Elasto-capillary coupling: piercing a liquid surface with an elastic rod

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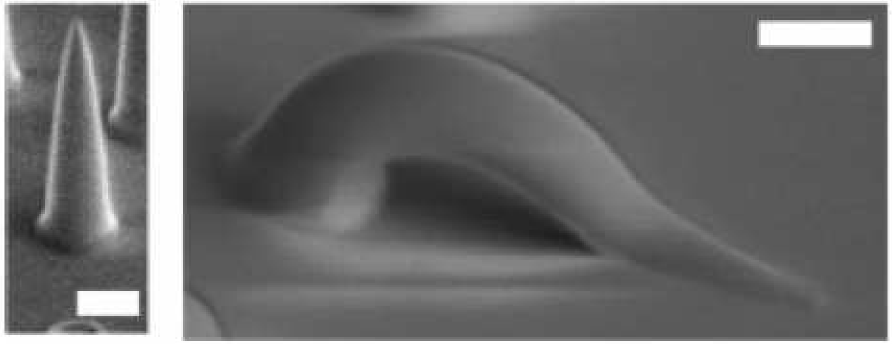
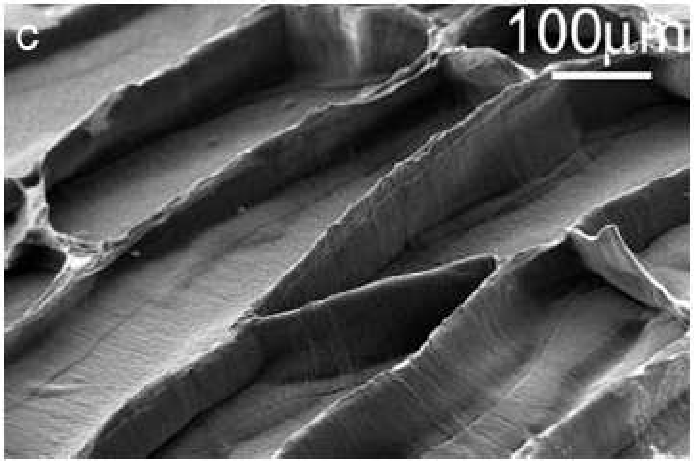
joint work with: José Bico & Benoît Roman (PMMH : CNRS & ESPCI)

(Some) Related works



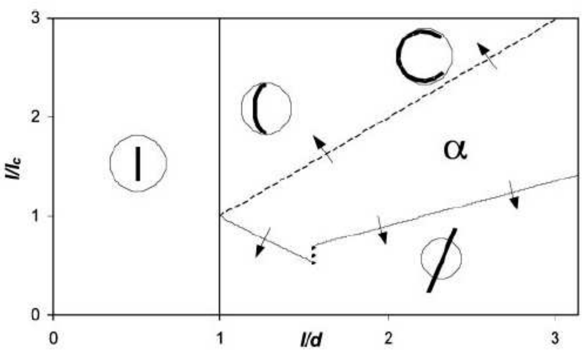
PDMS posts in water and ethanol
(SEM images)

P. Roca-Cusachs & al., *Langmuir* (2005)



Carbon nanotubes collapse into bundles
after liquid evaporation

N. Chakrapani & al., *Nature* (2004)

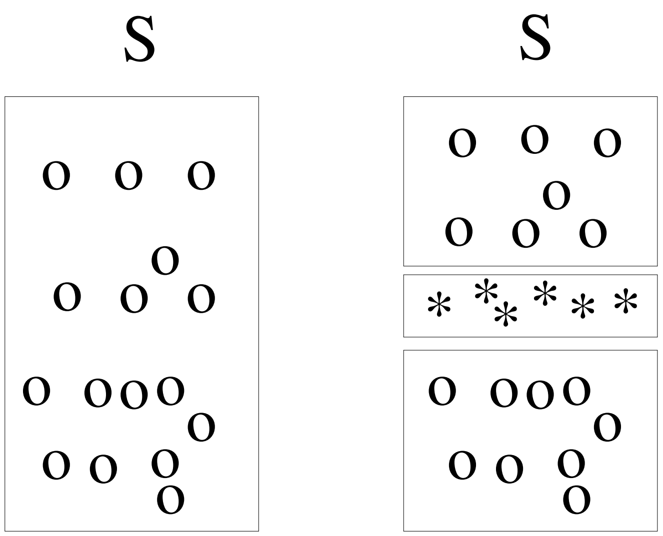


Rod in a bubble

A. Cohen & al., *PNAS* (2003)

Capillarity

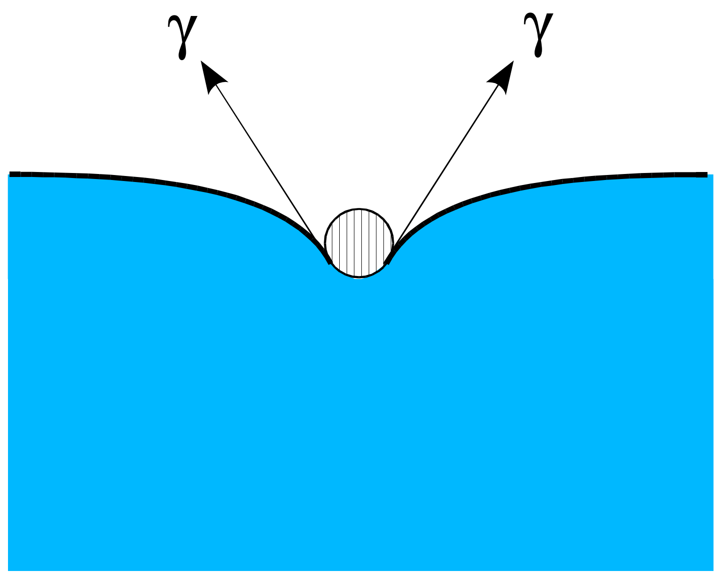
surface energy



$$E_{o} \quad - \quad E_{o,*} = -\gamma\,(2S)$$

energy cost: γ per surface area

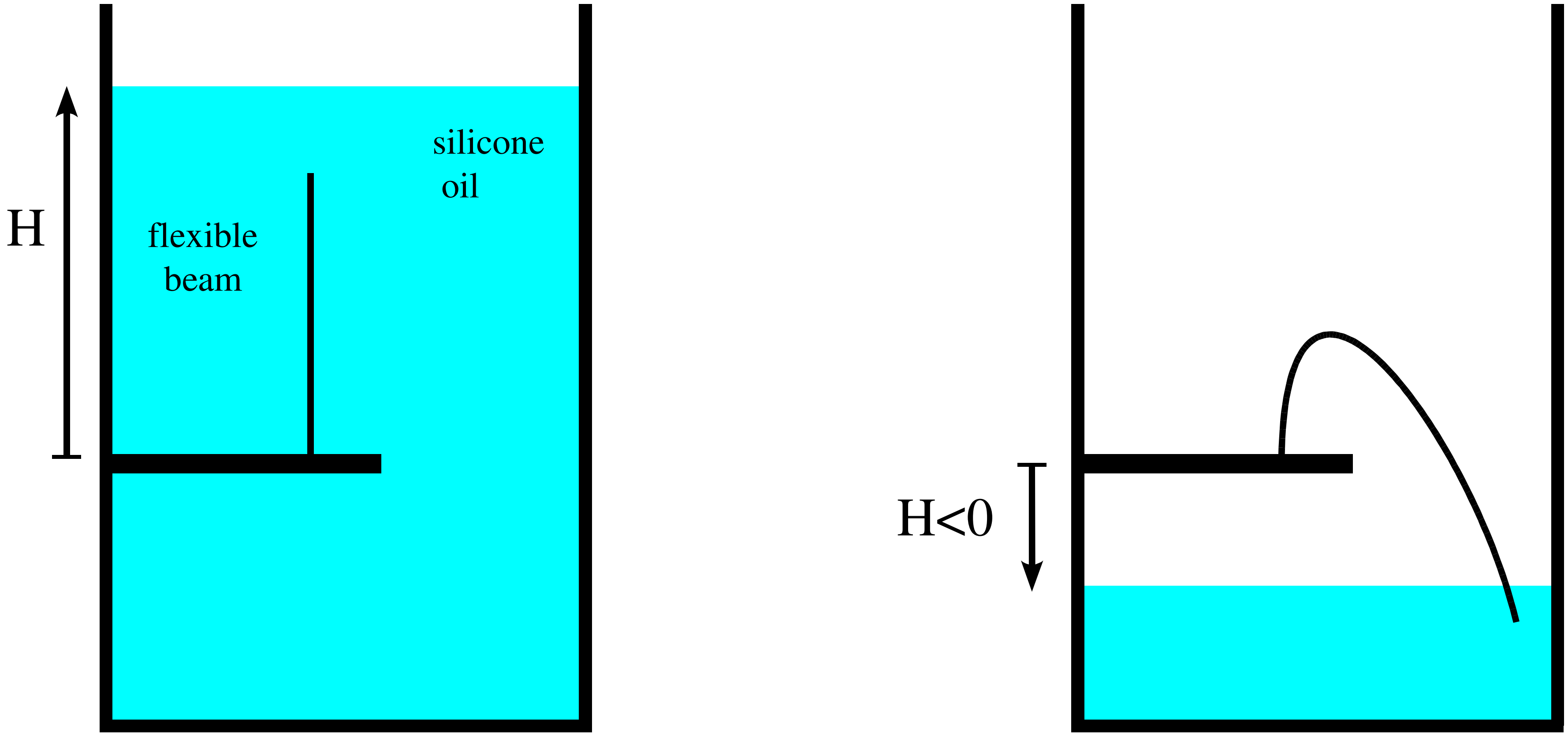
surface tension



force γ per unit length

perfect wetting : $\gamma = \gamma$ (water-air)

