

Nonlinear Spectral Synthesis of Soliton Gas in Deep-Water Surface Gravity Waves

Pierre Suret¹, Alexey Tikan¹, Félicien Bonnefoy², François Copie¹, Guillaume Ducrozet²,
 Andrey Gelash^{3,4}, Gaurav Prabhudesai⁵, Guillaume Michel⁶, Annette Cazaubiel⁷, Eric Falcon⁷,
 Gennady El⁸, and Stéphane Randoux^{1,*}

¹Univ. Lille, CNRS, UMR 8523—PhLAM—Physique des Lasers Atomes et Molécules, F-59 000 Lille, France

²École Centrale de Nantes, LHEEA, UMR 6598 CNRS, F-44 321 Nantes, France

³Institute of Automation and Electrometry SB RAS, Novosibirsk 630090, Russia

⁴Skolkovo Institute of Science and Technology, Moscow 121205, Russia

⁵Laboratoire de Physique de l'École normale supérieure, ENS, Université PSL, CNRS, Sorbonne Université,
 Université Paris-Diderot, 75005 Paris, France

⁶Sorbonne Université, CNRS, UMR 7190, Institut Jean Le Rond d'Alembert, F-75 005 Paris, France

⁷Université de Paris, Université Paris Diderot, MSC, UMR 7057 CNRS, F-75 013 Paris, France

⁸Department of Mathematics, Physics and Electrical Engineering, Northumbria University, Newcastle upon Tyne,
 NE1 8ST, United Kingdom



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Soliton gases represent large random soliton ensembles in physical systems that exhibit integrable dynamics at the leading order. Despite significant theoretical developments and observational evidence of ubiquity of soliton gases in fluids and optical media, their controlled experimental realization has been missing. We report a controlled synthesis of a dense soliton gas in deep-water surface gravity waves using the tools of nonlinear spectral theory [inverse scattering transform (IST)] for the one-dimensional focusing nonlinear Schrödinger equation. The soliton gas is experimentally generated in a one-dimensional water tank where we demonstrate that we can control and measure the density of states, i.e., the probability density function parametrizing the soliton gas in the IST spectral phase space. Nonlinear spectral analysis of the generated hydrodynamic soliton gas reveals that the density of states slowly changes under the influence of perturbative higher-order effects that break the integrability of the wave dynamics.

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Solitons are localized nonlinear waves that have been studied in many areas of science over the last decades [1–3]. Solitons represent fundamental nonlinear modes of physical systems described by integrable equations like the Korteweg–de Vries (KdV) equation or the one-dimensional nonlinear Schrödinger equation (1D-NLSE) [1,3–8]. Nowadays the dynamics of soliton interaction is so well mastered that ordered sets of optical solitons or their periodic generalizations, the so-called finite-gap potentials, are synthesized and manipulated to carry out the transmission of information in fiber optics communication links [9–12]. On the other hand, the question of collective dynamics of *large random* soliton ensembles represents a subject of active research in statistical mechanics and in nonlinear physics, most notably in the contexts of ocean wave dynamics and nonlinear optics, see, e.g., Refs. [13–25].

The concept of soliton gas (SG) as a large ensemble of solitons randomly distributed on an infinite line and elastically interacting with each other originates from the work of Zakharov [26], who introduced the kinetic equation for a nonequilibrium *diluted* gas of weakly interacting solitons of the KdV equation. Zakharov's kinetic equation

has been generalized to the case of a dense SG in Ref. [27] (KdV) and in Refs. [28,29] (focusing NLS). Additionally, the SG kinetic equation has recently attracted much attention in the context of generalized hydrodynamics for quantum many-body integrable systems; see Refs. [30–32] and references therein.

Within the inverse scattering transform (IST) formalism [5,7,33], each soliton in a gas is characterized by a discrete eigenvalue λ_i of the spectrum of the linear operator associated with the integrable evolution equation. The fundamental property of integrable dynamics is the preservation of the soliton spectrum under evolution. The central concept in SG theory, borrowed from quantum physics of disordered systems [34], is the density of states (DOS), which represents the distribution $u(\lambda, x, t)$ over the spectral eigenvalues, so that $u d\lambda dx$ is the number of soliton states found at time t in the element of the phase space $[\lambda, \lambda + d\lambda] \times [x, x + dx]$. In a spatially homogeneous (equilibrium) SG, the DOS is stationary in time, while in a nonhomogeneous SG, which is far from equilibrium, it evolves according to the kinetic equation [27–29].

