Thermal mixed convection induced locally by a step change in surface temperature in a Poiseuille Flow in the framework of Triple Deck theory.

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Abstract

The classical Lévêque solution of heat transfer induced by a small step change in the surface temperature in a shear flow (u linear in y) is revisited. To obtain the shear flow we rescale the laminar channel flow of a perfect gas at high Reynolds number in the Triple Deck scales, and we investigate the retroaction of the temperature on the basic Poiseuille profile (near the wall the profile is a linear in y). This retroaction is achieved by two means, first through the dependance of the viscosity and the density upon temperature and second through the gravity-induced transverse pressure gradient gauged by the inverse of the Froude number.

In the case of no transverse gradient a new self-similar solution is obtained showing that the skin friction at the lower wall is reduced by the heating while the one at the top wall is simultaneously increased.

In the general case with a Lower Deck based Froude number not infinite, the case of asymptotically small wall temperature variation allows a linearized solution which is solved with Fourier transform method. If the Froude number F is increased to infinity we recover the preceding self-similar solution with small temperature variation. If now F is decreased to zero we find that the leading term in $1/F$ of the solution shows that the skin friction at the lower wall is increased while that at the upper wall is decreased.

The conclusion is that the increase of temperature produces two opposite effects: first, the expandability of the gas causes an upward displacement of the streamlines and a pressure decrease (the preceding self-similar solution is recovered with small temperature variation); second, the buoyant effect produces the reverse effect of a downward displacement and a pressure increase which we believe may cause separation at the top wall in the non linear case (skin friction at the lower wall is increased whereas it is decreased at the upper wall). These two effects qualitatively explain the flow computed with full Navier-Stokes equation in a M.O.C.V.D. reactor.

Key words: Triple Deck; Mixed Convection; Boundary Layer

1 Introduction.

It is well known that the "Triple Deck theory", obtained from Navier Stokes equations in the limit of infinite Reynolds number, gives a good asymptotic description of the separation of the stationary laminar boundary-layer in external flows at any régime (subsonic, transonic, supersonic or hypersonic (Smith (25))) as well in pipe flows. In this last case Smith (24), revisited by Saintlos & Mauss (21), showed that the Triple Deck degenerates into a double one.

But, thermal effects had not been strictly introduced in treatments just cited: with a fixed wall temperature or an adiabatic wall the dynamical and thermal problems were decoupled (gravity being neglected). In hypersonic flows, however, a low temperature is responsible for an effect (Brown et al (2) or Neiland (20)) which explains the (Brown et al (3)) differences between experiments and theory (the temperature stratification comes from a low wall temperature or from the hypersonic entropy layer induced by the blunted nose of the plate(Lagrée (16))).

The problem of the thermal response of an incompressible Blasius boundary-layer has been posed by Zeytounian (31); Méndez et al (19) examined this problem but with the retroaction through a variable density (perfect gas) and viscosity (model fluid). Sykes (29) looked at buoyancy effects in a stratified flow, the stratification being in the perfect fluid. In these works, only the forced convection without gravity has been examined. Without external flow, the problem is a free convection problem driven by buoyancy, see Stewartson (27) and Gill et al (13), the conclusion being that the flow is heated and that it is pushed away from the hot plate (here we deal only with horizontal plates, see El Hafi (6) who investigated the natural convection flow along a vertical plate with a small bump leading to a special Triple Deck problem).

The mixed convection problem occurs when both effects of buoyancy and forced convection are present and compete. In the case of an incompressible buoyant fluid flowing over a horizontal plate at a colder temperature this leads to a singularity: Schneider & Wasel (23) or Daniels (4). Lagrée (17) and (18) introduces a very small stratification in the Blasius boundary layer showing the possibility of occurrence of a self induced solution in the Triple Deck framework. This "Lighthill eigenvalue" solution has been found simultaneously by Bowles (1) and by an alternative method by Steinrück (26). The latter shows the influence of the step size in the location of the singularity, the observed branching in fact being a "self induced" solution.

In this paper we look at the influence of a step change of the lower wall temperature in an established Poiseuille flow at high Reynolds Number and high Froude Number: the mixed convection is localized in a thin layer near the wall. We concentrate our investigation on a special range of longitudinal scales coherent with the Triple Deck technique. Because the basic flow is a shear flow near the wall, this is a step forward in the Lévêque (15) description, and, in certain respects, it is an extension of Méndez et al (19) and Lagrée (18).

Using the ideas of asymptotic analysis we present the equations obtained when Reynolds number and Froude number go to infinity and when there is gravity and variable density (perfect gas law). We solve this set of equations with two different techniques, first using self-similar variables, second using Fourier transform, in a sense finding the next order of the Lévêque solution. The choice of a Poiseuille flow instead of an external flow was motivated by a practical application, so, we conclude by a qualitative comparison of this theory with the flow occurring in a M.O.C.V.D. reactor computed with Fluent (8).

