Shape of attractors for three-dimensional dissipative dynamical systems

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We introduce a method to bound attractors of dissipative dynamical systems in phase and parameter spaces. The method is based on the determination of families of transversal surfaces (surfaces crossed by the flow in only one direction). This technique yields very restrictive geometric bounds in phase space for the attractors. It also gives ranges of parameters of the system for which no chaotic behavior is possible. We illustrate our method on different three-dimensional dissipative systems.

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I. INTRODUCTION

We shall consider ordinary differential equations defining time evolution of three-dimensional (3D) dissipative dynamical systems:

$$\dot{x}=P(x,y,z),\quad \dot{y}=Q(x,y,z),\quad \dot{z}=R(x,y,z),\qquad (1)$$

with $\partial P/\partial x+\partial Q/\partial y+\partial R/\partial z<0\,\forall(x,y,z)$. Usually the functions P, Q, and R are simple polynomials. These types of systems are dissipative: volumes in phase space contract under the flow because the divergence of the vector field (P,Q,R) is always negative. Hence, the attractor of the system is necessarily of dimension less than three (it may be an equilibrium point, a limit cycle, or a chaotic attractor). In this paper we are interested in the approximate location in phase space of the global attractor of the system, which contains all the dynamics evolving from all initial conditions. The global attractor is the set of points in phase space that can be reached from some initial condition set at an arbitrary long time in the past. The two fundamental properties of a global attractor are

(i) it is invariant under evolution;

(ii) the distance of any solution from it vanishes as $t\to+\infty$.

This last property may simply be interpreted thus: if the solution starts outside the global attractor, then it is attracted into it as $t\to+\infty$ and once inside it cannot escape. Whereas if the solution starts inside the global attractor, then it stays inside. The global attractor contains all the asymptotic motion for the dynamical system. It is common to talk of multiple attractors for a dynamical system and each of them may in its own right be considered as the attractor for initial conditions within its own basin of attraction. The notion of global attractor corresponds to the union of all such dynamically invariant attracting sets possible. In particular, it contains all possible structures such as equilibrium points, limit cycles, etc. The global attractor is sometimes contained in an absorbing ball in phase space and we want to obtain analytic estimates about its geometric shape. Moreover, this would enable us to find an upper bound for its Lyapunov dimension [2,3].

Until very recently, approximated locations of attractors in phase space have been obtained by the method of Lyapunov functions. The latter is a smooth positive definite function that decreases along trajectories. This type of function is a generalization of the energy function for mechanical systems: in the presence of friction or other dissipation, the energy decreases monotonically and the system stabilizes on an equilibrium state where the energy is minimal.

Let us consider, as an example, the Lorenz system [5] defined by:

$$\dot{x}=\sigma(y-x),\quad \dot{y}=rx-y-xz,\quad \dot{z}=xy-bz,\qquad (2)$$

where σ,r,b are positive parameters. For $r<1$ and σ and b arbitrary, every trajectory approaches the origin as $t\to+\infty$: the origin is globally stable. Hence there can be no limit cycle nor chaos for $r<1$. The proof of this important result can be obtained by constructing an adequate Lyapunov function. There is no systematic way to construct these Lyapunov functions, but often it is wise to try expressions involving sums of squares. Here we consider $V(x,y,z)=1/\sigma x^2+y^2+z^2$. The surfaces of constant V are concentric ellipsoids about the origin. The idea is to show that if $r<1$ and $(x,y,z)\neq(0,0,0)$, then $\dot{V}<0$ along all trajectories. This would imply that each trajectory keeps moving to lower V and hence penetrates smaller and smaller ellipsoids as $t\to+\infty$. But $V(x,y,z)$ is bounded below by 0, so $V(x(t),y(t),z(t))\to0$ and hence $(x(t),y(t),z(t))\to0$, as desired. Now we calculate

$$\begin{aligned}\dot{V}&=2\left(\frac{1}{\sigma}x\dot{x}+y\dot{y}+z\dot{z}\right)\\&=2(r+1)xy-2x^2-2y^2-2bz^2\\&=-2\left\{\left(x-\frac{r+1}{2}y\right)^2+\left[1-\left(\frac{r+1}{2}\right)^2\right]y^2+bz^2\right\}.\end{aligned}\qquad (3)$$

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This last quantity is strictly negative if $r<1$ and $(x,y,z)\neq(0,0,0)$. It is easy to see that the condition $\dot{V}(x,y,z)=0$ implies $(x,y,z)=(0,0,0)$. Therefore the origin is globally stable for $r<1$.

The powerful aspect of this method is that one does not need to integrate the equations to determine the qualitative behavior of the trajectories. On the other hand, the difficult feature of this technique is that there is no general way to find adequate expressions for $V(x,y,z)$, as said above. No general ansatz is known for this function.

More than proving the stability of the equilibrium point, this method also provides us with its basin of attraction. But when the system exhibits another type of attractor (limit cycle or chaotic attractor) the situation becomes more complicated. First, the position of the attractor cannot be determined as easily as in the case of an equilibrium point (where we only had to solve $P=Q=R=0$). We would like to use a method similar to the Lyapunov theorem to determine (at least roughly) the location of the attractor in phase space. Let us call A the set of points defining the global attractor. This attractor has an extension in phase space [A is bigger than the origin $O(0,0,0)$ which was the attractor in the former example]. In the general case, it will not be possible to find a function $V(x,y,z)$ such that $\dot{V}(x,y,z)<0$ for $(x,y,z)\in\mathbb{R}^3\setminus A$. In fact $\dot{V}$ is going to change sign and there will be a set of points for which $\dot{V}(x,y,z)\geqslant 0$. A first (naive) assumption is to say that the attractor is included in the region where $\dot{V}(x,y,z)\geqslant 0$ since V decreases outside. As mentioned in [4] and as we shall see below, this argument is not correct. To fully understand what happens here, one has to see things geometrically, defining regions in phase space that are globally attracting.

II. GEOMETRIC POINT OF VIEW

Let us consider the level surfaces of the function $V(x,y,z)$ defined by $V(x,y,z)=K$ in phase space. The quantity $\dot{V}$ defined by

$$\dot{V}=\frac{\partial V}{\partial x}P+\frac{\partial V}{\partial y}Q+\frac{\partial V}{\partial z}R \tag{4}$$

is the scalar product between the vector (P,Q,R) tangent to the trajectory at the point (x,y,z) and the vector $(\partial V/\partial x,\partial V/\partial y,\partial V/\partial z)$ normal to the surface at this point. Hence, in the region where $\dot{V}$ is of constant sign, the level surfaces of $V(x,y,z)$ are crossed by the flow in only one direction. If $\dot{V}$ is of constant sign on the whole surface $V(x,y,z)=K$, we call this surface a tranversal or a semipermeable surface.

