Vibrations planes de poutres: singularité de la limite inextensible

S. Neukirch, J. Frelat, C. Maurini
Institut d'Alembert , CNRS & UPMC Univ. Paris 6

A. Goriely
Centre for Applied Math. (OCCAM), Oxford, U.K.

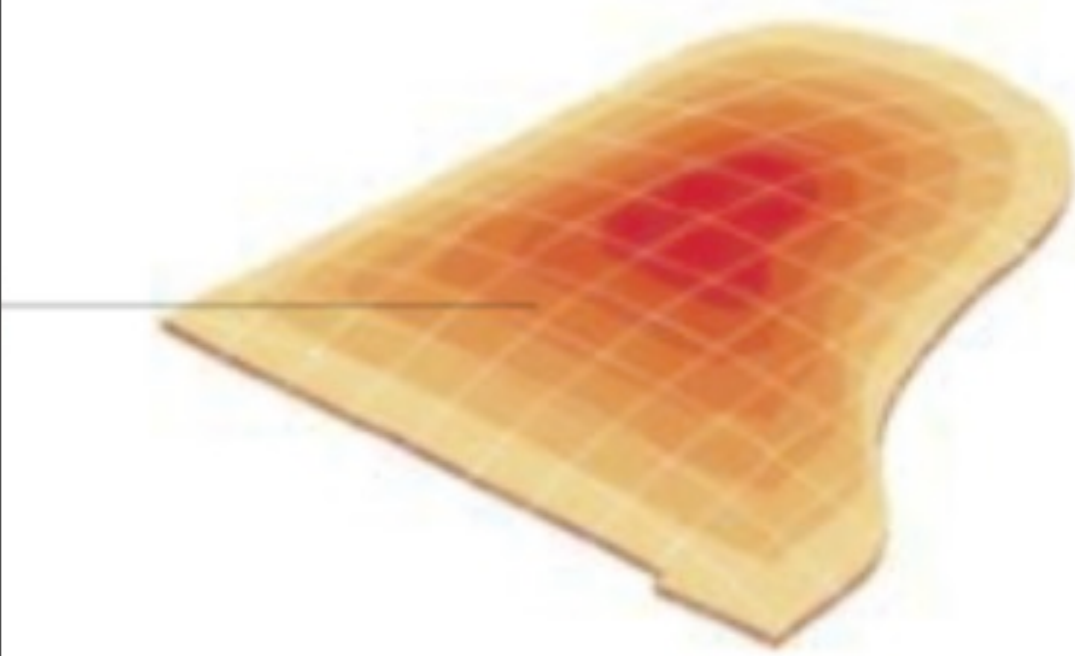
O. Thomas
Lab. de Mécanique des Structures et des Systèmes Couplés
CNAM - Paris

Piano soundboard

http://www.steinway.com/

Piano soundboard

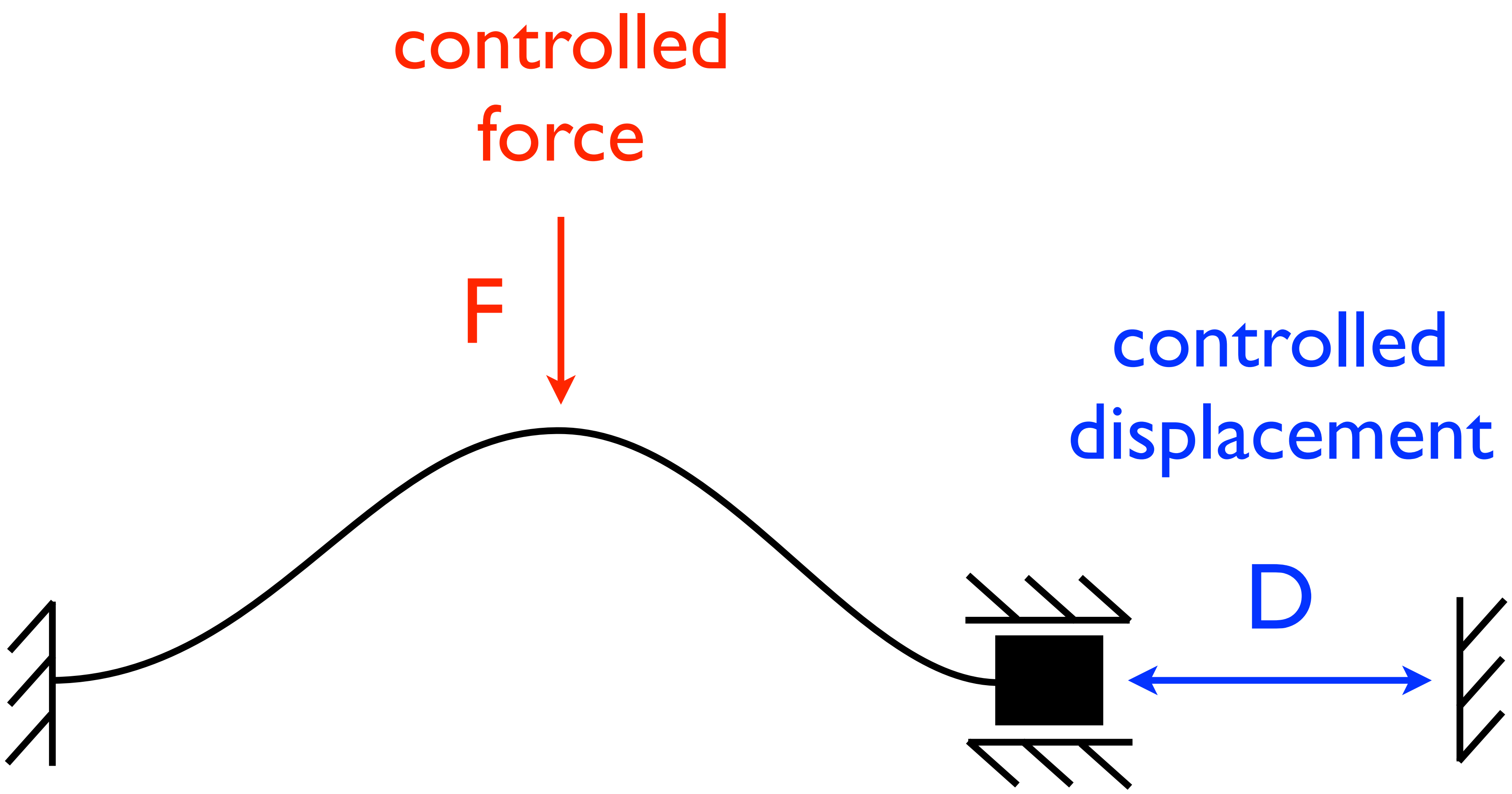
acoustic radiation from the soundboard (not the strings)



www.richardlipp.com.au

Model: pre-stressed beam

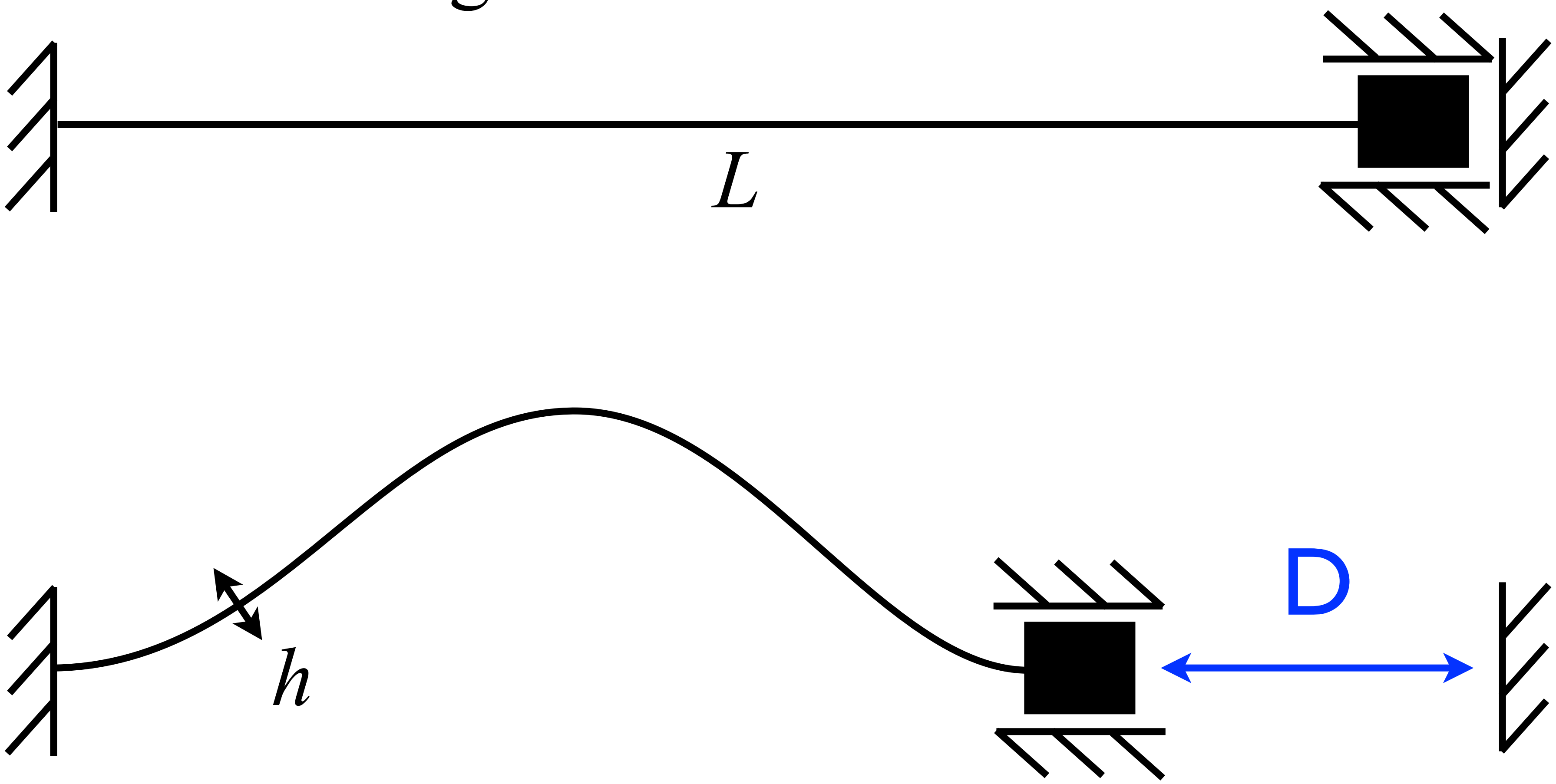
vibrations slender elastic beam in the plane



Influence of F, D
on the frequencies ?

Elastic beam in the plane

L : length in unstressed state



h : section thickness
w : section width

$$I = \frac{1}{12} h^3 w \qquad A = h\, w$$

Model : do we need extensibility ?

$$E_{\text{strain}} = \underbrace{\frac{1}{2}\int_0^L EI\,\kappa^2(s)ds}_{\text{curvature}} + \underbrace{\frac{1}{2}\int_0^L EA\,e^2(s)ds}_{\text{extension}}$$

$EI \rightarrow h^3\,w$

$EA \rightarrow h\,w$

h : section thickness
w : section width

$$\epsilon = \frac{I}{AL^2} = \frac{1}{12}\left(\frac{h}{L}\right)^2 \ll 1$$

$\epsilon = 0$ inextensible

$\epsilon > 0$ extension

Marigo Classification

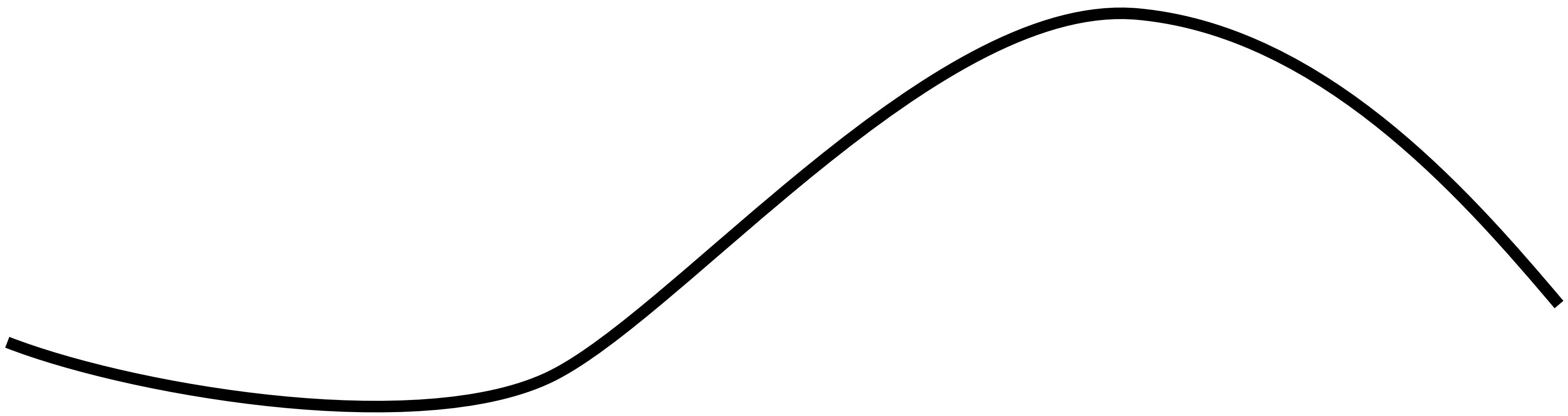
charge croissante

1. Poutres flexibles Petits Déplacements
2. Poutres flexibles Grands Déplacements
3. Fils inextensibles
4. Fils extensibles

JJM & Ghidouche & Sedkaoui, CRAS (1998)
JJM & Madani, CRAS (1998)
JJM & Meunier, Journal of Elasticity (2006)
JJM & Madani, Journal of Elasticity (2004)

