978), because he considers that the scale of temperature stratification is the scale of the upper deck, while we take it to be smaller and of order of the boundary layer thickness; it is different to from El Hafi (1994) who studies a vertical natural convection (previous papers referred to horizontal convection) boundary layer perturbed by a bump with similar asymptotic methods.

2. Basic Flow

We suppose, for the sake of simplicity, that the basic flow, driven by the free stream uniform velocity, is a classical Blasius boundary layer. This is not restrictive, because all we need is a basic velocity profile $U_O(y)$. We study a localised disturbance at the distance L of the leading edge. At this point, the boundary layer is of thickness $Re^{-1/2}L$. The incoming fluid is at temperature T_∞, and the plate at T_0, (say: $\Delta T_0 = T_0 - T_\infty$). Pure thermal convection is relevant as long as the transverse gradient from equation (1c) is small. It means that $1 >> J >> Re^{-1}$. So, under those hypotheses, the forced thermal boundary layer is of the same thickness as the dynamic one.

Let us remark that we impose the scale L, but Schneider & Wasel (1985) or Daniels & Gargaro (1993) do not because, in the real mixed convection problem, the natural longitudinal scale is built with Richardson number. It is the length that gives unit Richardson number. Nevertheless, the calculations they did, show that the breakdown occurs for a rather small abscissa, that means that the Richardson number built with this abscissa is smaller than one. It is then interesting to investigate small J behaviour.

3. Triple Deck interaction: the $J \sim Re^{-1/8}$ régime

3.1.1. Main Deck

The classical triple deck tool (Neiland (1970) and Brown *et al.* (1975)) is applied to cope with a localized disturbance gauged by $x_3 L$ longitudinally. Perturbation of speed in the main deck is εU_∞, where ε is unknown, so we recover:

$$U = U_0(y) + \varepsilon A(x) U_0(y), v = -\frac{\varepsilon\delta}{x_3} A(x) U_0(y), \quad and \quad T = \theta_0(y) + \varepsilon A(x)\theta_0'(y), \quad (2)$$

where $U_0(y)$ and $\theta_0(y)$ are of course the undisturbed basic speed and temperature profiles. For the temperature, as for the speed, there is a matching between the top of the main deck and the bottom of the upper deck, and it is the same for the lower deck. We see that the temperature behaves as the Stewartson S function in hypersonic flows (Brown *et al.* (1975)). We recover in this deck the unknown displacement function of the boundary layer $-A(x)$.

3.1.2. Crossing the main deck

This perturbation of temperature gives rise to a transverse change of pressure through the "main deck", we develop it in power of ε as (neglecting $0(\varepsilon^3)$ terms):

$$\frac{\partial p_0}{\partial y} + \varepsilon \frac{\partial p_1}{\partial y} + \varepsilon^2 \frac{\partial p}{\partial y} = \varepsilon \tilde{J}(\theta_0(y) + \varepsilon A(x)\theta_0'(y)), \quad (3)$$

where we have anticipated the fact that the pressure in the lower deck is of order ε^2, and that x_3 is of order ε^3 and where, for convenience, we have defined a reduced Richardson number: $\tilde{J} = \frac{J}{\varepsilon}$.

Looking at each power of ε, we see that the first term is zero (as we supposed); the second one shows that there is a pressure stratification coming from basic temperature profile ($\tilde{J} \int_0^\infty \theta_0(y) dy$), it does not depend on x at the short scale x_3, and it will then not be useful; the third one integrates (using $\theta_0(\infty) = 0; \theta_0(0) = 1$ by definition) as:

$$p(x, y \to \infty) - p(x, y \to 0) = \tilde{J} A(x)(\theta_0(\infty) - \theta_0(0)) = -\tilde{J} A(x), \quad (4)$$

where $p(x, y \to \infty)$ splices with upper deck and $p(x, y \to 0)$ with lower deck.

3.1.3. The upper deck

The matching with the "upper deck" for the pressure ($p(x, y \to \infty)$ is written for short $p(x)$) gives Hilbert integral (we recover the classical linear perturbed perfect fluid in the subsonic case): $\left(\frac{\varepsilon\delta}{\varepsilon^3}\right) \frac{1}{\pi} \int \frac{-A'}{x-\xi} d\xi$, the usual gauge $\varepsilon = \delta^{-1/4} = R^{-1/8}$ and the new pressure displacement relation (Lagrée (1994)) is:

$$\frac{1}{\pi} \int \frac{-A'}{x - \xi} d\xi = p - \tilde{J} A. \quad (5)$$

The effect of the temperature is to add a new term proportional to the displacement function A, it may be interpreted as a hydrostatic variation of pressure.