Let us consider, as an example, the case of a two-dimensional dynamical system. Here we must study the level curves $V(x,y)=K$, associated to a given function $V(x,y)$. Suppose that the level curves of V are all closed and that the value of V is increasing with the distance from the origin (in other words, V is a sink centered on the origin). Suppose now that $\dot{V}$ is negative for points far from the origin and positive for points near the origin. The level curves $V=K$ with large K are crossed inwards by the flow. If we reduce the value of K, these curves will still be crossed inwards by the flow as long as each one lies entirely in the region where $\dot{V}<0$. In Fig. 1 we have drawn the set of points where $\dot{V}=0$. The smallest curve to be entirely crossed inwards is the curve tangent to this set, $V=K_1$. Symmetricaly, we have drawn the biggest level curve to be entirely crossed outwards by the flow, the curve $V=K_2$ which is also tangent to the set $\dot{V}=0$.

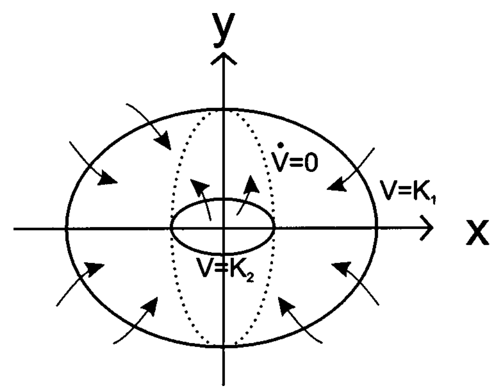

FIG. 1. Two level curves of a function $V(x,y)$ in a two-dimensional (2D) phase space. $V(x,y)=K_1$ is the lowest curve that is crossed by the flow inwards. $V(x,y)=K_2$ is the upper curve that is crossed by the flow outwards. The global attractor of the system lies between these two curves.

The time evolution of the $V(x(t),y(t))$ function for an initial condition far from the origin is shown in Fig. 2. The global attractor of the system is included in the region of phase space defined by $K_1<V(x,y)<K_2$. And if this region has no equilibrium point we know from the Bendixon-Poincaré theorem [1] that this attractor is a limit cycle. An analogous region for a three-dimensional (3D) system may contain limit cycles and/or chaotic attractors.

The region defined by $K_1<V(x,y)<K_2$ is an overestimation of the global attractor of the system. The method tells us where the attractor is but not what the attractor is. This means that the region $K_1<V(x,y)<K_2$ contains points that lie on the attractor but also points that are not on the attractor. If we were more clever (or equivalently if the attractor was not so complicated) we would find a better V function fitting the attractor more tightly. These considerations will be developed in Sec. VI.

In order to find the last entering curve $V(x,y)=K_1$ that will be the upper bound for the attractor, authors usually try

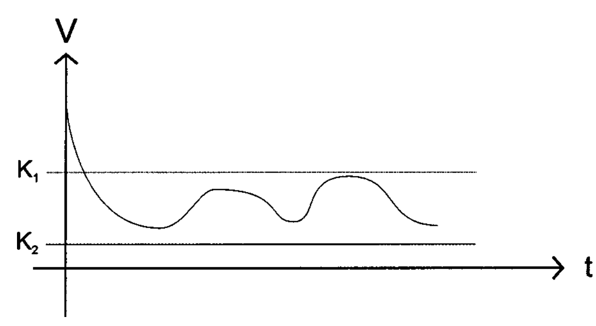

FIG. 2. The time evolution of the function $V(x(t),y(t))$ considered in Fig. (1). For some initial condition far from the origin, V(t) is decreasing at least until $V(x,y)=K_1$. Then $\underline{V}$ remains in the region $K_1<V(x,y)<K_2$.

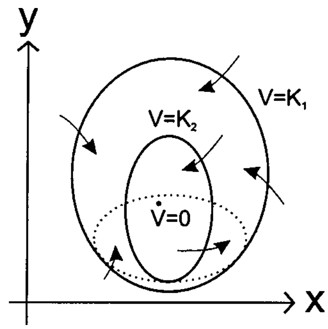

FIG. 3. The method of colinear gradients may sometimes be misleading. Here the two level curves $V=K_1$ and $V=K_2$ are tangent to the curve $\dot{V}=0$, but only the first one is semipermeable. Hence, tangency does not mean one-way crossing.

to find K_1 with the help of Lagrange multipliers [2,6]: they find the extrema of V on $\dot{V}=0$ by introducing the function $V-k\dot{V}$ (k is the Lagrange multiplier). This boils down to finding the points in phase space where the gradients of V and $\dot{V}$ are colinear. This method only works when the problem is simple because there are cases where tangency does not mean one way crossing. The level curve (or surface in 3D systems) may be tangent at some point but may cross the curve $\dot{V}=0$ at some other point(s), see Fig. 3.

Besides, for 3D systems, this method is more difficult to apply because it is then necessary to study the sign of the function $\dot{V}(x,y,z)$, which depends on three variables. It is relatively easy to find subsets of positive and negative sign for the $\dot{V}$ function, but it is rather difficult to find the level curves of V that are entirely included in each subset, since the parameters of the vector field are included in $\dot{V}$, together with the parameters of the function V. Hence, even if $V(x,y,z)$ is a polynomial, the problem is quite difficult.

III. SEMIPERMEABLE SURFACES METHOD

If we exploit further the geometric aspect of the problem, we notice that it is necessary for the function $\dot{V}(x,y,z)$ to be of constant sign only on the level set $V(x,y,z)=K$ and not in an entire space subset. It means that we have to study $\dot{V}|_{V=K}$ instead of $\dot{V}$ in the entire phase space. Thanks to the equality $V=K$, which permits us to replace one of the variables by the others, this new function will have only two variables. Of course, we restrict our studies to problems where this replacement is possible.[1] Then the analysis is much easier: we only have to study the sign of a two variable function when the variables vary on the entire surface (which means generally that we study the sign in all $\mathbb{R}^2$).

The semipermeable surfaces introduced in this context must have the two following properties.

Each surface (or, if it is not connected, each piece of the surface) must divide the phase space in two disconnected regions D_1 and D_2; either the surface is closed and then we can define an interior (D_1) and an exterior (D_2), or the surface is infinite, i.e., it separates also two regions D_1 and D_2 in phase space.

Each surface must be oriented; this means that the gradient must point toward the same region (D_1 or D_2) on the whole surface.