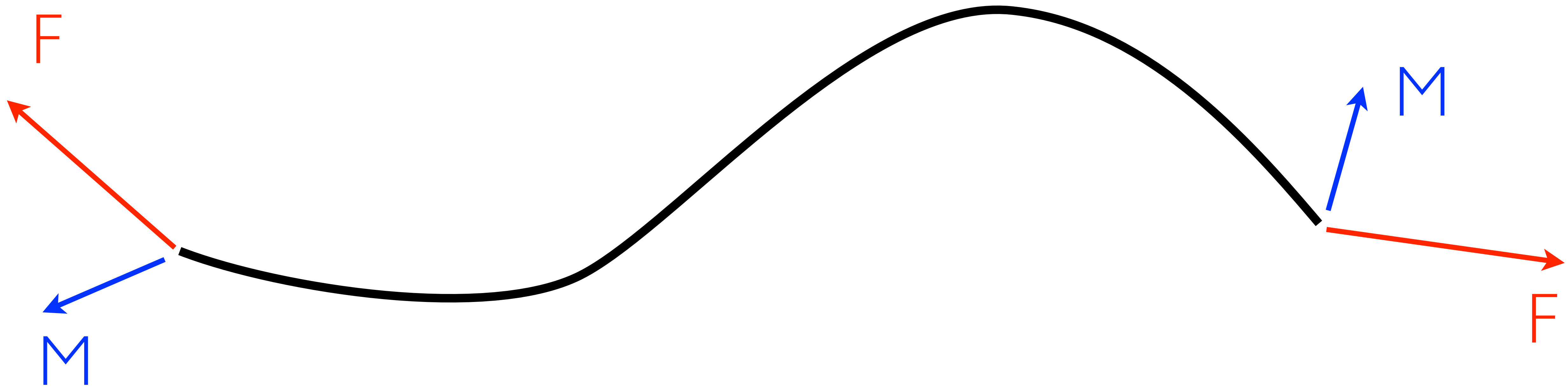
Kirchhoff equations

short
review

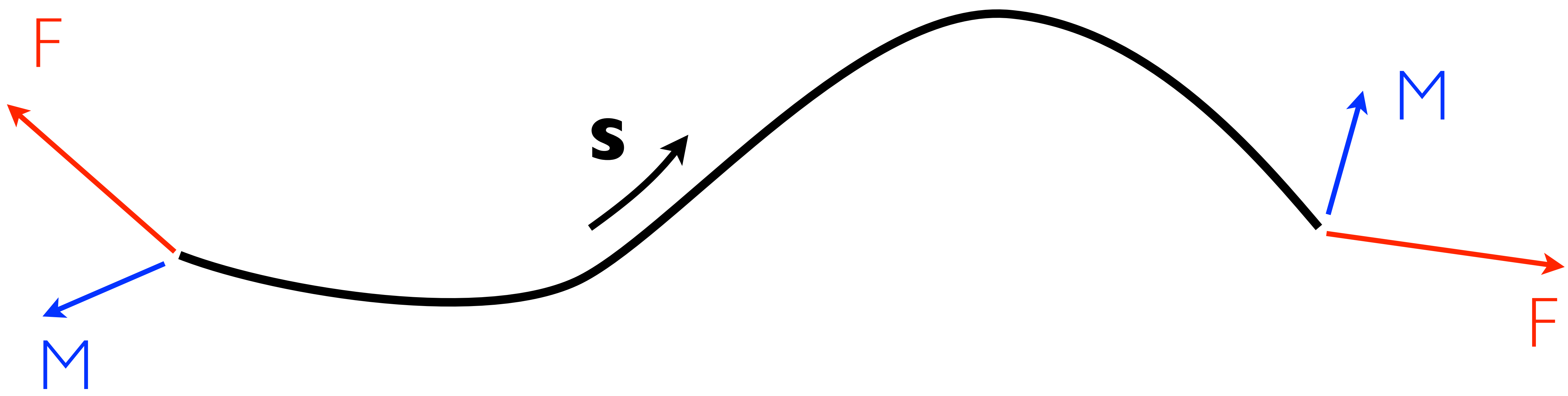


apply to :
- slender bodies
- not too bent

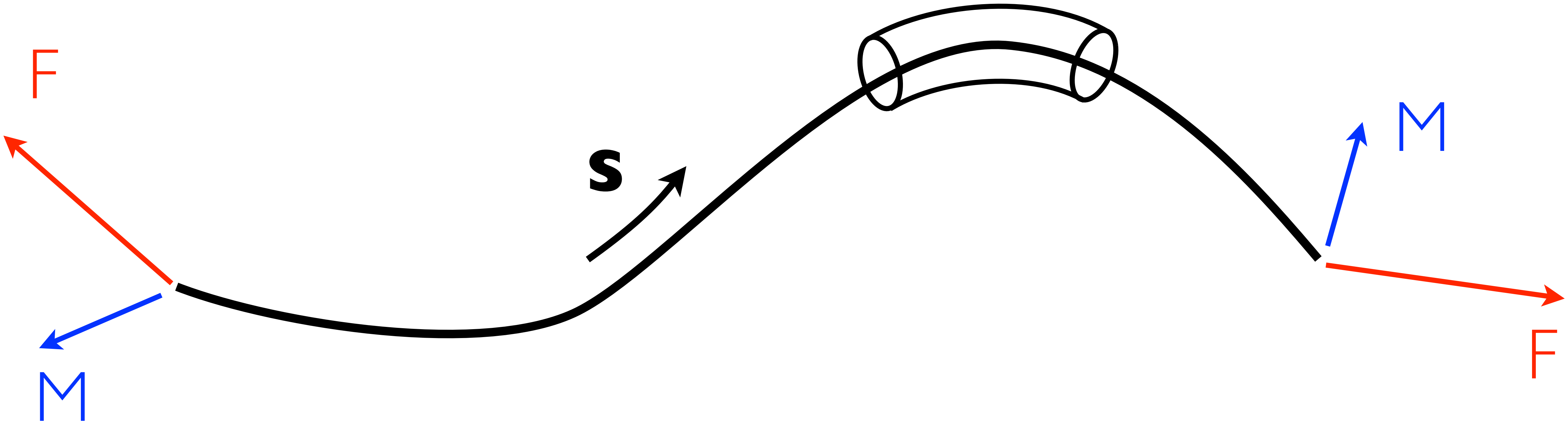
Kirchhoff equations



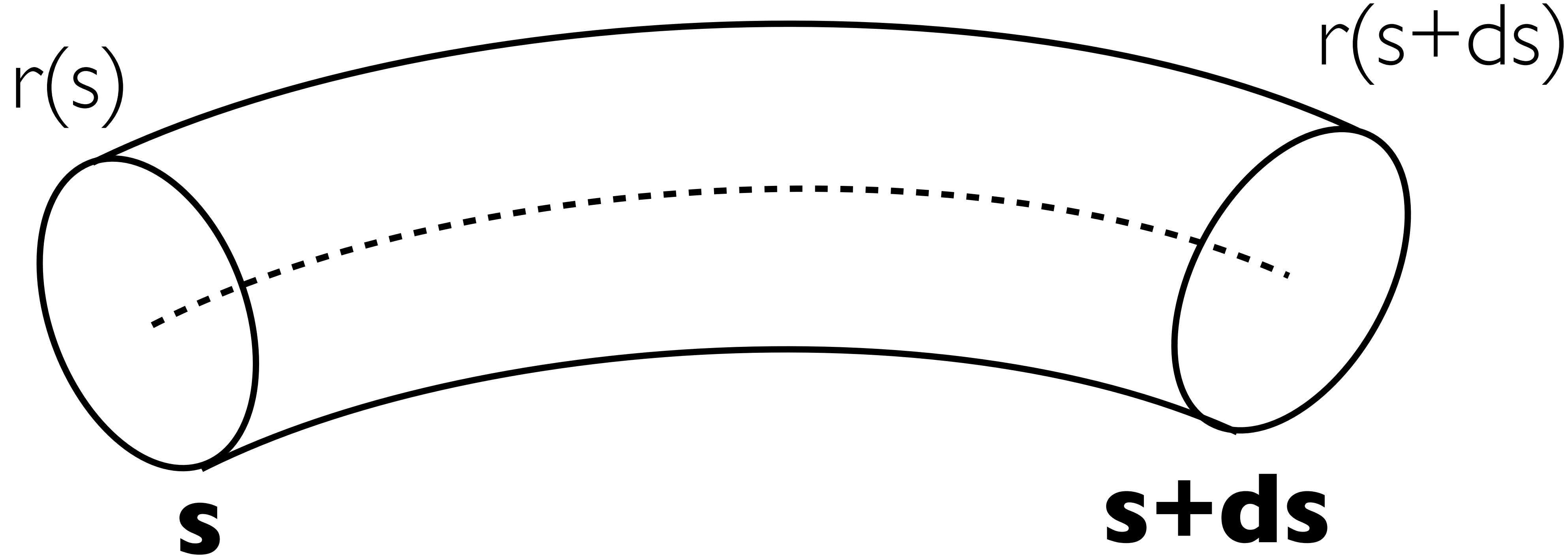
Kirchhoff equations



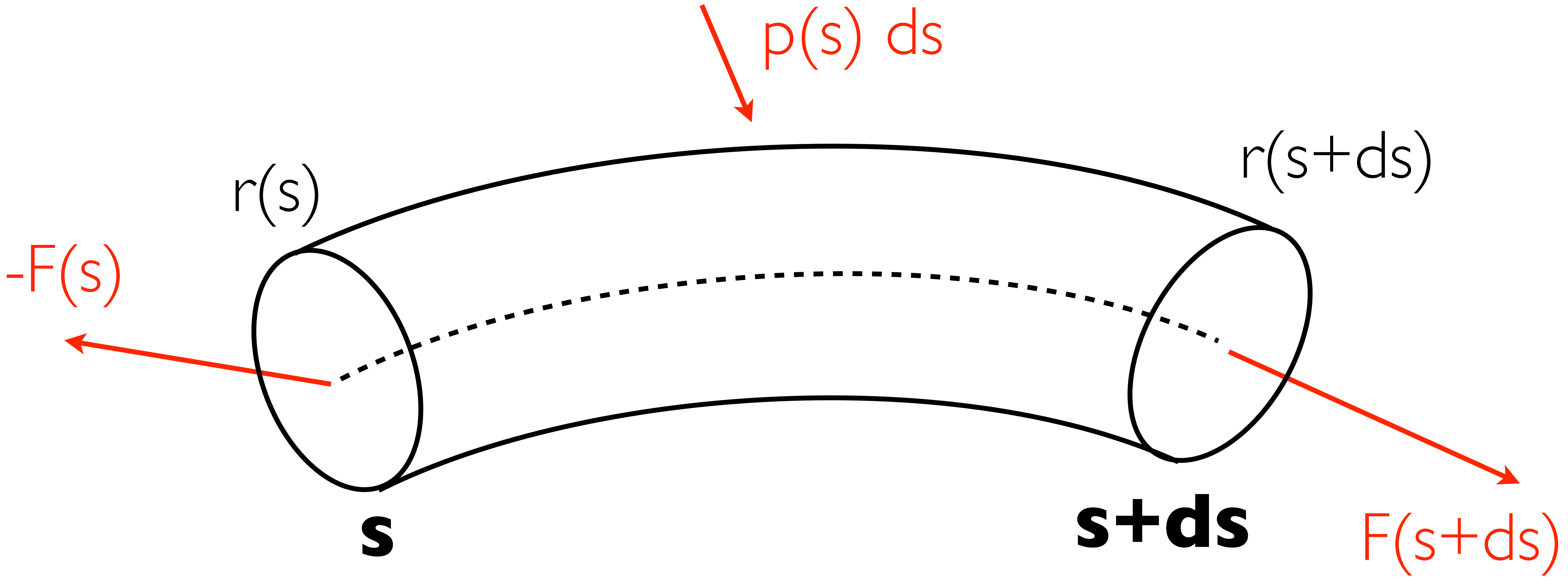
Kirchhoff equations



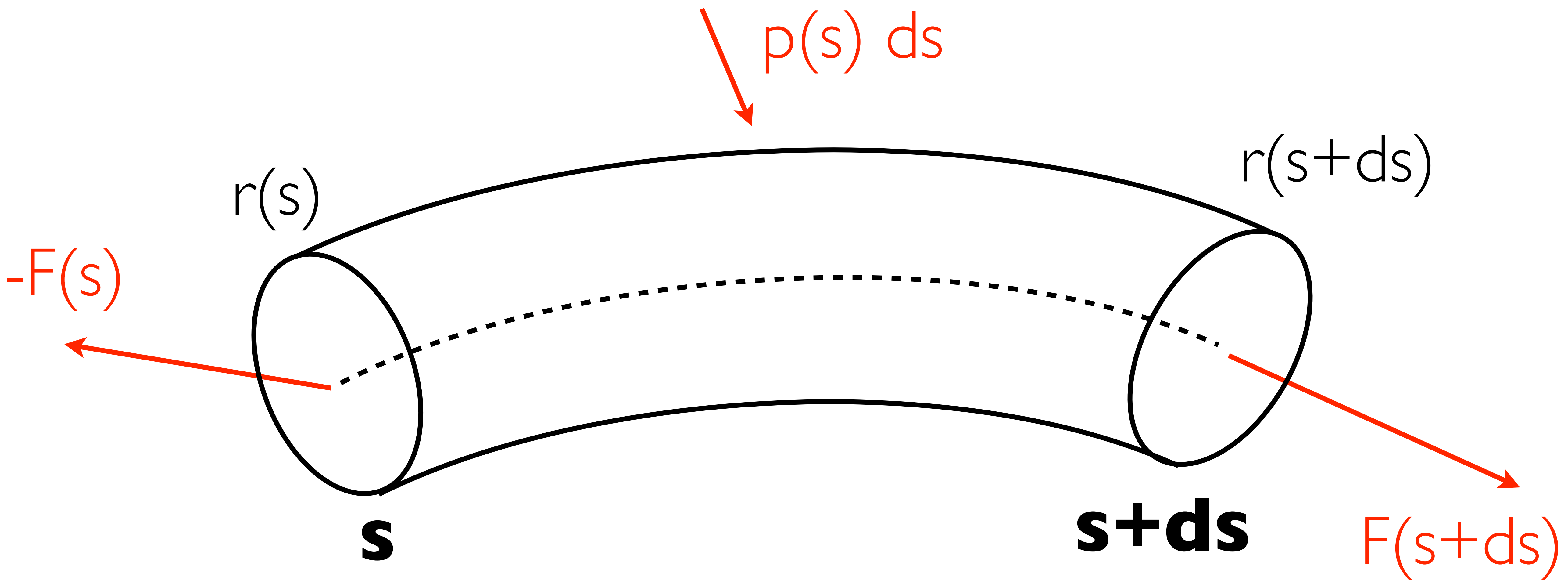
Kirchhoff equations



Kirchhoff equations



Kirchhoff equations

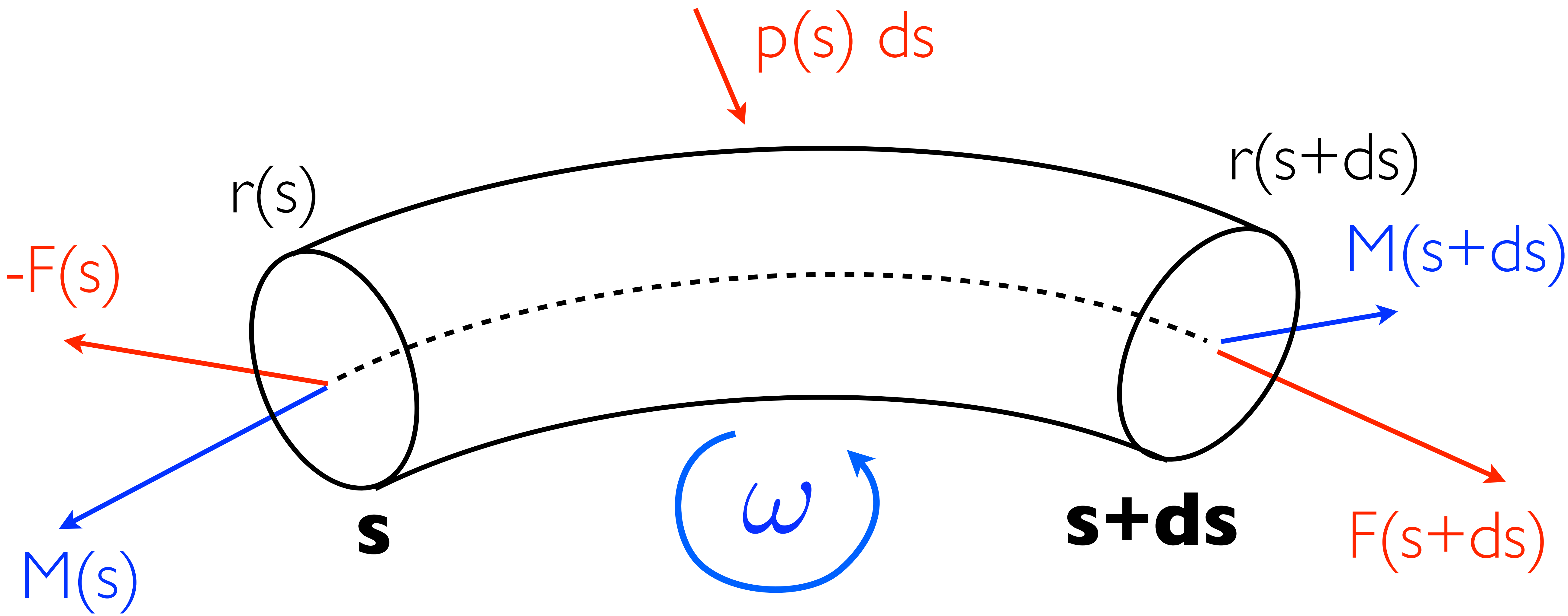


Dynamics (linear momentum):

$$\begin{aligned} F(s+ds,t) - F(s,t) + p(s,t)ds &= \rho\, A\, ds\, \ddot{r}(s,t) \\ F'(s,t) + p(s,t) &= \rho\, A\, \ddot{r}(s,t) \end{aligned}$$

$$\dot{} \equiv \frac{d}{dt} \qquad {}' \equiv \frac{d}{ds}$$

Kirchhoff equations

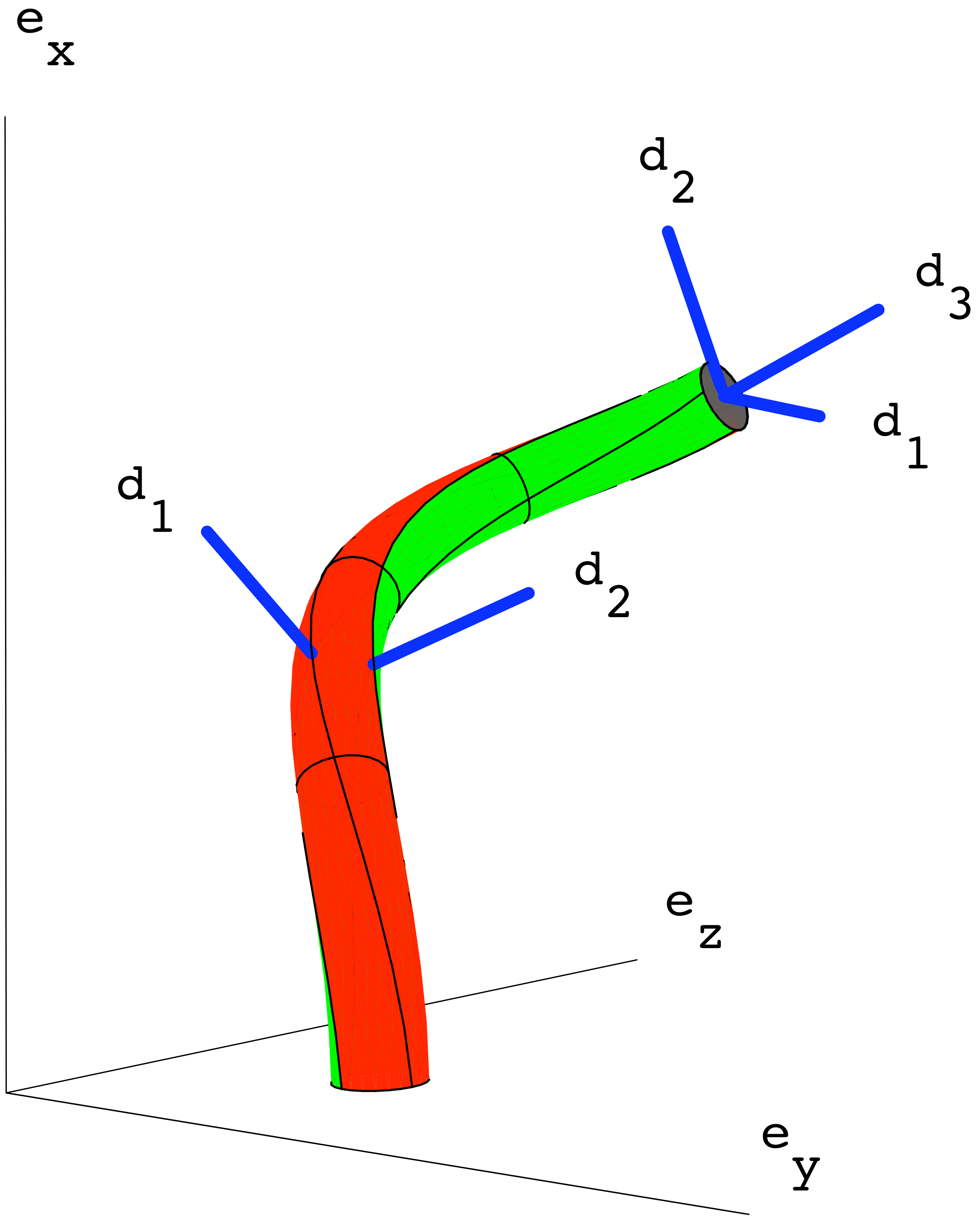


Dynamics (angular momentum):