3.2 Double deck interaction: the $Re^{-1/8} \ll J \ll 1$ régime

It is worth noticing that as the Richardson number increases, in order of magnitude, from $Re^{-1/8}$, the perturbations in the upper deck become smaller and smaller and the longitudinal gauge, while remaining much more smaller than one, grows; so we find that for moderate values of J, the good choice for ε is rather $\varepsilon = |J|$ and the relation becomes: $\left(|J|^{-4} Re^{-1/2}\right) \frac{1}{\pi} \int \frac{-A'}{x-\xi} d\xi = (p \pm A)$ and ultimately the pressure displacement relation degenerates in the form (Lagrée (1994)):

$$p = -A, \tag{6a}$$

for a cold wall, and in the form:

$$p = A, \tag{6b}$$

for a hot one, with: $Re^{-1/8} \ll |J| \ll 1$. This shows that the upper deck is not necessary for the interaction to take place, the same phenomenon exists in hypersonic flows (Brown *et al.* (1975)) for cold wall. The $p = -A$ relation is found in the birth of hydraulic jumps in Bowles & Smith (1992), Gajjar & Smith (1983) and in subsonic pipe flows by Ruban & Timoshin (1986) and leads to upstream influence. Note that the interaction relation $p = \pm A$ was independently found by Bowles (1994).

3.3. Lower Deck

There are two possibilities for the lower deck depending on the temperature of the spot either (i) it is of magnitude $\varepsilon\Delta T_0$ either (ii) bigger.

i) no spot, or weak one

If there is no spot (but a hump) or if it is weak, in the lower deck, classical gauges are used: u is of order ε, p of order ε^2, and x_3 is ε^3, temperature matching with main deck gives the order ε for the temperature, transverse variable is ε times the transverse variable in the main deck.

We recover the basic lower deck set:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})u = -\frac{d}{dx}p + \frac{\partial^2}{\partial y^2}u, \quad (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})T = \frac{\partial^2}{\partial y^2}T. \tag{7}$$

Boundary conditions are no slip at the wall, $A(-\infty) = 0$ and for $y \to \infty$, $u \to y + A$, and $T \to y + A$.

ii) spot

But, if the spot is warmer (or cooler), the temperature in the lower deck must be rescaled by the temperature of the spot T_w. Say that on a small distance scaled by x_3 the temperature of the plate changes from T_0 to $T_0 + T_w$; then it is better to use $((T_w/\Delta T_0)\Delta T_0) = T_w$, instead of $(\varepsilon(\Delta T_0))$ as a new gauge of temperature variation (and $T_w >> \varepsilon\Delta T_0$).

The convective/ diffusive temperature equation is still the same, but there is now a transverse pressure gradient in the lower deck ($\frac{\partial p}{\partial y} = j\theta$) if $j = \tilde{J}T_w/\Delta T_0$ is of order one (this is a second Richardson number):

$$-\frac{\partial p(x,y)}{\partial x} = -\frac{dp(x)}{dx} + j\int_y^\infty \frac{\partial \theta(x,y)}{\partial x}dy, \tag{8}$$

where $p(x)$ is the pressure at the top of the lower deck (at the bottom of the main deck). Temperature is gauged by: $T = T_\infty + (T_0 - T_\infty)(1 + \frac{T_w}{T_0 - T_\infty}\theta) = T_0 + T_w\theta$. The matching between the lower and main decks becomes: $1 + \frac{T_w}{T_0 - T_\infty}\theta(y \to \infty) = 1 + \varepsilon(y + A(x))\theta_0'$, and since the temperature is strong at the wall ($T_w, >> \varepsilon\Delta T_0$.), the temperature θ is then zero at infinity. Hence, we obtain the lower deck problem: (which is very similar to Zeytounian (1991)):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \quad (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})u = -\frac{d}{dx}p + j\int_y^\infty \frac{\partial \theta}{\partial x}dy + \frac{\partial^2}{\partial y^2}u, \quad (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\theta = \frac{\partial^2}{\partial y^2}\theta. \tag{9}$$

Boundary conditions are no slip at the wall, $A(-\infty) = 0$ and for $y \to \infty$, $u \to y + A$, but $\theta \to 0$.