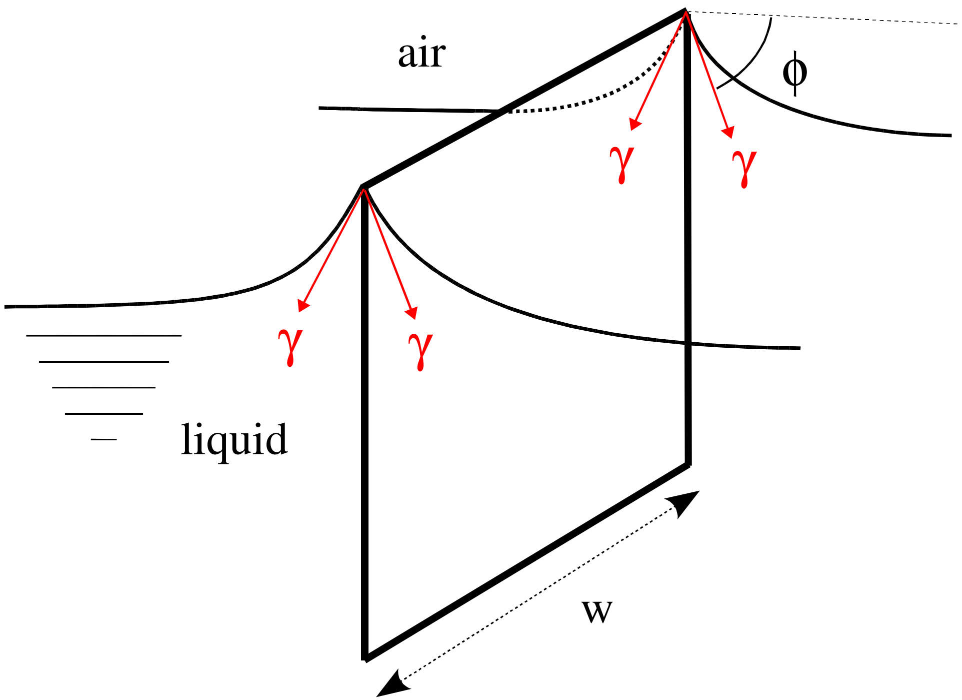
Experimental setup



oil perfectly wets the beam

beam: polyester

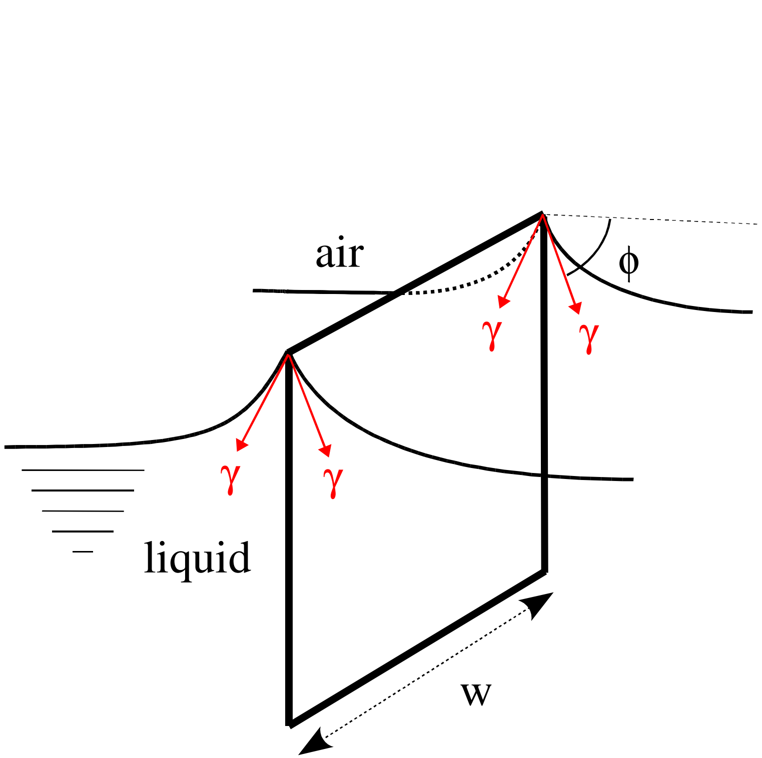
Capillary forces on an immersed beam



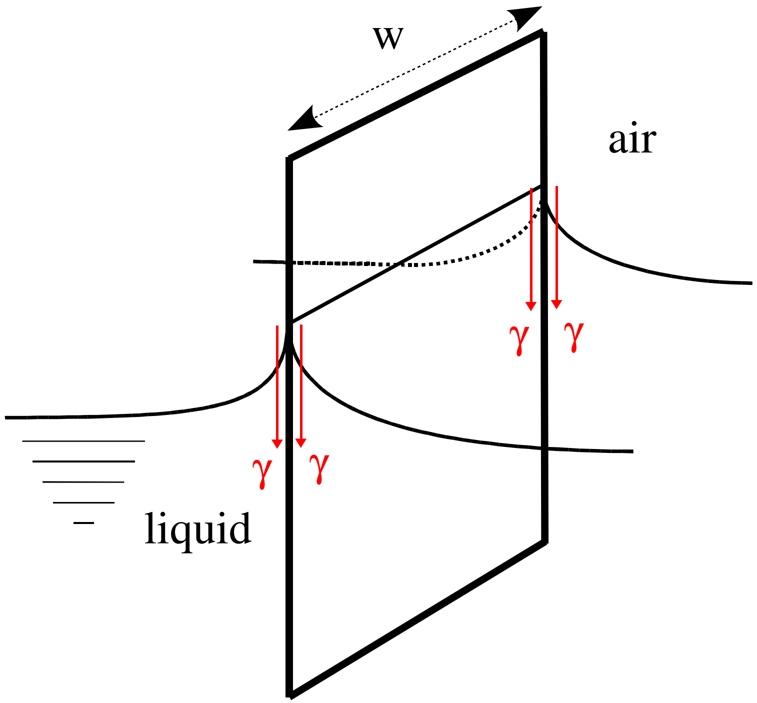
$$F_{tot} = 2\,\gamma\, w \sin\phi$$

(thickness omitted)

Capillary forces on an immersed beam



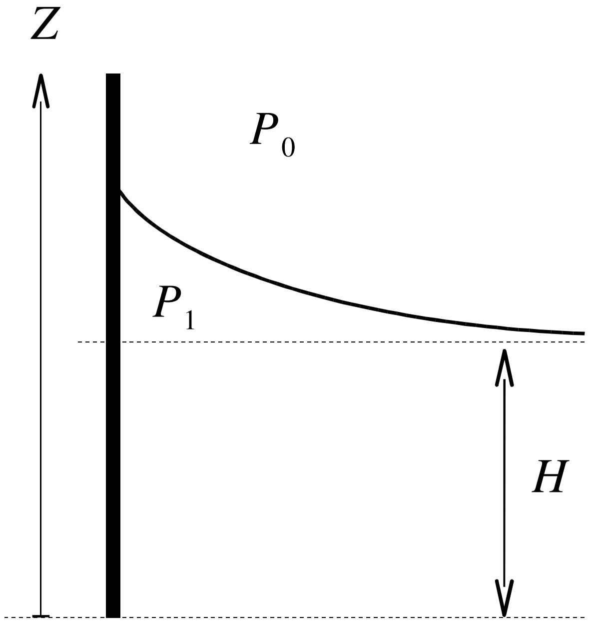
$$F_{tot} = 2\,\gamma\, w \sin\phi$$



$$F_{tot} = 2\,\gamma\, w$$

perfect wetting conditions

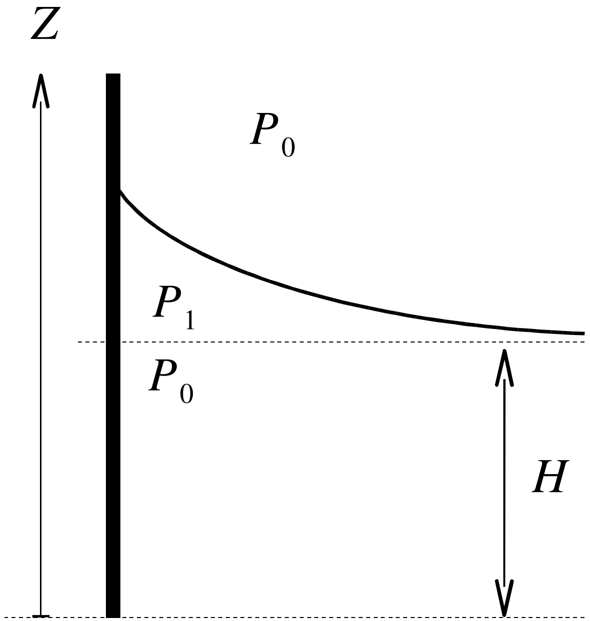
Meniscus shape (2D)



meniscus curvature

$$\Delta P = P_0 - P_1 = \gamma \kappa \qquad \text{(Laplace)}$$

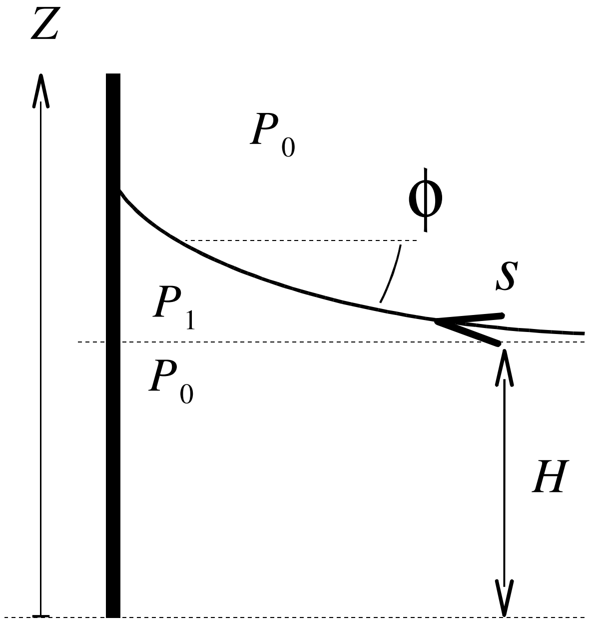
Meniscus shape (2D)



$$\Delta P = P_0 - P_1 = \gamma \kappa \qquad \text{(Laplace)}$$

$$\Delta P = \rho g (Z - H) \qquad \text{(hydrostatics)}$$

Meniscus shape (2D)



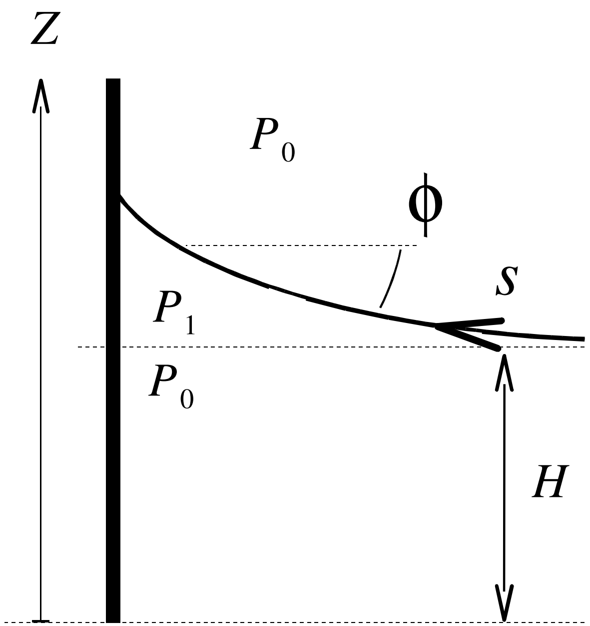
$$\Delta P = P_0 - P_1 = \gamma \kappa$$ (Laplace)

$$\Delta P = \rho g (Z - H)$$ (hydrostatics)

$$\frac{dZ}{ds} = \sin\phi$$ meniscus tangent

$$\kappa = \frac{d\phi}{ds}$$ meniscus curvature

Meniscus shape (2D)



$$\Delta P = P_0 - P_1 = \gamma\,\kappa$$ (Laplace)

$$\Delta P = \rho\, g\,(Z - H)$$ (hydrostatics)

$$\frac{dZ}{ds} = \sin\phi$$ meniscus tangent

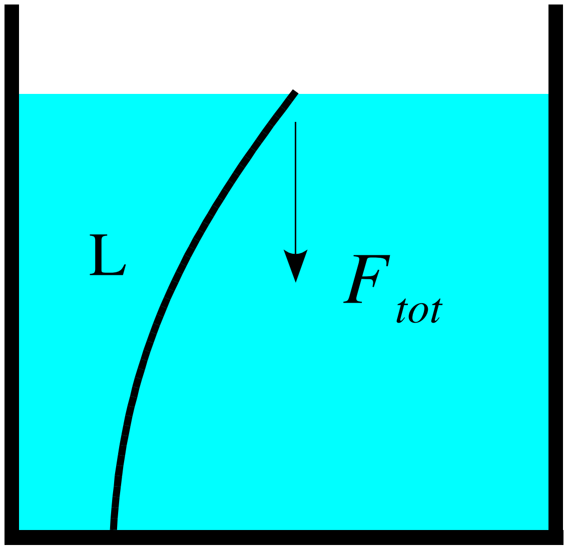
$$\kappa = \frac{d\phi}{ds}$$ meniscus curvature