$$M'(s,t) + r'(s,t) \times F(s,t) \;=\; \rho I \dot{\omega}(s,t)$$

$$\cdot \equiv \frac{d}{dt} \qquad ' \equiv \frac{d}{ds}$$

Kirchhoff equations: kinematics



Cosserat frame

$$d_1' = u \times d_1$$
$$d_2' = u \times d_2$$
$$d_3' = u \times d_3$$

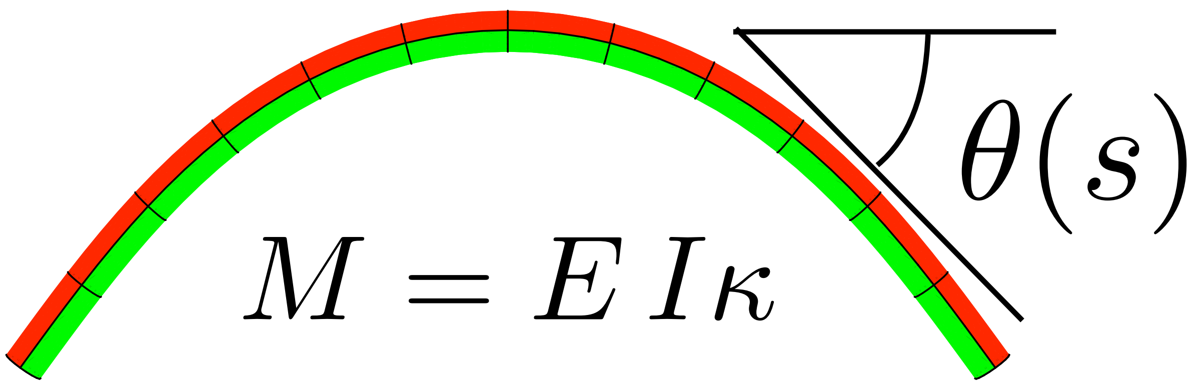
$$u = \{\kappa_1, \kappa_2, \tau\}_{d_i}$$

curvatures

twist

Kirchhoff equations: constitutive relations

curvature

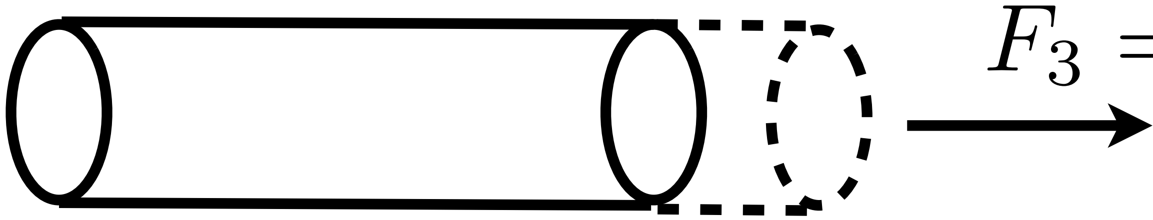


$$M = E\,I\kappa$$

with $\kappa(s) = \theta'(s)$

G	shear modulus
E	Young's modulus
I	second moment of area
A	area of section

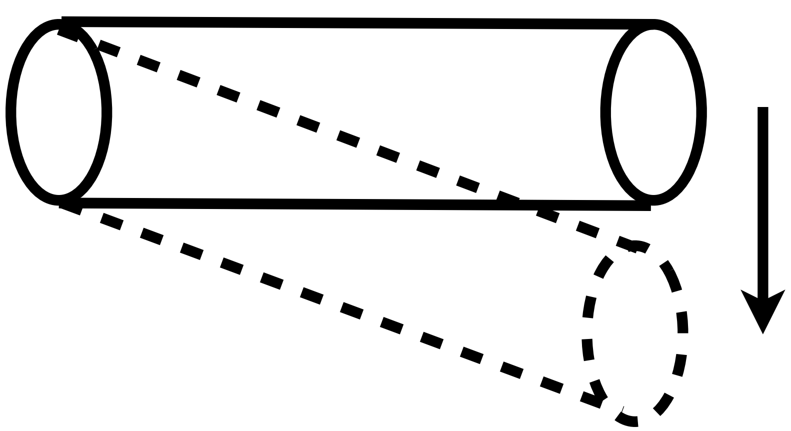
extension



$$F_3 = E A\,(v_3 - 1)$$

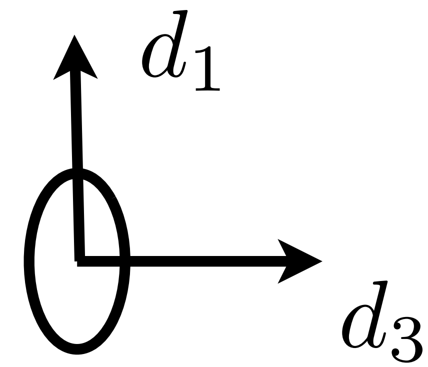
(v_1, v_3)

shear strains



shear

$$F_1 = G A\, v_1$$



Special Cosserat theory of rods

S. Antman, *Nonlinear problems of elasticity,* (2004).

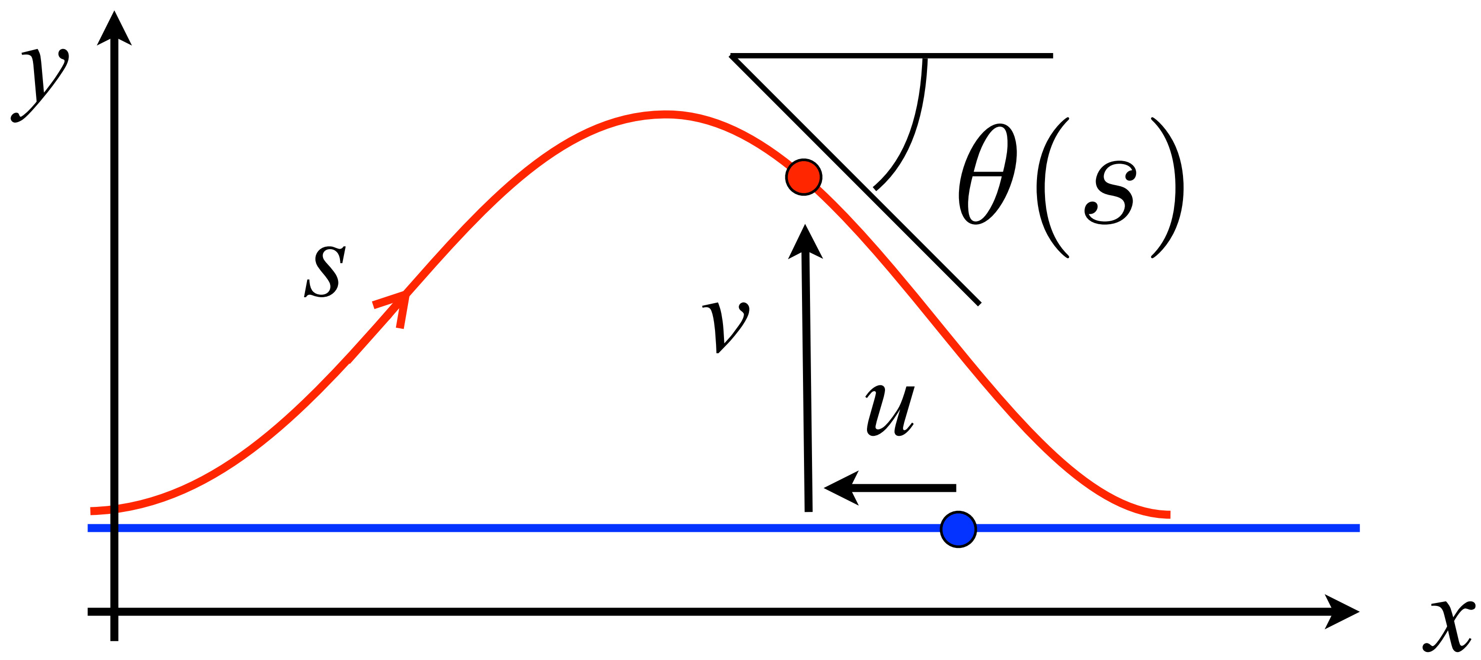
$$
\left.\begin{aligned}
F' &= \rho A \ddot{R} \\
M' &= F \times R' + \rho I \ddot{\theta}
\end{aligned}\right\} \text{dynamics}
$$

$$
R' = V = v_1 d_1 + v_3 d_3 \qquad \text{kinematics}
$$

$$
\left.\begin{aligned}
M &= EI\theta' \\
F \cdot d_1 &= GA\, v_1 \\
F \cdot d_3 &= EA\,(v_3 - 1)
\end{aligned}\right\} \text{constitutive relations}
$$

in the *(x,y)* plane

Strength of materials notations



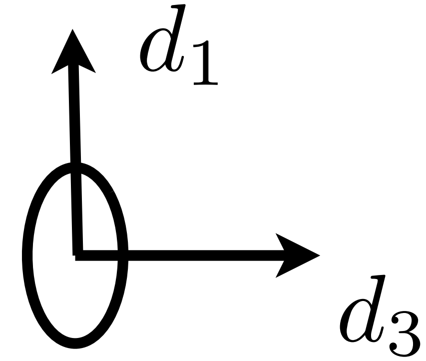
$$\begin{cases} x(s) = s - u(s) \\ y(s) = v(s) \end{cases}$$

$$\begin{cases} x'(s) = 1 - u'(s) \\ y'(s) = v'(s) \end{cases}$$

$$R'(s) = \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} = V(s) = v_1 d_1 + v_3 d_3$$

$$d_1(s) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \qquad d_3(s) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

with $\begin{cases} v_1(s) = (F \cdot d_1)/GA \\ v_3(s) = 1 + (F \cdot d_3)/EA \end{cases}$



Equilibrium equations (adim)

$$f = \frac{FL^2}{EI} \qquad m = \frac{ML}{EI} \qquad x = \frac{X}{L} \qquad s = \frac{S}{L}$$

ν : Poisson

$$\begin{cases} x' &= \cos\theta + \epsilon\left(f_3\cos\theta - 2(1+\nu)f_1\sin\theta\right) \\ y' &= \sin\theta + \epsilon\left(f_3\sin\theta + 2(1+\nu)f_1\cos\theta\right) \\ \theta' &= m \\ m' &= -f_1 + \epsilon\, f_1 f_3(1-2\nu) \\ f_x' &= 0 \\ f_y' &= 0 \end{cases}$$

boundary conditions

$$\begin{aligned} x(0) &= 0 & \\ y(0) &= 0 &= y(1) \\ \theta(0) &= 0 &= \theta(1) \end{aligned}$$

$$\epsilon = \frac{I}{AL^2} = \frac{1}{12}\left(\frac{h}{L}\right)^2 \ll 1$$

with $\begin{cases} f_1 &= -f_x\sin\theta + f_y\cos\theta \\ f_3 &= f_x\cos\theta + f_y\sin\theta \end{cases}$

$\epsilon = 0$ Euler-Bernoulli beam

$\epsilon > 0$ Timoshenko beam

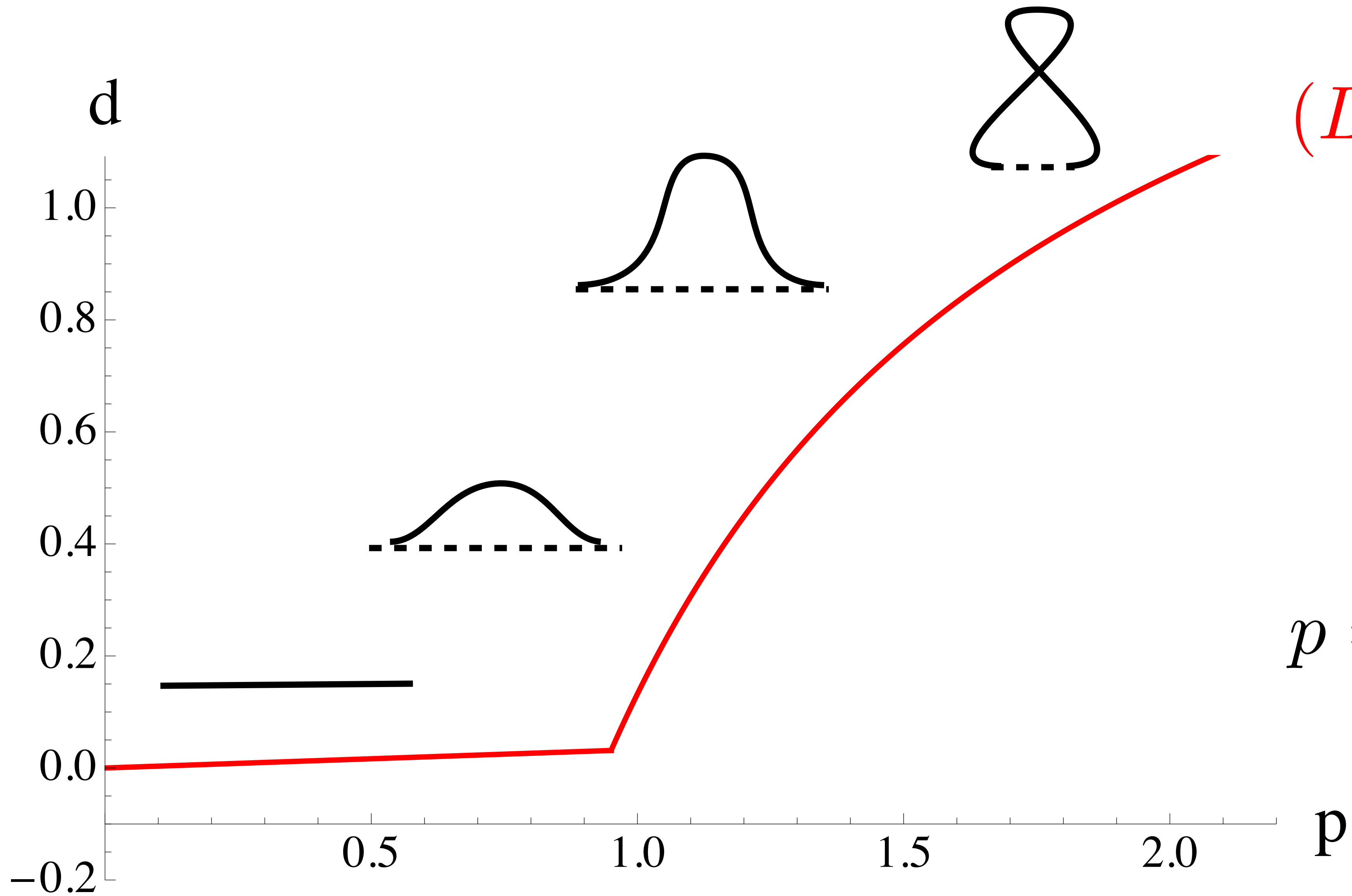
Resultats

Equilibrium (numerical study)