Following is an example that illustrates the superiority of the semipermeable surfaces method over the Lyapunov function method. Let us consider again the Lorenz system (2). In [2] the following surface is introduced:

$$-\frac{r}{\sigma}x^2+y^2+z^2=0, \tag{5}$$

which represents a certain bound (a double cone) for the attractor and is calculated by means of the Lyapunov function method. In [8], the following family of surfaces is introduced:

$$ax^2+y^2+z^2=R \quad \text{with} \quad R\leqslant 0, a<0. \tag{6}$$

For

$$\frac{-2\sigma r-(\sigma-1)^2-\sqrt{(\sigma-1)^4+4\sigma r(\sigma-1)^2}}{2\sigma^2}\leqslant a$$
$$\leqslant\frac{-2\sigma r-(\sigma-1)^2+\sqrt{(\sigma-1)^4+4\sigma r(\sigma-1)^2}}{2\sigma^2}, \tag{7}$$

the surfaces (6) are semipermeable and they define a better bound than surfaces (5) for the attractor of (2).

IV. METHODS TO OBTAIN SEMIPERMEABLE SURFACES

As said in Sec. III, it is easier to check whether a surface is semipermeable or not than to insure that a function possesses the Lyapunov property. But there is no general method to obtain these surfaces. In [8] we have determined several families of semipermeable surfaces for the Lorenz system guided by the time-dependent integrals of motion that exist for special values of the parameters of the system. Let us give an example of the application of this method: the Lorenz system (2) has the first integral $I(x,y,z,t)=(x^2-2\sigma z)e^{2\sigma t}$ when $b=2\sigma$ and σ and r are arbitrary (an easy calculation shows that $dI/dt\equiv 0$). Let us now consider the family of surfaces

$$V(x,z)=x^2-2\sigma z=K, \tag{8}$$

where K is an arbitrary constant. It is easy to show that $\dot{V}=-bK$. Hence, each surface of the family is transversal. The direction of crossing depends on the sign of the constant K. A particular surface of the family is obtained for $K=0$, and is invariant: an initial condition on this surface determines a trajectory that remains on the surface for all time. Besides, all the trajectories of the system are attracted by this invariant surface, as can be seen in Fig. 4. It is clear that the existence of these families of surfaces gives a lot of information about the dynamics of the system. The behavior of trajectories is extremely simple in all the phase space with the exception of the invariant surface $x^2=2\sigma z$. This surface

[1]For quadratic V functions, it has been shown in [7] that the study of $\dot{V}|_{V=K}$ was equivalent to the study of $\dot{V}$ in the entire phase space.

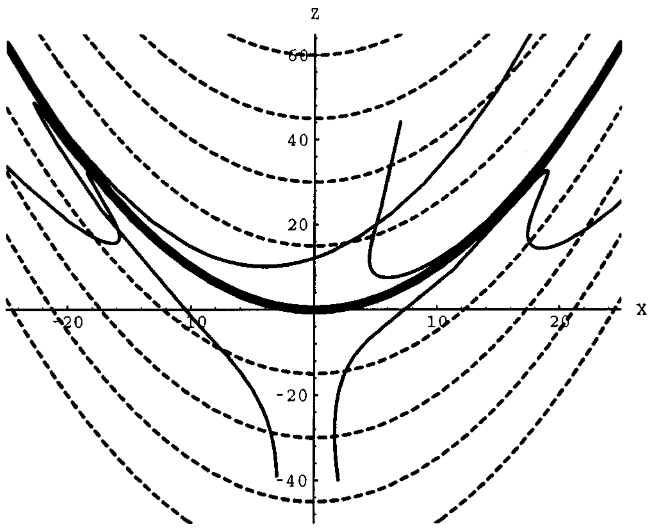

FIG. 4. The dashed curves represent semi-permeable surfaces (8) in the case $b=2\sigma$ for system (2). The bold curve is the invariant surface defined by Eq. (8) with $K=0$. We also show some trajectories of the system.

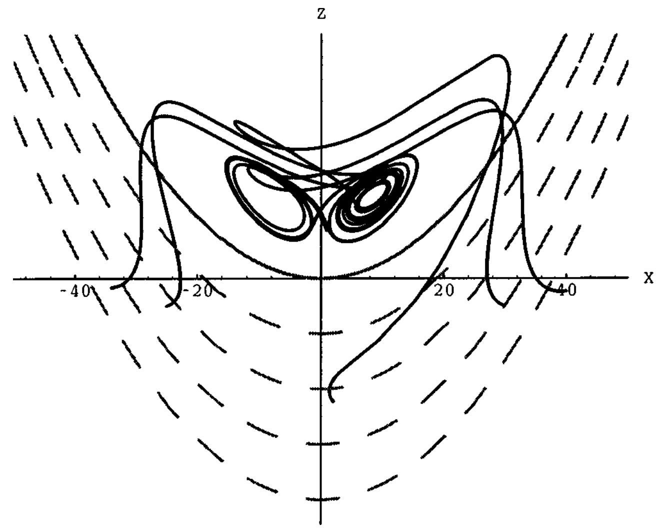

FIG. 5. Semipermeable surfaces (9) with $a_3<0$, $b<2\sigma$, $a_2=-2\sigma a_1$, $a_1=1$ for the system (2) together with its chaotic attractor.

contains the global attractor of the system for the case $b=2\sigma$.

The family of surfaces (8) obtained above enables us to characterize in a simple way this global attractor. The determination of this family of surfaces follows immediately from the existence of the integral of motion when $b=2\sigma$.

Now the natural question is when $b\neq 2\sigma$, is it still possible to find similar families of surfaces that the flow crosses in only one direction? In this case we shall no longer have at our disposal an integral of motion, and these semipermeable surfaces will not fill the phase space because in the general case the global attractor is not contained in a two-dimensional set. In order to find semipermeable surfaces in the general case, when $b\neq 2\sigma$, we proceed as follows. We first propose a surface of the same mathematical form as the integral of motion, but with arbitrary coefficients, i.e.,

$$V(x,z)=a_1x^2+a_2z+a_3. \tag{9}$$

Then we calculate $\dot{V}$ on the surface and obtain $\dot{V}_{|V=0}=(2a_1\sigma+a_2)xy+a_1(b-2\sigma)x^2+ba_3$. We now have an expression that depends only on two variables x and y. We must determine the coefficients a_1, a_2, and a_3 in such a way that this expression has the same sign for arbitrary values of x and y. We must hence set $a_2=-2\sigma a_1$, which yields

$$\dot{V}_{|V=0}=a_1(b-2\sigma)x^2+ba_3. \tag{10}$$

As a_1 must be nonzero we can take $a_1=1$ without loss of generality. If we consider $b>2\sigma$, we must set $a_3>0$ to get a first family of semipermeable surfaces and if we consider $b<2\sigma$, we must set $a_3<0$ to get a second family of semipermeable surfaces. We show the latter family as well as some trajectories of the system in Fig. 5.