L_c : capillary length (1.5mm for oil-air)

$$Z(s) = H + \sqrt{2}\,\sqrt{\frac{\gamma}{\rho\, g}}\,(1 - \cos\phi(s))$$

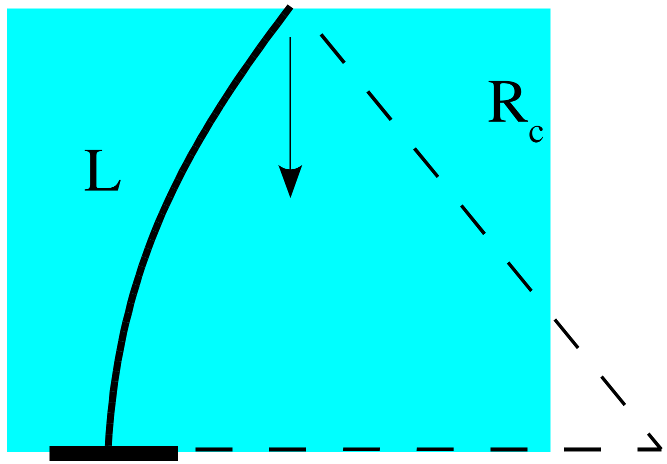
(max. meniscus height: $\sqrt{2}\,L_c$)

Capillarity versus elasticity



$$F_{tot} \sim \gamma w$$

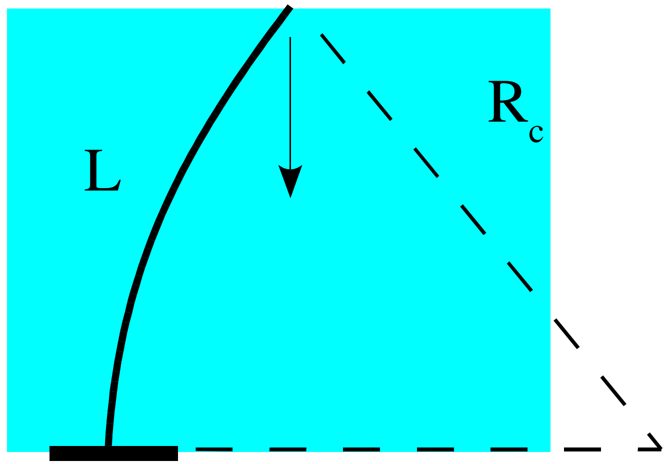
Capillarity versus elasticity



$$\gamma w \sim \frac{EI}{R_c} \frac{1}{L}$$

bending moment

Capillarity versus elasticity

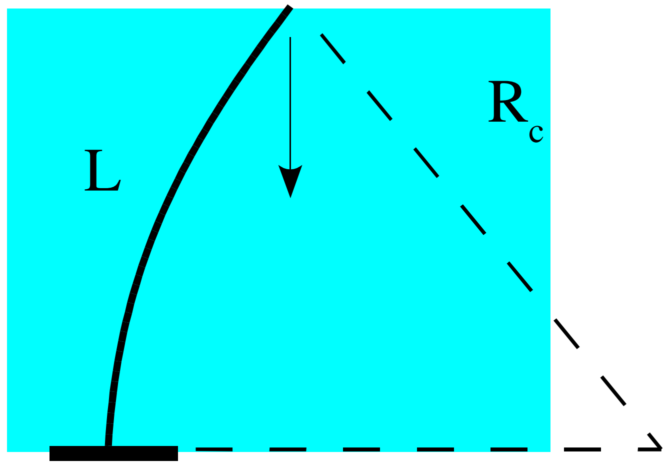


$$\gamma w \sim \frac{EI}{R_c}\frac{1}{L}$$

$$R_c \sim \sqrt{\frac{EI}{\gamma w}} \qquad (\text{ with } L \sim R_c)$$

Elasto-capillary length L_{EC}

Capillarity versus elasticity



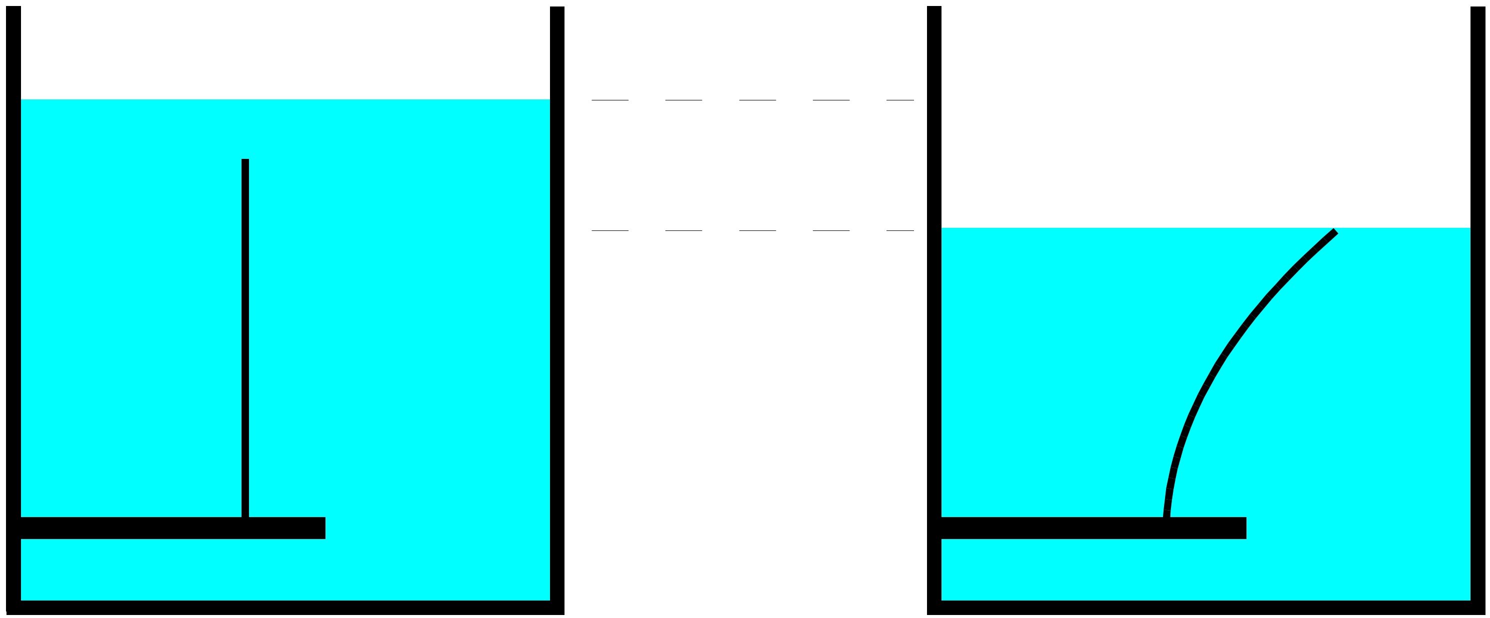
$$\gamma w \sim \frac{EI}{R_c}\frac{1}{L}$$

$$R_c \sim \sqrt{\frac{EI}{\gamma w}} \qquad (\text{ with } L \sim R_c)$$

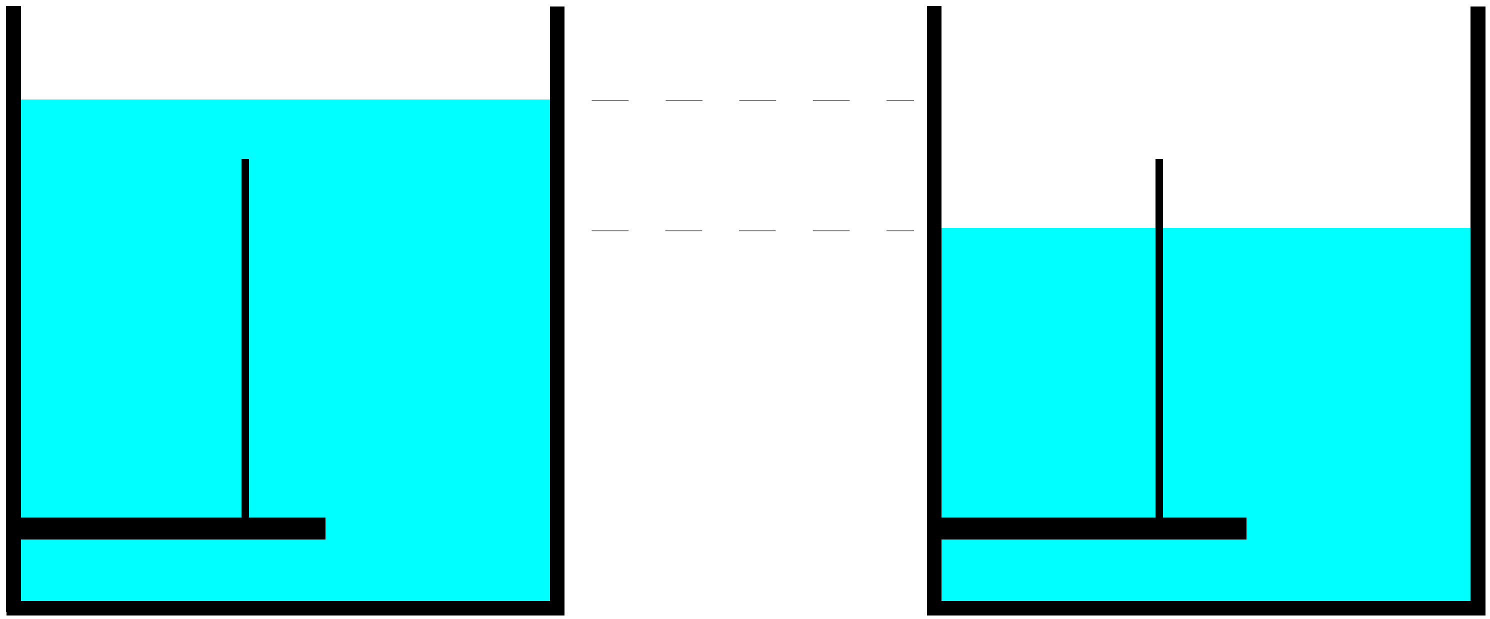
Elasto-capillary length L_{EC}

if L >~ L_{EC}, surface tension (largely) deflects the beam

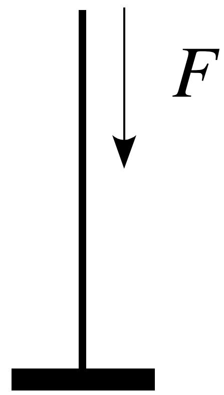
Long enough beam



Short beam

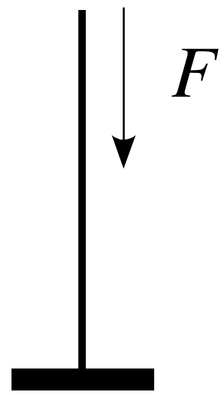


Limiting length



Euler buckling threshold $$F_{euler} = \left(\frac{\pi}{2}\right)^2 \frac{EI}{L^2}$$