2 Nomenclature

Roman symbols

-A the displacement function

Ai the Airy function

Bi the Airy Bi function

C Chapman constant, fixed to one for convenience

F reduced Froude number

F_0 natural Froude number

f similar function for stream-function

g similar function for temperature

G Fourier transformed wall temperature

H^* the height of the channel

L^* a longitudinal scale

p pressure

P_r Prandtl number, fixed to one for convenience

R Reynolds number

T Temperature

U_0^* characteristic velocity

x longitudinal coordinate measured from the beginning of the susceptor.

y transverse coordinate measured from the lower wall

Y the Howarth Doronitsyn variable

z transverse coordinate measured from the upper wall, positive downwards

Greek Symbols

α a small parameter

β a constant

β^* a constant

γ a constant

ε the small parameter

η a selfsimilar variable

θ a small parameter for the change of temperature: $T_w - 1$

μ viscosity

ν dynamical viscosity

ρ density

τ perturbation of the skin friction

ψ stream function

Subscripts

P Poiseuille

h at the upper wall

w at the lower wall

0 of the undisturbed flow §3.1 , or first expansion of linearized flow §4.1

1 perturbation in the Main Deck

Superscripts

$'$ derivative with respect to an obvious variable

$*$ with dimension

$-$ variables adimensionalized by H^*

2.1 Hypotheses

It is assumed that the fluid is Newtonian, steady, laminar, two-dimensional and that the Reynolds number is large. The basic profile is an established Poiseuille flow between two infinite horizontal flat plates. Figure 1 is a rough sketch of the physical problem. Having assumed a vanishingly small Eckert number (low Mach number flow) and a high Froude number, the temperature is uniform of constant value T_0^* (which is the wall temperature too) and the pressure is independent of y^* and decreases linearly with x^*.

The basic flow is then simply:

$$\begin{aligned} u^* &= U_0^* \frac{y^*}{H^*}(1 - \frac{y^*}{H^*}) = U_0^* U_p(\bar{y}), \\ p^* &= p_0^* - (\rho_0^* U_0^{*2})(2R^{-1}(x^*/H^*) - F_0^{-1}\ (y^*/H^*)). \end{aligned} \tag{1}$$

Here we have defined the Reynolds number $R = U_0^* H^*/\nu^*$ and the natural Froude Number $F_0 = U_0^{*2}/(g^* H^*)$.

At a certain place (the origin) the temperature is suddenly increased to the value T_w^*. If the variation is a step function, the temperature is T_w^* and remains constant but more generally it may be any function of x^* . The scale of variation must be consistent with the longitudinal scale x_3 which will be defined latter. Two effects are put in the model. First as in Méndez & al (19) the compressibility is accounted for by using a perfect gas model, and the dependance of viscosity with the temperature is modelled, as usual (Stewartson (28)), by a linear dependance (model fluid); the Prandtl number is one. Second, we will introduce the gravity in the transverse direction creating a mixed convection problem in a thin layer near the wall.

A simple analysis at infinite Reynolds number is that at scale H^* for x^* and y^* the change of wall temperature does not influence the core flow: $U_p(\bar{y})\partial\bar{T}/\partial\bar{x} = 0$. So $\bar{T} = 0$ everywhere at these scales. Hence a "boundary layer" is introduced near the wall of thickness $R^{-1/3}$ in order to recover $T(x > 0, y = 0) = 1$. In this thin wall layer we scale x^* with H^*, y^* with $R^{-1/3}H^*$ and u^* with $R^{-1/3}U_0^*$; then, assuming $P_r = 1$, no gravity, constant density and viscosity we have:

$$u = y, \qquad y\frac{\partial}{\partial x}T = \frac{\partial^2}{\partial y^2}T,$$

whose solution, in the case of $T(x < 0, y = 0) = 0$, $T(x, y = 0) = 0$ and $T(x > 0, y = 0) = 1$ is (Lévêque (15), Schlichting (22)):

$$T(x,y) = 1 - \frac{\int_0^{y/x^{1/3}} e^{-\frac{\zeta^3}{9}} d\zeta}{\int_0^\infty e^{-\frac{\zeta^3}{9}} d\zeta}.$$

This solution will be refered as the "Lévêque" solution. It may be written in terms of the incomplete Gamma function: $T(x,y) = \Gamma(1/3, y^3/x/9)/\Gamma(1/3)$. We will discuss in the following the scales that allow effects of gravity, density and viscosity variations and how they change this simple solution.

2.2 Equations near the bottom wall (Lower Deck)

As usual, the flow will be perturbed at small spatial scales at the vicinity of the wall where the speed is small. So, we look at the thin layer near the wall of gauge ε, whose value will be found next in studying the Main Deck (we anticipate that $\varepsilon = R^{-2/7}$, and that the associated longitudinal scale is $x_3 = R^{1/7}$; more exactly we will see that it is the smaller case satisfying least degeneracy leading to simple resolution for the complete set of equations, and so, we suppose that the temperature changes at this scale). We scale the equations to obtain the maximum of terms, we write:

$$(x^*, y^*, u^*, v^*) = (x_3 H^* x, \varepsilon H^* y, \varepsilon U_0^* u(x,y), \varepsilon^2 x_3^{-1} U_0^* v(x,y)),$$

the pressure is defined as the perturbation of the hydrostatic one:

$$p^* = p_0^* - 2(\rho_0^* U_0^{*2})\frac{x_3}{R}(x) + (-\frac{y}{F} + \tilde{p}(x,y))\varepsilon^2 \rho_0^* U_0^{*2}$$

and we put simply:

$$\rho^* = \rho_0^*\rho(x,y), \quad T^* = T_0^* T(x,y), \quad \mu^* = \mu_0^*\mu(T).$$

So $T_w = T^*(x^*, 0)/T_0^*$. We find relation between the transverse scale and the longitudinal one by least degeneracy: $x_3 = \varepsilon^3 R$, and we define here a new Froude number which is gauged by the new thickness εH^*: say $F = \varepsilon F_0$. The deduced system is then the classical one, but with a transverse pressure:

$$\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v = 0, \tag{2}$$

$$\rho u\frac{\partial}{\partial x}u + \rho v\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}\tilde{p} + \frac{\partial}{\partial y}\mu\frac{\partial}{\partial y}u, \tag{3}$$

$$0 = -\frac{\partial}{\partial y}\tilde{p} - \frac{\rho - 1}{F}, \tag{4}$$

$$\rho u\frac{\partial}{\partial x}T + \rho v\frac{\partial}{\partial y}T = P_r^{-1}\frac{\partial}{\partial y}\mu\frac{\partial}{\partial y}T, \tag{5}$$

$$\rho T = 1. \tag{6}$$

The boundary conditions are $u = v = 0$ in $y = 0$ (no slip at the wall) and $u \to y$ for $x \to -\infty$ (matching with the Poiseuille profile far upstream). For the temperature we have at the wall $T = 1$ for $x < 0$ and $T = T_w$ for $x > 0$. There are other boundary conditions for u, $\tilde{p}$ and T in $y \to \infty$ that will be given from the asymptotic matching between the two layers in the next paragraph.