As we can see from Fig. 5, in the region filled by the surfaces the dynamics of the system is very simple. The complex behavior can only occur in the region of phase space that is not occupied by these surfaces. The global attractor of the system must be located in the region $z>2\sigma x^2$.

Using the method explained above, we have determined, from the other known integrals of motion of the Lorenz system, several other families of semipermeable surfaces [8]. In the chaotic regime, only a bounded region of the phase space is not filled by these surfaces and the global attractor of the system must be contained in this region. In this way, we have obtained some information on the shape and location of the global attractor. These results are more restrictive than similar previous bounds that have been found by other authors, due to the method of Lyapunov functions [2].

The integrals of motion give us a hint that is of fundamental importance for obtaining semipermeable surfaces. In fact, when looking for these types of surfaces without having a previous idea of their mathematical expression we are faced with high algebraic difficulties. Nevertheless, some systems do not have integrals of motion, or at least, sufficiently simple integrals of motion to be found with the standard methods. In this paper we present an alternative method for determining semipermeables surfaces. This method is a variation of a technique introduced in [12] for finding integrals of motion. It can be applied to polynomial systems, i.e., systems where P,Q,R are polynomials in the three variables x,y,z.

We shall introduce the new method by analyzing a concrete example: the Lorenz system. We shall obtain again the semipermeable surfaces determined above from the alternative method. As the Lorenz system is linear with respect to each one of the three variables x,y,z, we propose a function $V(x,y,z)$ linear in z,

$$V=h_1(x,y)z+h_0(x,y), \tag{11}$$

where $h_0(x,y)$ and $h_1(x,y)$ are arbitrary functions of x and y. We define a function $M(x,y,z)$ as follows:

$$M(x,y,z)=\dot{V}+L(x,y,z)V, \tag{12}$$

where $L(x,y,z)$ is a polynomial of degree $n-1$ (n is the maximum degree of the polynomials P,Q,R). Since for the Lorenz system $n=2$, $L(x,y,z)$ will be of the form

$L(x,y,z)=\alpha_0+\alpha_1 x+\alpha_2 y+\alpha_3 z$, where the α_i are arbitrary parameters. The sign of $\dot{V}_{|V=0}$ is given by the sign of $M_{|V=0}$. In order to simplify the study of the sign of M in phase space, we shall impose conditions on the functions $h_0(x,y)$, $h_1(x,y)$ and on the parameters α_i. We shall obtain these conditions by imposing that M must be a function of only one variable, for instance, the variable x. The explicit expression of $M(x,y,z)$ is

$$\begin{aligned}M(x,y,z)=&\left(\alpha_3 h_1(x,y)-x\frac{\partial h_1}{\partial y}\right)z^2\\&+\left(\alpha_3 h_0(x,y)+(\alpha_0-b+\alpha_1 x+\alpha_2 y)h_1(x,y)\right.\\&\left.-x\frac{\partial h_0}{\partial y}+(rx-y)\frac{\partial h_1}{\partial y}+\sigma(y-x)\frac{\partial h_1}{\partial x}\right)z\\&+(\alpha_0+\alpha_1 x+\alpha_2 y)h_0(x,y)\\&+xyh_1(x,y)+(rx-y)\frac{\partial h_0}{\partial y}+\sigma(y-x)\frac{\partial h_0}{\partial x}.\end{aligned}\tag{13}$$

As the coefficient of z^2 must be zero we obtain the following expression for $h_1(x,y)$: $h_1(x,y)=g_1(x)\exp(\alpha_3 y/x)$, where $g_1(x)$ is an arbitrary function of x. Because we want to obtain a function V defined in all phase space, we take $\alpha_3=0$. The coefficient of z in the expression (13) must also be zero. This condition leads to the following equation: $(\alpha_0-b+\alpha_1 x+\alpha_2 y)g_1(x)+\sigma(y-x)g_1'(x)=x\partial h_0/\partial y$. The general solution of this equation is $h_0(x,y)=1/2x[2xg_0(x)+y(-2b+2\alpha_0+2\alpha_1 x+\alpha_2 y)g_1(x)+\sigma y(-2x+y)g_1'(x)]$, where $g_0(x)$ is an arbitrary function of x. Now the resulting expression of M is a function of x and y. We do not want to obtain the more general semipermeable surface of the form (11). Our aim is to give an example of the method explaining the differents steps of the algorithm. Hence, it is sufficient to consider $g_1(x)\equiv 1$, which yields

$$\begin{aligned}M(x,y)=&\frac{\alpha_2}{2x^2}(\alpha_2 x-\sigma)y^3+\{2b\sigma-2\alpha_0\sigma+[-2\alpha_2(b+1)\\&+3\alpha_0\alpha_2+\alpha_2\sigma]x+3\alpha_1\alpha_2 x^2\}\frac{1}{2x^2}y^2\\&+\{b-\alpha_0-b\alpha_0+\alpha_0^2-b\sigma+\alpha_0\sigma\\&+[\alpha_1(2\alpha_0-b-1)+\alpha_2 r]x\\&+(1+\alpha_1^2)x^2+\alpha_2 xg_0(x)+\sigma xg_0'(x)\}\frac{y}{x}\\&+\alpha_0 r-br+\alpha_1 rx+\alpha_0 g_0(x)+\alpha_1 xg_0(x)\\&-\sigma xg_0'(x).\end{aligned}\tag{14}$$

We want to obtain a function only of the variable x, so we take $\alpha_0=b$, $\alpha_2=0$, and $g_0(x)=-\alpha_1/\sigma(b-1)x-(1+\alpha_1^2)x^2/2\sigma+K_0$, where K_0 is an arbitrary constant. The resulting expression for M is

$$\begin{aligned}M(x)=&K_0 b+\alpha_1\left[(b-1)\left(1-\frac{b}{\sigma}\right)+k_0+r\right]x\\&+\left(1+\alpha_1^2-\frac{b}{2\sigma}+\frac{\alpha_1^2}{\sigma}-\frac{3b\alpha_1^2}{2\sigma}\right)x^2-\frac{\alpha_1}{2\sigma}(1+\alpha_1^2)x^3.\end{aligned}\tag{15}$$

For our purposes $M(x)$ must be of definite sign for arbitrary values of x, so we must take $\alpha_1=0$ and $M(x)$ becomes

$$M(x)=bk_0+\left(1-\frac{b}{2\sigma}\right)x^2.\tag{16}$$

The resulting expression for the function V is $V(x,z)=K_0-x^2/2\sigma+z$. The family of surfaces $V=0$ is semipermeable for $K_0>0$ if $b<2\sigma$ and for $K_0<0$ if $b>2\sigma$. If $b=2\sigma$, for $K_0=0$ we obtain the known invariant surface. In this way we arrive again at the results obtained with the help of an integral of motion of the Lorenz system. In Sec. IV we shall make use of both methods to find semipermeable surfaces. The first method has already been succesfully employed in [3,8,9] for the study of the Rabinovich, Lorenz, and Rikitake systems. For systems where we do not know any integral of motion, we shall use the alternative method. Both methods will yield bounds for the attractors in phase space, range of values of the parameters for which no chaotic behavior is possible, and make out part of the basin of attraction of the equilibrium points.

