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Soft beams: When capillarity induces axial compression

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We study the interaction of an elastic beam with a liquid drop in the case where bending and extensional effects are both present. We use a variational approach to derive equilibrium equations and constitutive relation for the beam. This relation is shown to include a term due to surface energy in addition to the classical Young's modulus term, leading to a modification of Hooke's law. At the triple point where solid, liquid, and vapor phases meet, we find that the external force applied on the beam is parallel to the liquid-vapor interface. Moreover, in the case where solid-vapor and solid-liquid interface energies do not depend on the extension state of the beam, we show that the extension in the beam is continuous at the triple point and that the wetting angle satisfies the classical Young-Dupré relation.

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I. INTRODUCTION

As with other surface effects, capillarity typically comes into play at small scales. Wetting phenomena and capillary effects have been classically studied in the context of instabilities and morphogenesis with menisci, bubbles, drops, and foams in the leading roles [1]. More recently, interactions between these structures and elastic solids have been considered and, when elastic forces are of the same order of magnitude as surface tension, rich behaviors have been found [2]. As surface tension is a rather small force, it has to be applied to flexible or soft systems, typically slender structures with low bending rigidity. In such setups, aggregation [3,4], folding [5], and snapping [6] have been demonstrated. In the case of low Young's modulus, surface tension may induce compression and wrinkling as well [7,8]. Apart from biological systems where capillarity and/or elasticity might be key players [9], elastocapillary interactions lie at the core of several engineering applications, for example, in the field of micro- and nanosystems [10,11]. In order to solve for the deformation of an elastic solid, one has to know the external applied forces and boundary conditions. In the case of a liquid drop sitting on a elastic plate or strip, pressure and meniscus forces act in concert and induce bending and compression [12]. The intensity and direction of such forces, as well as the resulting compression, have recently been discussed [13].

Here we derive, from energy principles, the equations that rule the equilibrium of such drop-strip systems, and we highlight unusual (constitutive) relations between forces and deformations in the presence of capillarity. We start in Sec. II with purely extensional setups. We first consider the case of a beam subject to both surface tension and external end load, and we show how the classical constitutive relation is modified. We then study the case of a beam in interaction with two symmetric droplets and compute the applied force at menisci together with the resulting extension in the beam. In Sec. III, we consider the case of a single drop deposited on the beam, which is a setup where both extension and bending occur. We compute the force at the meniscus and the resulting extension in the beam. In Appendix A, we consider the case where solid-liquid and solid-vapor interface energies depend on the strain state in the beam, and we list the differences and similarities with the results of Sec. II. Finally, in Appendix B, we recall equilibrium equations in the case of a beam subjected to classical external forces.

II. BEAMS IN PURE EXTENSION

A. Compression due to the environment

We start by considering a beam made out of a solid material of low Young's modulus E, typically $E \sim$ kPa [8]. For such low Young's modulus, the stretching energy

$$V_{\rm e} = \frac{1}{2}\int_0^L EA\, e^2(s)ds \tag{1}$$

is of the same order of magnitude as the surface energy

$$V_{\rm s} = \gamma_{\rm sv} P \int_0^L [1+e(s)]ds, \tag{2}$$

where L is the length of the beam in the reference state, A (P) is its cross-section area (perimeter), $e(s)$ is the extension strain with $e > 0$ ($e < 0$) corresponding to extension (compression), s is the arc length along the beam in the reference state, and $\gamma_{\rm sv}$ is the surface energy corresponding to the solid-vapor interface [14]. In a configuration where such a solid beam lies at equilibrium in a vapor phase, the extension e is uniform and is obtained by minimizing the total energy, $V(e) = V_{\rm e} + V_{\rm s}$. Imposing $V'(e) = 0$ yields

$$e = -\frac{\gamma_{\rm sv} P}{EA} < 0, \tag{3}$$

that is, $e = -0.02$ (2% compression) for a beam with $E =$ 10 kPa, $\gamma_{\rm sv} = 0.01$ N/m, and circular cross section of radius $h = 0.1$ mm. Such a beam is compressed due to surface tension, that is, its current length $(1+e)L$ is 2% less than in a situation where the beam would be surrounded by a solid phase of its own material, as in Fig. 1(a).

B. Tension-extension constitutive relation

We now consider a beam (solid phase S) surrounded by a phase i (in the following, this phase will either be liquid $i = \ell$ or vapor $i = V$). The beam is anchored at one extremity and

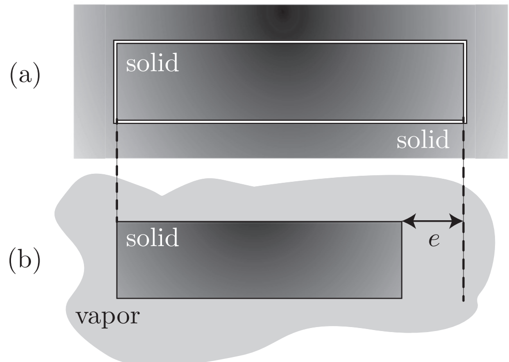

FIG. 1. (a) A beam surrounded by its own material. In this abstract configuration, there is no interface energy. (b) A beam (solid phase S) surrounded by air (vapor phase V). In this configuration, the interface energy is $\gamma_{\rm sv}$ per unit area.