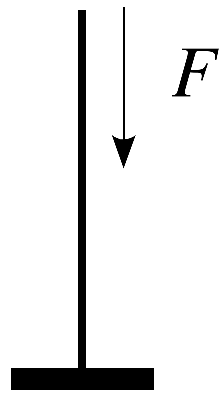
Limiting length



Euler buckling threshold $$F_{euler} = \left(\frac{\pi}{2}\right)^2 \frac{EI}{L^2}$$

max. capillary force (on straight beam) $$F_{tot} = 2\,\gamma\, w$$

Limiting length



Euler buckling threshold $$F_{euler} = \left(\frac{\pi}{2}\right)^2 \frac{EI}{L^2}$$

max. capillary force (on straight beam) $$F_{tot} = 2\,\gamma\, w$$

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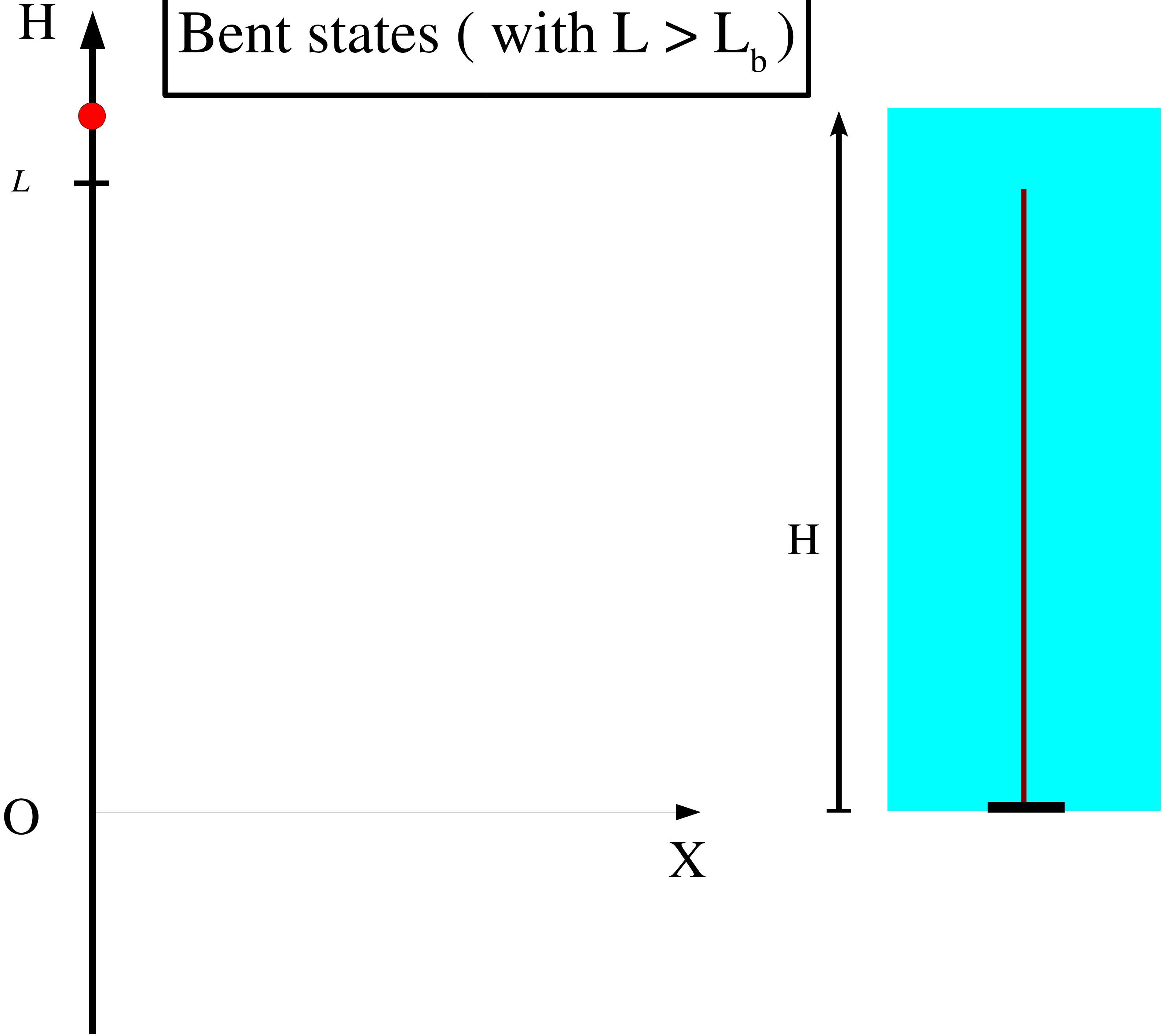
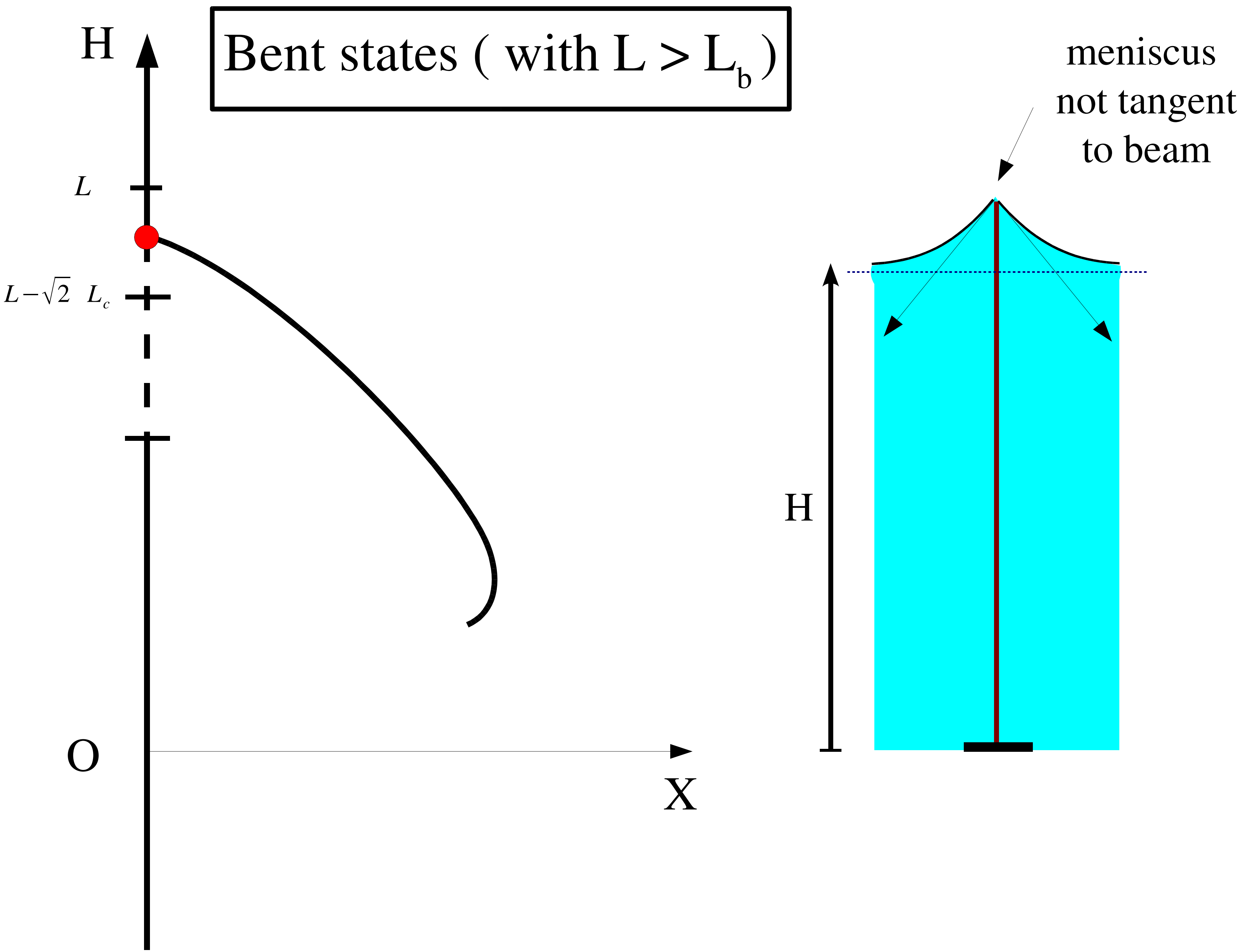
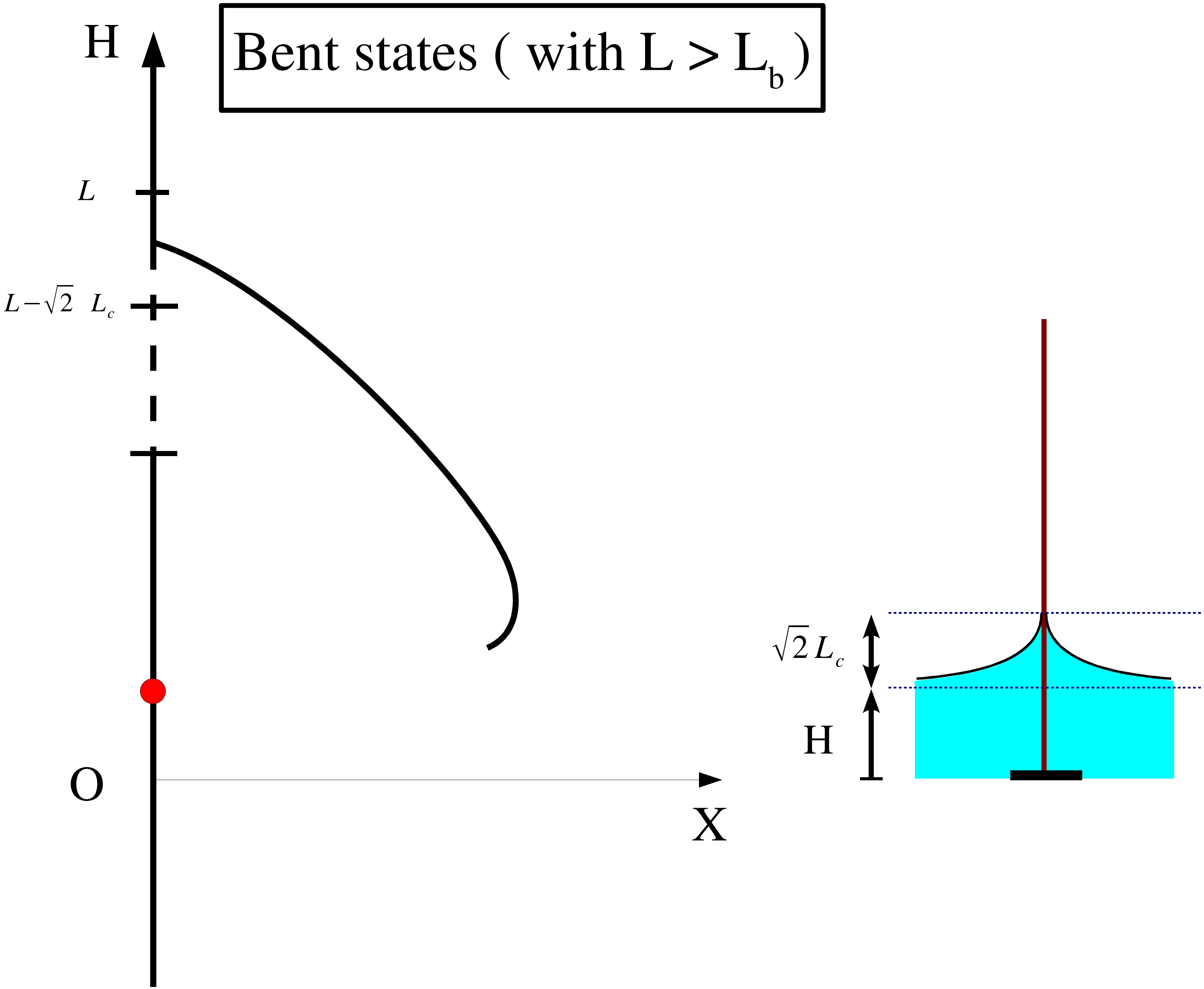
limiting length: $$L_b = \frac{\pi}{2\sqrt{2}} \sqrt{\frac{EI}{\gamma\, w}}$$