V. RESULTS ON PARTICULAR SYSTEMS

We shall first consider the system:

$$\dot{x}=-s(x+y),\quad \dot{y}=-y-sxz,\quad \dot{z}=v+sxy,\tag{17}$$

where s and v are positive parameters. This system has been introduced in the context of the qualitative study of the Lorenz attractor [10]. The divergence of the vector field is $\partial P/\partial x+\partial Q/\partial y+\partial R/\partial z=-s-1<0$. Hence, this system contracts volumes in all phase space. No integral of motion is known in the literature for this model. By appling the Painlevé method [11], we find that the quantity

$$I(x,y,z,t)=(x^4+8xy-4y^2+4x^2z)e^{4/3t}\tag{18}$$

is an integral of motion for the case $s=\frac{1}{3}$ and $v=0$. Using the method described in Sec. IV for the Lorenz model, we propose a family of surfaces of the form

$$V=d+cx^4+by^2+axy+ex^2z=0.\tag{19}$$

The expression of $\dot{V}$ on the surface $V=0$ is given by

$$\begin{aligned}\dot{V}_{|V=0}=&-x[ds(a+2e)x+ve^2x^3+cs(a-2e)x^5\\&+2ds(b+e)y+a(-e+as+es)x^2y\\&+s(2bc-2ce+e^2)x^4y\\&+(-2be+3abs+aes+2bes)xy^2\\&+2bs(b+e)y^3].\end{aligned}\tag{20}$$

Owing to the factor x in the above expression, we have to set $b=-e$ in order to obtain a function that does not change sign. After that, $\dot{V}_{|V=0}$ contains a common factor x^2 multiplied by a second degree polynomial in y^2. The discriminant of this polynomial is a polynomial of degree 6 in x which must be negative for all x. Since the coefficient $(e-4c)^2s^2$ of x^6 is positive, we must take $c=e/4$. After that, the discriminant is given by

$$\Delta=(as+es-e)[eds(a+2e)+(-a^2e+a^3s+a^2es+8e^3v)x^2+2e^2s(a-2e)x^4]. \tag{21}$$

If we set $e=0$ this expression cannot be negative. Therefore, without loss of generality we can set $e=-1$. So, the family of surfaces becomes

$$V=d-\frac{x^4}{4}+axy+y^2-x^2z=0. \tag{22}$$

Moreover, $\dot{V}|_{V=0}$ is given by

$$\dot{V}|_{V=0}=x^2\left(2[s(1-a)-1]y^2+a[s(1-a)-1]xy+s(a+2)\frac{x^4}{4}-vx^2+ds(2-a)\right) \tag{23}$$

and the discriminant Δ is

$$\Delta=-[s(1-a)-1](2s(a+2)x^4-\{8v+a^2[s(1-a)-1]\}x^2+8ds(2-a)). \tag{24}$$

Since Δ must be negative for all x, the coefficient $-2[s(1-a)-1]s(a+2)$ of x^4 must be negative, which is satisfied in each of the three following cases

(i) $-2<a<1$ and $s>1/1-a>0$,
(ii) $a<-2$ and $0<s<1/1-a$,
(iii) $-2<a$ and $1/1-a<s<0$.

After that, we must impose that Δ has no real root, which is satisfied in each one of the two following cases:

(iv) $d(a^2-4)s^2+\{v+\frac{1}{8}a^2[s(1-a)-1]\}^2<0$,
(v) $d(4-a^2)>0$ and $s(a+2)\{v+\frac{1}{8}a^2[s(1-a)-1]\}<0$.

Therefore, in order to have Δ negative for all x, we may combine any one of the three cases (i), (ii), (iii) with anyone of the two cases (iv), (v). We then have six different cases to consider. Two cases are particulary interesting: (i), (iv) and (ii), (v). In the case (i), (iv) the chaotic attractor is bounded by the semi-permeable surfaces (22) as shown in Fig. 6. In the case (ii), (v), the semipermeable surfaces are crossed by the flow in the upper direction. If we set $d=0$ then the surfaces divide the phase space in three disconnected regions and the two equilibrium points (which are attracting here) are separated by these surfaces. This means that, for values of s and v that satisfy (ii), (v), we know a part of the basin of attraction of each one of the two points. This also means that trajectories cannot wander from one equilibrium point to another and hence there is no chaos for these values of the parameters (see Fig. 7).

Continuing the study of system (17), we have looked for new integrals of motion but we have not been able to find any. In consequence, we have applied the alternative method introduced in Sec. IV. We have obtained the following family of surfaces with this method:

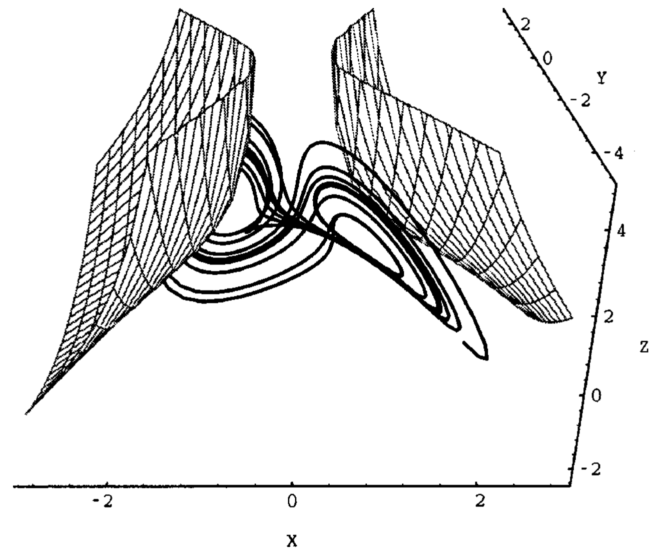

FIG. 6. Chaotic attractor of system (17) with $v=5/2$, $s=3$. The attractor is bounded by surface (22) with $a=-1$ and $d=1$ (condition (i), (iv)).