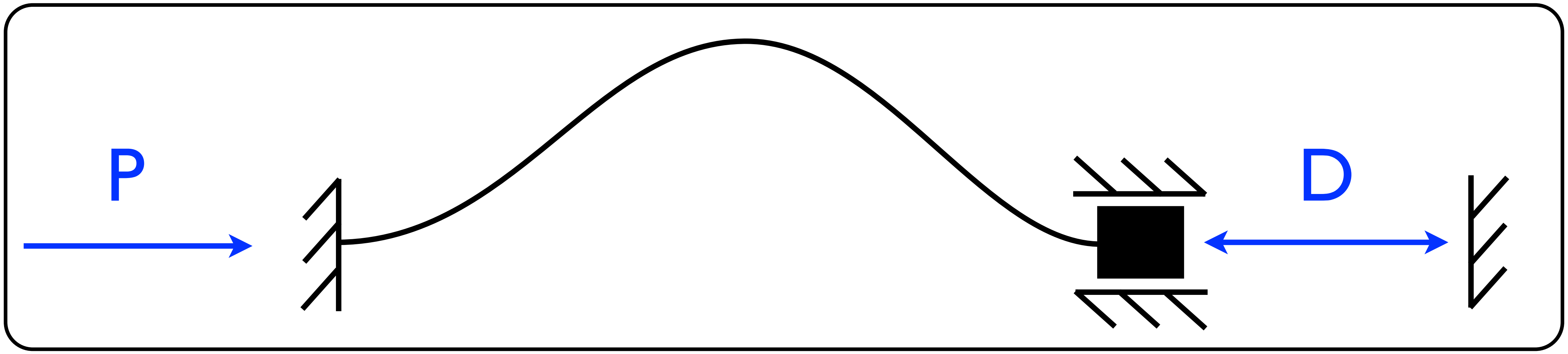
$$d = \frac{D}{L}$$

$$\epsilon = \frac{1}{1200}$$

$$(L = 10h)$$

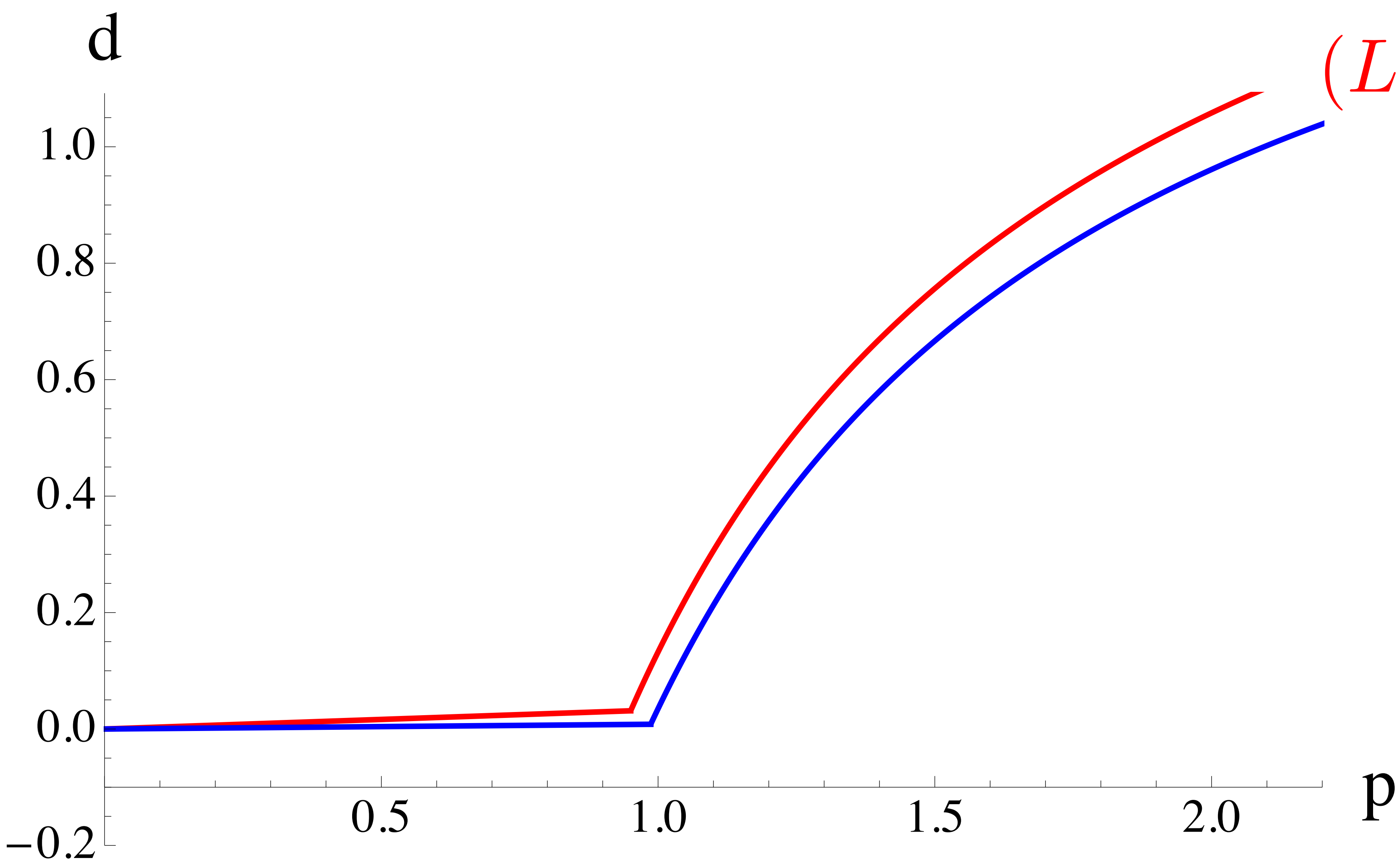


$$p = \frac{PL^2}{4\pi^2 EI}$$



Equilibrium (numerical study)

$$d = \frac{D}{L}$$

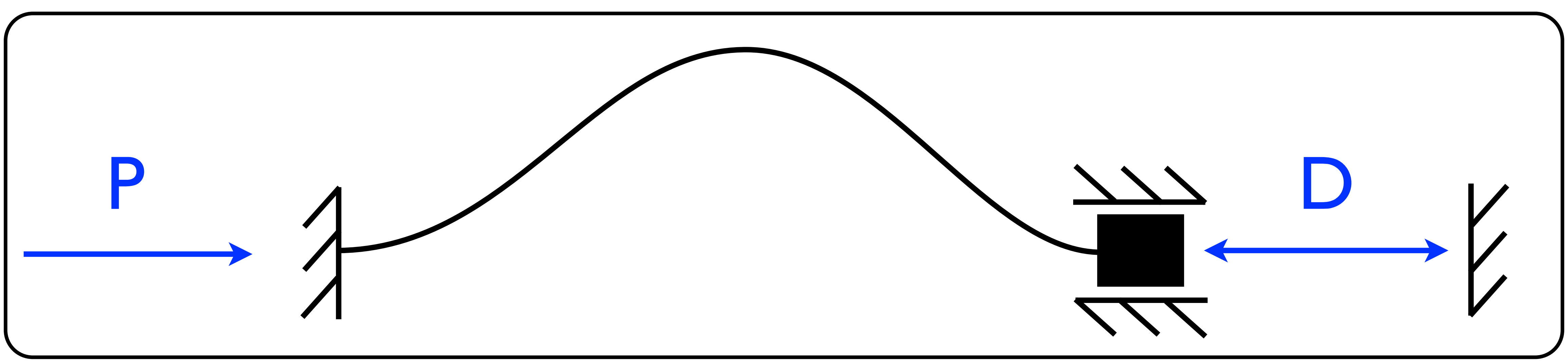


$$\epsilon = \frac{1}{1200}$$

$$(L = 10h)$$

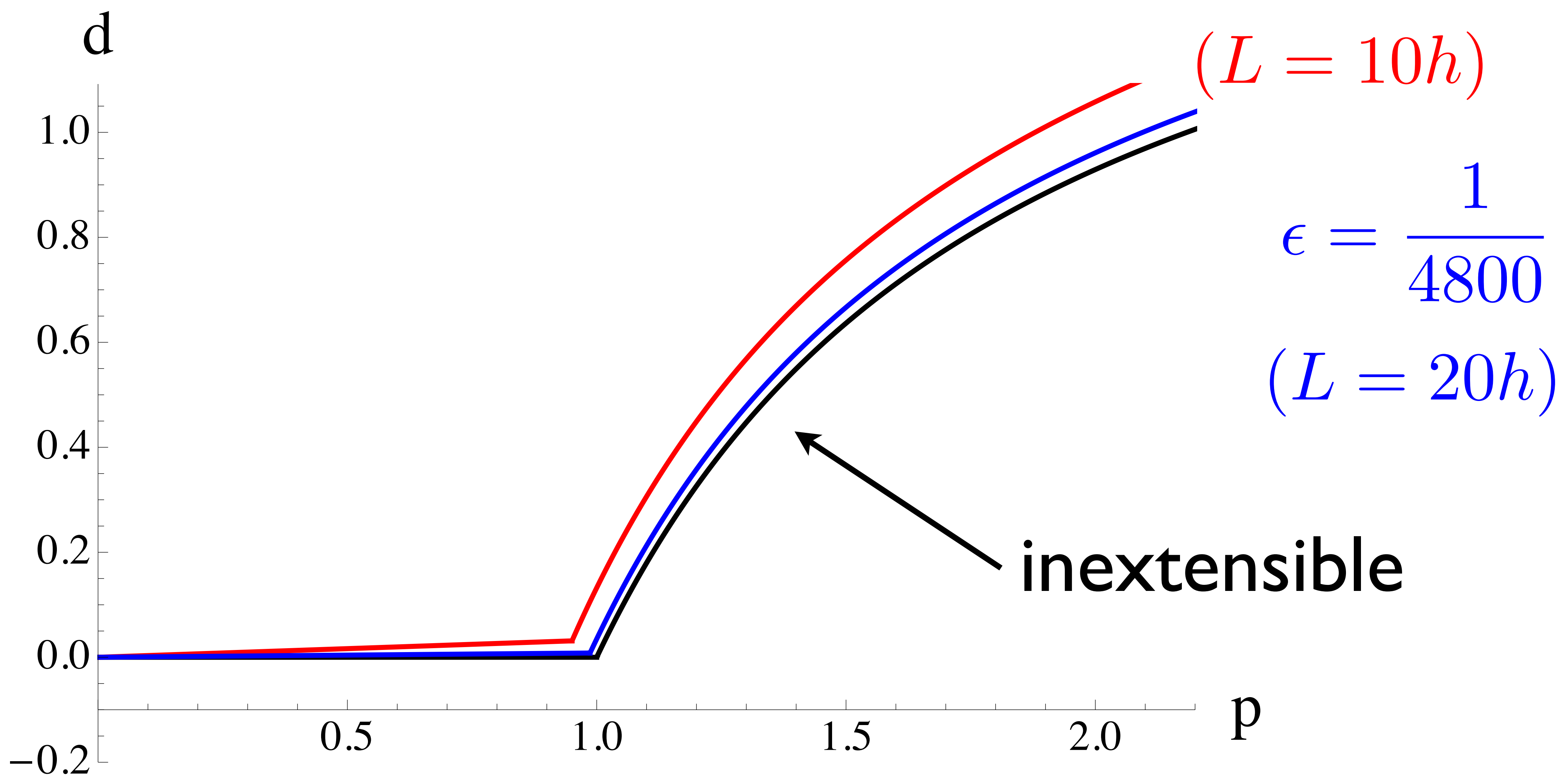
$$\epsilon = \frac{1}{4800}$$

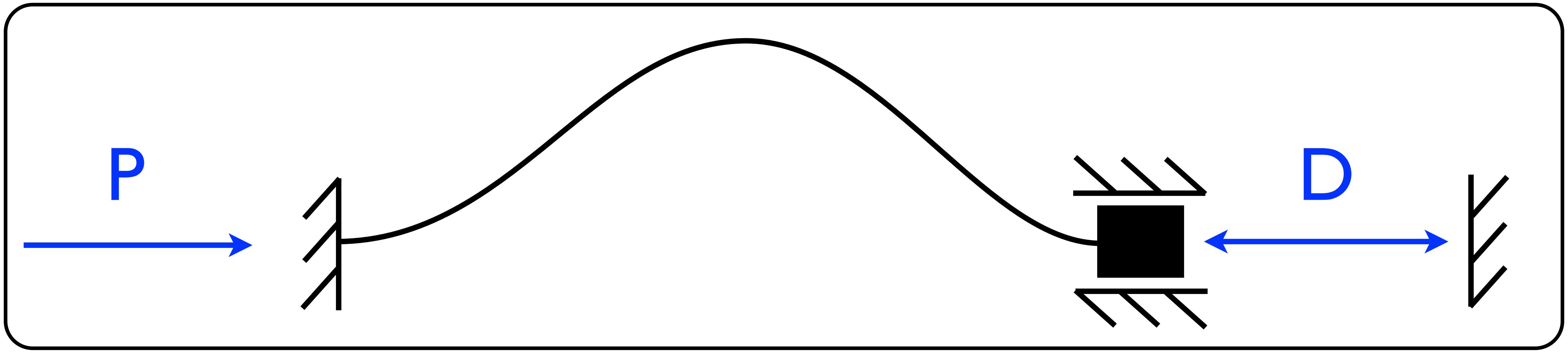
$$(L = 20h)$$



Equilibrium (numerical study)

$$d = \frac{D}{L}$$





Dynamics

with shear, extension, and rotational inertia

$\epsilon > 0$ Timoshenko

$$0 < \epsilon = \frac{I}{AL^2} = \frac{1}{12}\left(\frac{h}{L}\right)^2 \ll 1$$

$$\begin{cases} x' &= \cos\theta + \epsilon\left(f_3\cos\theta - 2(1+\nu)f_1\sin\theta\right) \\ y' &= \sin\theta + \epsilon\left(f_3\sin\theta + 2(1+\nu)f_1\cos\theta\right) \\ \theta' &= m \\ m' &= -f_1 + \epsilon\, f_1 f_3(1-2\nu) + \epsilon\,\ddot{\theta} \\ f_x' &= \ddot{x} \\ f_y' &= \ddot{y} \end{cases}$$

ν : Poisson

with $\begin{cases} f_1 &= -f_x\sin\theta + f_y\cos\theta \\ f_3 &= f_x\cos\theta + f_y\sin\theta \end{cases}$

boundary conditions

$x(0,t) = 0$	$x(1,t) = 1 - d$
$y(0,t) = 0$	$y(1,t) = 0$
$\theta(0,t) = 0$	$\theta(1,t) = 0$

Vibrations

small amplitude vibrations around pre or post-buckled equilibrium

$$x(s,t) = x_e(s) + \delta\, \bar{x}(s)\, e^{i\omega t} \text{ with } |\delta| \ll 1$$

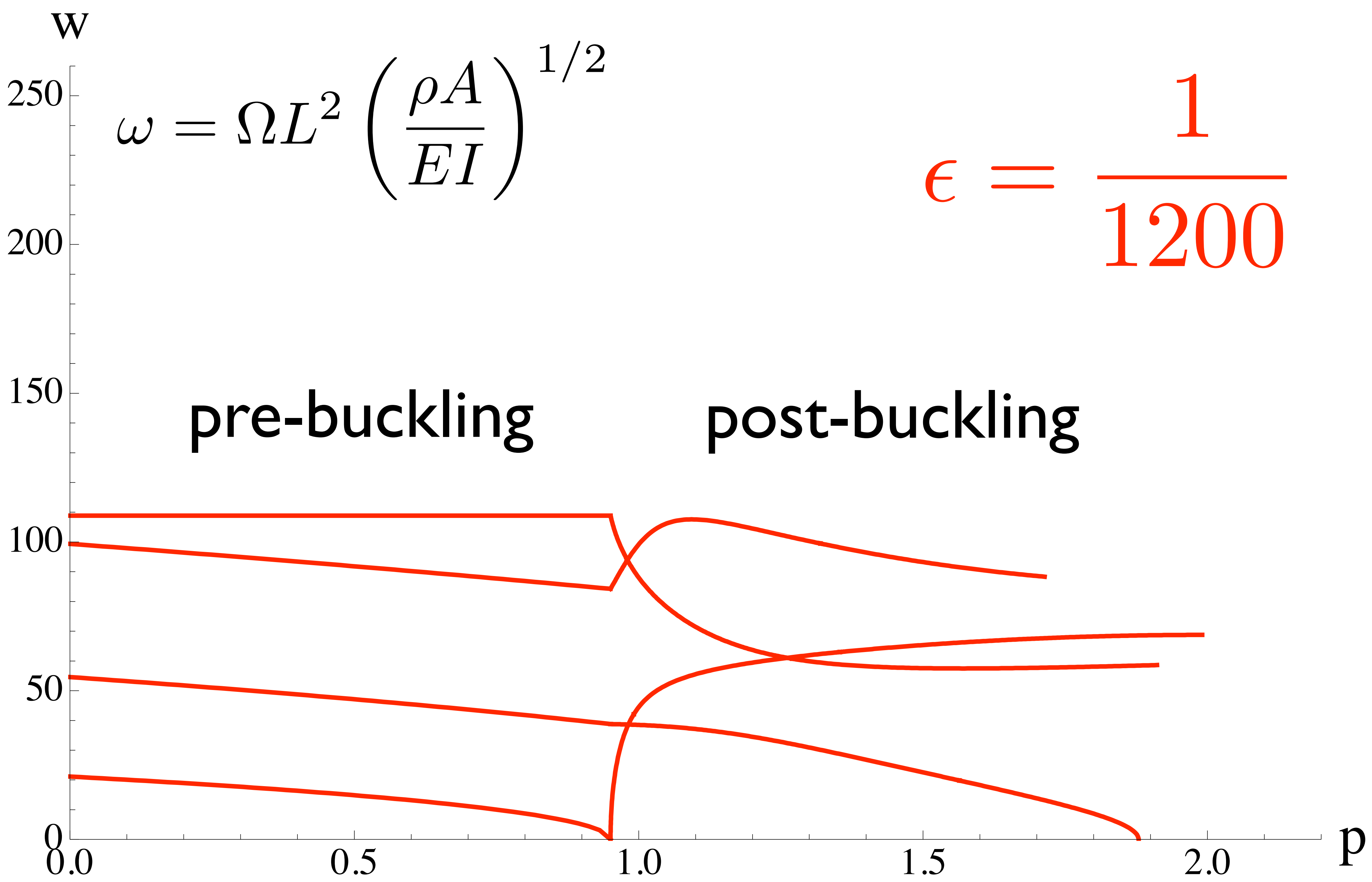
$$\begin{cases} \bar{x}'(s) &=& -\bar{\theta}\, \sin\theta_e + \epsilon \left(\bar{f}_1\, \cos\theta_e - 2(1+\nu)\, \bar{f}_2 \sin\theta_e\right) + \epsilon\, \bar{\theta}\, (-f_{1e}\, \sin\theta_e - 2(1+\nu)\, f_{2e} \cos\theta_e) \\ \bar{y}'(s) &=& +\bar{\theta}\, \cos\theta_e + \epsilon \left(\bar{f}_1\, \sin\theta_e + 2(1+\nu)\, \bar{f}_2 \cos\theta_e\right) + \epsilon\, \bar{\theta}\, (+f_{1e}\, \cos\theta_e - 2(1+\nu)\, f_{2e} \sin\theta_e) \\ \bar{\theta}'(s) &=& \bar{m} \\ \bar{m}'(s) &=& -\bar{f}_2 + \epsilon \left((1+2\nu)(\bar{f}_1\, f_{2e} + f_{1e}\, \bar{f}_2) - \omega^2 \bar{\theta}\right) \\ \bar{f}_x'(s) &=& -\omega^2 \bar{x} \\ \bar{f}_y'(s) &=& -\omega^2 \bar{y} \end{cases}$$

with
$$\bar{f}_1 = +\bar{f}_x \cos\theta_e + \bar{f}_y \sin\theta_e + \bar{\theta}\, (-f_{xe} \sin\theta_e + f_{ye} \cos\theta_e)$$
$$\bar{f}_2 = -\bar{f}_x \sin\theta_e + \bar{f}_y \cos\theta_e + \bar{\theta}\, (-f_{xe} \cos\theta_e - f_{ye} \sin\theta_e)$$