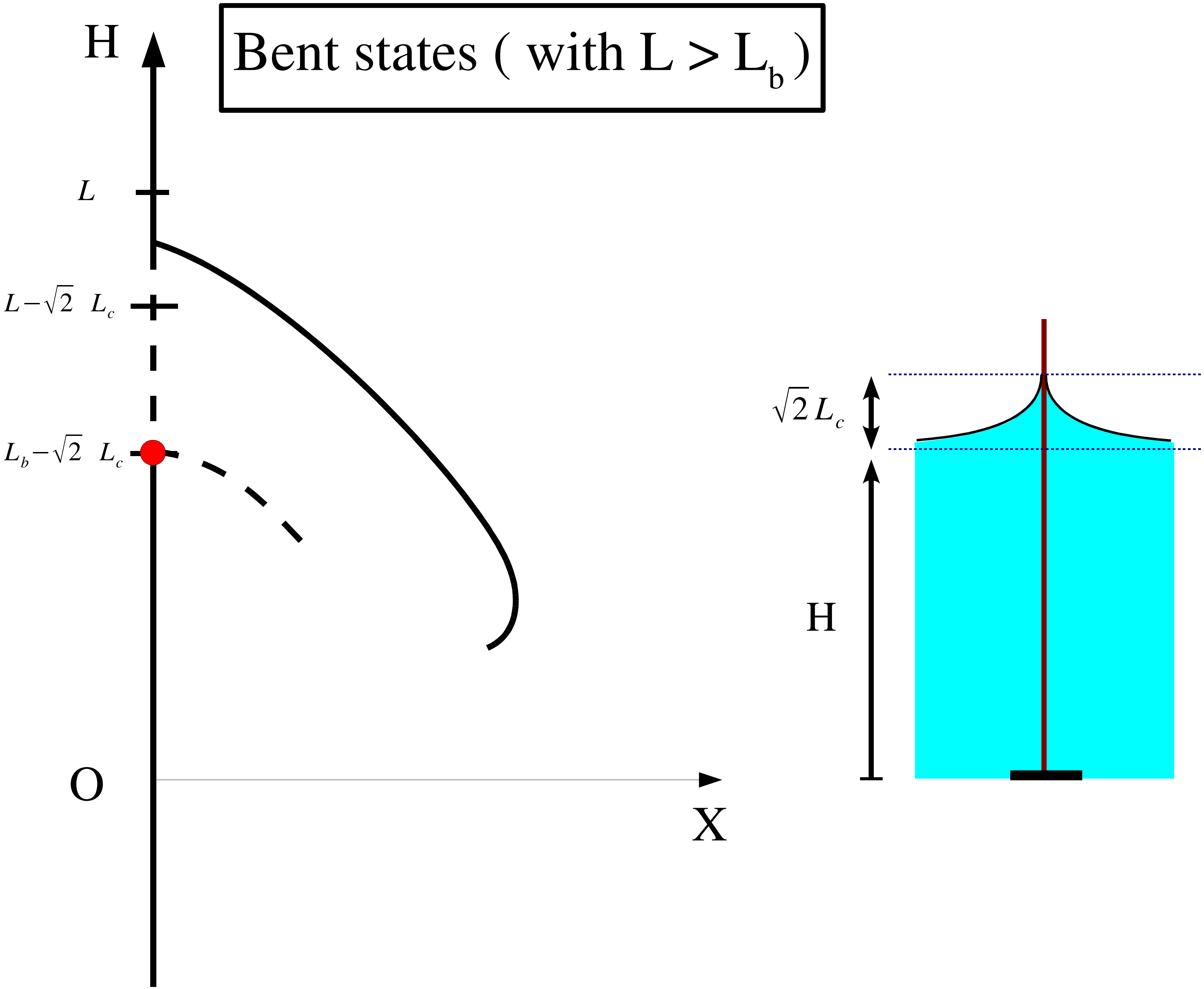
Elasto-capillary length L_{EC}

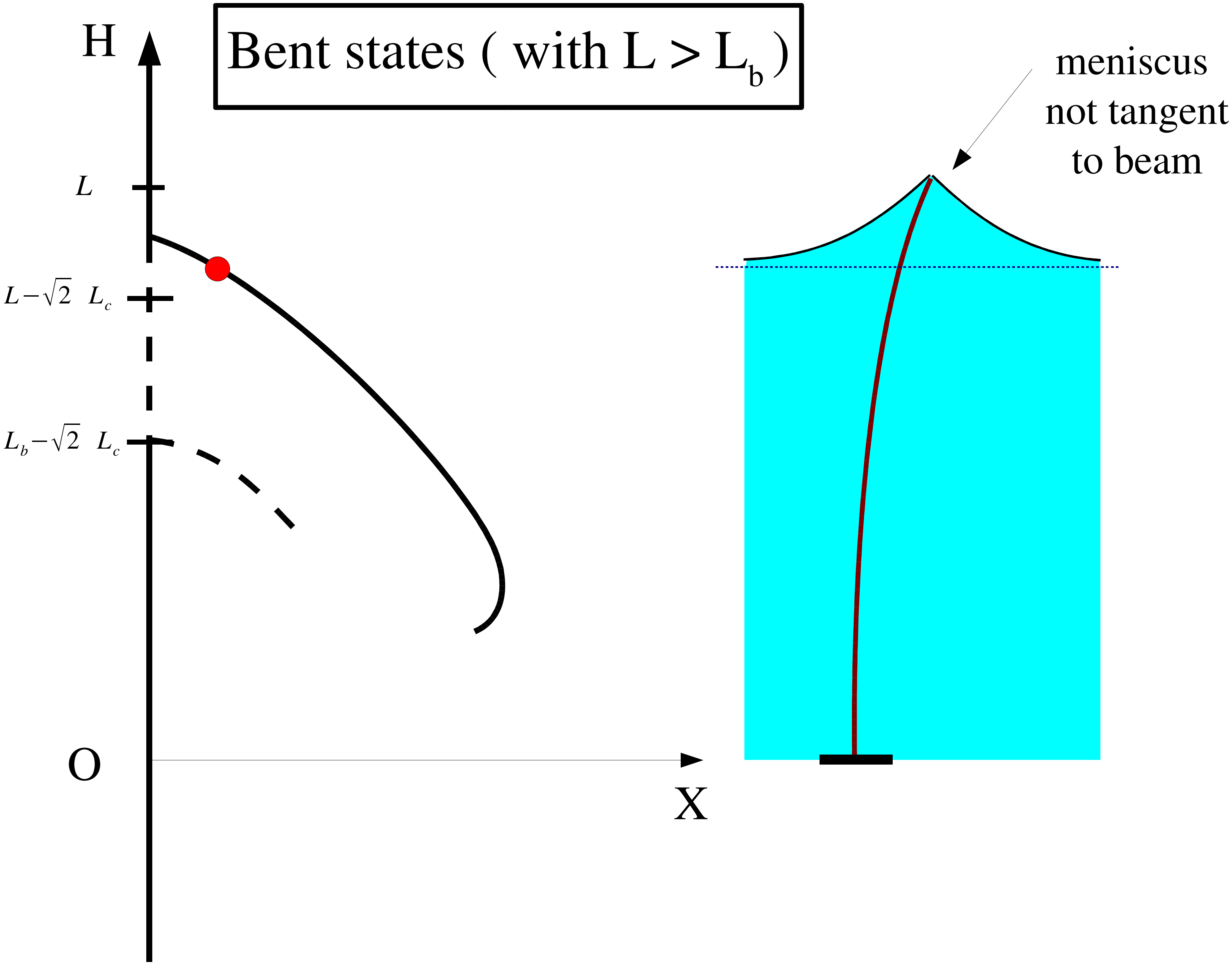
Bent states (with $L > L_b$)