subjected to an external tension at the other end; see Fig. 2. As in the former section, the beam has Young's modulus E, cross-section area A and perimeter P, and reference length L. To the internal energy $V_{\rm e} + V_{\rm s}$ [see Eqs. (1) and (2)], we add the work done by the external load T, i.e., $W_T = -T[x(L) - x(0)] = -T\int_0^L x'(s)ds$, to obtain the total potential energy $V = V_{\rm e} + V_{\rm s} + W_T$ that depends on two unknown functions $e(s)$ and $x(s)$. The definition of strain $e(s) = x'(s) - 1$ implies that these two unknowns are linked by a continuous constraint. We therefore introduce a continuous Lagrange multiplier $\nu(s)$ and work with

$$\mathcal{L}[x(s),e(s)] = V_{\rm e} + V_{\rm s} + W_T + \int_0^L \nu(s)\{x'(s) - [1 + e(s)]\}ds. \quad (4)$$

We introduce perturbations $x \to x + \varepsilon\bar{x}$, $e \to e + \varepsilon\bar{e}$ and we compute the expansion

$$\mathcal{L}(x + \varepsilon\bar{x}, e + \varepsilon\bar{e}) = \mathcal{L}(x,e) + \varepsilon \left.\frac{d\mathcal{L}}{d\varepsilon}\right|_{\varepsilon=0} + \cdots . \quad (5)$$

A necessary condition for V to be minimum at (x,e) under the constraint $x'(s) = 1 + e(s)$ is that the first variation $(d\mathcal{L}/d\varepsilon)_{\varepsilon=0}$ vanishes at (x,e) for all $(\bar{x},\bar{e})$. After derivation and integration by parts, we obtain

$$\left.\frac{d\mathcal{L}}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^L (EAe + P\gamma_{si} - \nu)\,\bar{e}(s)ds - \int_0^L \nu'(s)\,\bar{x}(s)ds - [(T-\nu)\bar{x}]_0^L, \quad (6)$$

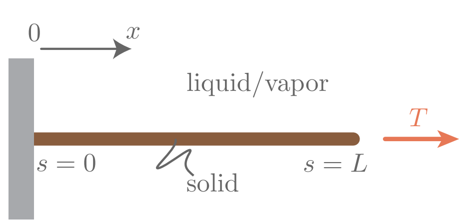

FIG. 2. (Color online) A beam (solid phase S) surrounded by a phase i (liquid or vapor). The beam is subjected to an external tension T.

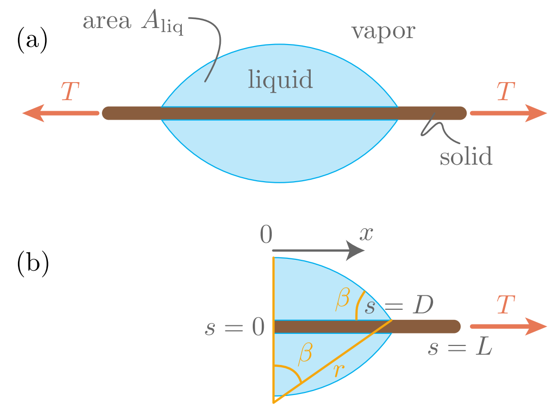

FIG. 3. (Color online) (a) An extensible beam bears two liquid drops. In addition, the beam is subjected to an external tension T. (b) Using symmetry, we study the right-half problem and fix the midpoint of the beam $s = 0$ at the origin.

with γ_{si} the surface energy for the interface between the solid beam and the surrounding phase i. Boundary condition $x(0) = 0$ yields $\bar{x}(0) = 0$, but $\bar{x}(L)$ is arbitrary. Consequently, requiring (6) to vanish for all $[\bar{x}(s),\bar{e}(s)]$ brings equations

$$\nu'(s) = 0, \quad (7)$$

$$\nu(s) = EAe(s) + P\gamma_{si}, \quad (8)$$

and the natural boundary condition

$$\nu(L) = T. \quad (9)$$

In the light of this last relation, we interpret the Lagrange multiplier $\nu(s)$ as the beam internal tension $N(s)$. Using (8), we obtain the following constitutive relation:

$$N(s) = EA\,e(s) + P\gamma_{si}, \quad (10)$$

between the extension $e(s)$ and internal tension $N(s)$. The total tension N is seen as the sum of the bulk force $EAe(s)$ and the surface stress $P\gamma_{si}$. This is comparable to Hooke's law in thermoelasticity where local stress is created by both strain and temperature change. The deformation due to interface energy is analogous to the classic deformation observed when heating a beam away from its fabrication temperature, with the surface energy γ playing the role of a negative thermal expansion coefficient [15].

Note that the situation of Sec. II A is regained by setting $T = 0$.