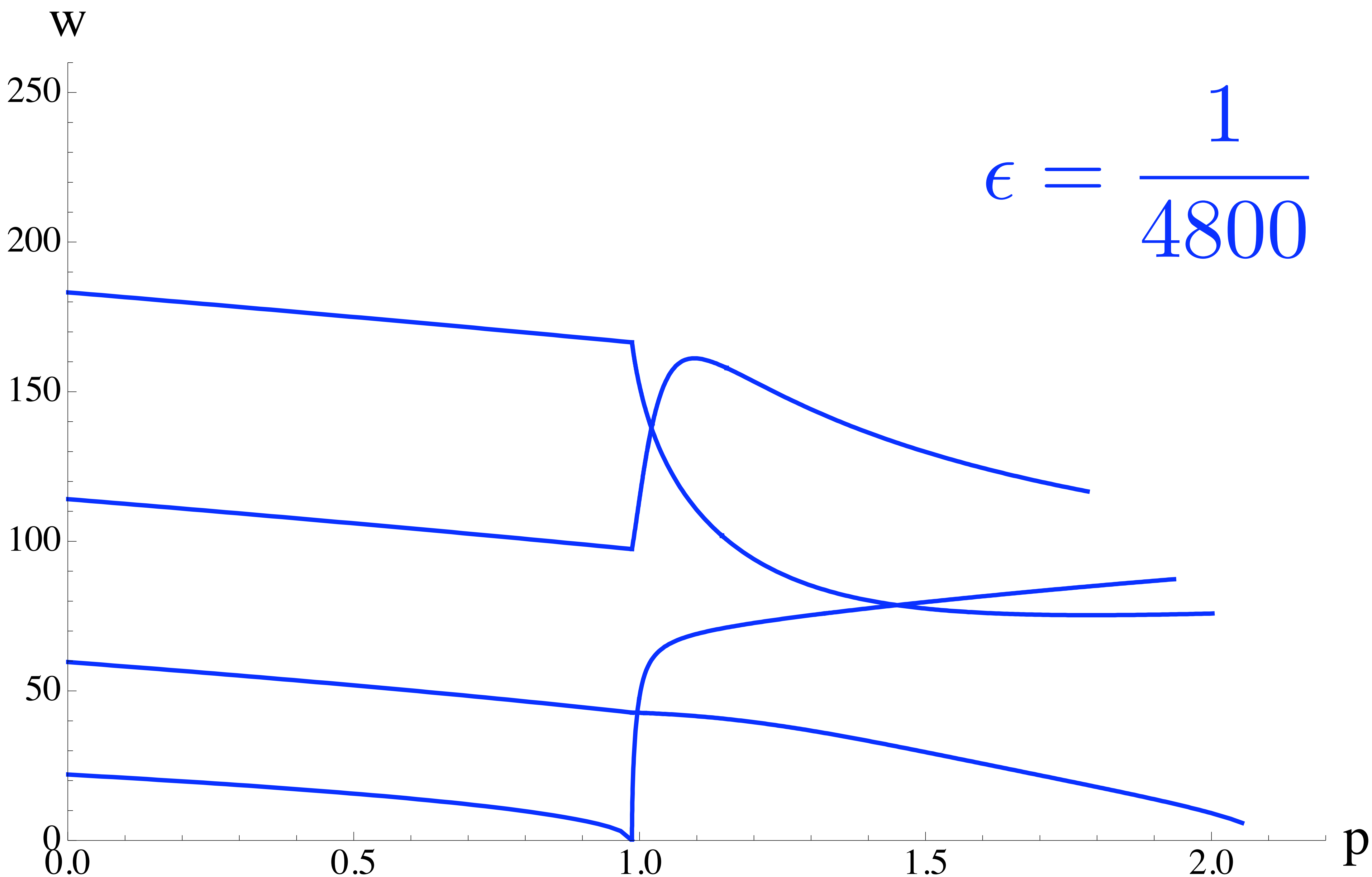
boundary conditions

$$\bar{x}(0) = 0 = \bar{x}(1)$$
$$\bar{y}(0) = 0 = \bar{y}(1)$$
$$\bar{\theta}(0) = 0 = \bar{\theta}(1)$$

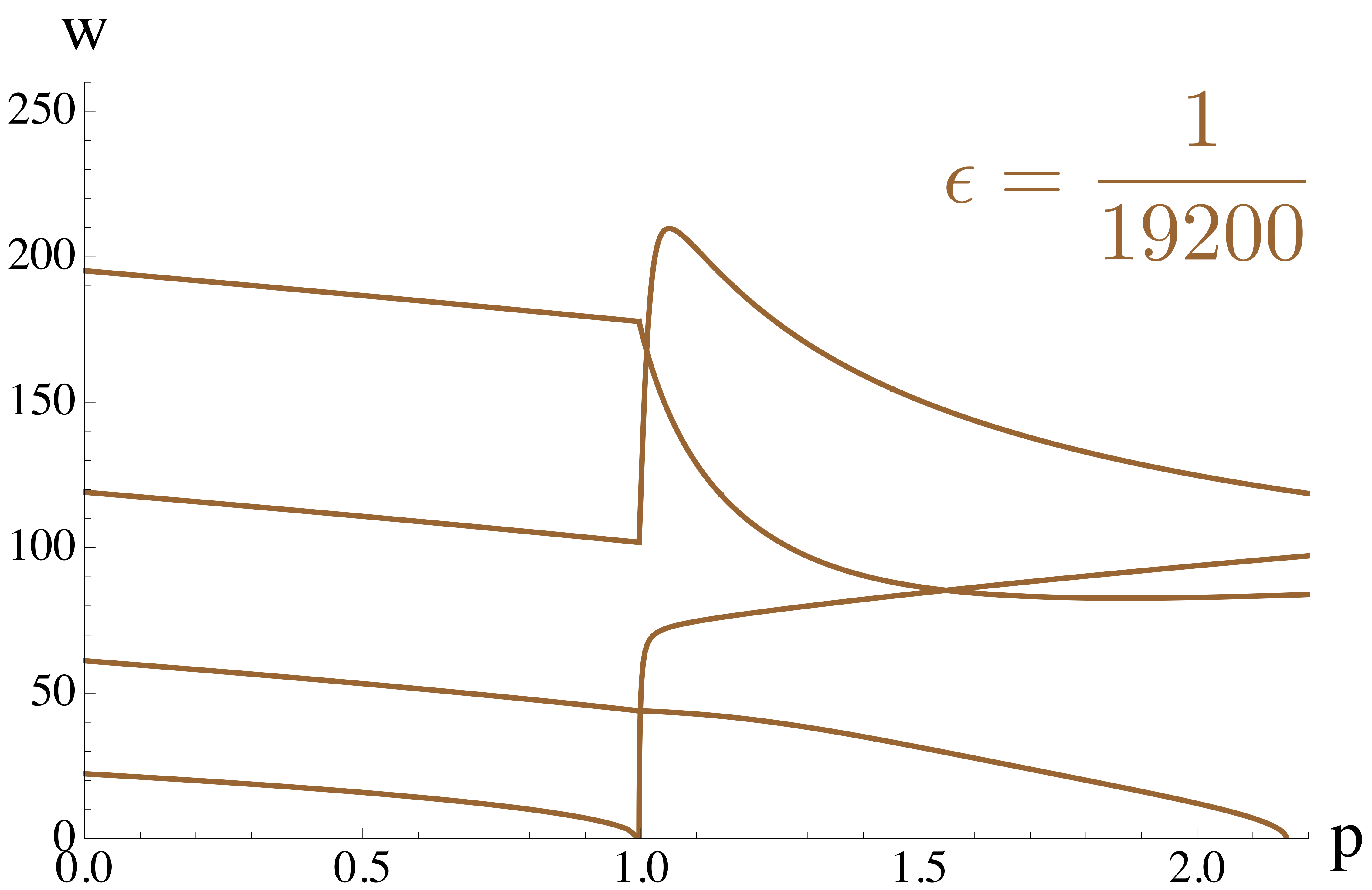
Vibrations (extensible case)



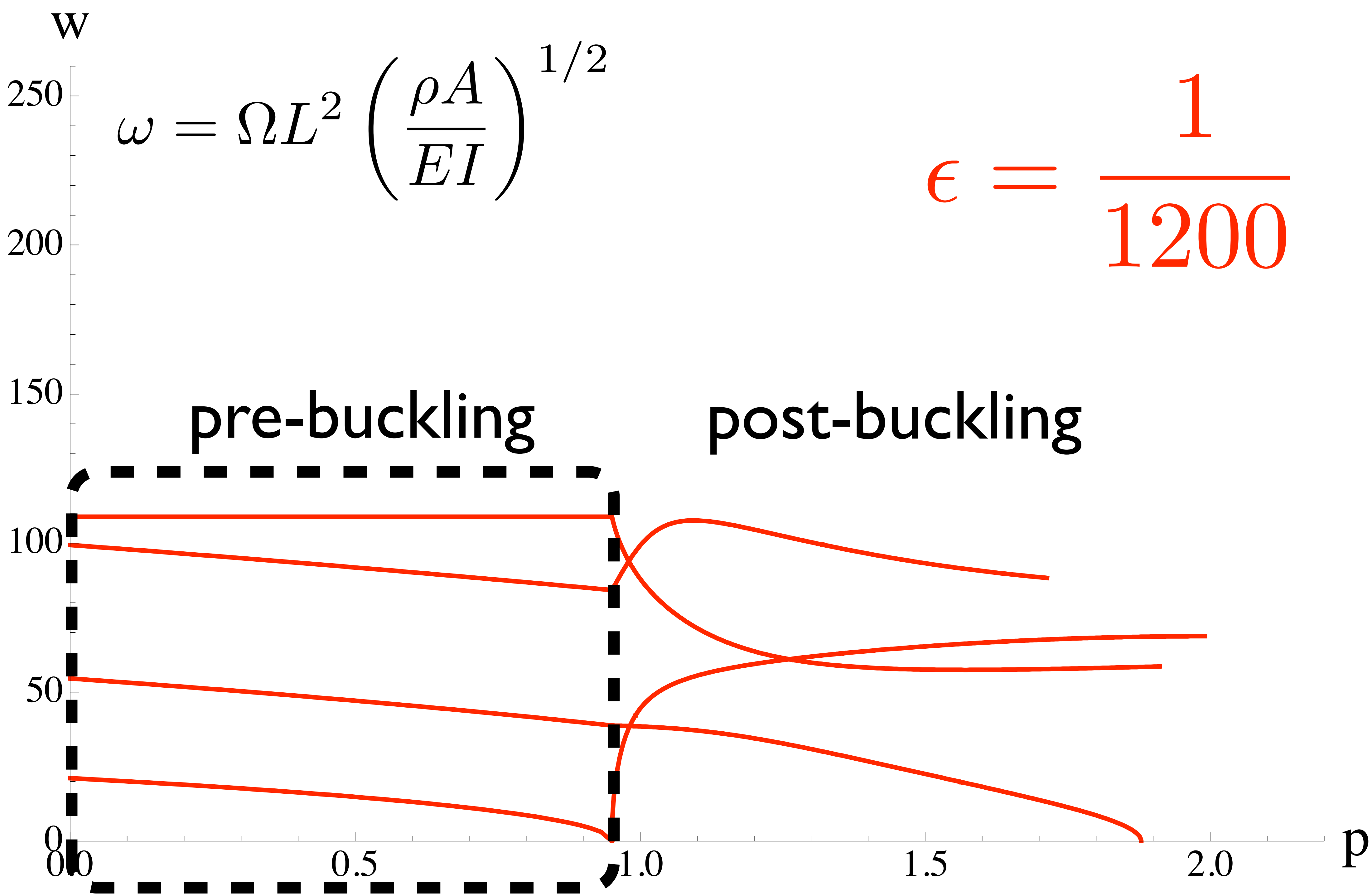
Vibrations (extensible case)



Vibrations (extensible case)

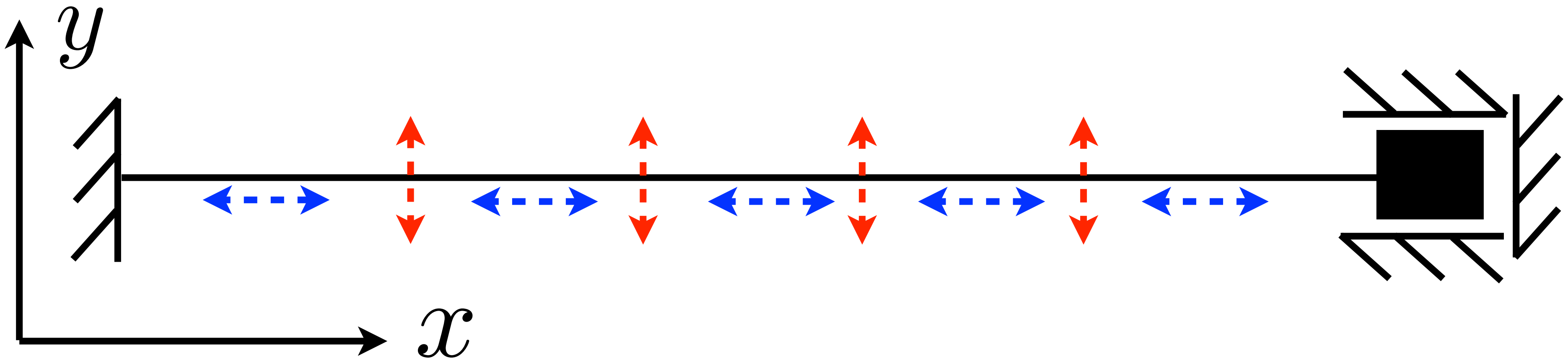


Vibrations (extensible case)



Vibrations (extensible case)

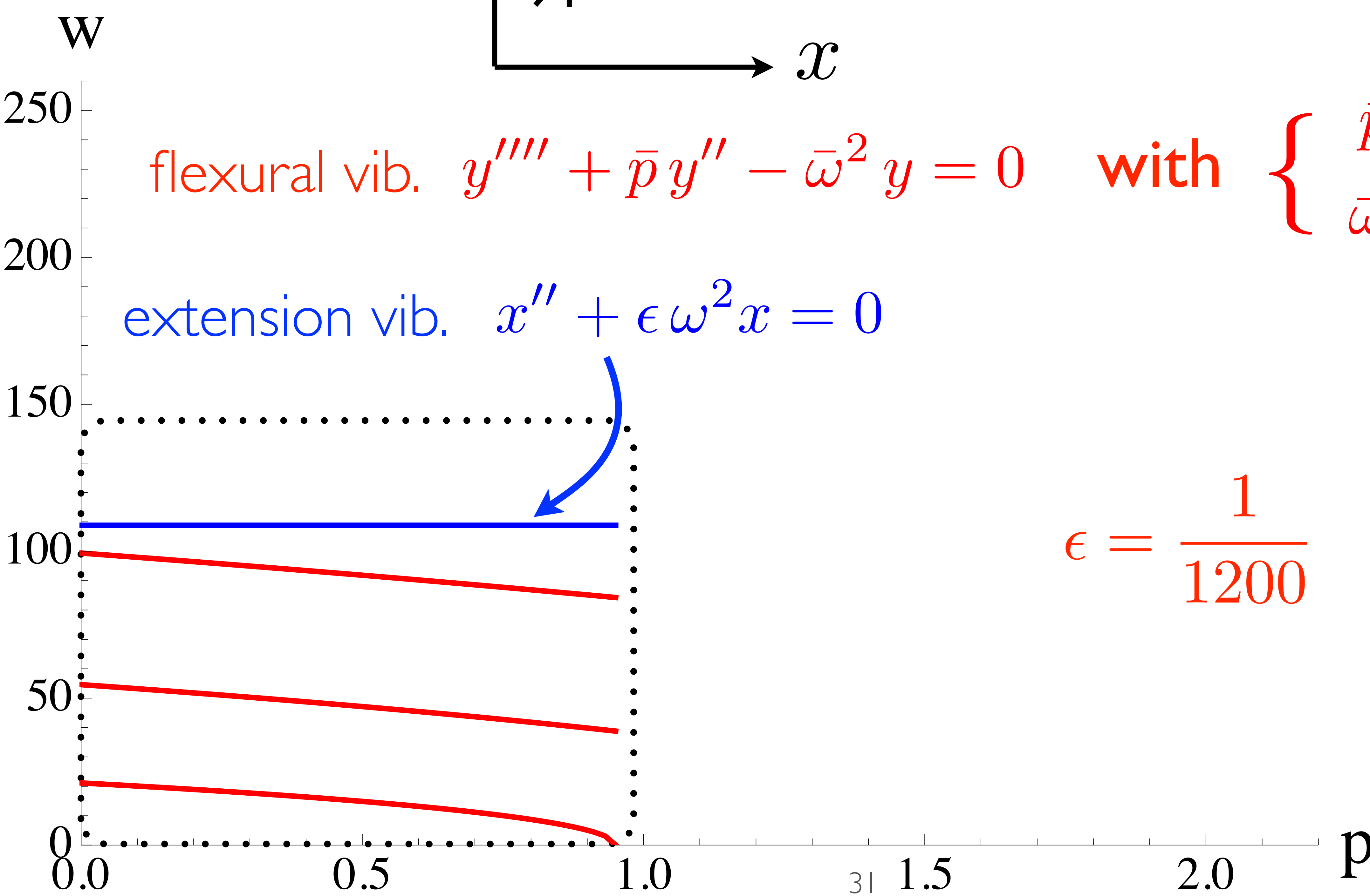
pre-buckling



flexural vib. $y'''' + \bar{p}\, y'' - \bar{\omega}^2\, y = 0$ **with** $\begin{cases} \bar{p} = p(1 - \epsilon p) \\ \bar{\omega} = \omega(1 - \epsilon p) \end{cases}$