2.3 Equations in the core flow (Main Deck)

2.3.1 Pressure displacement relation

This layer should have been examined at first (Smith (25)) and its examination would have let to the conclusion that the boundary conditions are not fulfilled at the walls, implying the existence of the preceding Lower Deck. In the Main Deck the longitudinal scale is small $x^* = x_3 H^* x$, (this is one of the key ideas of the Triple Deck: a quick longitudinal scale to explain abrupt changes in the boundary layer) and the transverse scale remains the natural one: H^* so $y^* = H^*\, \bar{y}$.

The velocity is written as the perturbation of the Poiseuille flow: $u^* = U_0^*(U_p(\bar{y}) + \varepsilon u_1(x, \bar{y}))$ and $v^* = \varepsilon x_3^{-1} U_0^* v(x, \bar{y})$. The pressure is $p^* = p_0^* - 2(\rho_0^* U_0^{*2})\frac{x_3}{R}(x) + (\frac{\bar{y}}{F_0} + p_1(x, \bar{y})\varepsilon^2)\rho_0^* U_0^{*2}$. And we have:

$$\rho^* = \rho_0^*(1 + \varepsilon\rho_1),\ T^* = T_0^*(1 + \varepsilon T_1),\ \mu^* = \mu_0^*(1 + \varepsilon\mu_1)$$

Navier-Stokes equations reduce then to an inviscid perturbation:

$$\rho_0^{-1}\frac{\partial}{\partial x}\rho_1 + \frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial \bar{y}}v_1 = 0, \tag{7}$$
$$U_p\frac{\partial}{\partial x}u_1 + v_1\frac{d}{d\bar{y}}U_p = 0, \tag{8}$$
$$U_p\frac{\partial}{\partial x}v_1 = -\frac{\partial}{\partial y}p_1, \tag{9}$$
$$U_p\frac{\partial}{\partial x}T_1 = 0. \tag{10}$$

So there is a transverse pressure-variation induced by the transverse velocity. The solution is straightforward, because of the lack of initial thermal stratification (see Lagrée (17) where it is present),

$$u_1(x,\bar{y}) = A(x)U_p'(\bar{y}), v_1(x,\bar{y}) = -A'(x)U_p(\bar{y}).$$

If we write the stream function: $\psi(x,\bar{y}) = \psi_p(x,\bar{y}) + \varepsilon\ A(x)\frac{\partial}{\partial \bar{y}}\psi_p(x,\bar{y})$, so:

$$\psi(x,\bar{y}) = \psi_p(x,\bar{y}+\varepsilon A(x)).$$

The physical explanation is that the streamlines are deflected from $-\varepsilon A$, that is the reason why we will call $-A$ the displacement function. The pressure plays a key role because its possible variation across the deck gives the correct value of ε. See Saintlos & Mauss 96 (21) for systematic derivation. The scale x_3 is in fact chosen here, if $\varepsilon = R^{-2/7}$, and $x_3 = R^{1/7}$ we have:

$$p_1(x,1) - p_1(x,0) = \frac{1}{30}A''(x) \tag{11}$$

2.3.2 Other choices of scales

The discussion of other scales may be found in Saintlos & Mauss 96 (21). One other choice for the longitudinal scale $x_3 \gg R^{1/7}$ would have led to a total transmission of pressure:

$$p_1(x,1) - p_1(x,0) = 0. \tag{12}$$

This simple relation will be used to find self- similar results. The other interesting possibility is in the range $R^{1/7} \gg x_3$: no displacement is induced at first order, the displacement is a second order effect induced by the pressure. The smallest scale is bounded by a Navier-Stokes region. Because of the fact that there is no displacement for $R^{1/7} \gg x_3$, we concentrate on $x_3 \geq R^{1/7}$ scales.

2.3.3 Matching

The asymptotic matching between the main deck and the lower deck gives the lacking boundary condition for the velocity in Lower Deck: $u \to y + A$ for $y \to \infty$ (at the lower boundary) and for the pressure $\tilde{p}(x, \infty) = p_1(x, 0)$. Far upstream there is no displacement of the streamlines in the Main Deck, so we have $-A(\infty) = 0$. As well the matching of temperatures gives $T \to 0$ for $y \to \infty$, the perturbed temperature is localized in the Lower Deck: the mixed convection problem is local.

This layer transmits the perturbation induced in the Lower Deck through it up to the upper boundary where no slip condition is violated too: the Upper Lower Deck. For this deck the pressure will match with $p_1(x, 1)$ and the longitudinal velocity with:

$$U_p(\bar{y}) + \varepsilon u_1 \to (1 - \bar{y}) - \varepsilon A. \tag{13}$$

There are no temperature variations at all for it.