C. Force jump at the contact line

Next we consider a beam subjected to capillary interactions. A beam with a rectangular cross section (thickness h, width $w \gg h$) bears a liquid drop on its upper surface and a similar drop on its lower surface; see Fig. 3(a). An external tension is applied at both extremities. We look for the equilibrium equations for such a beam under the following assumptions. Due to the large aspect ratio $w \gg h$ of the cross section, we work in a two-dimensional setup, considering the problem invariant along the third dimension. Consequently, we neglect end effects, write $P = 2w$, and consider the liquid-vapor interfaces to be cylinder arcs; see [16]. We also assume that the simultaneous presence of two liquid drops prevents the beam from bending and we only deal with extensional deformations. We break the translation invariance by fixing the beam middle point $s = 0$ at the origin: $x(s = 0) = 0$. Consequently, we solve the planar half problem of Fig. 3(b).

As capillary forces will be applied on the beam at the contact line, where the three phases meet, we anticipate the possibility of a discontinuity in the extension and therefore introduce the extension $e_\mathrm{i}(s)$ inside the drops, and $e_\mathrm{o}(s)$ outside the drops. The stretching energy of the beam is now

$$V_\mathrm{e} = \frac{1}{2}\int_0^D EAe_\mathrm{i}^2(s)ds + \frac{1}{2}\int_D^L EAe_\mathrm{o}^2(s)ds. \tag{11}$$

The work of the external load is $W_T = -Tx(L)$, with $x(L) = \int_0^D x_\mathrm{i}'(s)ds + \int_D^L x_\mathrm{o}'(s)ds$. The sum of the interface energies is

$$\begin{aligned} V_\mathrm{s} &= 2w\gamma_{\mathrm{s}\ell}\, x(D) + 2w\gamma_\mathrm{sv}[x(L) - x(D)] + 2w\gamma_{\ell\mathrm{v}} r\beta \\ &= 2w\gamma_{\mathrm{s}\ell}\int_0^D [1 + e_\mathrm{i}(s)]ds \\ &\quad + 2w\gamma_\mathrm{sv}\int_D^L [1 + e_\mathrm{o}(s)]ds + 2w\gamma_{\ell\mathrm{v}} r\beta, \end{aligned} \tag{12}$$

where r is the radius of both circular liquid-vapor interfaces and β is the wetting angle.

We want to minimize the total potential energy $V = V_\mathrm{e} + V_\mathrm{s} + W_T$ under the following constraints. First, as in the previous section, we have relations linking unknown functions $x(s)$,$e_\mathrm{i}(s)$,$e_\mathrm{o}(s)$: for s in $(0,D)$, $x_\mathrm{i}'(s) = 1 + e_\mathrm{i}(s)$, and for s in (D,L), $x_\mathrm{o}'(s) = 1 + e_\mathrm{o}(s)$. Second, the total liquid volume $\mathcal{V} = A_\mathrm{liq} w$ being fixed, we have

$$wr^2(\beta - \sin\beta\cos\beta) = A_\mathrm{liq} w, \tag{13}$$

where A_liq is the area of the surface lying in between the beam and the liquid-vapor interface. Finally, the position of the contact line imposes

$$x(D) = r\sin\beta. \tag{14}$$

We therefore will compute the first variation of

$$\begin{aligned} \mathcal{L} &= V_\mathrm{e} + V_\mathrm{s} + W_T - \mu w[r^2(\beta - \sin\beta\cos\beta) - A_\mathrm{liq}] \\ &\quad - \eta\left[\int_0^D x_\mathrm{i}'(s)ds - r\sin\beta\right] + \int_0^D \nu_\mathrm{i}[x_\mathrm{i}' - (1 + e_\mathrm{i})]ds \\ &\quad + \int_D^L \nu_\mathrm{o}[x_\mathrm{o}' - (1 + e_\mathrm{o})]ds, \end{aligned} \tag{15}$$

where $[D,\beta,r,e_\mathrm{i}(s),e_\mathrm{o}(s),x_\mathrm{i}(s),x_\mathrm{o}(s)] = X$ are unknowns and $[\mu,\eta,\nu_\mathrm{i}(s),\nu_\mathrm{o}(s)]$ are Lagrange multipliers. We therefore introduce the perturbation $X \to X + \varepsilon\bar{X}$, and we look for the conditions for which $d\mathcal{L}(X + \varepsilon\bar{X})/d\varepsilon|_{\varepsilon=0} = 0$. We perform integration by parts to get rid of the $\bar{x}_\mathrm{i}'(s)$ and $\bar{x}_\mathrm{o}'(s)$ terms, and we obtain