Despite various developments of SG theory (see, e.g., Refs. [35–42]) and the existence of an unambiguous

characterization of SG through the concept of DOS, the experimental or observational results in this area are quite limited. Costa *et al* have reported in 2014 the observation of random wave packets in shallow water ocean waves that have been analyzed using numerical IST tools and interpreted as randomly distributed solitons that might be associated with KdV SG [43]. In 2015 large ensembles of interacting and colliding solitons have been observed in a levitating rectilinear water cylinder [44]. Recently Redor *et al.* have used the process of fission of a sinusoidal wave train to generate an ensemble of bidirectional shallow water solitons in a 34-m long flume and they have analyzed the generated SG using (linear) Fourier transform [45]. In optics, the SG terminology has been used to describe experiments where light pulses were synchronously injected in a passive fiber ring cavity [46]. Another recent experimental observation of complex nonlinear wave behavior attributed to SG dynamics was reported in Ref. [47], where the formation of an incoherent optical field has been observed in the evolution of a square pulse in a focusing medium [48]. At the same time, to our knowledge, there is no existing experiment where SG would have been unambiguously identified using some well-defined, quantitative characteristics and, in particular, where the measurement and control of the DOS of the SG would have been achieved.

In this Letter, we report experiments fully based on the IST method where we generate and study hydrodynamic deep-water dense and spatially homogeneous SGs. We take advantage of the recently developed methodology for the effective numerical construction of the so-called N -soliton solutions of the focusing 1D-NLSE with N large (Ref. [49]), to create an incoherent wave field having a dominant and controlled solitonic content characterized by a measurable DOS. We show that the nonlinear wave field in the generated SG may undergo some complex space-time evolution while the discrete IST spectrum is found to be nearly conserved, albeit being perturbed by higher-order effects.

Our experiments were performed in a wave flume 148 m long, 5 m wide, and 3 m deep. Unidirectional waves are generated at one end with a computer assisted flap-type wave maker and the flume is equipped with an absorbing device strongly reducing wave reflection at the opposite end. As in Ref. [50], the setup comprises 20 equally spaced resistive wave gauges that are installed along the basin at distances $Z_j = j \times 6$ m, $j = 1, 2, \dots, 20$ from the wave maker located at $Z = 0$ m. This provides an effective measuring range of 114 m.

In our experiment, the water elevation at the wave maker reads $\eta(Z = 0, T) = \text{Re}[A_0(T)e^{i\omega_0 T}]$, where $\omega_0 = 2\pi f_0$ is the angular frequency of the carrier wave. $A_0(T)$ represents the complex envelope of the initial condition. Our experiments are performed in the deep-water regime, and they are designed in such a way that the observed dynamics is described at leading order by the focusing 1D-NLSE:

$$\frac{\partial A}{\partial Z} + \frac{1}{C_g} \frac{\partial A}{\partial T} = i \frac{k_0}{\omega_0^2} \frac{\partial^2 A}{\partial T^2} + i \alpha k_0^3 |A|^2 A, \quad (1)$$

where $A(Z, T)$ represents the complex envelope of the water wave that changes in space Z and in time T [51]. k_0 represents the wave number of the propagating wave [$\eta(Z, T) = \text{Re}[A(Z, T)e^{i(\omega_0 T - k_0 Z)}]$], which is linked to ω_0 according to the deep water dispersion relation $\omega_0^2 = k_0 g$, where g is the gravity acceleration. $C_g = g/(2\omega_0)$ represents the group velocity of the wave packets and α is a dimensionless term describing the small finite-depth correction to the cubic nonlinearity [50].

The first important step of the experiment consists in generating an initial condition $A_0(T)$ in the form of a random wave field having a pure solitonic content. To achieve this, we move to the ‘‘IST-friendly’’ canonical dimensionless form of the 1D-NLSE:

$$i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = 0, \quad (2)$$

where $\psi(x, t)$ represents the normalized complex envelope of the water wave. Connections between the physical variables of Eq. (1) and dimensionless variables in Eq. (2) are given by $t = Z/L_{\text{NL}}$, $x = (T - Z/C_g)\sqrt{g/(2L_{\text{NL}})}$ with the nonlinear length being defined as $L_{\text{NL}} = 1/(\alpha k_0^3 \langle |A_0(T)|^2 \rangle)$, where the angle brackets denote the average over time.