3.4. various fundamental problems

Depending on the values of the Richardson number we may build various "fundamental problems of triple deck" by mixing the above lower deck problem and the pressure displacement relations we found.

3.5. linearized solution, eigenvalue

The system admits the Blasius solution $u = y$ as a basic one. Perturbing it by a small amount, and taking the Fourier transform of the result, gives:

$$i\alpha\tilde{u} + \frac{\partial}{\partial y}\tilde{v} = 0, \quad i\alpha y\tilde{u} + \tilde{v} = -i\alpha\tilde{p} + j\int_y^\infty i\alpha\tilde{T}dy + \frac{\partial^2}{\partial y^2}\tilde{u}, \quad i\alpha y\tilde{T} = \frac{\partial^2}{\partial y^2}\tilde{T}. \tag{10}$$

The temperature ($\tilde{T} = Ai(y(i\alpha)^{1/3})$) is an Airy function. Perturbation of skin friction $\tilde{\tau} = \frac{\partial \tilde{u}}{\partial y}$ verifies an Airy differential equation forced by an Airy function, this is solved as:

$$\frac{\partial}{\partial y}\tilde{u}(0) = 1 + \frac{(i\alpha)^{2/3}\tilde{p}}{Ai'(0)} - j(i\alpha)^{1/3}(\frac{Ai(0)}{3Ai'(0)} - Ai'(0)), \tag{11}$$

then we find the displacement function ($\tilde{F}$ is the Fourier transform of the bump):

$$\tilde{A} + \tilde{F} = \frac{(i\alpha)^{1/3}\tilde{p}}{3Ai'(0)} - j(Ai(0) + \frac{1}{9Ai'(0)}). \tag{12}$$

In this expression (written for short $\tilde{A} + \tilde{F} = \beta^*\tilde{p} + j\beta_j$, where β^* is the standard triple deck coefficient) with temperature effect, we see that, for a negative j, there is a decrease of skin friction and a positive displacement $-\tilde{A}$, so an increase of the boundary layer. The Fourier transform of the new pressure displacement relation reads:

$$\tilde{p} = (|\alpha| + \tilde{J})\tilde{A}. \tag{13}$$

Then we deduce, for the pressure at the top of the lower deck, the following expression:

$$\tilde{p} = \frac{(|\alpha| + \tilde{J})}{1 - (|\alpha| + \tilde{J})\beta^*}(j\beta_j - \tilde{F}). \tag{14}$$

This expression will be used to compute the linearized response of the stratified boundary layer to a bump or/and a thermal spot as we will see in section 3.7.

On the other hand, the linear approach gives the behaviour for $x \to -\infty$. In the case of cold wall, ($p = -A$) we recover the same behaviour as in hypersonic flows (Brown *et al.* (1975) and Gajjar & Smith (1983)), thus, the Lighthill eigenvalue (from the growing exponential) may be found, which shows that there is upstream influence: $p = ae^{kx}$, with $k = (-3Ai'(0))^3$. This upstream influence may describe the phenomenon of "blocking" which is observed in stratified flows.

Note that in Daniels & Gargaro (1993) breakdown, in the vicinity of the singular point, the perturbed quantities behave with a $p = -A$ law.

3.6. Stability

Classical triple deck technique is applied, and we find that the hot wall case ($p = A$) is linearly unstable and marginal stability curve is given by $\omega = 2.29\alpha^{2/3}$ and $\tilde{J} = 1.001\ \alpha^{-1/3} - |\alpha|$ (by analogy with Tollmien- Schlichting waves with triple deck scales deduced by Smith (1979)), so there is a non viscous mode for $\tilde{J}$ negative and large, and a viscous mode of large wave length for $\tilde{J}$ positive and large (which corresponds to $p = A$).

3.7. Numerical results

The behaviour, computed with those relations will be shown in the case of a bump or a spot of shape $(1 - x^2)^2$, with various values of $\tilde{J}$. It shows upstream influence (as predicted), that may be interpreted, in stratified flows (Tritton (1988)), as "blocking", and downstream we observe small oscillations, interpreted as "lee waves", positive values of $\tilde{J}$ show indeed lowering and cancelling of upstream influence (see figure 1 and 2). Non linear results are presented on figure 3 and 4.