$$\begin{aligned} \left.\frac{d\mathcal{L}}{d\varepsilon}\right|_{\varepsilon=0} &= \bar{\beta}[2w\gamma_{\ell\mathrm{v}} r - \mu w r^2(1 - \cos 2\beta) + \eta r\cos\beta] \\ &\quad + \bar{r}\,[2w\gamma_{\ell\mathrm{v}}\beta - \mu w r(2\beta - \sin 2\beta) + \eta\sin\beta] \\ &\quad + \int_0^D (EAe_\mathrm{i} + 2w\gamma_{\mathrm{s}\ell} - \nu_\mathrm{i})\,\bar{e}_\mathrm{i} ds \\ &\quad + \int_D^L (EAe_\mathrm{o} + 2w\gamma_\mathrm{sv} - \nu_\mathrm{o})\,\bar{e}_\mathrm{o} ds + [(\nu_\mathrm{i} - T - \eta)\bar{x}]_0^D \\ &\quad - \int_0^D \nu_\mathrm{i}'\bar{x} ds + [(\nu_\mathrm{o} - T)\bar{x}]_D^L - \int_D^L \nu_\mathrm{o}'\bar{x} ds \\ &\quad + \bar{D}\left\{\frac{1}{2}EAe_\mathrm{i}^2(D) + 2w\gamma_{\mathrm{s}\ell}[1 + e_\mathrm{i}(D)] - (T + \eta)x_\mathrm{i}'(D)\right. \\ &\quad \left. - \frac{1}{2}EAe_\mathrm{o}^2(D) - 2w\gamma_\mathrm{sv}[1 + e_\mathrm{o}(D)] + Tx_\mathrm{o}'(D)\right\}, \end{aligned} \tag{16}$$

where we have used $\int_0^{D+\varepsilon\bar{D}} f(s)ds = \int_0^D f(s)ds + \varepsilon\bar{D}f(D) + O(\varepsilon^2)$. Note that since the position $x(s)$, and hence $\bar{x}(s)$, has to be continuous, we do not use any subscript for $x(s)$ or $\bar{x}(s)$.

We first examine the conditions for the first variation (16) to vanish for all $\bar{x}(s)$. From the boundary condition $x(0) = 0$, we have $\bar{x}(0) = 0$, but $\bar{x}(D)$ and $\bar{x}(L)$ are arbitrary. The conditions are then

$$\nu_\mathrm{i}'(s) = 0 \text{ and } \nu_\mathrm{o}'(s) = 0, \tag{17a}$$

$$\nu_\mathrm{o}(L) = T, \tag{17b}$$

$$\nu_\mathrm{o}(D) - \nu_\mathrm{i}(D) + \eta = 0. \tag{17c}$$

Here, again, we interpret $\nu_\mathrm{i}(s)$ and $\nu_\mathrm{o}(s)$ as the internal force in the beam, which experiences a jump of amplitude η at the contact line. The conditions for the first variation (16) to vanish for all $\bar{\beta}$ and $\bar{r}$ are

$$\mu = \gamma_{\ell\mathrm{v}}/r, \tag{18a}$$

$$\eta = -2w\gamma_{\ell\mathrm{v}}\cos\beta, \tag{18b}$$

where we see that the Lagrange multiplier μ, enforcing volume, is the Laplace pressure inside the liquid drop $\gamma_{\ell\mathrm{v}}/r$. From (17c) and (18b), the Lagrange multiplier η, associated to the constraint on the position of the contact line, is interpreted as the external force applied on the beam at the contact line (from the liquid and vapor phases). For each drop, the force on the beam is of intensity $\gamma_{\ell\mathrm{v}} w$ and is oriented along the liquid-vapor interface. This result is not changed in the case where surface energies depend on strains; see Appendix A. The conditions for the first variation (16) to vanish for all $\bar{e}_\mathrm{i}(s)$ and $\bar{e}_\mathrm{o}(s)$ are

$$\nu_\mathrm{i}(s) = EAe_\mathrm{i}(s) + 2w\gamma_{\mathrm{s}\ell}, \tag{19a}$$

$$\nu_\mathrm{o}(s) = EAe_\mathrm{o}(s) + 2w\gamma_\mathrm{sv}, \tag{19b}$$

which are interpreted as constitutive relations between extension e_i (e_o) and internal force ν_i (ν_o) in each region of the beam. Finally, the condition for the first variation (16) to vanish for all $\bar{D}$ is

$$[1 + e_\mathrm{i}(D)]^2 - [1 + e_\mathrm{o}(D)]^2 = 0 \tag{20a}$$

$$\text{or}\quad [e_\mathrm{i}(D) - e_\mathrm{o}(D)]\,[e_\mathrm{i}(D) + e_\mathrm{o}(D) + 2] = 0. \tag{20b}$$

As the vanishing of the second term of (20b) would require oversized extension values, this equality can only be fulfilled if

$$e_{\rm i}(D) - e_{\rm o}(D) = 0, \tag{21}$$

that is, the extension is continuous at the contact line. Considering (17c), (18b), (19a), (19b), and (21), we obtain

$$\gamma_{s\ell} - \gamma_{sv} + \gamma_{\ell v}\cos\beta = 0, \tag{22}$$

which means that the wetting angle satisfies the Young-Dupré relation.

We conclude that in the case where an extensible beam is in capillary interaction with a liquid drop, we have the following three properties: (i) the constitutive relation linking internal force and extension is modified and includes a surface tension term, (ii) the internal force inside the beam experiences a jump at the contact line, corresponding to the force coming from the liquid-vapor interface, with this force being oriented along the interface, and (iii) the extension of the beam is continuous at the triple line.