The nonlinear wave field $\psi(x, t)$ satisfying Eq. (2) can be characterized by the so-called scattering data (the IST spectrum). For the *localized*, i.e., decaying to zero as $|x| \rightarrow \infty$ wave field the IST spectrum consists of a discrete part related to the soliton content and a continuous part related to the dispersive radiation. A special class of solutions, the N -soliton solutions (N-SSs), exhibit only a discrete spectrum consisting of N complex-valued eigenvalues λ_n , $n = 1, \dots, N$ and N complex parameters $C_n = |C_n|e^{i\phi_n}$, called norming constants, defined for each λ_n . In all the experiments described below, the phases ϕ_n of the norming constants C_n characterizing the generated N-SS are randomly and uniformly distributed over $[0, 2\pi)$, while their modulus $|C_n|$ is chosen to be equal to unity. As shown in Refs. [49,52], such an N -soliton statistical ensemble is a good model for a homogeneous dense SG.

In our first experimental run, we used numerical methods described in Ref. [49] to generate an N-SS of Eq. (2), hereafter denoted $\psi_{16}(x, t)$, with $N = 16$ eigenvalues chosen arbitrarily within some domain of the complex spectral plane, as shown with blue points in Fig. 1(b). A relatively small number of solitons in this random soliton ensemble prevents its proper macroscopic spectral characterization and the identification with SG. However, it is important as a first step in our experiment to establish a robust protocol for the generation of random soliton ensembles in a spectrally controlled way.

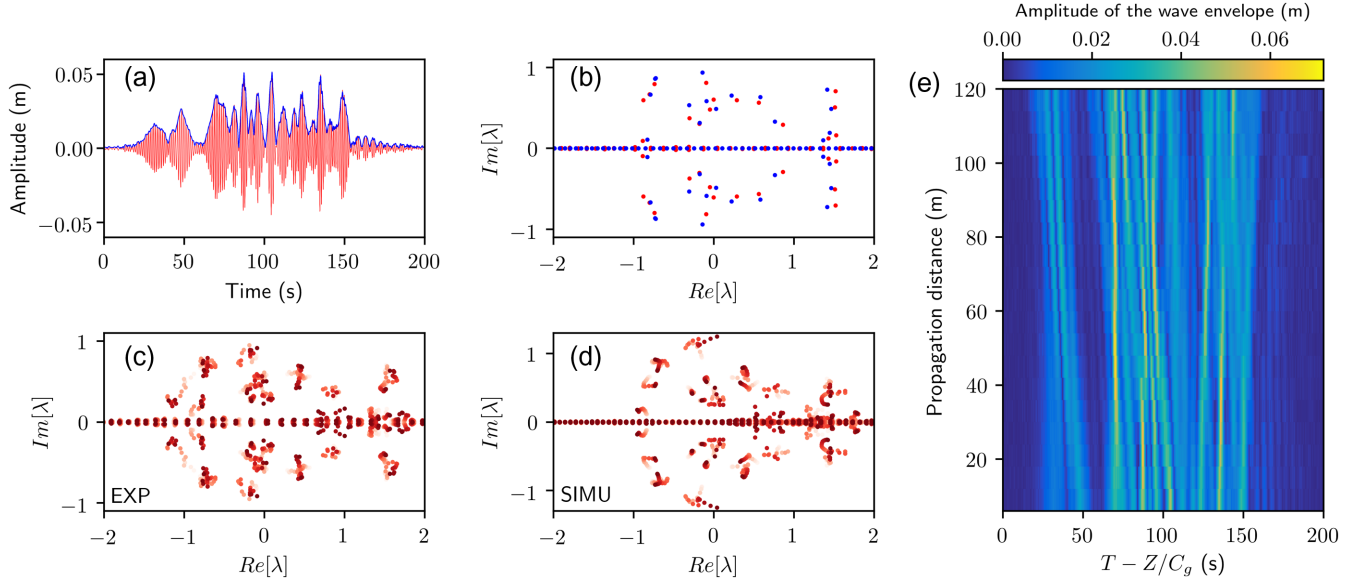


FIG. 1. Ensemble of $N = 16$ solitons propagating in the 1D water tank. (a) Water elevation (red line) and modulus of the wave envelope measured at $Z_1 = 6$ m, close to the wave maker. (b) Blue points represent the discrete IST spectrum of the numerically generated N-SS $\psi_{16}(x, t = 0)$ and red points represent the discrete IST spectrum measured at $Z_1 = 6$ m by using the signal plotted in (a). (c) Space evolution of the discrete IST spectra measured along the tank from $Z_1 = 6$ (light red) to $Z_{20} = 120$ m (dark red). (d) Same as in (c) but obtained from numerical simulations of a modified (not integrable) 1D-NLSE including higher-order effects; see Supplemental Material [53]. (e) Space-time evolution of modulus of the wave envelope recorded by the 20 gauges regularly spaced along the tank. Physical parameters characterizing the experiment are $f_0 = 0.9$ Hz, $k_0 = 3.26$ m $^{-1}$, $\alpha = 0.895$, $L_{NL} = 210$ m ($\langle |A_0(T)|^2 \rangle = 1.53 \times 10^{-4}$ m 2).