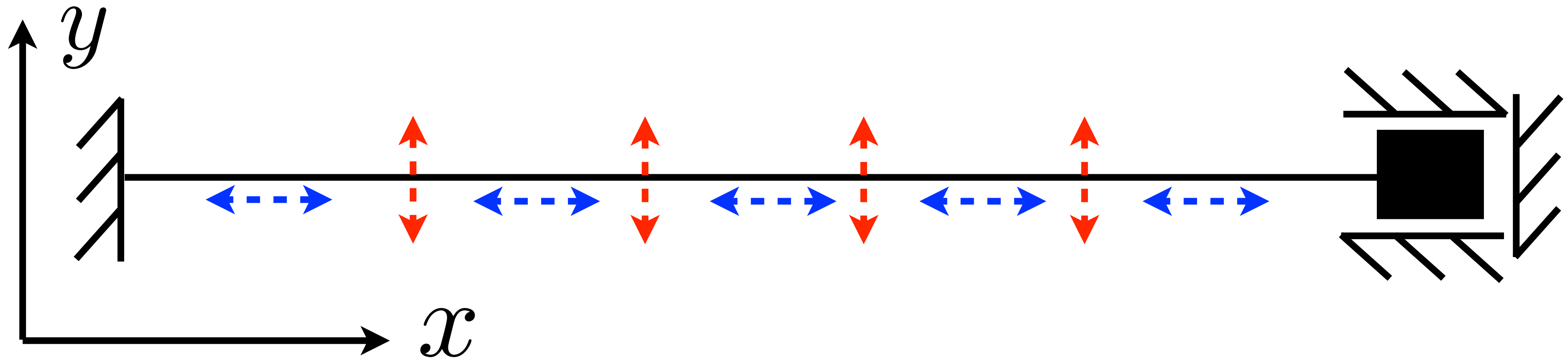
extension vib. $x'' + \epsilon\, \omega^2 x = 0$

$\epsilon = \dfrac{1}{1200}$



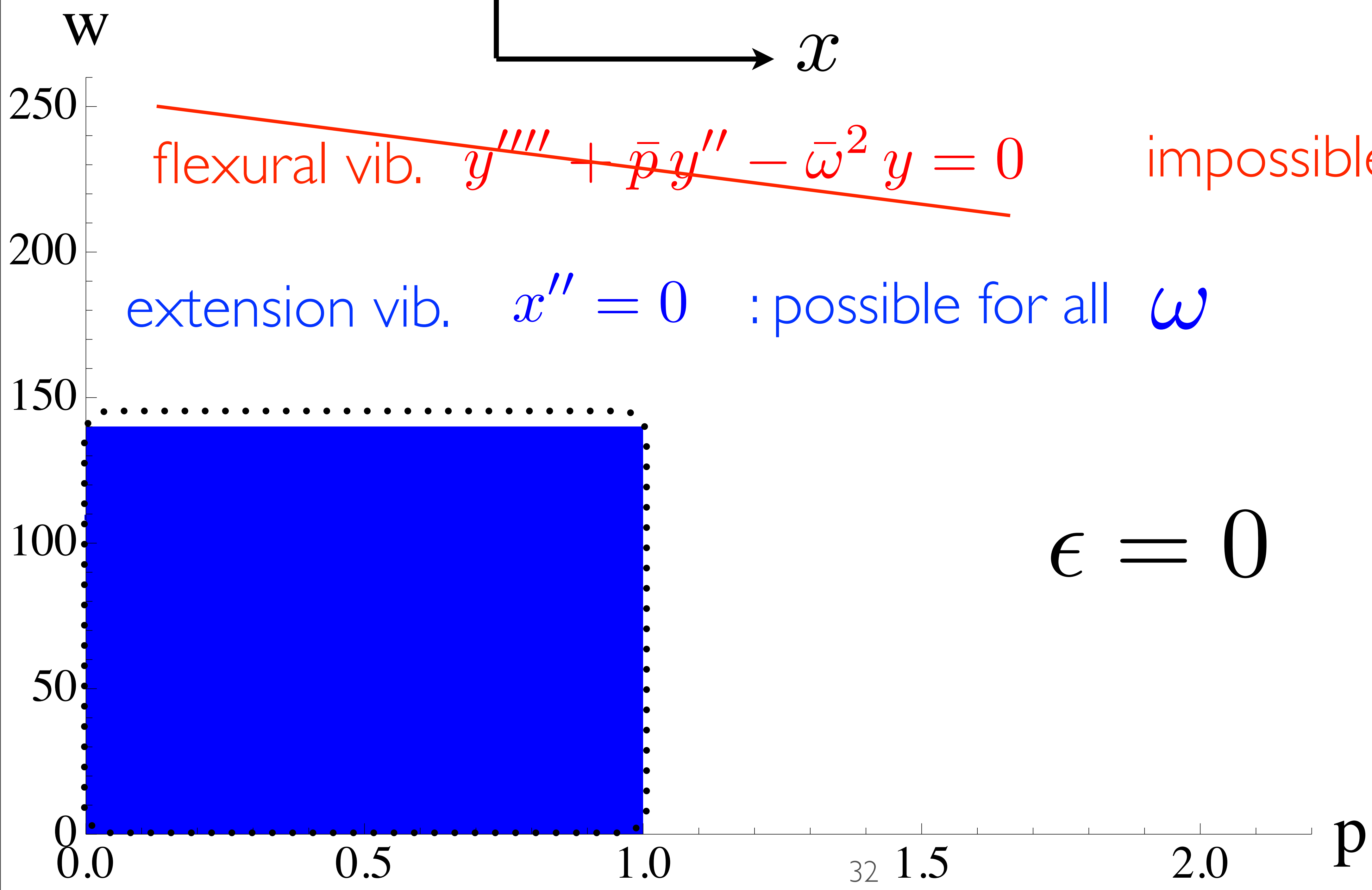
Vibrations (inextensible)

pre-buckling



flexural vib. $y'''' + \bar{p}\,y'' - \bar{\omega}^2\,y = 0$ impossible

extension vib. $x'' = 0$: possible for all ω

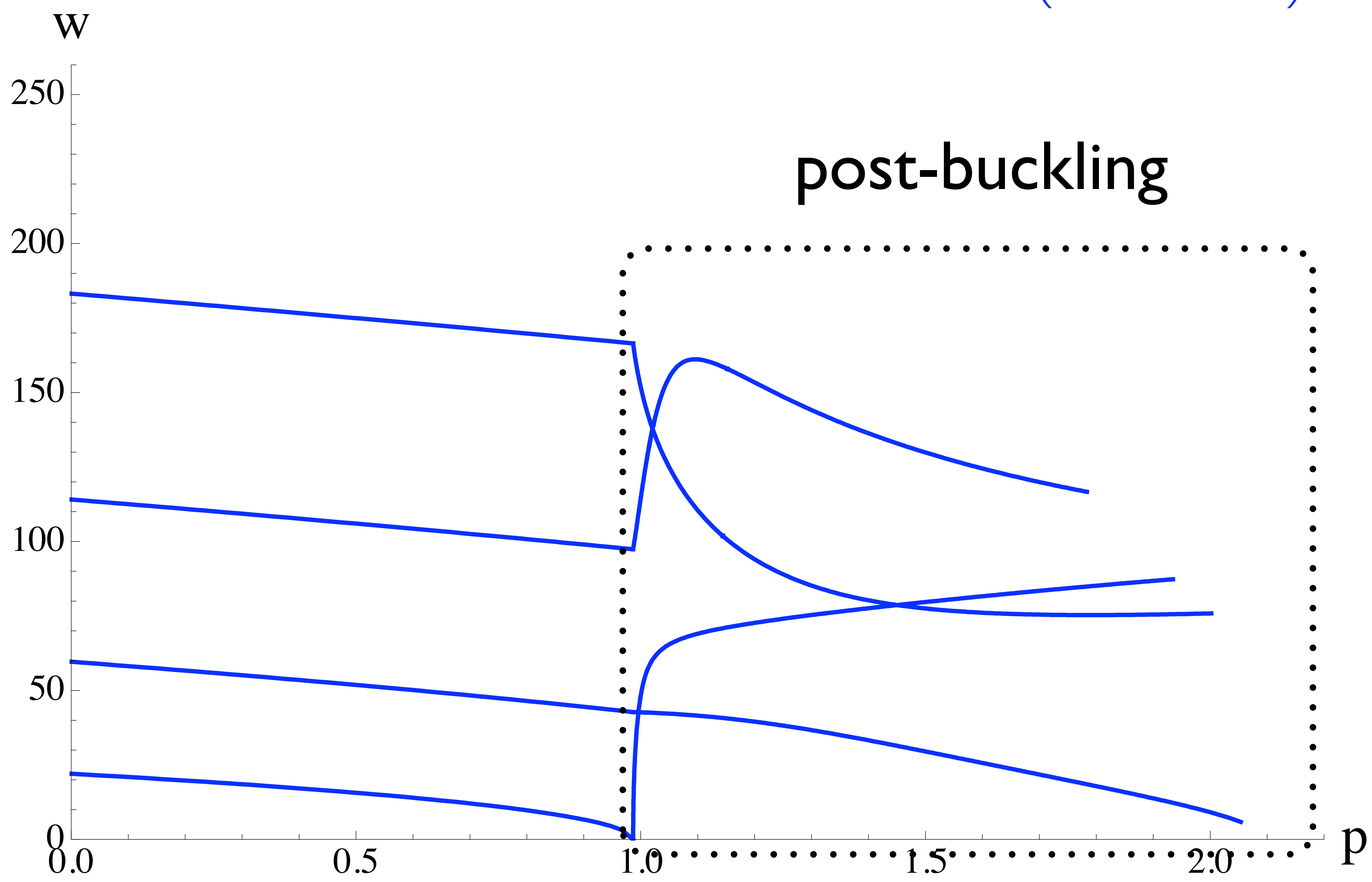


$\epsilon = 0$

Vibrations

$$\epsilon = \frac{1}{4800}$$

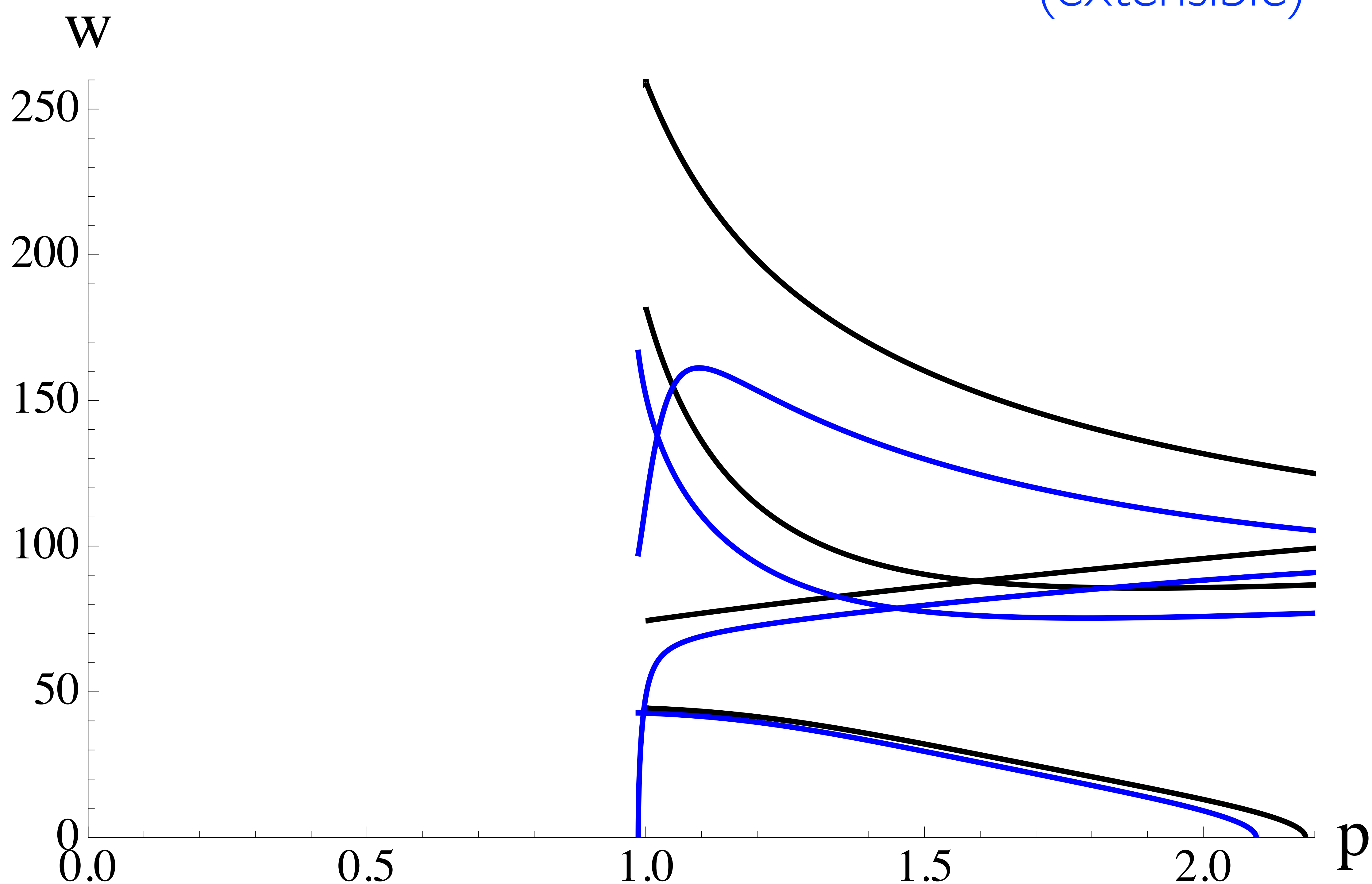
(extensible)



Vibrations

$\epsilon = 0$
(inextensible)

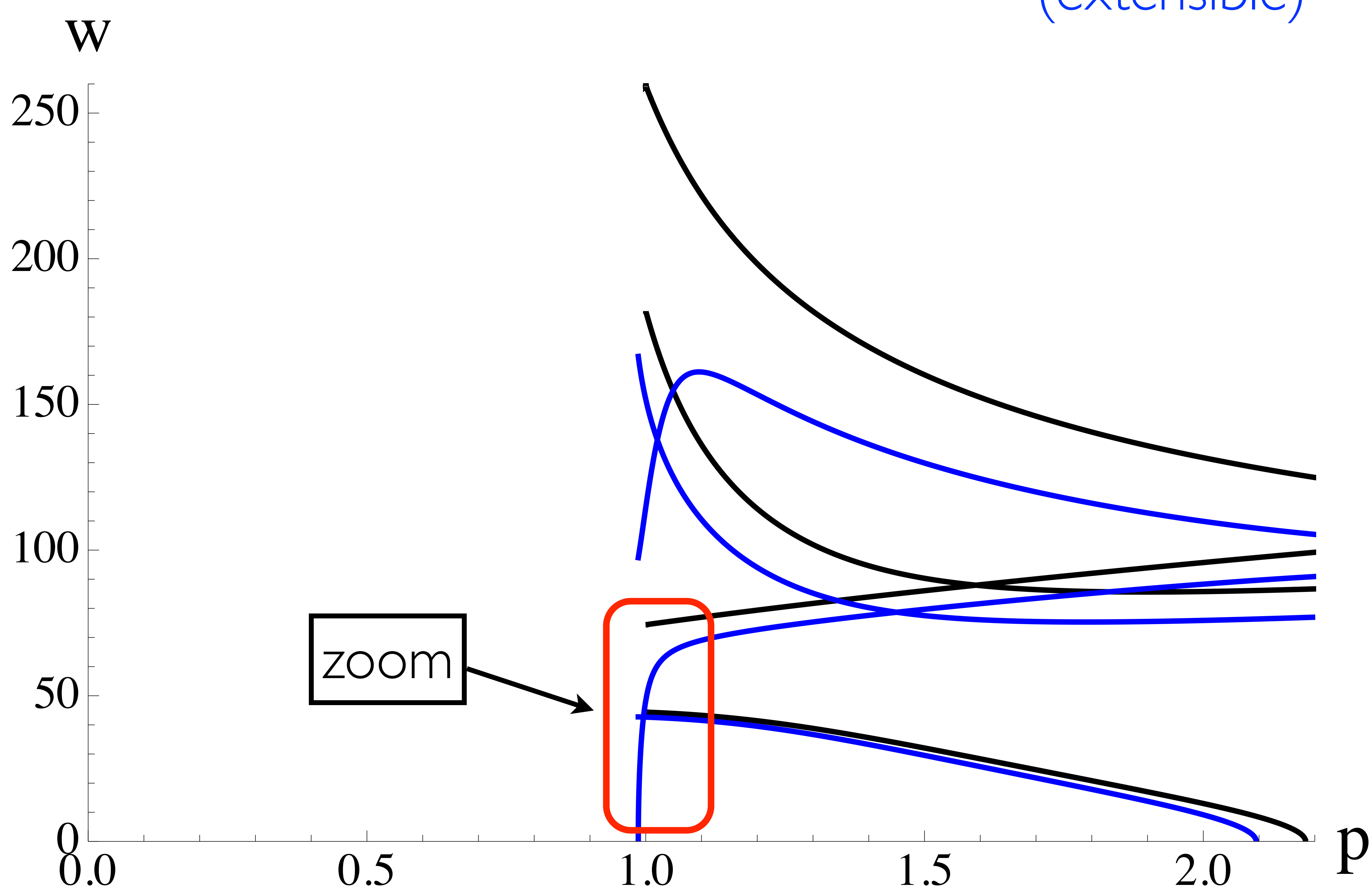
$\epsilon = \frac{1}{4800}$
(extensible)



Vibrations

$\epsilon = 0$
(inextensible)