III. BEAMS EXPERIENCING BOTH BENDING AND EXTENSION

We now consider the case where a liquid drop sits on the top of a flexible and extensible beam and we look for equilibrium equations; see Fig. 4. As in the former section, we use a beam of rectangular cross section (thickness h, width w) and set $P = 2w$ and $A = hw$. We focus on one-half of the system, $s \in [0;L]$, and work with the following boundary conditions:

$$x(0) = 0, \quad y(0) = 0, \quad \theta(0) = 0. \tag{23}$$

As in the previous sections, s is the arc length of the beam in its reference state. Consequently, once the beam is deformed, it may no longer have total contour length L. To the stretching energy $V_{\rm e}$, given by Eq. (11), we add the bending energy,

$$V_\kappa = \frac{1}{2}EI\int_0^D [\theta_{\rm i}'(s)]^2 ds + \frac{1}{2}EI\int_D^L [\theta_{\rm o}'(s)]^2 ds, \tag{24}$$

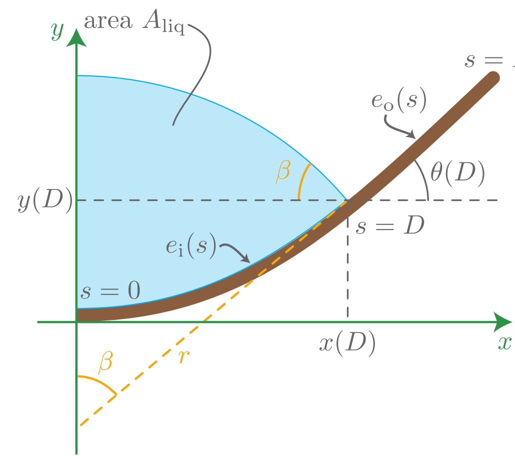

FIG. 4. (Color online) A beam in interaction with a liquid drop. In the deformed state, both bending and extension are present.

where EI is the bending rigidity of the strip (E is Young's modulus of the beam material and $I = h^3w/12$ is the second moment of area of the section of the beam).

Ignoring constant terms, the sum of the interfaces energies is [17]

$$V_{\rm s} = w\gamma_{s\ell}\int_0^D [1 + e_{\rm i}(s)]ds + w\gamma_{sv}\int_D^L [1 + e_{\rm o}(s)]ds + w\gamma_{\ell v} r\beta. \tag{25}$$

We minimize $V_{\rm e} + V_\kappa + V_{\rm s}$ under the following constraints. First, with the liquid volume $\mathcal{V} = A_{\rm liq}w$ being fixed, we have

$$A_{\rm liq} = \frac{1}{2}r^2(\beta - \sin\beta\cos\beta) + x(D)\,y(D) - \int_{x(0)}^{x(D)} y\,dx. \tag{26}$$

Second, we still have the geometric constraint (14). And, finally, the relations linking the unknown functions $x(s), y(s), \theta(s), e_{\rm i}(s), e_{\rm o}(s)$ are for s in $(0,D)$: $x_{\rm i}'(s) = [1 + e_{\rm i}(s)]\cos\theta(s)$ and $y_{\rm i}'(s) = [1 + e_{\rm i}(s)]\sin\theta(s)$; and for s in (D,L): $x_{\rm o}'(s) = [1 + e_{\rm o}(s)]\cos\theta(s)$ and $y_o'(s) = [1 + e_{\rm o}(s)]\sin\theta(s)$.

We therefore compute the first variation of

$$\begin{aligned}\mathcal{L} = {} & V_\kappa + V_{\rm e} + V_{\rm s} - \eta\left[\int_0^D x_{\rm i}'ds - r\sin\beta\right] - \mu w\left[\frac{r^2}{2}(\beta - \sin\beta\cos\beta) + \int_0^D x_{\rm i}'ds\int_0^D y_{\rm i}'ds - \int_0^D y x_{\rm i}'ds\right] \\ & + \int_0^D \nu_{\rm i}(s)[x_{\rm i}' - (1+e_{\rm i})\cos\theta]ds + \int_D^L \nu_{\rm o}(s)[x_{\rm o}' - (1+e_{\rm o})\cos\theta]ds \\ & + \int_0^D \lambda_{\rm i}(s)[y_{\rm i}' - (1+e_{\rm i})\sin\theta]ds + \int_D^L \lambda_{\rm o}(s)[y_o' - (1+e_{\rm o})\sin\theta]ds,\end{aligned} \tag{27}$$

where $\mathcal{L} = \mathcal{L}(x,y,e_{\rm i},e_{\rm o},\theta,\beta,r,D)$. Computing the first derivative, we obtain