2.4 Equations near the top wall (Upper Lower Deck)

We now look at the thin layer near the upper wall of gauge ε, as for the Lower Deck this layer is necessary to obtain the no slip condition violated by (13). Again $x^* = x_3 H^* x$, but now we put $y^* = H^* - \varepsilon H^* z$. The other quantities follow:

$$u^* = \varepsilon U_0^* u(x, z), \qquad v^* = \varepsilon^2 x_3^{-1} U_0^* v(x, z),$$

$$\rho^* = \rho_0^*, \qquad T^* = T_0^*,$$

$$p^* = p_0^* - 2(\rho_0^* U_0^{*2}) \frac{x_3}{R}(x) + (-\varepsilon \frac{1 - \varepsilon z}{F} + p_h(x)\varepsilon^2)\rho_0^* U_0^{*2}.$$

The problem is here completely incompressible, non-buoyant and isothermal (there is no confusion with the u and v in the two Lower Decks, so only the pressure is subscripted):

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial z} v = 0; \tag{14}$$

$$u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial z} u = -\frac{d}{dx} p_h + \frac{\partial^2}{\partial z^2} u; \tag{15}$$

From the matching of velocity with the Main Deck ($\bar{y} \to 1$, equation (13)) we have the behaviour $u \to z - A$ for $z \to \infty$ and $u \to z$ for $x \to -\infty$. The no slip condition is $u = v = 0$ in $z = 0$. The pressures match as $p_h(x) = p_1(x, 1)$.

2.5 Final equations

2.5.1 Classical form

The final problem is then made by solving the Lower Deck problem (equations (2-6)) with its boundary conditions, the Upper Lower Deck problem (equations (14-15)) with its boundary conditions, and the coupling pressure relation (equation (12) or (11), depending on the chosen longitudinal scale).

2.5.2 Howarth Doronitsyn form

A classical trick is used in order to write the equations in an incompressible form (Méndez & al (19) or Stewartson (28)). The longitudinal variable remains the same, but the transversal one is changed in noticing that the equations simplify if we put $dY = \rho dy$. So $y = \int_0^Y T(Y')dY'$, which may be written as:

$$y = Y + \int_0^Y (T(x, Y') - 1)dY'.$$

Next we define V as $\rho v + u\frac{\partial Y}{\partial x}$. The final simplification is that the fluid is a model fluid $\mu = CT$, and for sake of simplification C is 1 and P_r is 1 too. Finally we write the pressure in the Lower Deck with the help of the Upper Deck's one as:

$$p(x, y) = p_1(x, 0) - \frac{1}{F}\int_\infty^y (\frac{1}{T} - 1)dy'.$$

With those final assumptions we write the "fundamental Triple Deck problem" of locally induced mixed convection in a Poiseuille flow. The Lower Deck problem reads:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial Y}V = 0, \tag{16}$$

$$u\frac{\partial}{\partial x}u + V\frac{\partial}{\partial Y}u = -T(\frac{\partial}{\partial x}p_1 - \frac{1}{F}\int_\infty^Y \frac{\partial}{\partial x}(1 - T)dY' - \tag{17}$$

$$-\frac{1}{F}\frac{\partial Y}{\partial x}(1-T)) + \frac{\partial^2}{\partial Y^2}u, \tag{18}$$

$$u\frac{\partial}{\partial x}T + V\frac{\partial}{\partial Y}T = \frac{\partial^2}{\partial Y^2}T, \tag{19}$$

with the boundary conditions: $u = V = 0$ in $Y = 0$, and $u \to Y + \int_0^Y (T - 1)dY' + A(x)$ for $Y \to \infty$. The temperature is prescribed at the wall $Y = 0$: $T = 1$ for $x < 0$ and $T = T_w$ for $x > 0$.

The Upper Lower Deck problem is unchanged:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial z}v = 0; \tag{20}$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial z}u = -\frac{d}{dx}p_h + \frac{\partial^2}{\partial z^2}u; \tag{21}$$

$u \to z - A$ for $z \to \infty$ and $u \to z$ for $x \to -\infty$. The no slip condition is $u = v = 0$ in $z = 0$. The pressures match as:

$$p_h(x) - p_1(x) = \frac{1}{30}A''(x),$$

or as (12) if $x_3 \gg R^{1/7}$. Notice the links between the skin frictions:

$$\frac{\partial u}{\partial y}_{y=0} = \frac{\partial u}{\partial Y}_{Y=0}\frac{1}{T_w T(x, Y = 0)}. \tag{22}$$

The full non linear resolution should be done with an appropriate technique to catch separation of the boundary-layer, if it exists. Here, we will only look at self-similar results in a simplified case and at linearized results for small $\theta = T_w - 1$, so we will only see a small decrease at first order in θ in the skin friction.

3 Fourier transformed linear resolution

3.1 Linearized equations

Here we suppose that the variation of temperature at the wall is very small but not necessary a step, say: $T = 1 + \theta T_0(x, 0)$ where $T_0(x, 0)$ is the imposed wall temperature variation, it is the Heaviside function in the case of a step. We linearize the variables:

$y = Y + \theta \int_0^Y T_0(x, Y')dY'$, $u = Y + \theta u_0, V = \theta V_0$ and so on...

The linearized equations in the Lower Deck are then:

$$\frac{\partial}{\partial x}u_0 + \frac{\partial}{\partial Y}V_0 = 0, \tag{23}$$

$$Y\frac{\partial}{\partial x}u_0 + V_0 = -\frac{\partial}{\partial x}(p_{10} + \frac{1}{F}\int_\infty^Y T_0 dY') + \frac{\partial^2}{\partial Y^2}u_0, \tag{24}$$

$$Y\frac{\partial}{\partial x}T_0 = \frac{\partial^2}{\partial Y^2}T_0. \tag{25}$$

We notice that at first order the classical Lévêque problem of diffusion of temperature in a linear shear flow is of course recovered (as seen on §3.1).