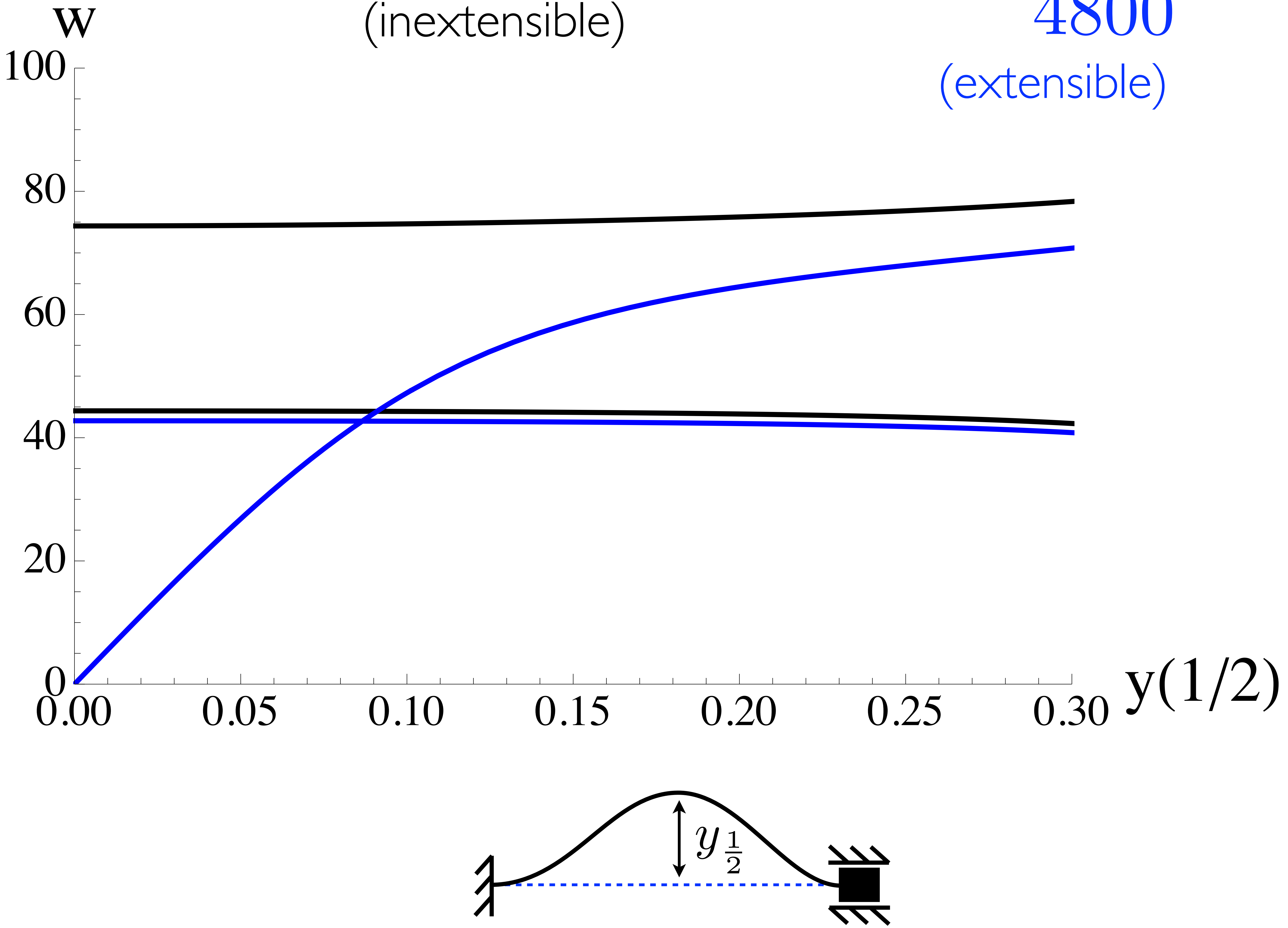
$\epsilon = \frac{1}{4800}$
(extensible)



Vibrations

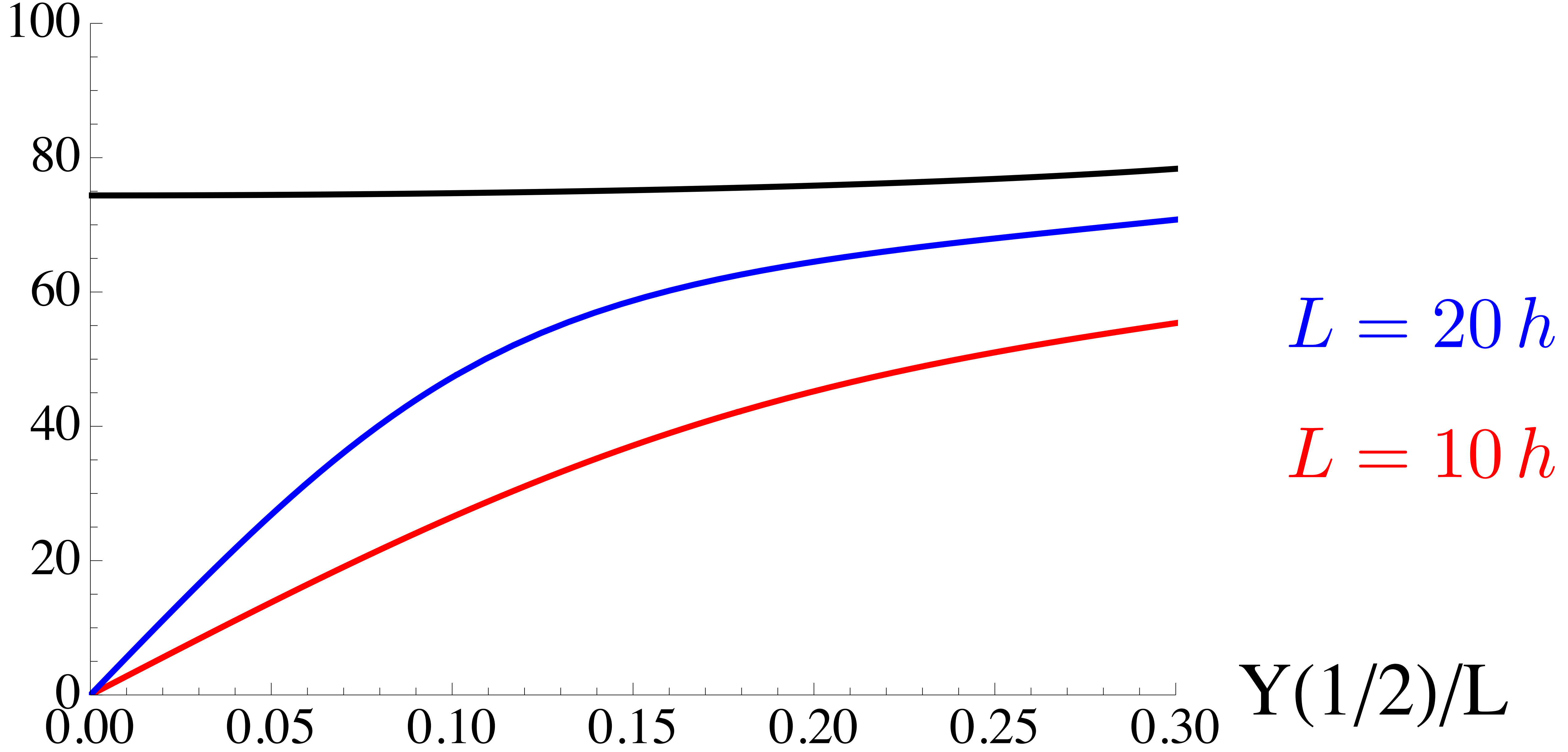
$\epsilon = 0$ (inextensible)

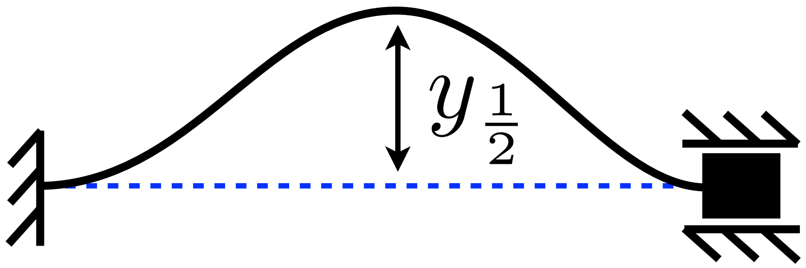
$\epsilon = \frac{1}{4800}$ (extensible)



Vibrations

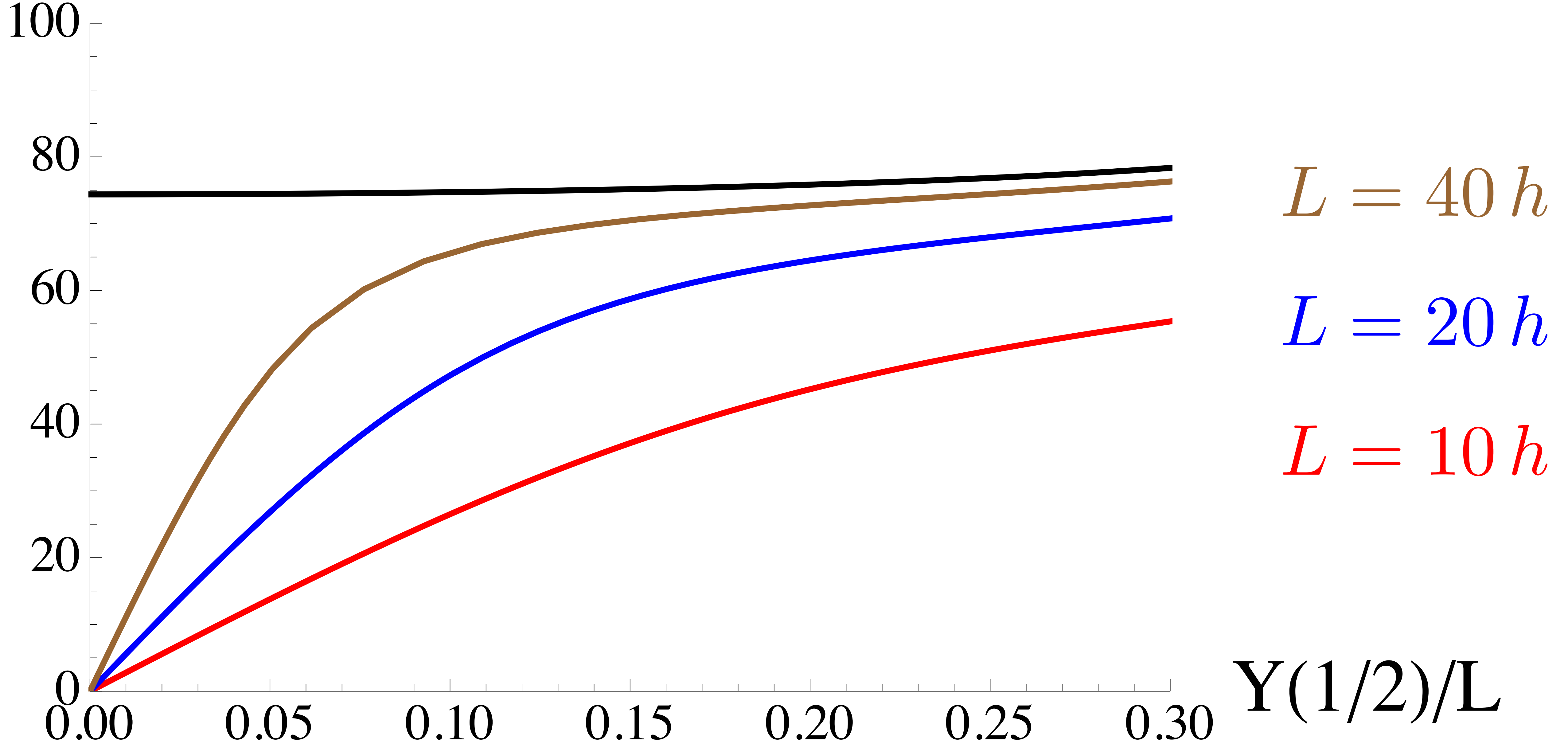
$$\omega = \Omega L^2 \left(\frac{\rho A}{EI}\right)^{1/2}$$

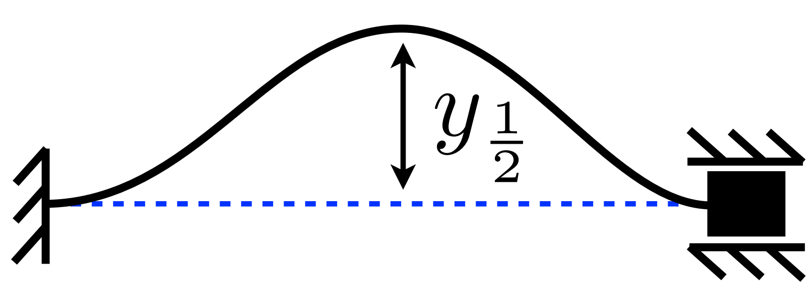




Vibrations

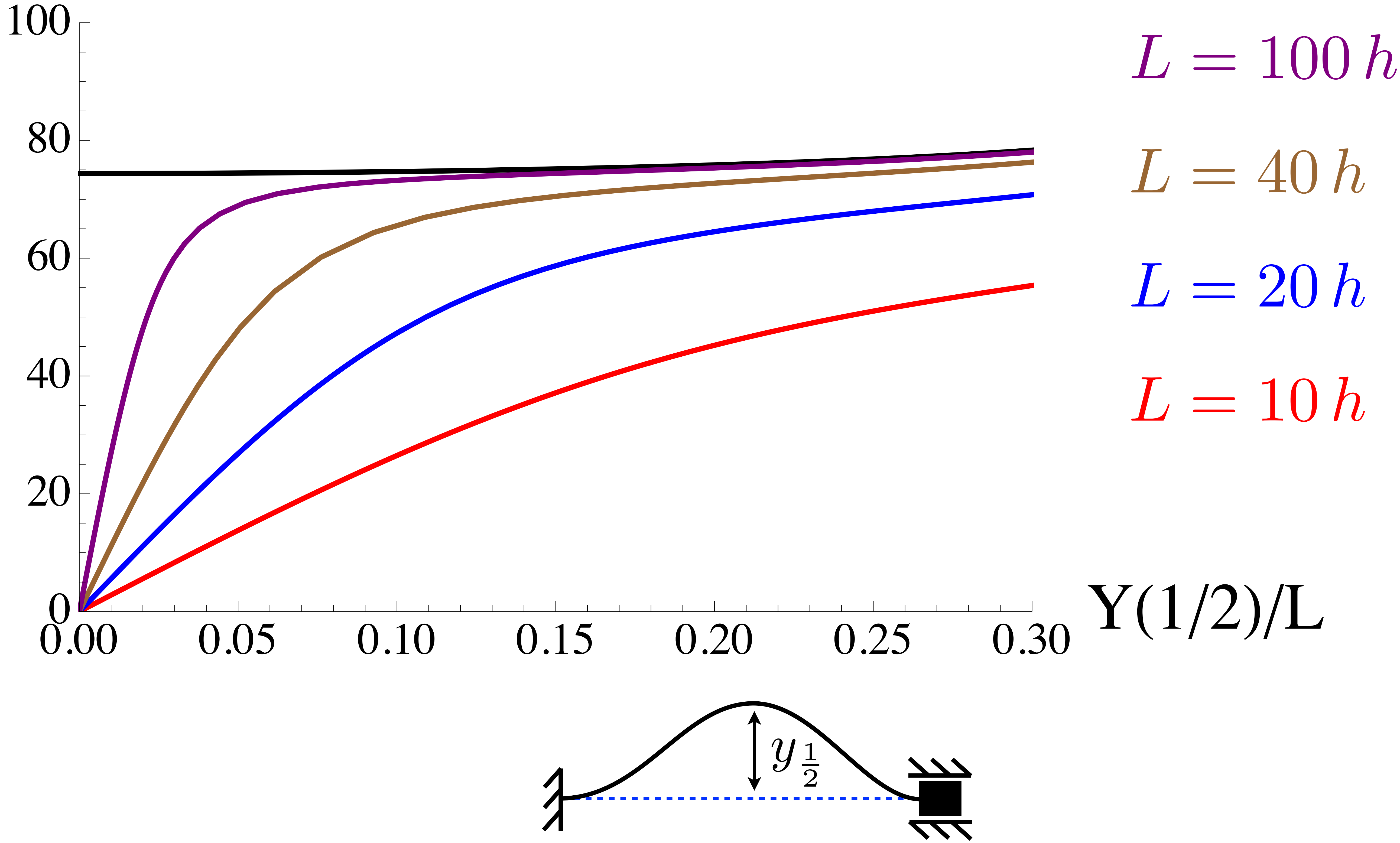
$$\omega = \Omega L^2 \left(\frac{\rho A}{EI}\right)^{1/2}$$