$$\begin{aligned}\left.\frac{d\mathcal{L}}{d\varepsilon}\right|_{\varepsilon=0} = {} & EI\int_0^D \theta_{\rm i}'\bar{\theta}_{\rm i}'ds + \frac{1}{2}EI\bar{D}[\theta_{\rm i}'(D)]^2 + EI\int_D^L \theta_{\rm o}'\bar{\theta}_{\rm o}'ds - \frac{1}{2}EI\bar{D}[\theta_{\rm o}'(D)]^2 + EA\int_0^D e_{\rm i}\bar{e}_{\rm i}ds + \frac{1}{2}EA\bar{D}e_{\rm i}^2(D) \\ & + EA\int_D^L e_{\rm o}\bar{e}_{\rm o}ds - \frac{1}{2}EA\bar{D}e_{\rm o}^2(D) + w\gamma_{s\ell}\int_0^D \bar{e}_{\rm i}ds + w\gamma_{s\ell}\bar{D}[1 + e_{\rm i}(D)] + w\gamma_{sv}\int_D^L \bar{e}_{\rm o}ds - w\gamma_{sv}\bar{D}[1 + e_{\rm o}(D)]\end{aligned}$$

$$+ w\gamma_{\ell v}(\bar{r}\beta + r\bar{\beta}) - \mu w \left[r\bar{r}\left(\beta - \frac{1}{2}\sin 2\beta\right) + \frac{\bar{\beta} r^2}{2}(1-\cos 2\beta)\right]$$
$$- \mu w \left[\int_0^D \bar{x}_i' ds \int_0^D y_i' ds + \int_0^D x_i' ds \int_0^D \bar{y}_i' ds + \bar{D} x_i'(D) y(D) + \bar{D} x(D) y_i'(D) \right]$$
$$+ \mu w \left[\int_0^D \bar{y} x_i' ds + \int_0^D y\, \bar{x}_i' ds + \bar{D} y(D) x_i'(D)\right] - \eta \left[\int_0^D \bar{x}_i' ds + \bar{D} x_i'(D) - \bar{r}\sin\beta - r\bar{\beta}\cos\beta \right]$$
$$+ \int_0^D \nu_i(s)[\bar{x}_i' - \bar{e}_i\cos\theta + \bar{\theta}(1+e_i)\sin\theta] ds + \int_D^L \nu_o(s)[\bar{x}_o' - \bar{e}_o\cos\theta + \bar{\theta}(1+e_o)\sin\theta] ds$$
$$+ \int_0^D \lambda_i(s)[\bar{y}_i' - \bar{e}_i\sin\theta - \bar{\theta}(1+e_i)\cos\theta] ds + \int_D^L \lambda_o(s)[\bar{y}_o' - \bar{e}_o\sin\theta - \bar{\theta}(1+e_o)\cos\theta] ds. \quad (28)$$

As $x(s)$, $y(s)$, and $\theta(s)$ are assumed to be continuous, so are their variations $\bar{x}(s)$, $\bar{y}(s)$, and $\bar{\theta}(s)$. Consequently, these variables do not carry any subscript. As in the previous section, the conditions for the first variation (28) to vanish for all $\bar{\beta}$ and $\bar{r}$ yields

$$\mu = \gamma_{\ell v}/r \quad \text{and} \quad \eta = -w\gamma_{\ell v}\cos\beta. \quad (29)$$

Requiring (28) to vanish for all $\bar{x}$ yields, after integration by parts,

$$[\{-\mu w[y(D) - y] - \eta + \nu_i\}\bar{x}]_0^D - \int_0^D (\mu w y_i' + \nu_i')\bar{x} ds$$
$$+ [\nu_o \bar{x}]_D^L - \int_D^L \nu_o' \bar{x} ds = 0. \quad (30)$$

The fact that we have $\bar{x}(0) = 0$, but arbitrary $\bar{x}(D)$ and $\bar{x}(L)$, implies

$$\nu_o(L) = 0, \;\; \nu_o(D) - \nu_i(D) = -\eta,$$
$$\nu_o'(s) = 0, \;\; \nu_o'(s) = -\mu w y_i'(s). \quad (31)$$

Requiring (28) to vanish for all $\bar{y}$ similarly implies

$$\lambda_o(L) = 0, \;\; \lambda_o(D) - \lambda_i(D) = -\mu w x(D),$$
$$\lambda_o'(s) = 0, \;\; \lambda_i'(s) = \mu w x_i'(s). \quad (32)$$

Here again we identify $\boldsymbol{N}_i(s) = [\nu_i(s), \lambda_i(s)]$ and $\boldsymbol{N}_o(s) = [\nu_o(s), \lambda_o(s)]$ as the internal force in the beam. We see that Laplace pressure μ generates an outward normal distributed force $\mu w(y_i', -x_i')$ that causes the internal force $\boldsymbol{N}_i(s)$ to vary. In addition, using (29), we see that at the contact line $s = D$, the force experiences a jump of amplitude $w\gamma_{\ell v}$ and is oriented along the liquid-air interface:

$$\boldsymbol{N}_o(D) - \boldsymbol{N}_i(D) = -\gamma_{\ell v} w \begin{pmatrix} -\cos\beta \\ \sin\beta \end{pmatrix}. \quad (33)$$

Requiring (28) to vanish for all $\bar{\theta}$ yields, after integration by parts,

$$EI[\theta_i'\bar{\theta}]_0^D + \int_0^D [-EI\theta_i'' + (1+e_i)(\nu_i\sin\theta - \lambda_i\cos\theta)]\bar{\theta} ds$$
$$+ EI[\theta_o'\bar{\theta}]_D^L + \int_D^L [-EI\theta_o'' + (1+e_o)$$
$$\times (\nu_o\sin\theta - \lambda_o\cos\theta)]\bar{\theta} ds = 0. \quad (34)$$