$$V=a_1x^2+y^2+(z+a_1)^2-a_4=0. \tag{25}$$

The scalar product on the surface is given by

$$\dot{V}|_{V=0}=(s-1)y^2+sz^2+z(v+2sa_1)+a_1^2s+a_1v-a_4s \tag{26}$$

and it is of definite sign when the following conditions are satisfied:

$$s>1,\quad 4a_4s^2+v^2\leqslant 0,\quad a_1<0. \tag{27}$$

We have a family of x-axis hyperboloid of revolution and each surface consists of two separated pieces. In Fig. 8 we see the chaotic attractor of system (17) and one of the semipermeable surfaces. For each negative value of a_1, the opti-

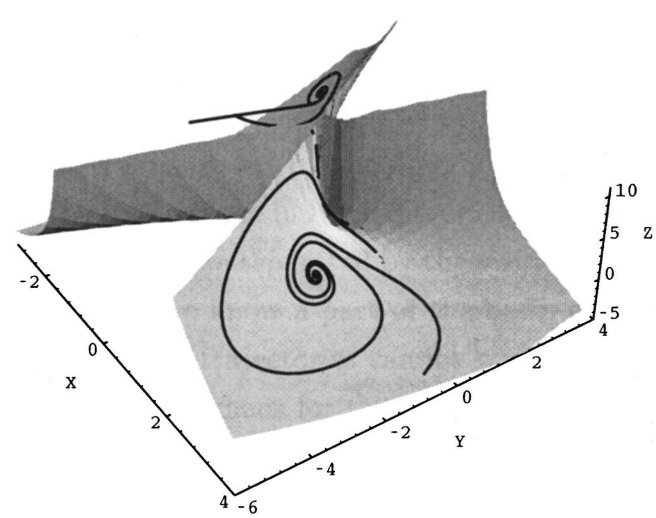

FIG. 7. System (17) with $v=1$, $s=1/4$. The surface (22) with $d=0$, $a=-5/2$ reveals part of the basin of attraction of each one of the equilibrium points. No chaotic behavior is possible in this case.

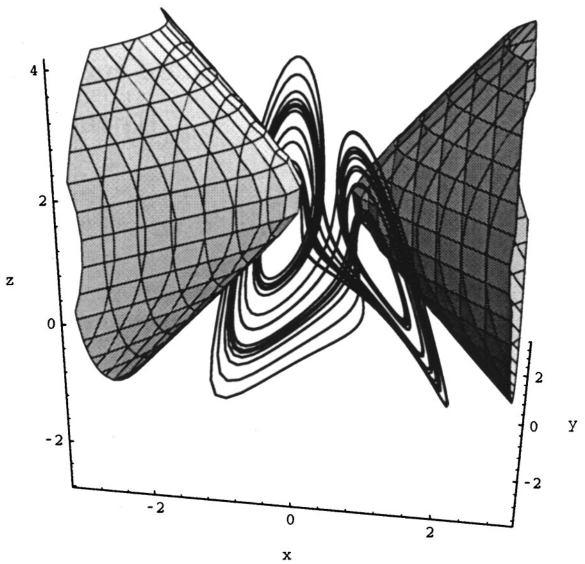

FIG. 8. Chaotic attractor of system (17) with $v=5/2$ and $s=3$ and a semipermeable surface (25) with $a_1=-1$ and $a_4=-v^2/4s^2$.

mal surface is obtained for $a_4=-v^2/4s^2$. For this value of a_4, varying a_1 within negative values, we have a monoparametric family of semipermeable surfaces. The optimal surface is the envolvent of the family, defined by

$$V=0, \quad \frac{\partial V}{\partial a_1}=0, \tag{28}$$

i.e.,

$$\frac{v^2}{4s^2}-\frac{x^4}{4}+y^2-x^2z=0. \tag{29}$$

It is remarkable that this last surface is a particular case of the family (22) with $a=0$ and $d=v^2/4s^2$, satisfying conditions (i), (iv).

We now consider the system

$$\dot{x}=y, \quad \dot{y}=z, \quad \dot{z}=-Az+y^2-x, \tag{30}$$

where A is a constant parameter. This system has been recently introduced [13] as the simplest system (since it has only one nonlinear quadratic term in the vector field) exhibiting chaotic behavior (for $A\simeq 2$). By applying different methods, we have not been able to find integrals of motion for this system. As the system is linear with respect to the z variable, we propose for the family of surfaces a function V linear in z, of the form

$$V=g_1(x,y)z+g_0(x,y), \tag{31}$$

where $g_0(x,y)$ and $g_1(x,y)$ are arbitrary functions of x and y. Following the method introduced in Sec. IV for the Lorenz system, we find that the following family of surfaces

$$V=z-ax+\left(A+\frac{1}{a}\right)y-d=0, \tag{32}$$

with the scalar product on the surface given by

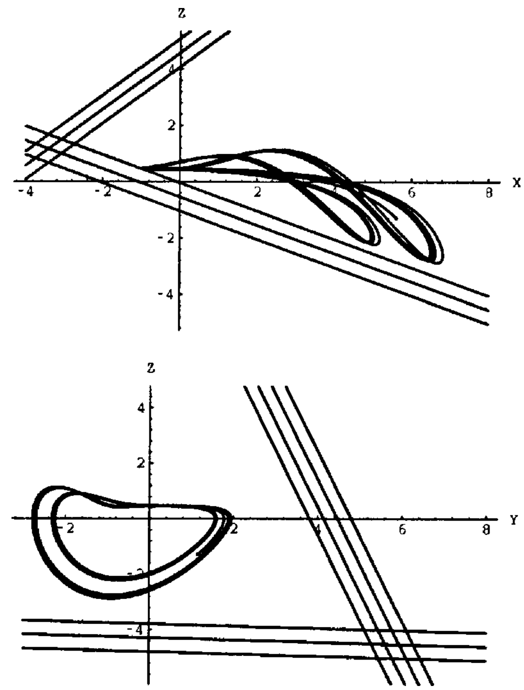

FIG. 9. The chaotic attractor of system (30) with $A=2.04$ contained in between planes ($a=1$ and $d>4.08$) and ($a=-\frac{1}{2}$ and $d<-0.04$). Note that the chaotic attractor is winding around the z axis.

$$\dot{V}|_{V=0}=a^2y^2-y(1+aA+a^3)+ad \tag{33}$$

is semipermeable if

$$\Delta=(1+aA+a^3)^2-4a^3d<0. \tag{34}$$

This condition yields two different cases.

$A\leqslant -(\frac{27}{4})^{1/3}\simeq -1.88$. Here there exist values of a for which $(1+aA+a^3)=0$ and for these values there are semipermeable planes $\forall d$. The z axis is surrounded by these planes. The chaotic attractor, when it exists, turns around this axis. Now, the semipermeable planes prevent this situation from occuring, so the chaotic attractor cannot exist in this case.