3.2 *Fourier resolution*

3.2.1 *Method*

The classical technique is used (Smith (25)):

$$u_0 = f'(Y)e^{i\alpha x}, V_0 = -i\alpha f(Y)e^{i\alpha x}, p_{10} = Pe^{i\alpha x}, T_0 = T(Y)e^{i\alpha x}.$$

and we find first that the temperature satisfies an Airy equation:

$$(\frac{d^2}{d\eta^2} - \eta)T = 0,$$

with $\eta = (i\alpha)^{1/3}Y$, so $T = G(\alpha)\frac{\mathrm{Ai}(\eta)}{\mathrm{Ai}(0)}$, where $G(\alpha)$ is the Fourier transform of the heating at the wall (if the heating is a step, this expression is the Fourier transform of the generalized incomplete Gamma function leading to the Lévêque solution). Differentiating twice the velocity, we obtain an Airy equation for the transformed perturbation of skin friction $\tau = f''(Y)$ this equation being forced by the temperature which is an Airy function:

$$(\frac{d^2}{d\eta^2} - \eta)\tau = \frac{1}{F}(i\alpha)^{1/3}T.$$

By chance the particular solution is simply proportional to Ai'. Notice that a blind application of standard technique leads to the following expression which is then equivalent to $\mathrm{Ai}'(\eta)$:

$$-\pi(\int_0^\eta \mathrm{Ai}(\xi)\,\mathrm{Bi}(\xi)d\xi)\,\mathrm{Ai}(\eta)+\pi(\int_\infty^\eta \mathrm{Ai}^2(\xi)d\xi)\,\mathrm{Bi}(\eta)+$$

$$+(\mathrm{Ai}'(0)-\pi\,\mathrm{Bi}(0)(\int_\infty^0 \mathrm{Ai}^2(\xi)d\xi)\,\mathrm{Ai}(\eta)/\,\mathrm{Ai}(0).$$

3.2.2 Results

So we find the perturbation of skin friction:

$$f''(0)=3\beta^*\,\mathrm{Ai}(0)P-\frac{(i\alpha)^{1/3}}{F}\gamma G(\alpha), \tag{26}$$

next we obtain the displacement function:

$$A=\beta^*P+\frac{\beta G(\alpha)}{F}+BG(\alpha), \tag{27}$$

where the coefficients, β^*, β, γ and B are defined as follows:

$\beta^*=\frac{(i\alpha)^{1/3}}{3\,\mathrm{Ai}'(0)}$ is the classical standard triple deck response,

$\beta=-(1+\frac{1}{9\,\mathrm{Ai}(0)\,\mathrm{Ai}'(0)})$ and $\gamma=(\frac{1}{3\,\mathrm{Ai}'(0)}-\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)})$ are due to the transverse pressure variation,

$B=-\frac{1}{3\,\mathrm{Ai}(0)(i\alpha)^{1/3}}$ comes from the density expansion.

The pressure displacement relation (11) gives $P_h-P=\frac{-\alpha^2}{30}A$ and the response of the Upper Lower Deck is simply $P_h=-\frac{A}{\beta^*}$ (because there is no β no γ and no B, and the upper displacement function is the opposite of the lower one). The final pressure displacement relation which includes the Main and Upper Lower Decks is then $P=ZA$, (if we define Z as $-\frac{1}{\beta^*}+\frac{\alpha^2}{30}$).

So, for a given Fourier mode α, the final linear response in pressure depending of the heating at the wall is:

$$P=-\frac{\beta/F+B}{\beta^*-1/Z}G(\alpha) \tag{28}$$

With this we obtain next the displacement $-A$ (in substituting (28) in (27)) and the perturbation of the skin friction τ (in substituting (28) in (26) to obtain the "real" skin friction we have to substract $G(\alpha)$ as seen in (22)). This solves completely the linearized problem of retroaction of heating on the basic flow with compressibility and gravity effects at Triple Deck scales in any case of heating at the wall consistent with the Fourier method.

4 Self-similar resolution

In this paragraph we look at the response of the three decks to a step in temperature occurring at the lower wall. This may be obtained directly (in the linearized case) by inverse Fourier techniques (see Gittler (12) who does the systematic search of self induced solution in the "standard" Lower Deck problem), but here we prefer to search from scratch for a self similar solution. To simplify we disconnect the two effects, we will first find a non-linear self-similar solution of the non buoyant effect, and second a linear self-similar solution of the strongly buoyant effect. To obtain the similarity we have to look at the equations at a longitudinal scale greater than $R^{1/7}$ (pressure is equation (12)).

4.1 No buoyancy

Here we suppose that the gravity is negligible. The pressure is constant across all the decks (equation (12)), and it is easy to observe that it is then possible to have a self-similar solution in the Lower Deck (eq. (16)-(19)). With the new variables x and $\eta = Y/x^{1/3}$ we have:

$$\psi(x,Y) = x^{2/3} f(\eta),\ u(x,Y) = x^{1/3} f'(\eta),\ V(x,Y) = \frac{1}{3} x^{-2/3}(\eta f'(\eta) - 2f(\eta))$$

$$p(x,Y) = x^{2/3} P_0,\ T(x,Y) = 1 + \theta g(\eta).$$

Recall that here θ is not necessary small. As well in the Upper Lower Deck, we put in (20)-(21) the following variables:

$$p_h(x,z) = x^{2/3} P_0,\ \psi(x,z) = x^{2/3} f_h(\eta),$$

$$u(x,z) = x^{1/3} f_h'(\eta),\ v(x,z) = \frac{1}{3} x^{-2/3}(\eta f_h'(\eta) - 2f_h(\eta)),$$

with here an other self similar variable $\eta = z/x^{1/3}$ (there is no confusion).