Boundary condition $\theta(0) = 0$ imposes $\bar{\theta}(0) = 0$, but $\bar{\theta}(D)$ and $\bar{\theta}(L)$ are arbitrary. Consequently, (34) yields

$$\theta_i'(D) = \theta_o'(D), \quad (35a)$$

$$\theta_o'(L) = 0, \quad (35b)$$

$$EI\theta_i''(s) = (1+e_i)[\nu_i\sin\theta - \lambda_i\cos\theta], \quad (35c)$$

$$EI\theta_o''(s) = (1+e_o)[\nu_o\sin\theta - \lambda_o\cos\theta]. \quad (35d)$$

That is the curvature $\theta'(s)$ (and hence the bending moment) is continuous as s goes through D and it vanishes at the $s = L$ extremity. Moreover, we recognize in the last two equations the moment equilibrium equations along the beam. Requiring (28) to vanish for all $\bar{e}_i(s)$ and $\bar{e}_o(s)$ yields the following two relations:

$$EA\, e_i(s) + w\gamma_{s\ell} = \nu_i\cos\theta + \lambda_i\sin\theta, \quad (36a)$$

$$EA\, e_o(s) + w\gamma_{sv} = \nu_o\cos\theta + \lambda_o\sin\theta, \quad (36b)$$

that we interpret as constitutive relations linking the extension $e(s)$ to the internal tension $\boldsymbol{N}(s)\cdot\boldsymbol{t}(s)$, where $\boldsymbol{t}(s)$ is the tangent to the beam $\boldsymbol{t}(s) = [\cos\theta(s), \sin\theta(s)]$. Using (31) and (32), we see that $EAe_o(s) = -w\gamma_{sv}$ for all s, and that

$$\frac{EA}{w}[e_o(D) - e_i(D)] = \gamma_{s\ell} - \gamma_{sv} + \gamma_{\ell v}\cos[\theta(D) + \beta]. \quad (37)$$

Finally, requiring (28) to vanish for all $\bar{D}$ yields

$$\frac{1}{2}EAe_i^2 - \frac{1}{2}EAe_o^2 + w\gamma_{s\ell}[1 + e_i(D)] - w\gamma_{sv}[1 + e_o(D)]$$
$$- \mu w x(D) y_i'(D) - \eta x_i'(D) = 0. \quad (38)$$

Using (14), (29), and (37), this is simplified to

$$[1 + e_i(D)]^2 - [1 + e_o(D)]^2 = 0 \quad (39a)$$

or

$$[e_i(D) - e_o(D)]\,[e_i(D) + e_o(D) + 2] = 0. \quad (39b)$$

Consequently, we see that even in the presence of bending, the extension is continuous as s go through D and the Young-

Dupré relation holds for the wetting angle:

$$e_i(D) - e_o(D) = 0, \tag{40}$$

$$\gamma_{s\ell} - \gamma_{sv} + \gamma_{\ell v} \cos[\theta(D) + \beta] = 0. \tag{41}$$

IV. CONCLUSION

We have shown that the constitutive relation between tension and extension in an elastic beam is altered by surface energies, leading to a modification of Hooke's law. In a setup where three phases are involved (solid, liquid, and vapor), we have shown the following properties, in the case where surface energies do not depend on the strain state in the solid: (p_1) the extension in the beam is continuous at the triple line, (p_2) the wetting angle satisfies the Young-Dupré relation, and (p_3) the external force applied on the beam is along the liquid-vapor interface. This last property (p_3) has been shown to hold even if strain dependence is introduced in the surface energies. Properties (p_2) and (p_3) have already been established in [18] in the pure bending case, and we have verified here that in a setup where bending and extension are both present, these properties still hold. The constitutive relations presented here [Eq. (19) or Eq. (A1)] between tension and extension compel one to be careful when analyzing experimental results, as a jump in extension no longer implies the same jump in tension. For example, in Sec. II C, we find continuous extension and a discontinuous tension. Experimental results reported in [13] indicate that the extension changes sign at the contact line. The present derivations show that this is not possible under the classical hypotheses used in Sec. II C and that, for example, strain dependence of surface energies has to be invoked, as also stated in [19]. Moreover, in order for $e_i(D)$ and $e_o(D)$ to have opposite sign, we see that the derivatives $\gamma'_{s\ell}(e)$ and $\gamma'_{sv}(e)$ have to be of the same order as $\gamma_{s\ell}$ and γ_{sv}. The sign change of the extension at the contact line was attributed to the presence of a tangential component in the force at the contact line [13], derived from a microscopic model of capillarity in [20] and observed in molecular dynamics simulations [19,21]. Nevertheless, the present results indicate that no tangential component, in the external force applied on the beam at the contact line, is needed in order to have a sign change of the extension.

As a matter of fact, here this external force is found to classically lie along the liquid-vapor interface, even in the case of strain-dependent surface energies.