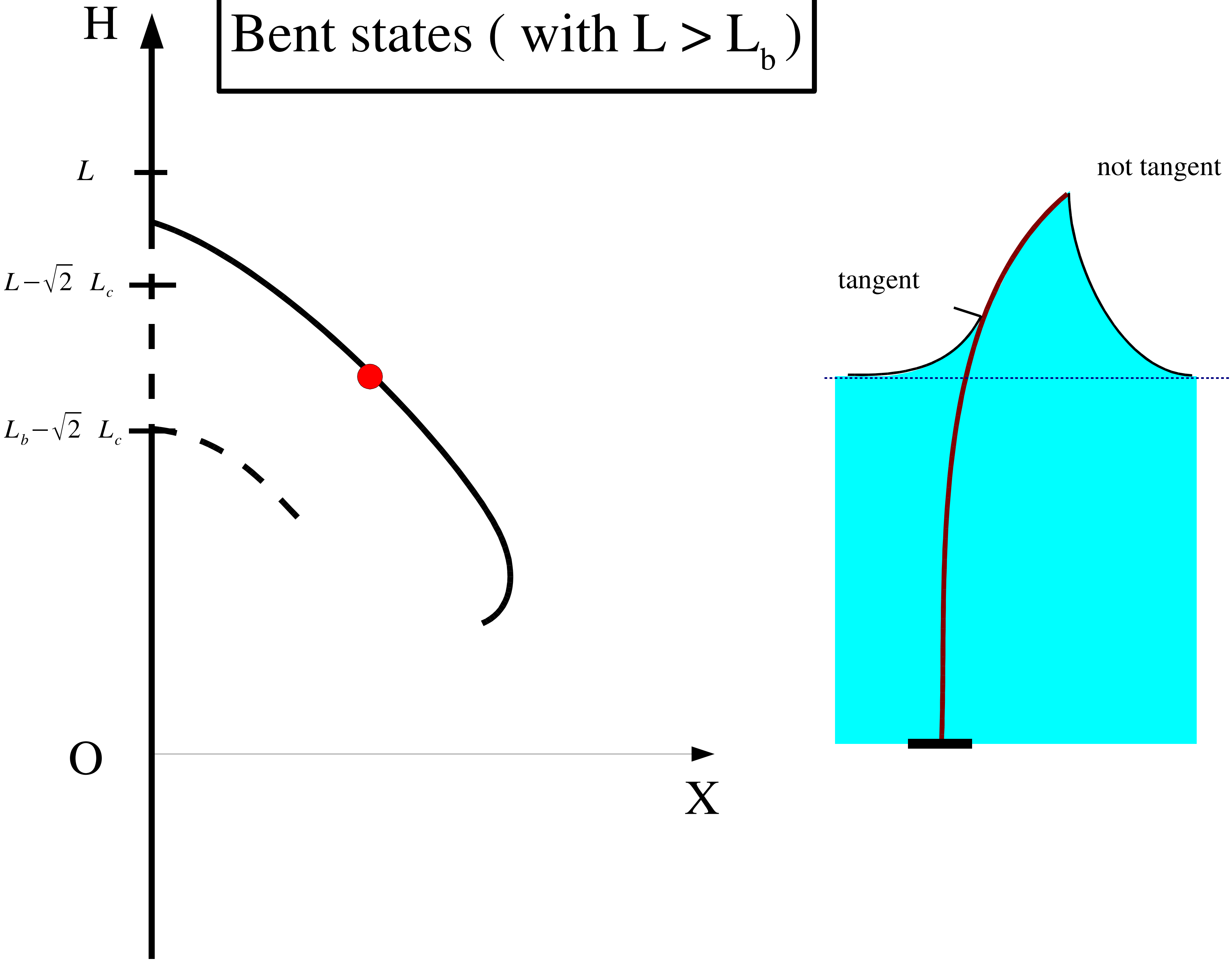

Bent states (with L > L
b
)
H
L
L − √2 L
c
O
X
meniscus
not tangent
to beam
H

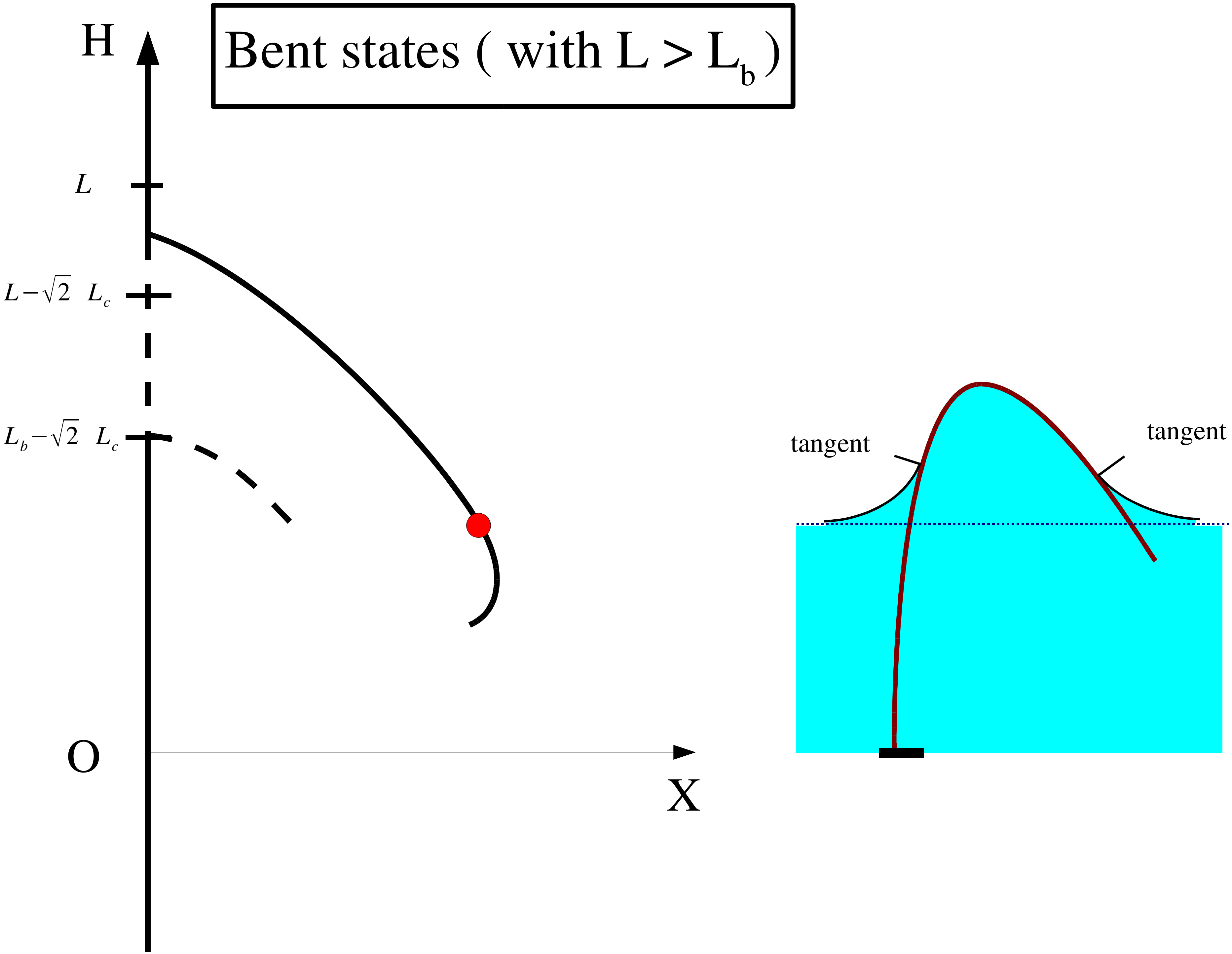

Bent states (with L > L_b)
H
L
$L-\sqrt{2}\ L_c$
O
X
$\sqrt{2}\,L_c$
H


Bent states (with L > L_b)
H
L
L−√2 L_c
L_b−√2 L_c
O
X
√2 L_c
H


Bent states (with L > L_b)
H
L
L − √2 L_c
L_b − √2 L_c
O
X
meniscus
not tangent
to beam

Bent states (with $L > L_b$)




Bent states (with L > L_b)
H
L
$L-\sqrt{2}\ L_c$
$L_b-\sqrt{2}\ L_c$
O
X
tangent
tangent

Planar elastica with external loads

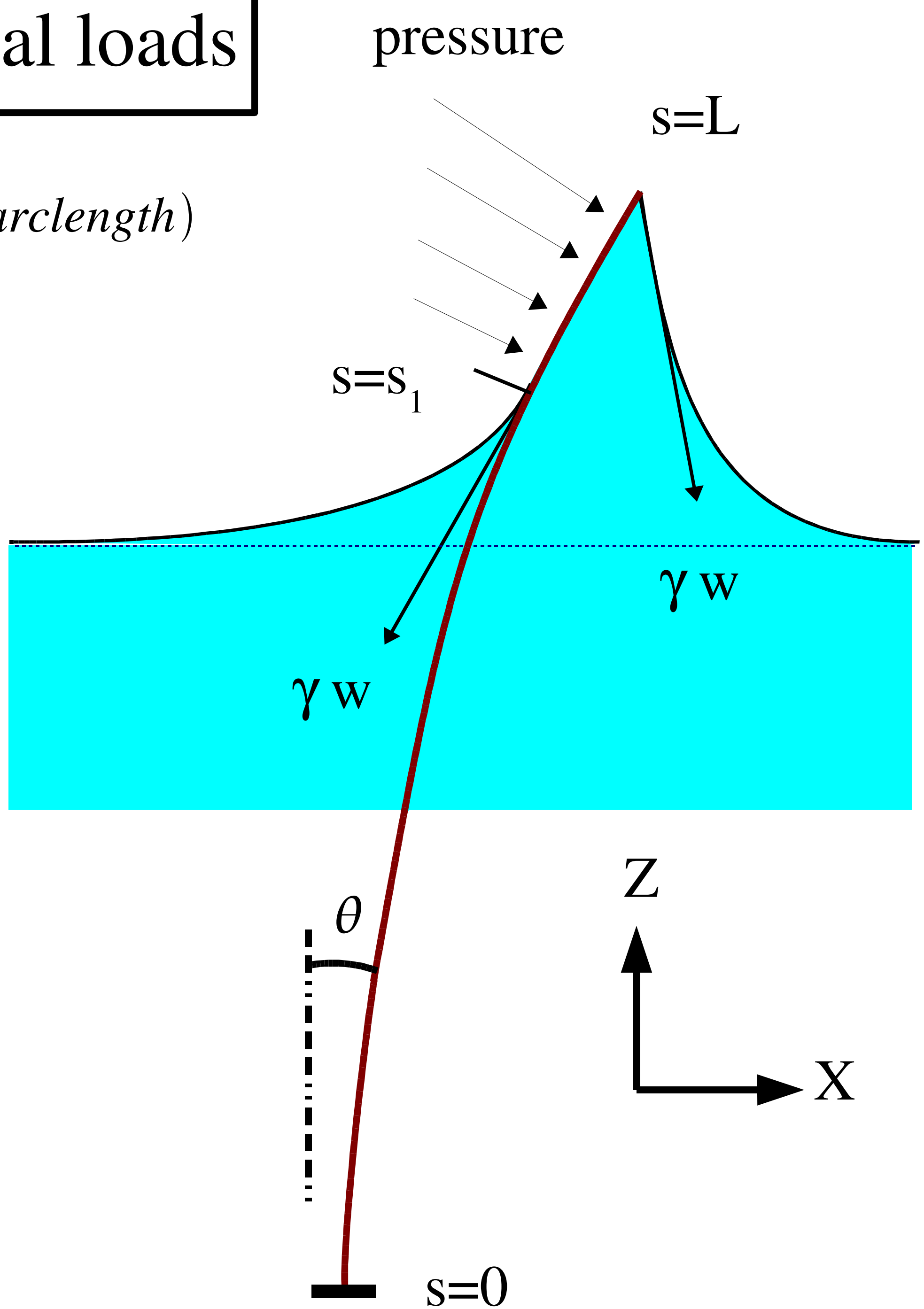
$$\begin{cases} N_x' = -P_x \\ N_z' = -P_z \\ M_y' = N_z \sin\theta - N_x \cos\theta \\ X' = \sin\theta \\ Z' = \cos\theta \\ \theta' = M_y \ / \ (EI) \end{cases}$$

$$' \equiv \frac{d}{ds} \quad ; \quad (s : arclength)$$

$s = 0 \ \rightarrow \ s = s_1$

$$\begin{cases} P_x = 0 \\ P_z = -(\rho_b - \rho) g w e \ < \ 0 \end{cases}$$

buoyancy



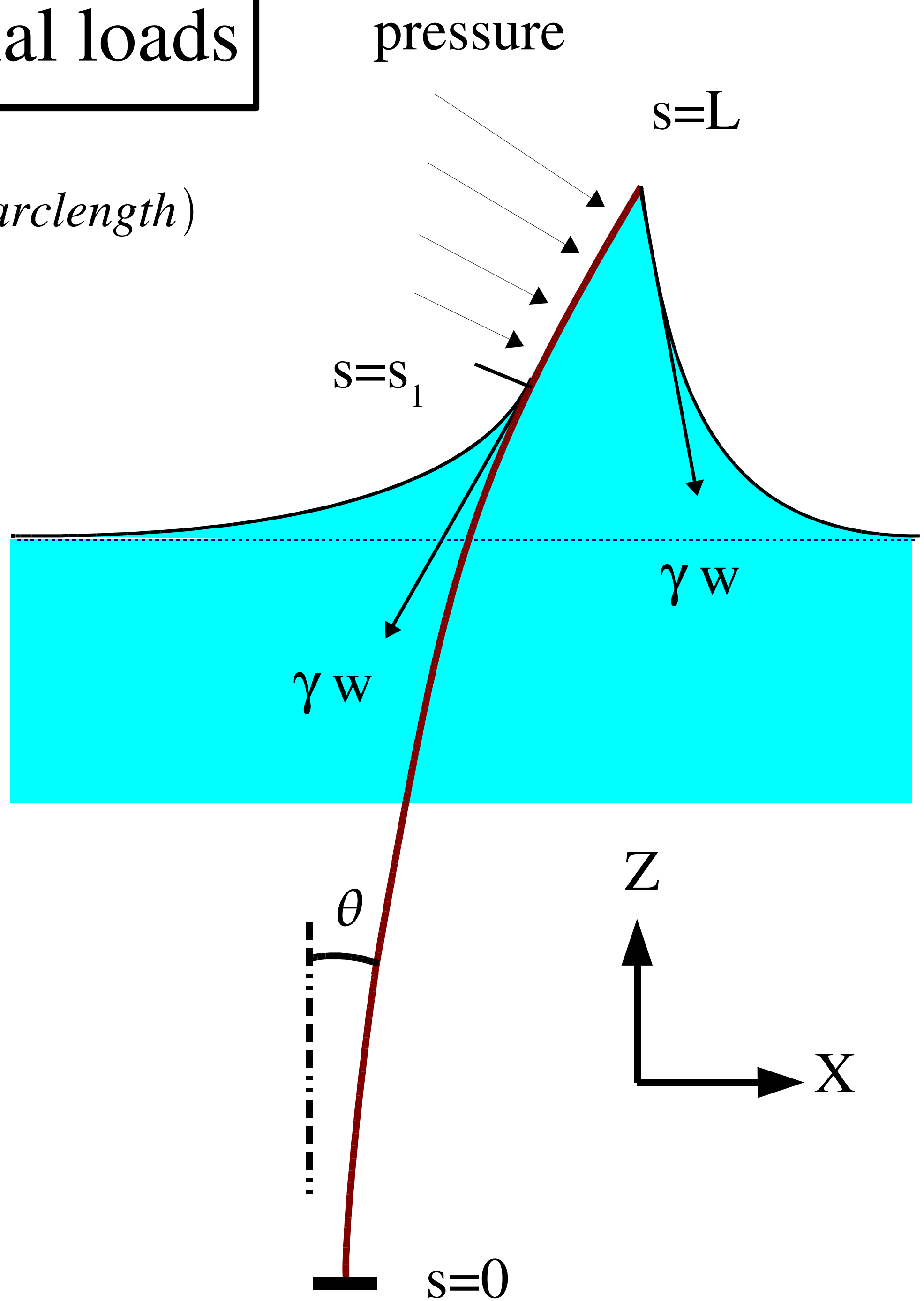
Planar elastica with external loads

$$\begin{cases} N_x{}' = -P_x \\ N_z{}' = -P_z \\ M_y{}' = N_z \sin\theta - N_x \cos\theta \\ X' = \sin\theta \\ Z' = \cos\theta \\ \theta' = M_y / (EI) \end{cases}$$