$A>-(\frac{27}{4})^{1/3}$. The chaotic attractor may exist in this case and when it exists, it is stuck in between two families of semipermeable planes, one above it ($d>0$) and one below ($d<0$) (see Fig. 9).

Now we consider once again the classical Lorenz system (2) for which several families of semipermeable surfaces have been found in [8]. By using the alternative method, we have found an interesting family of surfaces that gives important information about the behavior of the orbits on the chaotic attractor. We propose the following form for the family of surfaces:

$$V(x,y,z)=g_1(x,z)y+g_0(x,z)=0, \tag{35}$$

where $g_1(x,z)$ and $g_0(x,z)$ are arbitrary functions of x and z. Following the method employed above, we find

$$g_1(x,z)\equiv 1 \quad \text{and} \quad g_0(x,z)=a_1x^3-2a_1\sigma xz+a_2x, \tag{36}$$

which yields

$$V(x,y,z)=y+a_1x^3-2a_1\sigma xz+a_2x. \tag{37}$$

If we write these surfaces as

$$z=\frac{1}{2a_1\sigma}\left(a_1x^2+a_2-\frac{y}{x}\right), \tag{38}$$

the scalar product in this case could be of constant sign, but the surfaces (which are disconnected) are not oriented: the gradient vector does not point toward the same space subset for $x>0$ and for $x<0$ (this is due to the $-y/x$ term). Whereas if we write V as

$$y=-x(a_1x^2-2a_1\sigma z+a_2), \tag{39}$$

the surfaces are connected and oriented and the scalar product on the surfaces is

$$\dot{V}|_{V=0}=x[-4a_1^2\sigma^3z^2+zf(x)+g(x)], \tag{40}$$

where

$$f(x)=-1-2a_1\sigma+2a_1b\sigma+2a_1\sigma^2+4a_1a_2\sigma^2+4a_1^2\sigma^2x^2,$$

$$g(x)=a_2+r-a_2\sigma-a_2^2\sigma+(a_1-3a_1\sigma-2a_1a_2\sigma)x^2 -a_1^2\sigma x^4. \tag{41}$$

We see that (40) changes sign at $x=0$. Therefore this family of surfaces is not strictly semipermeable. Nevertheless, we shall obtain some important information from it. Hence, we shall study the cases in which the function $-4a_1^2\sigma^3z^2+zf(x)+g(x)$ holds the same sign $\forall(x,z)\in\mathbb{R}^2$. This happens when the two following conditions are satisfied:

$$2a_1\sigma(2\sigma-b)+1\geqslant 0, \tag{42}$$

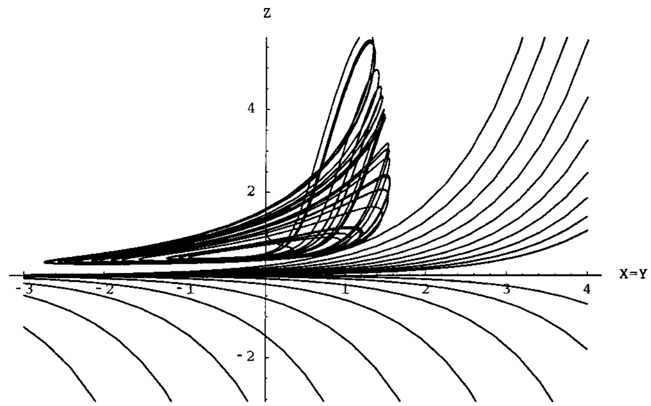

FIG. 11. Chaotic attractor of system (44) with $a=\frac{2}{5}$, $b=2$, $c=4$, and the $z>0$ sheet of the semipermeable surface (45) with $k_1\simeq-3.5$ and $k_2\simeq 1$.

$$1+4a_1\sigma(1-b-\sigma)+8a_1a_2\sigma^2(2ba_1\sigma-1) +4a_1^2\sigma^2[(b-1)^2+\sigma(4r-2+\sigma+2b)]<0. \tag{43}$$

When $x<0$ the surfaces (37) are crossed by trajectories in one way and when $x>0$ they are crossed by trajectories in the opposite way. Hence, these surfaces do not represent an external bound for the chaotic attractor when it exists.

We recall that the Lorenz attractor is formed by the addition of two wings, each wing lying around the equilibrium points C^+ and C^-, respectively. Therefore each surface of the family separates the attractor in two winding regions. One region is contained in $x>0$ and the other one is contained in $x<0$ (see Fig. 10).

Let us study the behavior of a trajectory around the ''positive'' wing (around the equilibrium point C^+). The trajectory wanders around C^+ until it ''decides'' to cross the x

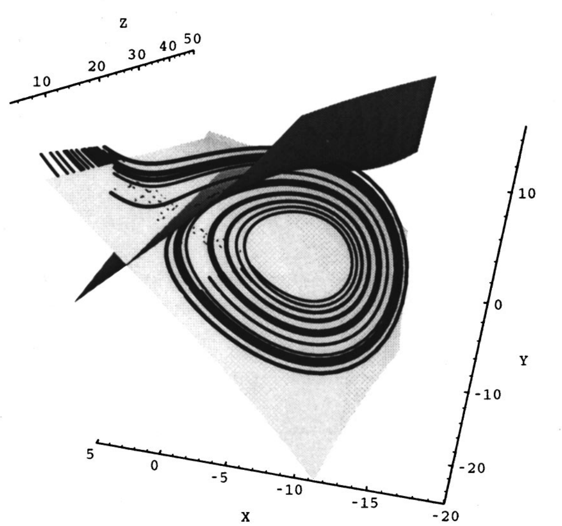

FIG. 10. Two surfaces (37) with ($a_1=-1/500$; $a_2=-1109/498$) and ($a_1=1/500$; $a_2=292/201$) represented for negative x. The wing of the attractor around the equilibrium point C^- is restricted in the region between the two surfaces. The trajectories that cross the above surface (a_1 and a_3 positive) from right to left are the trajectories that go to the other wing (in $x>0$).