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APPENDIX A: STRAIN-DEPENDENT SURFACE ENERGY

We investigate here how the results of Sec. II C change if surface energies $\gamma_{s\ell}$ and γ_{sv} depend on the strain of the elastic material, that is, $\gamma_{s\ell} = \gamma_{s\ell}(e_i)$ and $\gamma_{sv} = \gamma_{sv}(e_o)$ [19,22]. First, results in (17) and (18) remain unchanged. Consequently, the Lagrange multiplier ν is still interpreted as the internal force, and the force applied on the beam at the contact line is still of intensity $\gamma_{\ell v} w$ and still oriented along the liquid-vapor interface, in contradiction to what is argued in [19]. Second, the constitutive relation between tension and extension, formerly (19a) and (19b), now comprises one more term:

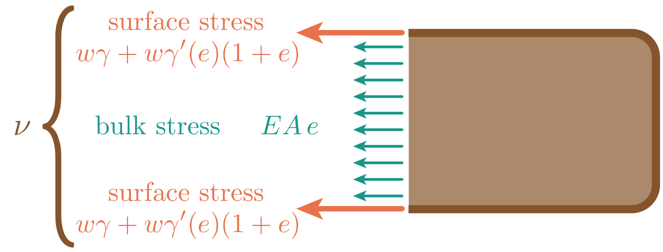

FIG. 5. (Color online) Decomposition of the beam internal tension into bulk and surface stresses.

$$\nu_i(s) = EAe_i(s) + 2w\gamma_{s\ell} + 2w\gamma'_{s\ell}(e_i)[1 + e_i(s)], \tag{A1a}$$

$$\nu_o(s) = EAe_o(s) + 2w\gamma_{sv} + 2w\gamma'_{sv}(e_o)[1 + e_o(s)]. \tag{A1b}$$

As depicted in Fig. 5, these relations are the beam mechanics version of Shuttleworth's equation [23]. In this case, Eq. (20a) is modified and now reads

$$[1 + e_o(D)]^2 \left\{1 + \frac{4\gamma'_{sv}[e_o(D)]}{EA/w}\right\} = [1 + e_i(D)]^2 \left\{1 + \frac{4\gamma'_{s\ell}[e_i(D)]}{EA/w}\right\}, \tag{A2}$$

and shows that the extension now has a discontinuity at the contact line. Third, relation (22), giving the wetting angle, is also changed and becomes

$$\gamma_{s\ell} - \gamma_{sv} + \gamma_{\ell v}\cos\beta = \frac{EA}{2w}[e_o(D) - e_i(D)] + \gamma'_{sv}(e_o)[1 + e_o(D)] - \gamma'_{s\ell}(e_i)[1 + e_i(D)]. \tag{A3}$$

If we now assume that $(EA/w)\,e$ and $\gamma'(e)$ are both of the order of γ, and if we neglect $O(e^2)$ terms, we find

$$e_o(D) - e_i(D) = \frac{2}{EA/w}[\gamma'_{s\ell}(e_i) - \gamma'_{sv}(e_o)] + O(e^2), \tag{A4}$$

$$\gamma_{s\ell} - \gamma_{sv} + \gamma_{\ell v}\cos\beta = O(e^2), \tag{A5}$$

as also found in [19].

APPENDIX B: CLASSICAL BEAM EQUATIONS

In order to illustrate our interpretation of the Lagrange multipliers ν and η in Sec. II, we recall here the classical

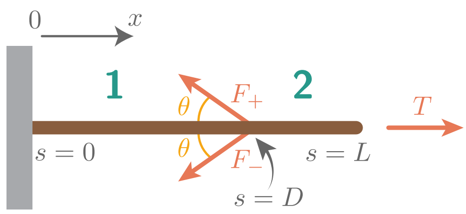

FIG. 6. (Color online) A beam subjected to an external tension T and two compressive forces $\boldsymbol{F}_\pm$.

equilibrium equations for a beam subjected to external forces. In Fig. 6, we show a beam anchored at $s = 0$ and subjected to an external tension T at $s = L$. Moreover, two external forces are applied at $s = D$: $\boldsymbol{F}_\pm = F(-\cos\theta, \pm\sin\theta)$. In this case, equations for the longitudinal internal beam force $N(s)$ are

$$N_1'(s) = 0 \text{ and } N_2'(s) = 0, \quad \text{(B1a)}$$

$$N_2(L) = T, \quad \text{(B1b)}$$

$$N_2(D) - N_1(D) - 2F\cos\theta = 0, \quad \text{(B1c)}$$

where the index 1 (2) refers to the beam region $s \in (0; D)$ [$s \in (D; L)$]. Equation (B1a) is the local force equilibrium for a beam with no distributed load (e.g., gravity). Equation (B1b) is the boundary condition stating that the external applied force at $s = L$ is tension T. Equation (B1c) is the force equilibrium at $s = D$. A comparison of Eqs. (B1) with Eqs. (17) and (18b) naturally leads to the interpretation of the Lagrange multiplier ν in Sec. II C as the internal force in the beam, of $w\gamma_{\ell v}$ as the intensity of the external force at $s = D$, and of β as its orientation.

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