$$' \equiv \frac{d}{ds} \quad ; \quad (s : arclength)$$

$at\ s = s_1$

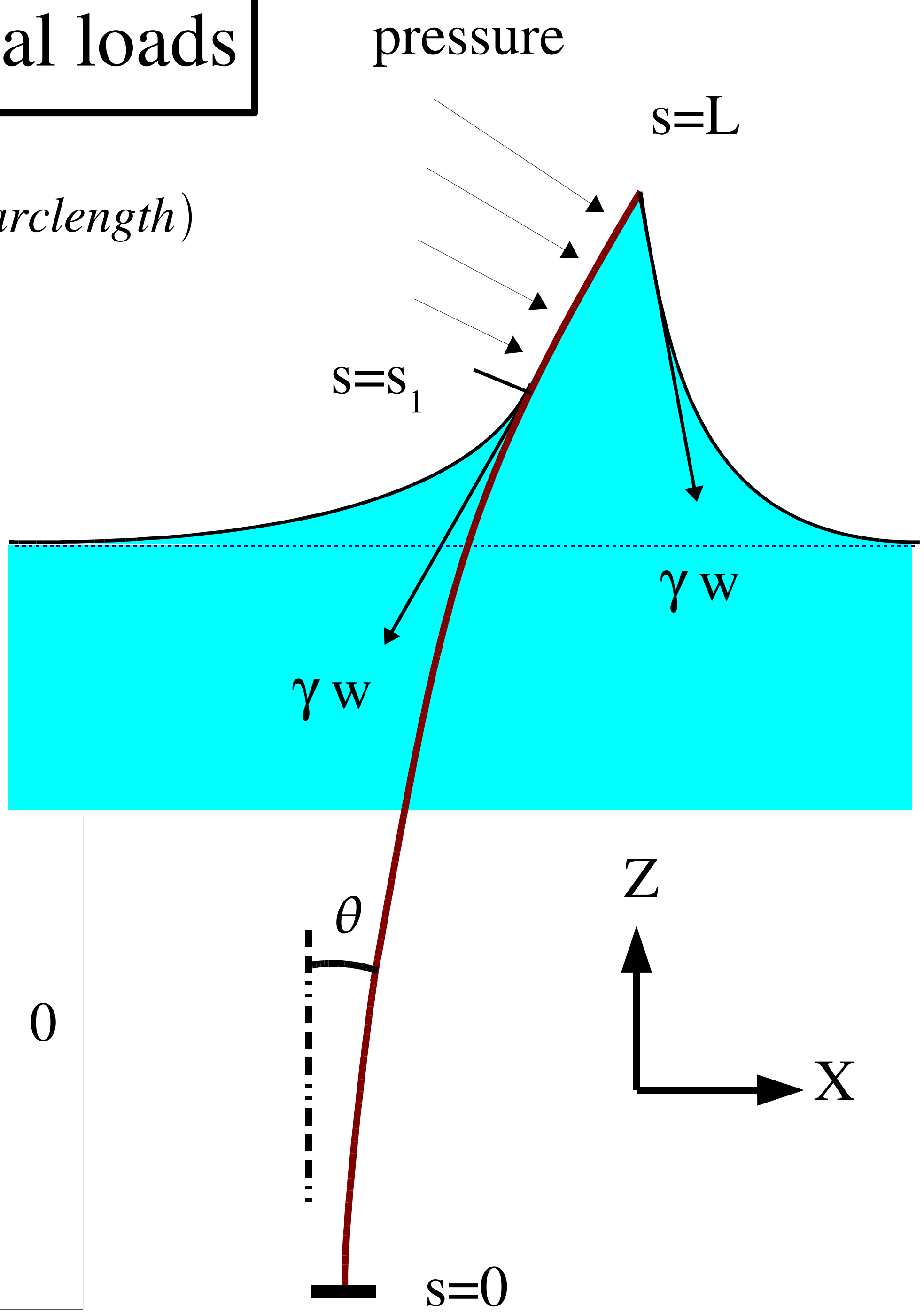
$$\begin{cases} N_x(s_1^+) - N_x(s_1^-) = \gamma w \sin\theta(s_1) \\ N_z(s_1^+) - N_z(s_1^-) = \gamma w \cos\theta(s_1) \end{cases}$$



Planar elastica with external loads

$$
\begin{cases}
N_x{}' = -P_x \\
N_z{}' = -P_z \\
M_y{}' = N_z \sin\theta - N_x \cos\theta \\
X' = \sin\theta \\
Z' = \cos\theta \\
\theta' = M_y \ / \ (EI)
\end{cases}
$$

$$' \equiv \frac{d}{ds} \ ; \ (s : arclength)$$



$s = s_1 \ \rightarrow \ s = L$

$$
\begin{cases}
P_x = +\rho g (Z - H) w \cos\theta \ > \ 0 \\
P_z = -\rho g (Z - H) w \sin\theta - \rho_b g w e \ < \ 0
\end{cases}
$$

pressure

weight

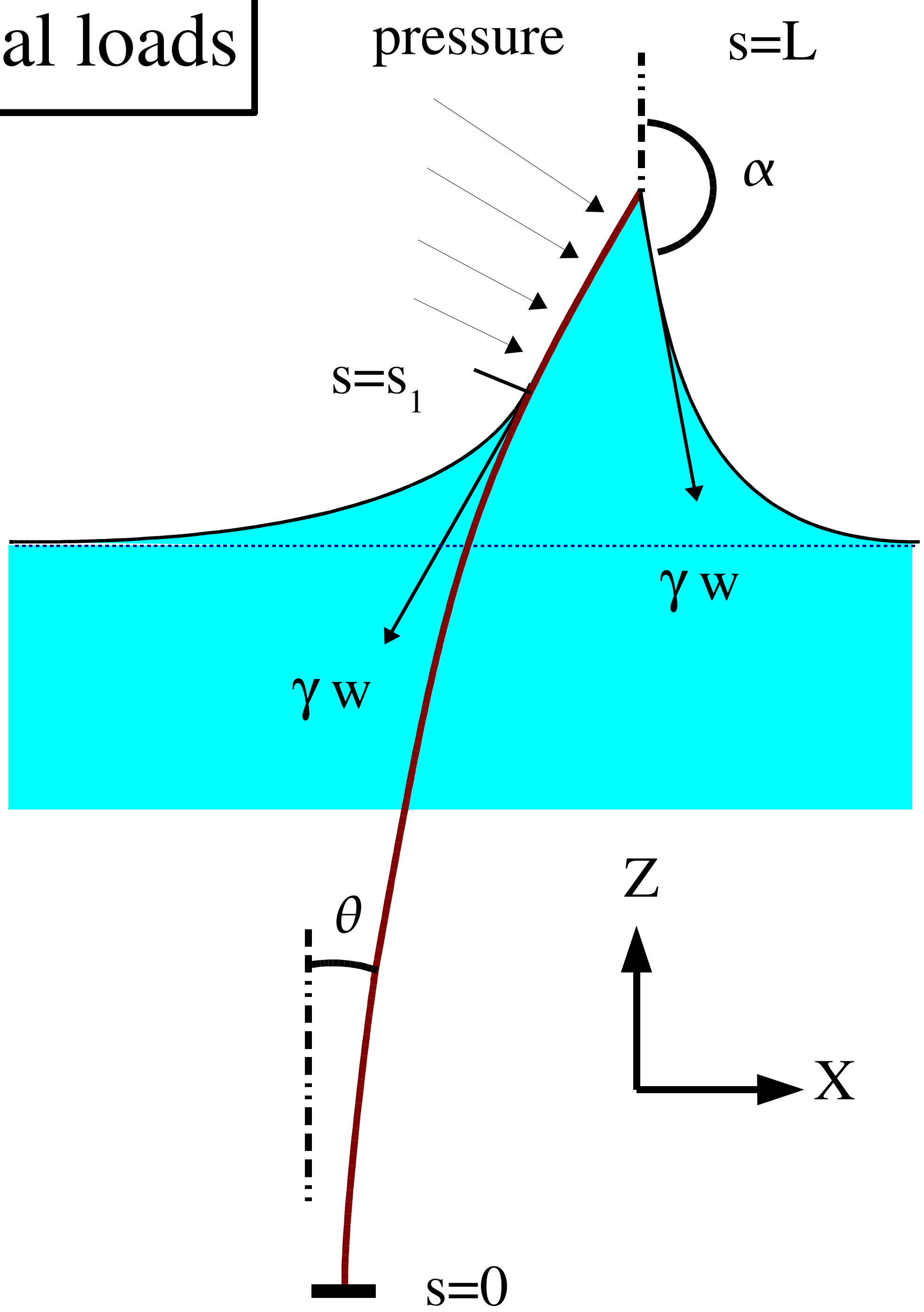
Planar elastica with external loads

Boundary conditions

$$s = 0\ :$$
$$X(0) = 0 = Z(0);\quad \theta(0) = 0$$

$$s = s_1\ :$$
$$Z(s_1) = H + \sqrt{2}\, L_c \sqrt{1 - \sin\theta(s_1)}$$

$$s = L:$$
$$M_y(L) = 0$$
$$N_x(L) = \gamma\, w \sin\alpha$$
$$N_z(L) = \gamma\, w \cos\alpha$$
$$Z(L) = H + \sqrt{2}\, L_c \sqrt{1 - \sin\alpha}$$