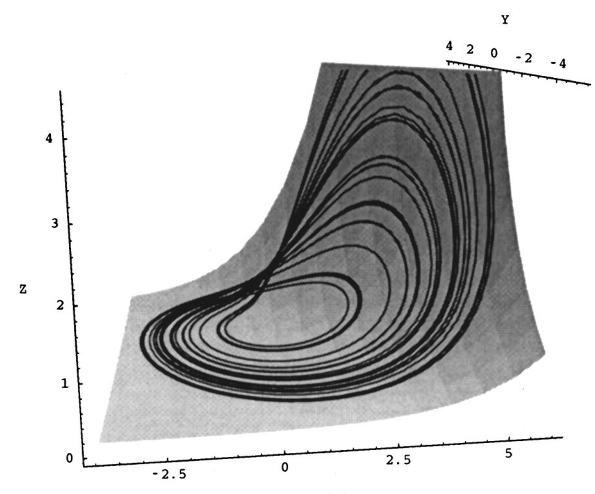

FIG. 12. Semipermeable surfaces in the $z<0$ half-space ($k_1\in[-12.7;10]$, $k_2=1$) and in the $z>0$ half-space ($k_1\in[-10;-3.5]$, $k_2\simeq-1$) together with a projection of the chaotic attractor ($a=\frac{2}{5}$, $b=2$, $c=4$) on the plane $x=y$. We see that all the trajectories initially in the $z<0$ half-space eventually cross the $z=0$ plane. Hence, the asymptotic behavior takes place in the $z>0$ half-plane.

$=0$ plane and goes wandering around the other equilibrium point. All the essence of complexity in the system comes from the fact that we do not know when the trajectory ''decides'' to jump to the other side of the plane $x=0$. Here is the interesting feature of this family of surfaces (37). The $x>0$ side of this surface is placed between the $x=0$ plane and the ''positive'' wing of the attractor. Because (37) is semipermeable in the $x>0$ half-space, once the trajectory has crossed this surface, it cannot go on wandering around the point C^+ and it is compelled to go winding around the other equilibrium point C^-. We may consider such surfaces as a separation between the two wings of the attractor. Besides, surfaces (37) give a bound in phase space for the period-one limit cycles around each equilibrium point.

The last example we shall consider is the classical Rössler system:

$$\dot{x}=-y-z, \quad \dot{y}=x+ay, \quad \dot{z}=b+z(x-c), \tag{44}$$

where a,b,c are positive parameters. For certain values of these parameters, this system has a chaotic attractor (Fig. 11). Moreover, it has two equilibrium points when $c^2 \geqslant 4ab$. One of the points (P_{in}) is nested inside the chaotic attractor and the other one (P_{out}) is outside the chaotic region. This system has a nonconstant divergence and there are no known integrals of motion for it. Nevertheless, using the new method, we find the following family of semipermeable surfaces:

$$V=y+k_1+(a+k_2)x-(1+ak_2+k_2^2)\ln|z|=0. \tag{45}$$

Each one of these surfaces consists of two disconnected sheets. One sheet lies entirely in $z>0$ and the other one in $z<0$. The two sheets are obtained from the expression

$$z=\pm\exp\left(\frac{y+k_1+(a+k_2)x}{1+ak_2+k_2^2}\right). \tag{46}$$

The scalar product on the surface is given by

$$\begin{aligned}\dot{V}|_{V=0}&=-b(1+ak_2+k_2^2)\frac{1}{z}+[k_1k_2+c(1+ak_2+k_2^2)]\\&\quad-(a+k_2)z-k_2(1+ak_2+k_2^2)\ln|z|\\&\stackrel{\text{def}}{=}f(z).\end{aligned}$$

The function $f(z)$ must be of constant sign on each sheet (46), i.e., for $z>0$ and for $z<0$, respectively. From the study of this one variable function, we find that the necessary and sufficient conditions for each sheet to be semipermeable are

$$\begin{gathered}f(z_i^\star)z_i^\star(a+k_2)<0 \quad \text{with} \quad i=1,2,\\ b(1+ak_2+k_2^2)(a+k_2)>0,\end{gathered} \tag{47}$$

with

$$z_1^\star=-\frac{k_2(1+ak_2+k_2^2)-\sqrt{k_2^2(1+ak_2+k_2^2)^2+4b(1+ak_2+k_2^2)(a+k_2)}}{2(a+k_2)}, \tag{48}$$

$$z_2^\star=-\frac{k_2(1+ak_2+k_2^2)+\sqrt{k_2^2(1+ak_2+k_2^2)^2+4b(1+ak_2+k_2^2)(a+k_2)}}{2(a+k_2)}. \tag{49}$$

These conditions can be satisfied when the chaotic attractor exists (for example, when $a=\frac{2}{5},c=4,b=2$). In fact the $z<0$ half-space is filled by semipermeable surfaces crossed by the flow upward. Hence, this proves that for such values of the parameters, the asymptotic ($t\to+\infty$) behavior takes place in the $z>0$ half-space, where the chaotic attractor must lie entirely (see Fig. 12). Moreover, there are also semipermeable surfaces lying in the $z>0$ half-space, bounding the chaotic attractor quite more tightly (see Fig. 11).

VI. GETTING CLOSER TO THE ATTRACTOR

So far we have introduced a method to get geometric bounds on the attractors of dissipative systems. These bounds are sometimes tight and sometimes loose. The natural question that arises is, can we get closer to the attractor?

If we consider each point on a semipermeable surface surrounding an attractor as an initial condition ($t=0$) and integrate numerically, the set of points at $t>0$ will define another semipermeable surface (with a different shape). As $t\to+\infty$, the surface will merge with the attractor (for a simple example see [14], p. 42).

Some authors define entering regions as a combination of different functions [15]. The surface surrounding the attractor is then defined by multiple equations, each one valid for a precise region in phase space. This is a way to tackle the complexity of the attractor. As regards our method, when more than one surrounding surface is known, one has to consider the composition of the different surfaces. This yields a tighter bound for the attractor. It is likely that by considering many more equations of surfaces, we could get even nearer to the attractor. To stick to the attractor (chaotic or limit cycle), we should consider an infinite combination of surfaces.

If we want to bound the attractor with only one type of equation and we want this bound to get tighter and tighter, we shall have to refine the equation of the surface at each step. This is what is done in [16,17] for the van der Pol system, where the attractor is a limit cycle (which equation is given by an unknown transcendental function). At each step of the procedure, the curve bounding the limit cycle is defined by a particular level curve of a polynomial function in two variables, of increasing degree. This curve (which is semipermeable) is getting closer and closer to the limit cycle. Taking the limit, the curve (defined by an infinite series in two variables) seems to merge with the limit cycle.

VII. CONCLUSIONS

We have shown that the method introduced in [8] for the Lorenz system works for other 3D chaotic dynamical systems. We have also introduced an alternative method to find semipermeable surfaces and applied it to several chaotic dynamical systems, showing that semipermeable surfaces enable us to bound the chaotic attractor in phase space or reveal ranges of parameters' values for which no chaotic behavior is possible in these dissipative systems. This last aspect of the method represents an important theoretical progress in the study of 3D dissipative dynamical systems.

ACKNOWLEDGMENTS

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