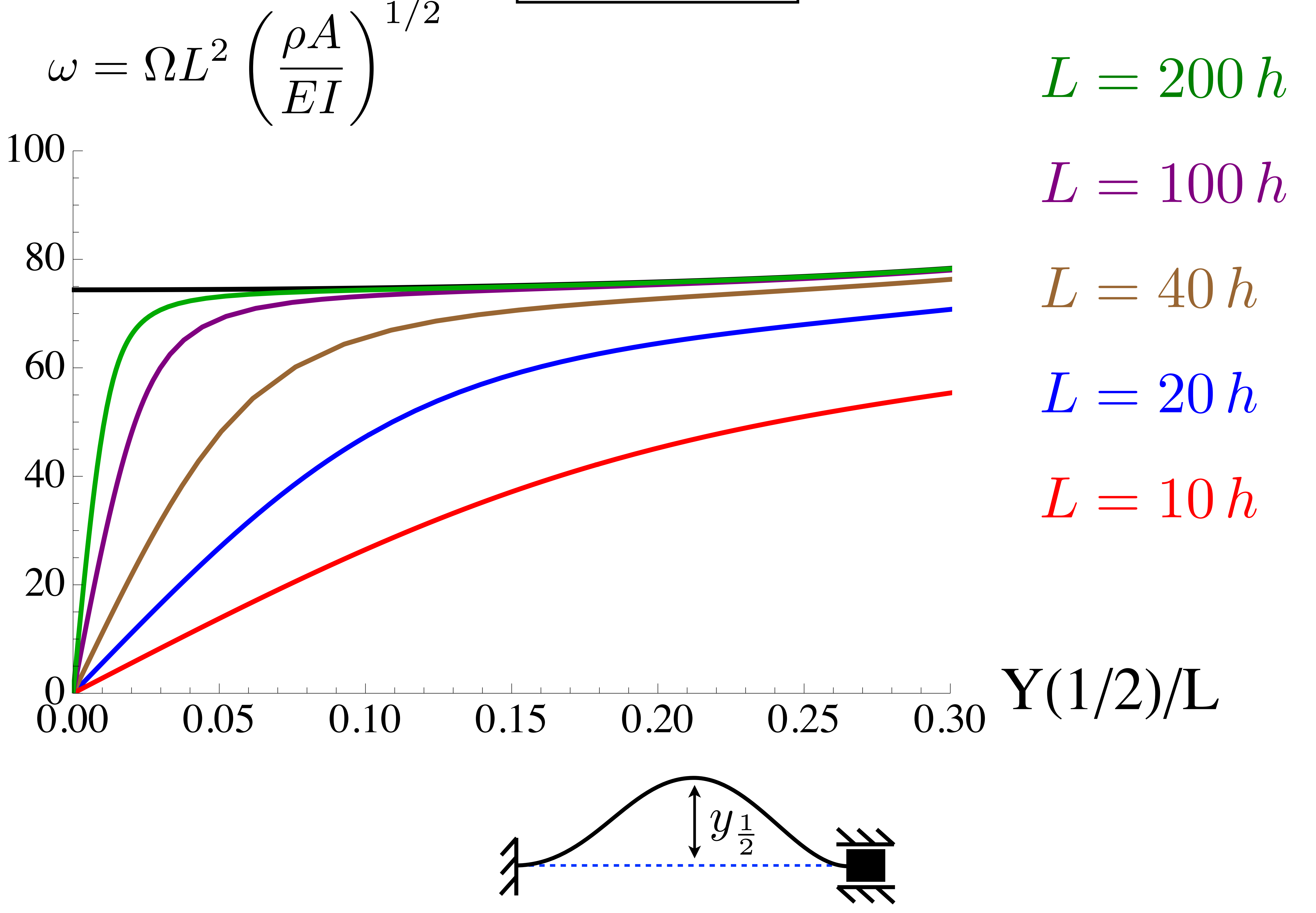


Vibrations

$$\omega = \Omega L^2 \left(\frac{\rho A}{EI}\right)^{1/2}$$

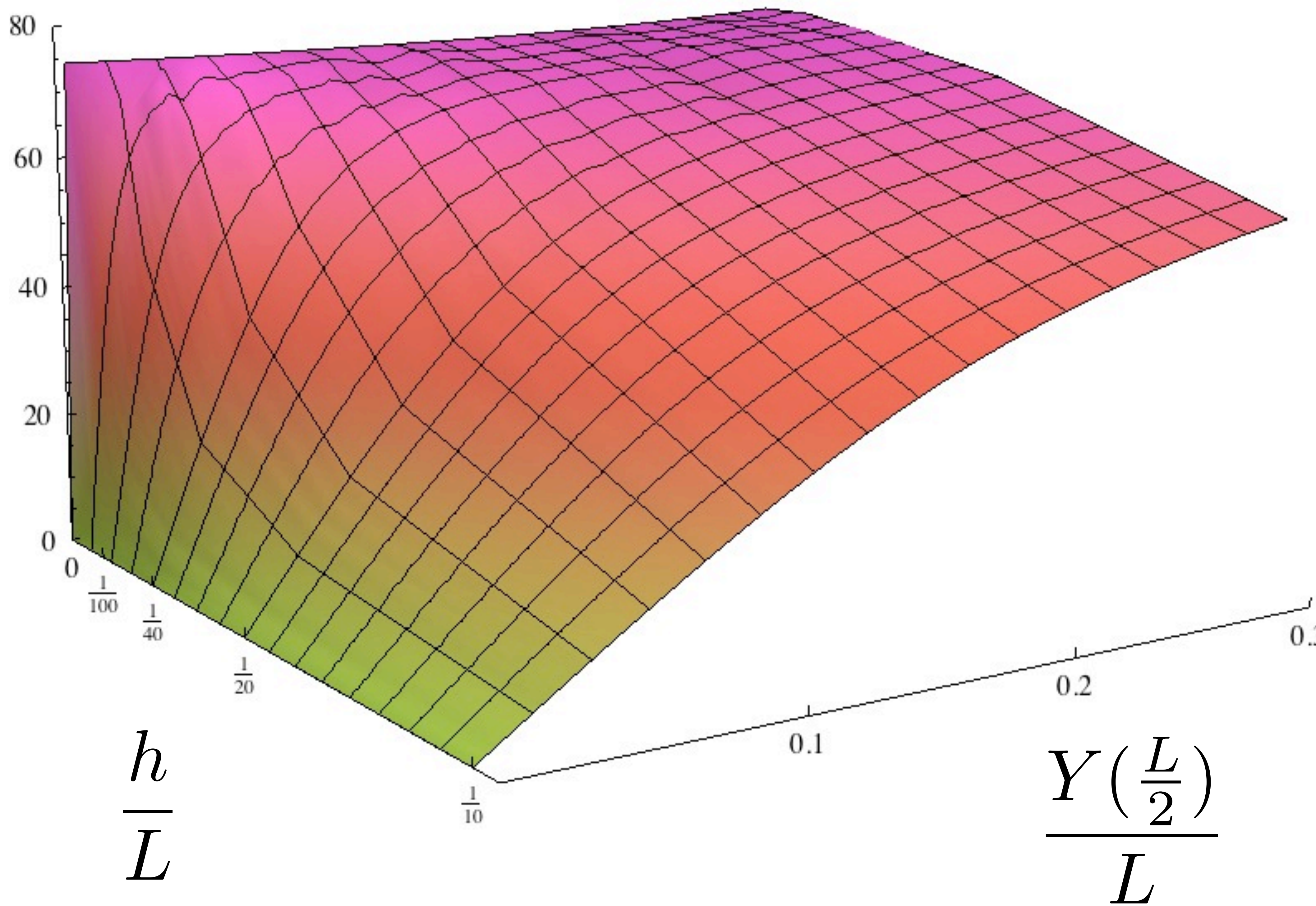


Vibrations

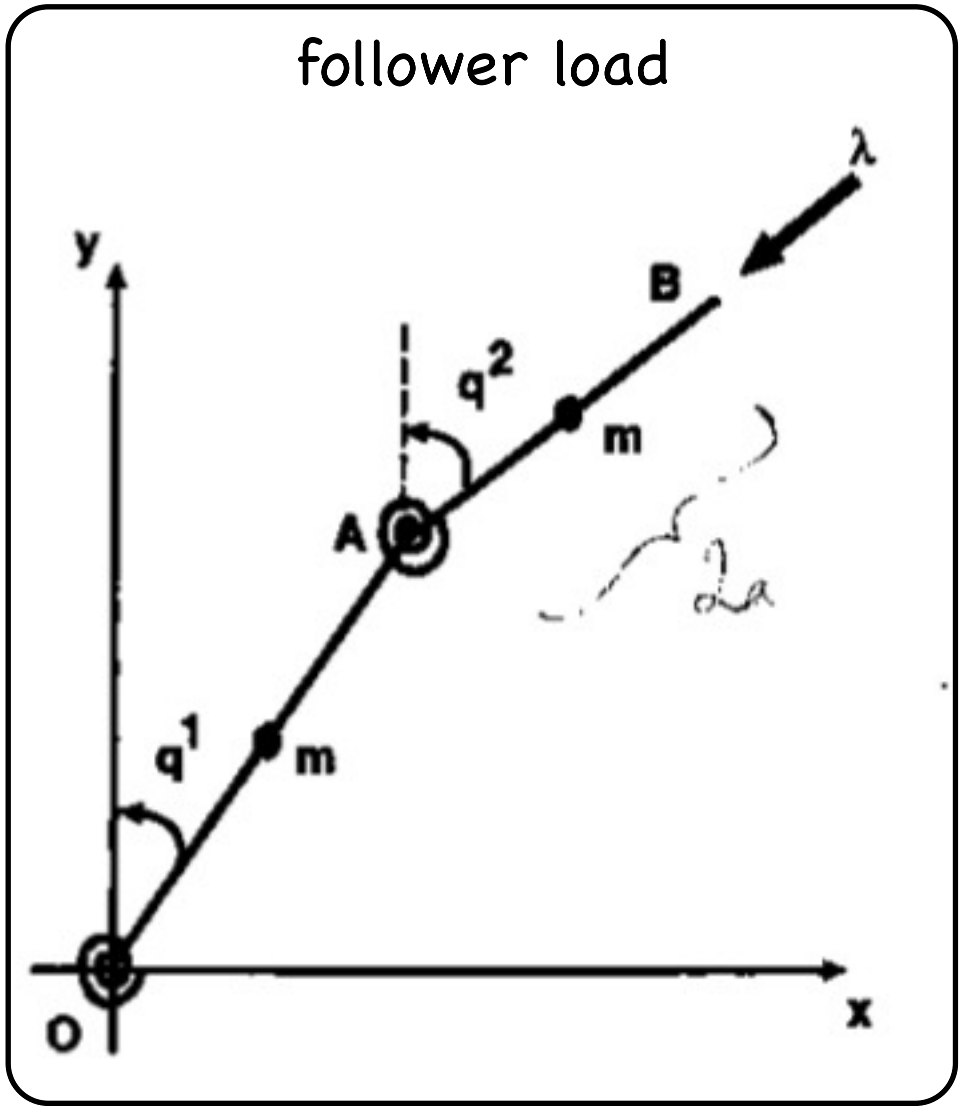

$\omega = \Omega L^2 \left(\frac{\rho A}{EI}\right)^{1/2}$
$L = 200\,h$
$L = 100\,h$
$L = 40\,h$
$L = 20\,h$
$L = 10\,h$
100
80
60
40
20
0
0.00
0.05
0.10
0.15
0.20
0.25
0.30
Y(1/2)/L
$y_{\frac{1}{2}}$

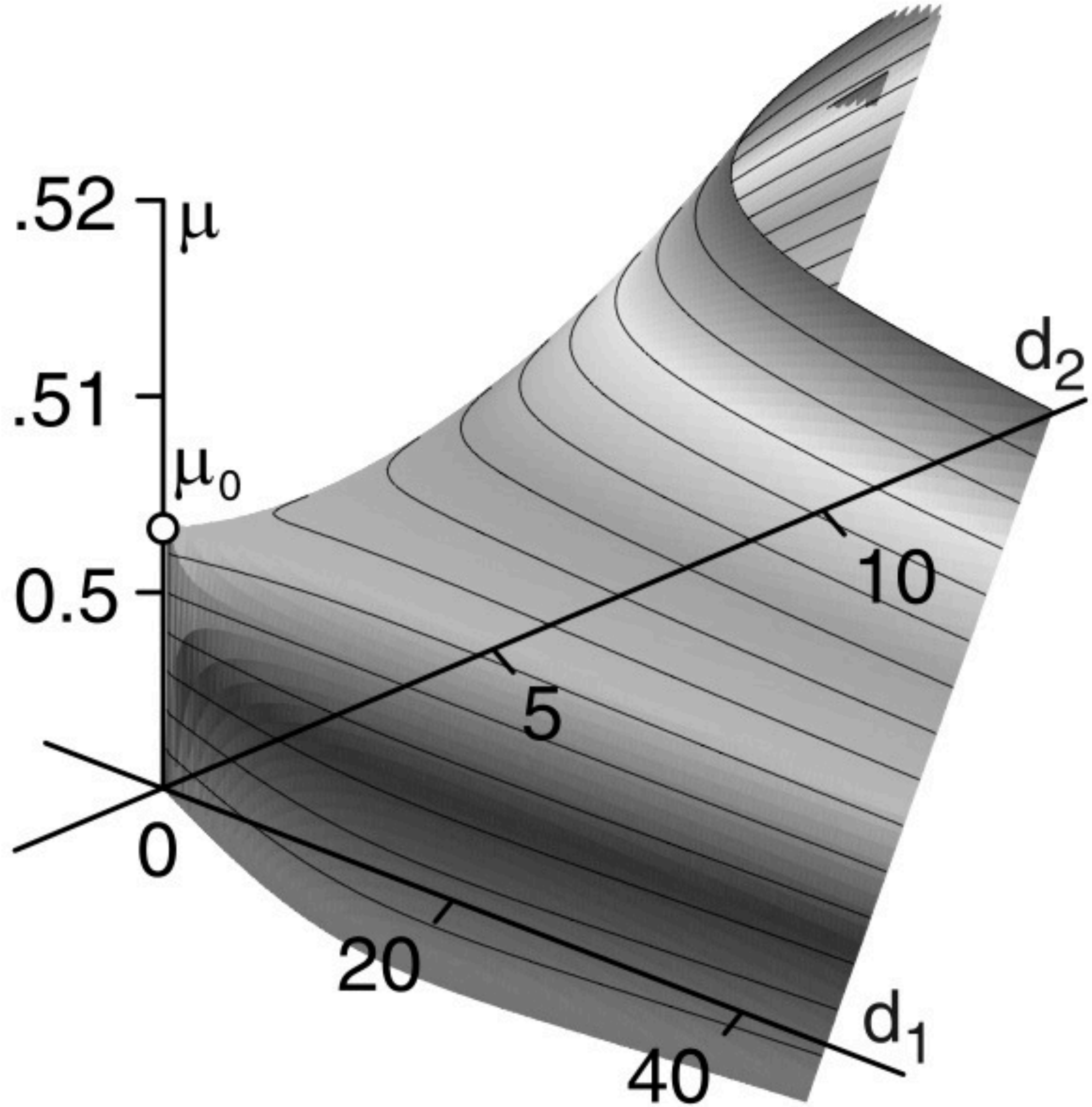
Vibrations

$$\omega = \Omega L^2 \left(\frac{\rho A}{EI}\right)^{1/2}$$



Ziegler Paradox

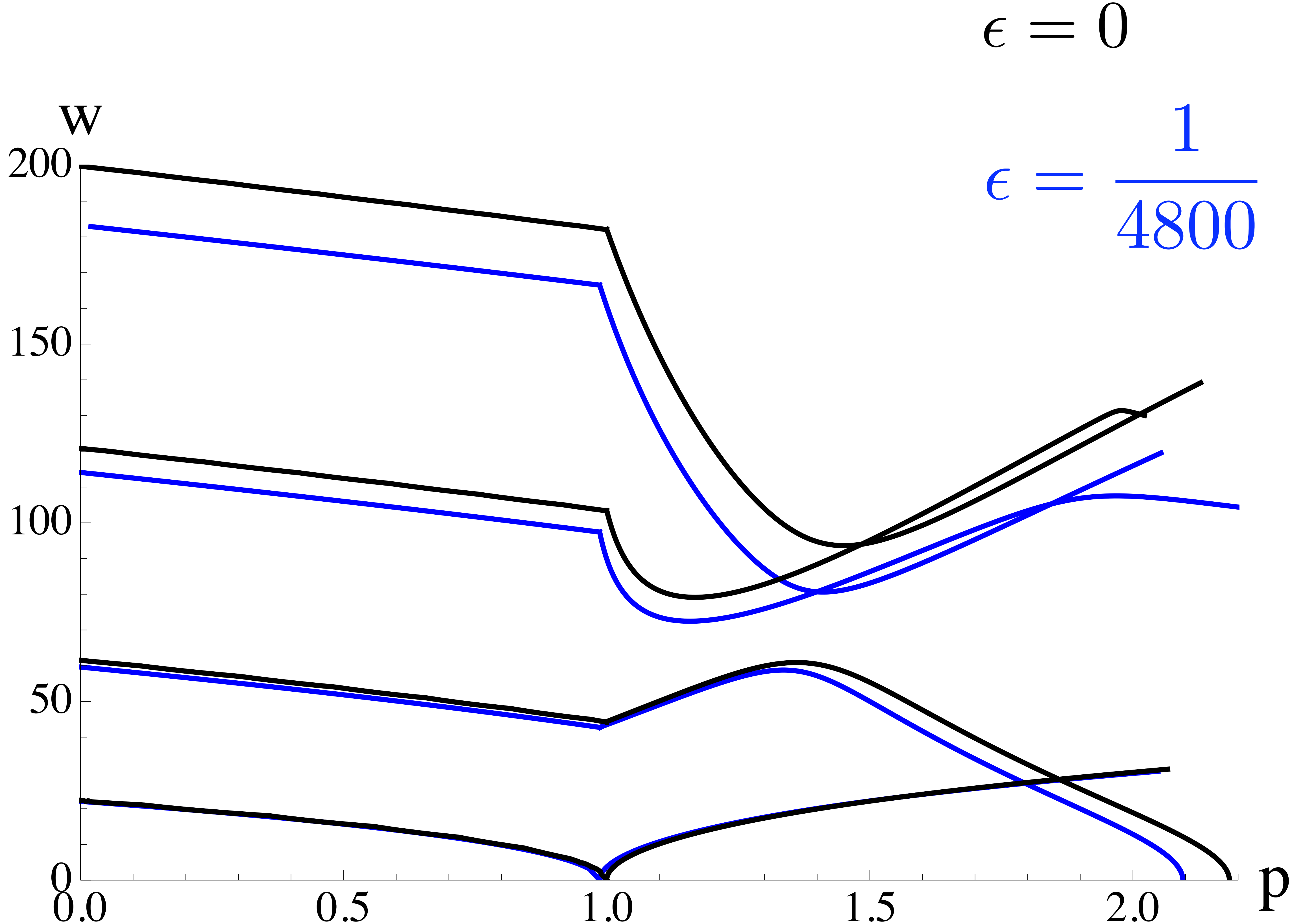




O. Bottema. Indagationes Math. 1956

O. Kirillov & F. Verhulst. ZAMM 2010

Vibrations : dead load



Thank you