After appropriate scaling, the generated dimensionless wave field $\psi_{16}(x, t = 0)$ is converted into the physical complex envelope $A_{16}(Z = 0, T) = A_0(T)$ of the initial condition, which is generated by the wave maker. Figure 1(a) shows the water elevation measured at $Z_1 = 6$ m together with the modulus of the envelope $|A_{16}(Z_1, t)|$ computed using standard Hilbert transform techniques [51]. The generated wave field with pure solitonic content spreads over approximately 140 s and exhibits large amplitude fluctuations due to the random phase distribution. Figure 1(b) shows the discrete IST spectrum that is computed from the signal recorded by the first gauge and plotted in Fig. 1(a). The measured eigenvalues plotted in red points in Fig. 1(b) are close to the discrete eigenvalues (blue points) that we have selected to build $\psi_{16}(x, t = 0)$, the N-SS under consideration. This demonstrates that the process of generation of the N-SS solution is well controlled in our experiments.

As shown in Fig. 1(e), the space-time evolution of the generated wave packet measured with 20 gauges distributed along the tank reveals complex dynamics with multiple interacting coherent structures. Despite the apparent complexity of the observed wave evolution, the measured discrete IST spectra, compiled and superimposed in Fig. 1(c), are nearly conserved over the whole propagation distance.

The fact that the isospectrality condition perfectly fulfilled in a numerical simulation of the 1D-NLSE (see Supplemental Material [53]) is not exactly verified in

the experiment arises from perturbative higher-order effects that break the integrability of the wave dynamics [50,61,62]. As shown in Fig. 1(d), numerical simulations of a modified NLSE including higher-order effects (see Supplemental Material [53]) reveals that each discrete eigenvalue follows an individual trajectory in the complex plane under the influence of higher-order effects. In the experiment, these trajectories are not resolved because of measurement inaccuracies; compare Fig. 1(c) and Fig. 1(d). Nevertheless, the results of the nonlinear spectral analysis reported in Fig. 1(c) show that the dynamical features observed for the wave field composed of 16 solitons are nearly integrable.

We now take advantage of the above method of the controlled generation of multiple-soliton, random phase solutions of the 1D-NLSE to generate a random N -soliton ensemble that can be identified as SG. It is clear from the definition of DOS in SG theory that to achieve that, the number of solitons N should be sufficiently large. Figure 2 shows the dynamical and spectral features characterizing the experimental evolution of an ensemble of $N = 128$ solitons with random spectral (IST) characteristics. The important difference with the first example is that, due to a large number of solitons generated, we are now able to characterize the soliton ensemble by a DOS $u(\lambda)$; see Fig. 3. Specifically, we generate a SG with eigenvalues $\lambda_i \in \mathbb{C}$ distributed nearly uniformly on a rectangle in the upper half-plane of the complex IST spectral plane (and the