So we have to solve the following problem: given θ find P_0 such as the relation of conservation of the $-A$ function is true. What we have to verify is: $\check{a} =$

$\theta \int_0^\infty g d\eta - a_h$, with $\breve{a}$ and a defined as follows. First in the Lower Deck the resolution of

$$-f^{'''} - \frac{2}{3} f f^{''} + f^{'2} + \frac{2}{3} P_0 (1 + \theta g) = 0, \;\; g^{''} + \frac{2}{3} f g = 0,$$

with $f(0) = 0$, $f^{'}(0) = 0$ and $f''(\infty) = 1$. $g(0) = 1$, and $g(\infty) = 0$, gives a pseudo displacement $\breve{a} = \lim_{\eta\to\infty}(f' - \eta)$.

Second, in the Upper Lower Deck the resolution of

$$-f_h^{'''} - \frac{2}{3} f_h f_h^{''} + \frac{f^{'2}}{3} + \frac{2}{3} P_0 = 0$$

with $f_h(0) = 0$, $f_h^{'}(0) = 0$ and $f_h''(\infty) = 1$, gives the opposite of the real displacement: $a_h = \lim_{\eta\to\infty}(f_h' - \eta)$.

All the self-similar calculations are performed with a classical Runge-Kutta 4 integration with shooting (on the condition at infinity). We display now on figure 2 the value of P_0 as a function of θ : the more the flow is heated, the more there is a pressure expansion, the line is the linearized result $-.4507\ \theta$ (Gittler (12)). The full linearized expression of pressure is then $-.4507\ \theta x^{2/3}$. On figure 3 are plotted the skin frictions functions of θ , the line with a positive slope is again the Gittler linearized result $(1 + .558\theta\ )$ which corresponds to the linearized solution for the upper wall, there the problem is a "standard Triple Deck" one. The conclusion is then natural for a favourable pressure gradient: the more the pressure decreases (when θ increases), the more the skin friction $f_h''(0)$ increases from the unit value. On the other hand, we note that the physical skin friction at the lower wall $f''(0)/(1 + \theta)$ is always lower than 1, the value of Poiseuille. The full linearized results for the small step are $1 - .442\theta x^0$ for the skin friction at the lower wall and the corresponding displacement is $-.526\theta x^{1/3}$.

5 Governing equations

The profiles are plotted below (on figure 4) in the arbitrary case $\theta = 1$. The physical velocity Lower Deck profile is $f'(\eta)/(1 + \theta g(\eta))$ function of $y/x^{1/3} = \eta + \theta \int_0^\eta g(\xi) d\xi$, and the temperature profile is g function of the same $y/x^{1/3}$. The Upper Deck velocity is $f_h'(z/x^{1/3})$. We see the upward displacement in the Lower Deck transmitted to the Upper Lower Deck; the straight line is the Poiseuille flow, linear at those scales.

5.1 Strong buoyancy

In this case there is a transverse pressure variation but $\theta\ /F << 1$ and $\theta << 1$; it is impossible to find a complete non linear self similar solution. So we simply look at a linearized similar solution. The temperature is the classical incomplete Gamma solution, his integral disappears from the displacement equation, so we have for the pressure:

$$P(x,y) = \theta F^{-1}(\int_{\infty}^{\eta} T(\eta)d\eta + \Pi)x^{1/3}.$$

The problem is to find Π in order to have in the Lower Deck where: $u = \eta x^{-1/3} + \theta/F f'(\eta)$

$$-f''' - \frac{\eta^2}{3}f'' + \frac{\eta}{3}f' + \frac{1}{3}\int_{\infty}^{\eta} T(\eta)d\eta - \frac{\eta}{3}T(\eta) + \frac{\Pi}{3} = 0,$$

satisfying $f'(\infty) = a$, $f'(0) = 0$, $f''(\infty) = 0$, and in the Upper Lower Deck where: $u = \eta x^{-1/3} + \theta/F f_h'(\eta)$

$$-f_h''' - \frac{\eta^2}{3}f_h'' + \frac{\eta}{3}f_h' + \frac{1}{3}f_h + \frac{\Pi}{3} = 0,$$

satisfying $f_h'(\infty) = -a$, $f_h'(0) = 0$. $f_h''(\infty) = 0$.

So we find after numerical (again Runge-Kutta 4 with shooting) integration of the systems that $\Pi = 0.091$ and $a = 0.105$. The slopes are $f''(0) = 0.330$ and $f_h''(0) = -0.082$. So the pressure in the Main Deck behaves as $0.091\ \frac{\theta}{F}x^{1/3}$ and the displacement function $-A$ is constant over the heated region of value $-0.105\frac{\theta}{F}$ (there is an abrupt downward discontinuous displacement at those scales). The skin friction ($\partial u/\partial y$) at the lower wall is $1 + 0.330\frac{\theta}{F}x^{-1/3}$ while it is $1 - 0.082\frac{\theta}{F}x^{-1/3}$ on the upper wall (note that they are singular in $x = 0$ at those scales). We plot the perturbations of the velocity profiles in the Lower and Upper Lower Deck $f'(\eta)$ and $f_h'(\eta)$ on figure 5. We see that conclusions are reversed in comparison with the previous case with no gravity. The upper profile is with a smaller slope, the lower profile has an increased skin friction and it presents an over shoot: in "falling down" the fluid creates a small jet.

6 Comparisons in the step case and discussion

We now compare the two linearized methods (self-similar and the Fourier transform). For the Fourier method we use a "door-function" increase of temperature ($T = 1$ for $x > 1$ and $x < -1$, and $T = 1 + \theta$ for $-1 < x < 1$). The comparison of the two methods is meaningful only near the first discontinuity of temperature (in $x = -1$, be aware that in the previous sections $x = 0$ was the origin of thermal change). We take 16384 points for the standard FFT code we use and the domain lies between -50 and 50. The algebraic decay of the quantities is problematic in the FFT code and the domain should be enlarged for better results. To compare the two methods with make two shifts, first a shift of value -1 for the origin of x for the self-similar results, and second we shift the ordinate by the calculated FFT value obtained just before -1 (the tail after $+1$ is interfering in -1 because of the periodicity coming from the FFT method).