4. Single deck interaction?: the $J \sim 1$ régime

Numerical calculations have clearly shown that there is a singularity in the self interaction of the boundary layer at $J = O(1)$. We believe that this singularity is similar to the "branching solutions" obtained in supersonic inviscid- viscous interacting flows (Le Balleur (82)). An over simplification of the problem with integral methods, may be done as follow. First, let us suppose that the pressure is averaged through the boundary layer, $p \approx 1 - JRe^{1/2}\delta_1\alpha$ (introducing an integral parameter $\alpha \approx 1.7$ for $J = 0$). Second, the classical Kármán momentum equation with standard notations reads:

$$U_0^2(\frac{d(\delta_1/H)}{dx}) + \delta_1(1 + \frac{2}{H})U_0\frac{dU_0}{dx} = \frac{f_2 H}{Re\delta_1}U_0. \tag{15}$$

Combining those two equations, and without forgetting that H may be written as a function of $\Lambda_1 = \delta_1^2 Re \frac{dU_0}{dx}$, the Pohlhausen description (near the Blasius solution $(\delta_0 Re^{-1/2}, \delta_0 \approx 1.7)$) gives $H \approx H_0 - H_p \frac{dU_0}{dx}$, with $H_0 \approx 2.58$, & $H_p \approx 0.533(1.7)^2$. We obtain:

$$J\alpha\delta_0 \frac{H_p}{H_0^2}\frac{d^2\delta_1}{dx^2} + \frac{d\delta_1}{dx}(\frac{1}{H_0} + J\alpha Re^{1/2}\delta_1(1+\frac{2}{H_0})) = \frac{f_{20}H_0}{Re\delta_1}. \tag{16}$$

This development is exactly Schneider & Wasel (1985) ones, however we have added the second derivative term which is the most dominant one coming from the derivative $\partial(1/H)/\partial x$. When this term is omitted is easy to see that the forward integration leads to a singularity at a fixed abscissa whose value $x_s = (6H_0^4(\alpha J)^2(1+2/H_0)^2 f_{20})^{-1}$ is not so bad an estimation. But the forgotten second derivative term means, in our opinion, that another boundary condition must be imposed. This equation may be solved without difficulty with δ_1 prescribed at each boundary of the domain. This over simplified heuristical approach may be a guideline for the numerical resolution of the single deck equations: they must be solved with a global technique with the two end values prescribed (work in this direction is in progress, Bowles (1994)), not by a marching technique from downstream to upstream.

5. Conclusion

We have presented the response of a boundary layer with small buoyant effects to a thermal spot or a bump. This is an extension of the triple deck approach, and we have seen that if the Richardson number is large enough, this is a double deck interaction. Stability and the understanding of the singularity could be investigated. We did not clear the singularity up, but we have shown that even at small values of J there is upstream influence (from downstream) and self induced solutions which may be interpreted as birth of some thing being downstream which may be simply seen on the over simplified partial equation.

Note

Recently, H. Steinruck ("Mixed convection over a cooled horizontal plate: non-uniqueness and numerical instabilities of the boundary layer equations", *J. Fluid Mech.*, vol 278, pp. 251-265) has shown that the mixed convection problem presents branching solutions (when numericaly solved in marching from up to downstream) associated to an unbounded sequence of eigenvalues that he found by asymptotic expansion (the one he obtains for small x, is exactly the Lightill eigenvalue coming from $p = -A$).

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$$U_0^2(\frac{d(\delta_1/H)}{dx}) + \delta_1(1+\frac{2}{H})U_0\frac{dU_0}{dx} = \frac{f_2 H}{Re\delta_1}U_0.$$

Combining those two equations, and without forgetting that H may be written as a function of $\Lambda_1 = \delta_1^2 Re\frac{dU_0}{dx}$, near the Blasius solution $(\delta_0 Re^{-1/2}, \delta_0 \approx 1.7)$ the Pohlhausen description gives $H \approx H_0 - H_p\frac{dU_0}{dx}$, with $H_0 \approx 2.58$, & $H_p \approx 0.533(1.7)^2$. We obtain, after forgetting some terms:

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