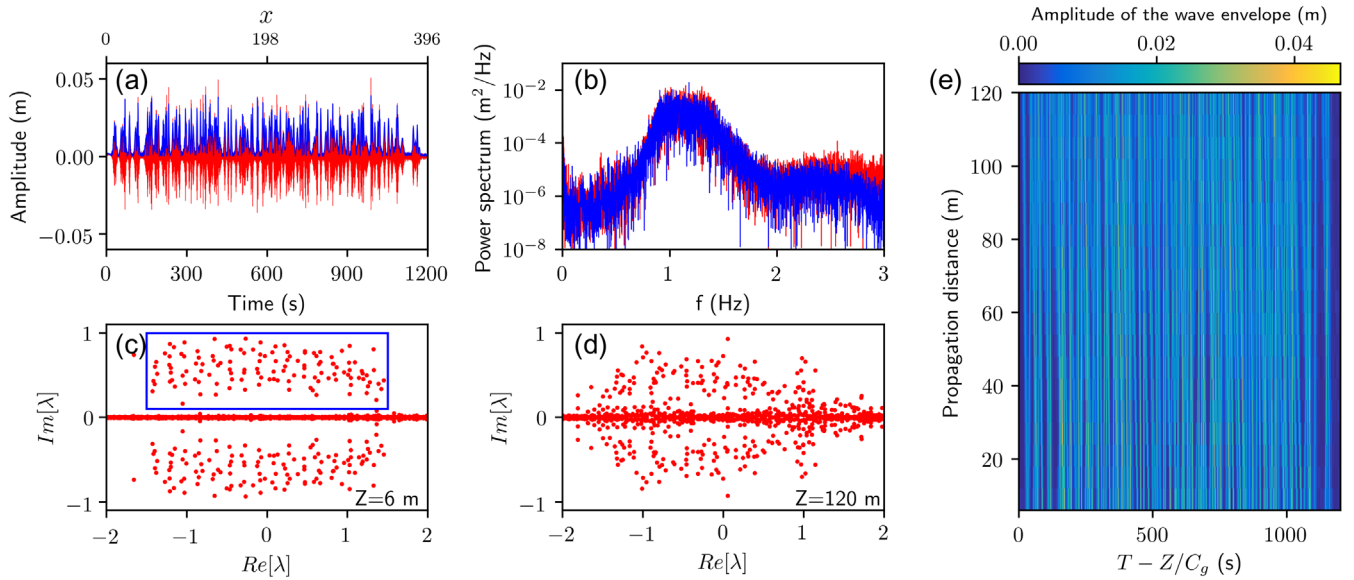


FIG. 2. Gas of $N = 128$ solitons propagating in the 1D water tank. (a) Water elevation (red line) and modulus of the wave envelope measured at $Z_1 = 6$ m, close to the wave maker. (b) Fourier power spectra of wave elevation at $Z_1 = 6$ m (blue line) and at $Z_{20} = 120$ m (red line). (c) Discrete IST spectrum measured at $Z_1 = 6$ m. (d) Discrete IST spectrum measured at $Z_{20} = 120$ m. (e) Space-time evolution of modulus of the wave envelope recorded by the 20 gauges regularly spaced along the tank. Physical parameters characterizing the experiment are $f_0 = 1.15$ Hz, $k_0 = 5.32$ m⁻¹, $\alpha = 0.936$, $L_{NL} = 45$ m ($\langle |A_0(T)|^2 \rangle = 1.58 \times 10^{-4}$ m²).

c.c. rectangle in the lower half plane) and the DOS $u(\lambda) = u_0$ being nearly constant within the rectangle, see Fig. 2(c). In other words, we generate a (approximately) *homogeneous* SG.

Figure 2(a) shows that the generated SG has the form of a random wave field spreading over $\Delta T = 1200$ s which corresponds to a range $\Delta x = 396$ in the dimensionless

variables of Eq. (2). Clearly the generated SG does not represent a *diluted* SG composed of isolated and weakly interacting solitons but rather a *dense* SG which cannot be represented as superposition of individual solitons. Figure 2(b) shows that the propagation of the generated SG is not accompanied by any significant broadening of Fourier power spectrum.