6.1 No gravity

With those values, on figure 6 we compare favorably the upward displacement without gravity in the case of the self similar resolution ($-.526(1+x)^{1/3}$ note the translation in -1) and in the FFT method (equation (28) with no $-A''$ term). The case of transverse pressure variation introduces some upstream influence (the A'' in term (11) which was not present before) smoothing the displacement. On figure 7 is plotted the analytic pressure $-.4507(1+x)^{2/3}$ compared with his Fourier counterpart, again the agreement is excellent (if again we shift the analytic pressure to the value of the calculated one just before -1). The upstream induced pressure grows before the heated region, it is coherent with the upward displacement. Far away from the step we note that the influence of A'' becomes negligible, which is coherent with the fact that there is no pressure variation if we look at the phenomena at a bigger longitudinal scale.

The skin friction is plotted on figure 8, the two methods agree in the lowering on the lower wall and increase on the upper one, with always the self induced smoothing when A'' is taken.

6.2 With gravity

Next we look at the buoyant case. Here we compare the Fourier result (28) with $1/F = 100$ to the self similar solution. Figure 9 shows the skip (in the (12) case) in displacement from 0 to $0.105/F$: the streamlines are deflected downwards. Again, this skip is smoothed when the upstream influence term A'' (11) is put in the FFT code. The pressure grows 0.091 $\frac{1}{F}x^{1/3}$ on figure 10, with a small drop if upstream influence is allowed. The small discrepancy comes from the fact that the full (28) is solved, and not his limit expression at $1/F \to \infty$. Of course the introduction of $-A''$ in the pressure relation introduces some upstream influence, but far from the discontinuity it is again negligible.

The skin friction is increased at the lower wall and diminished at the upper wall. Here, on figure 11, we have plotted too the exponential e^{Kx} growth of the departure of the skin friction from the Poiseuille flow before the heated region, showing the upstream influence. We checked that the birth of this upstream influence is in exponential e^{Kx} in every case. In the gravity dependant case, at the upper wall P and $-A$ decrease and the skin friction increases as e^{Kx}. The value of K comes from the search of an eigensolution of the system. $K = (-180\ Ai'(0))^{3/7} (= 5.188\ )$ is the "Lighthill eigenvalue" occurring here in this expansive free interaction. In the preceding case with no gravity this result is valid again, pressure decreases and displacement increases at upper wall, but $-A$ increases (because of the integral of the temperature term which is negligible in the strong buoyant case).

The skin friction remains discontinuous even when there is upstream influence (the same with the elliptic law of Méndez et al (19)). A new smaller scale must be introduced to tackle with this.

7 Conclusion and discussion.

7.1 results

We have stated the problem of a change of wall temperature in a Poiseuille flow of a model perfect gas in the framework of Triple Deck (*i.e.* laminar Newtonian flow at Reynolds infinite). We have found the full linearized solution in Fourier space and we have found with self-similar variables a linearized solution and even a nonlinear solution when possible. We have plotted the displacement, the pressure and the skin friction in the two limiting cases (F infinite or small), any intermediate value of the Froude number may be plotted but we whishe to focus on the competition of the expandability of the gas against the gravity effect.

As a result a physical explanation of the flow may be given. On the one hand, we observe that in the case of no gravity, heating increases the volume of the gas so, by the expansion of the volume, the pressure decreases and the displacement $-A$ of the streamlines is positive; because of this upward defection, the skin friction increases on the top wall but decreases on the bottom one. On the other hand, the case of strong gravity implies a favorable pressure gradient induced by the temperature stratification in the Lower Deck and a negative displacement of the streamlines associated with an increase of pressure in the Main and Upper Decks. So the skin friction is increased at the bottom and decreased at the top, a sort of jet is created near the wall. It is likely that this adverse pressure gradient may cause separation at the top wall, to confirm this a full nonlinear resolution must be done. The heuristical explanation of this phenomenon would be that "the cold gas falls down", this is a dramatic change to the common belief that states that "the warm gas goes up"! Nevertheless it is consistent with the counter pressure induced by the heating which tends to brake the flow.

7.2 Discussion

A discussion of this behaviour may be done if we look at the free convection flow problem over a step in temperature. Scales are now any L^* for x, αL^* for y, deviation from hydrostatic pressure is scaled by $\alpha\rho_0 g L^*$, density and viscosity by (ρ_0, μ_0),and velocities (u, v) by $((\alpha g L^*)^{1/2}, \alpha(\alpha g L^*)^{1/2})$, with the small parameter α obtained by least degeneracy: $\alpha = (gL^{*3}/\nu^2)^{-1/5}$. Where gL^{*3}/ν^2 is a kind of Grashoff number.