Planar elastica with external loads

Counting the shooting parameters / equations

Once the beam characteristics
(L, Lb, Lc) are fixed

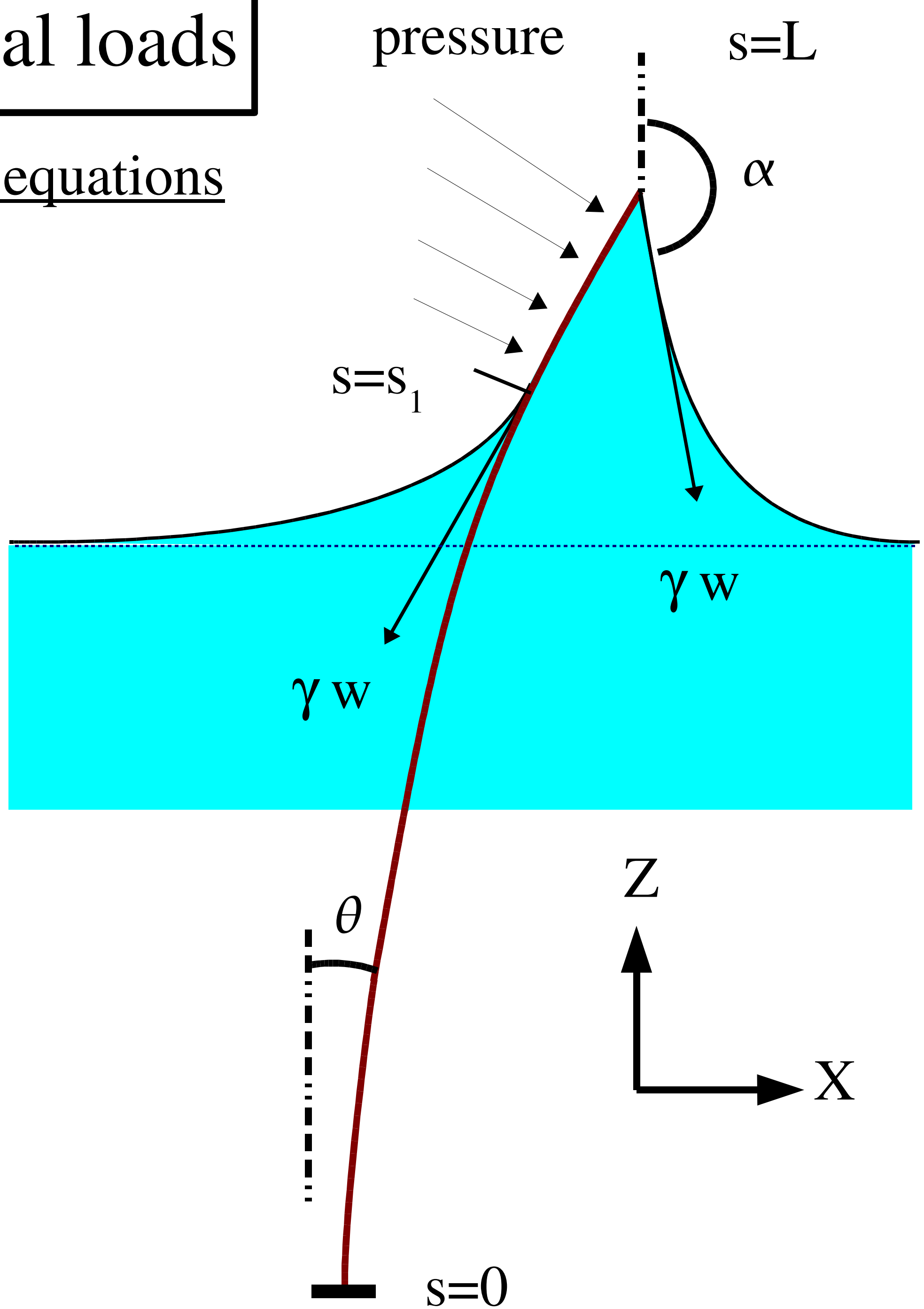
5 variables

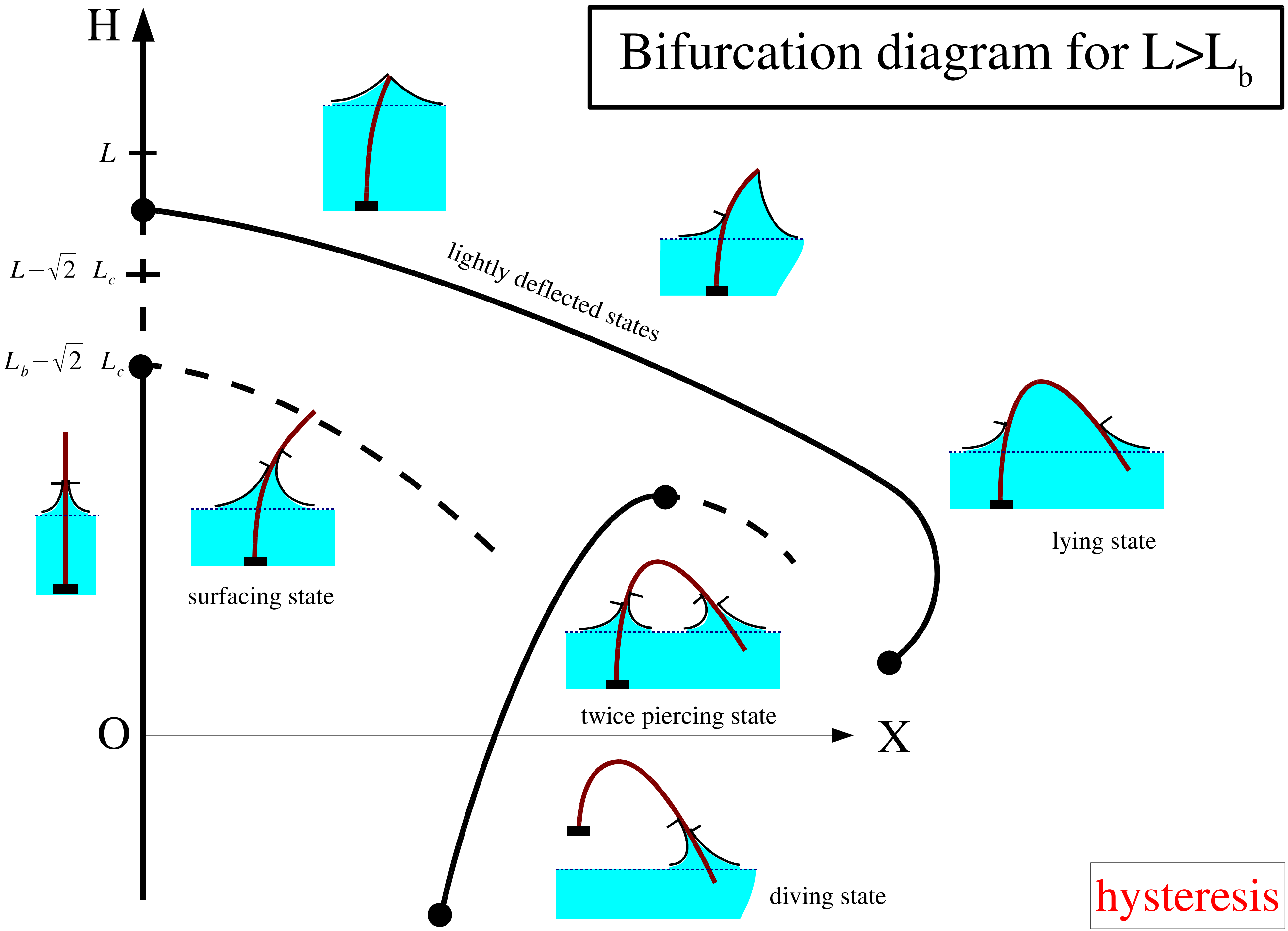
$$M_y(0), N_x(0), N_z(0), s_1, \alpha$$

4 equations

$$M_y(L)=0\,,\; N_x(L)=\ldots, N_z(L)=\ldots,$$
$$Z(s_1)-\sqrt{2}\,L_c\sqrt{1-\sin\theta(s_1)}=$$
$$Z(L)-\sqrt{2}\,L_c\sqrt{1-\sin\alpha}$$

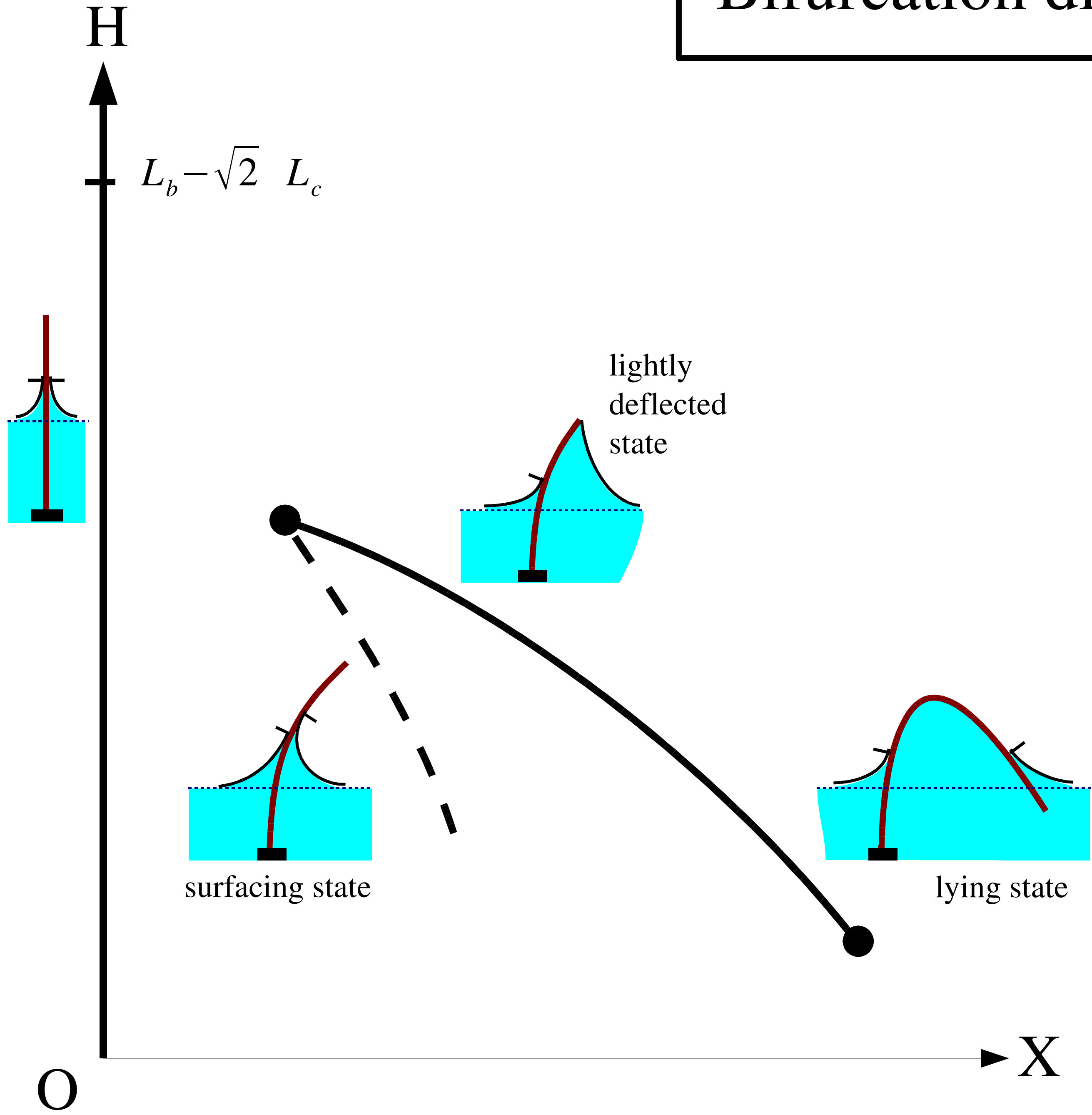
=> 1D solution manifold





Bifurcation diagram for $L>L_b$
H
L
$L-\sqrt{2}\ L_c$
$L_b-\sqrt{2}\ L_c$
O
X
lightly deflected states
surfacing state
lying state
twice piercing state
diving state
hysteresis

Bifurcation diagram for $L<L_b$



Phase diagram

polyester: w=2cm, e=25μm,
Lb=1.9cm, Lc=1.5mm (silicone oil)