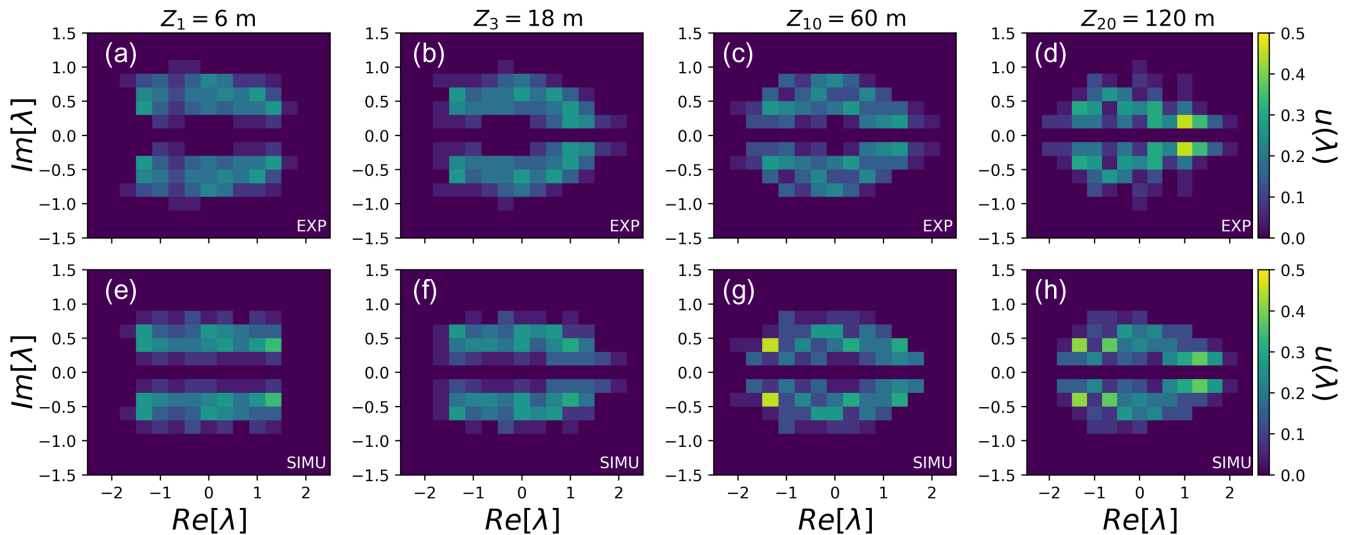


FIG. 3. Statistical analysis of discrete IST spectra of the gas of 128 solitons showing the slow evolution of the DOS $u(\lambda)$ (the probability density function of the discrete IST eigenvalues in the complex plane) as a function of propagation distance in the water tank: (a),(e) $z_1 = 6$; (b),(f) $z_3 = 18$; (c), (g) $z_{10} = 60$; (d), (h) $z_{20} = 120$ m. The upper row (a)–(d) represents the DOS measured in the experiment while the lower row represents the DOS computed in numerical simulation of Euler's equations; see Supplemental Material for details [53].

Figure 2(c) shows the discrete IST spectrum of the wave field measured at $Z_1 = 6$ m. A set of $N = 128$ eigenvalues is now measured within a rectangle in the upper complex plane. Similarly to the features reported in Fig. 1, the perturbative higher-order effects influence the observed dynamics and the discrete spectrum measured at $Z_{20} = 120$ m is not identical to the one measured at Z_1 , see Fig. 2(d). Even though the isospectrality condition characterizing a purely integrable dynamics is not exactly satisfied in our experiment, the measured discrete spectrum remains confined to a well-defined region of the complex plane. Moreover, the large number of eigenvalues distributed with some density within this limited region of the complex plane justifies the introduction of a statistical description of the spectral (IST) data, which represents the key point for the analysis of the observed wavefield in the framework of the SG theory.

In the context of the 1D-NLSE (2) the DOS $u(\lambda)$, where $\lambda = \beta + i\gamma$, represents the density of soliton states in the phase space, i.e., $u d\beta d\gamma dx$ is the number of solitons contained in a portion of SG with the complex spectral parameter $\lambda \in [\beta, \beta + d\beta] \times [\gamma, \gamma + d\gamma]$ over the space interval $[x, x + dx]$ at time t (corresponding to the position Z in the tank). Considering that the generated SG is homogeneous in space, the DOS represents the probability density function of the complex-valued discrete eigenvalues normalized in such a way that $\int_{-\infty}^{+\infty} d\beta \int_0^{+\infty} d\gamma u(\lambda) = N/\Delta x$, where N represents the number of eigenvalues found in the upper complex plane and Δx represents the spatial extent of the gas. Figure 3 (upper row) displays the normalized DOS experimentally measured at different propagation distances in the water tank while Fig. 3 (lower row) displays the normalized DOS computed in a numerical simulation of Euler's equations; see Supplemental Material [53] for details. The experiments and numerical simulations reveal a slow evolution of the DOS along the tank occurring over a characteristic length scale determined by $L_{NL}\Delta f/f_0$, where Δf represents the spectral bandwidth of the wave field (see Supplemental Material [53]). This slow evolution is not due to the gas nonhomogeneity, but mainly originates from the presence of perturbative higher-order effects. Results reported in Fig. 3 suggest that the incorporation of higher-order perturbative physical effects in the theory of SG represents a theoretical question of significant interest.

In this Letter, we have reported hydrodynamic experiments demonstrating that a controlled synthesis of a dense SG can be achieved in deep-water surface gravity waves. We show that the generated homogeneous SG is characterized by a measurable spectral DOS, which provides an essential first step towards experimental verification of the kinetic theory of nonequilibrium SGs. We hope that our work will stimulate new experimental and theoretical research in the fields of statistical mechanics and nonlinear random waves.

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*stephane.randoux@univ-lille.fr

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