$$\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v = 0, \tag{29}$$

$$\rho u\frac{\partial}{\partial x}u + \rho v\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}p + \frac{\partial}{\partial y}\mu\frac{\partial}{\partial y}u, \tag{30}$$

$$0 = -\frac{\partial}{\partial y}p - (\rho - 1), \tag{31}$$

$$\rho u\frac{\partial}{\partial x}T + \rho v\frac{\partial}{\partial y}T = P_r^{-1}\frac{\partial}{\partial y}\mu\frac{\partial}{\partial y}T, \tag{32}$$

$$\rho T = 1. \tag{33}$$

Now, if we suppose the temperature variation gauge θ small: $T = 1 + \theta\tilde{T}$, so $(\rho - 1) = -\,\theta\tilde{T}$, and variations of μ are of order θ. So if we rescale y by Δ, the pressure is rescaled by $\theta\Delta$, the longitudinal velocity by $(\theta\Delta)^{1/2}$, the convective diffusive equilibrium gives $\Delta = \theta^{-1/5}$. The final transverse scale is then $(g\theta L^{*3}/\nu^2)^{-1/5}L^*$. The final system, where we omit the $O(\theta)$ variations of ρ and μ, is exactly the Stewartson (27) free convection problem on a plate in the Boussinesq approximation (here we neglect the influence of the upper wall). It will produce the same conclusion than in Gill & al (13): the fluid will go up heated by the plate. This solution will destroy the mixed one when the Lower Deck velocity (εU_0) is smaller than the free convection velocity ($\theta^{2/5}(gL^{*3}/\nu^2)^{-1/10}(gL^*)^{1/2}$) at the chosen longitudinal scale x_3 (so L^* is chosen as H^*x_3), this gives the criterion $1 \ll \frac{\theta}{F}$, which is really out of range ofour study. That is the reason why common sense predicts an upward deviation, but our problem of mixed convection (at $\theta/F \leq 1$) cannot be considered as a superposition of a Poiseuille flow plus this solution because of the nonlinearity present.

7.3 Qualitative application to M.O.C.V.D

An application of this asymptotic analysis may be found in M.O.C.V.D. flows. "Metal Oxide Chemical Vapor Deposition" is a state-of-the-art method of Epitaxy for creation of semi-conductors for electronics and optoelectronics. The reactor is often a pipe, at the entry the fresh reactant gases arrive and the chemical process takes place on the heated zone called the "susceptor". The challenge is to obtain the growth of very thin layers of atoms on the susceptor. Of course complete Navier-Stokes solvers are now very accurate (Fotiadis & al (9), Ern & al (7)) to compute all the fields. In fact the problems are not in the dynamics (the flow is laminar and stationary) but in the chemistry, because there are many complex reactions. Using Fluent (8), we evaluate roughly the flow with strong (figure 12) and small buoyancy (figure 13) in a typical simplified reactor. Here calculations are a bit crude and the values of Reynolds and Froude are $R = 227$ and $F_0 = .7$.

The first figure is very similar to those calculated and plotted by Holstein (14) (the adimensionalized number he uses is Grashoff divided by R^2 which is the inverse of our Froude number) and it clearly shows the negative deflection of the fluid flow, and a separation near the upper wall; the second figure is without gravity and shows the positive deflection of the fluid flow. But approximate methods are always interesting: for example, it has been shown by Van de Ven et al (30) or Ghandi & Field (11) that the linear profile approximation is good enough to predict the growth rate; our purpose was to investigate the next term in this expansion, which was done using the Triple Deck Theory. The hypotheses that we needed in this paper are unfortunately a bit far from experimental problems occurring in real M.O.C.V.D. reactors, an increase of the velocity by factor 10 would move the non-dimensional numbers in a better range. Nevertheless, even if the operating Reynolds number is often less than 200 and the Froude Number is smaller than one, the preceding theory seems to agree qualitatively (we put no ticks on the graphs) with numerical phenomena occurring in the reactor. With no gravity, the expansion of the gas pushes up the streamlines (fig 6 and 13), with gravity the cold gas is falling down (fig 9 and fig 12) and a recirculating region is created.

This comparison is only a rough one, and a better one would involve a careful examination of more Navier-Stokes calculations rescaled with the scales that we exhibited. Then a full nonlinear triple deck calculation would be necessary to calculate the supposed (as observed in the full N.S. computation) heating-induced upper-wall separation (incidentally the case of cold spot instead of a hot one could be investigated). Finally, and most interesting for the M.O.C.V.D. community, the transport equation may be simply included in this theory, at first order, it remains the same as in (30) or (11): the heating induces changes of velocity inducing themselves, at second order, a modification of the growth rate on the susceptor.

This work is dedicated to Marguerite Delepierre née Rogier.

8 Titles and Abstracts in foreign languages

8.1 title and abstract in french

titre: À propos de la convection thermique mixte, induite localement par une variation brusque de température dans un écoulement de Poiseuille et étudiée dans le cadre de la "triple couche".

Résumé: La solution classique de Lévêque décrivant la variation de température induite par une variation brusque du chauffage de la paroi pour un écoulement de cisaillement pur est réexaminée dans le cas d'un écoulement de Poiseuille plan horizontal d'un gaz parfait pesant et visqueux près de la paroi.

La rétroaction de l'élévation de température sur le profil de base est effectuée de deux manières: premièrement par la dépendance de la densité et de la viscosité avec la température, deuxièmement par le gradient de pression transverse induit par la gravité (et jaugé par l'inverse du nombre de Froude), la démarche adoptée est celle de la "Triple Couche".

Outre la solution nonlinéaire autosemblable obtenue dans le cas non pesant, des solutions linéarisées sont présentées. Si la gravité n'est pas prise en compte, on constate que les lignes de courant sont déportées vers le haut (de par la dilatibilité du gaz): le frottement pariétal diminue en bas et augmente en haut. En revanche, si la gravité est incluse et qu'elle est importante, le phénomène inverse se produit; cela pourrait causer un courant de retour près de la paroi supérieure.

Ces deux effets ont été observés qualitativement par simulation directe des équations de Navier Stokes complètes dans un réacteur pour épitaxie en phase gazeuse.

8.2 title in german

Gemischter Konvektion infolgue eines Sprungs der Oberflächentemperatur in eines Poiseuille- Strömung mit Dreier- Deck- Theorie.

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