

Vibrations of post-buckled rods: the singular inextensible limit

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Abstract

The small-amplitude in-plane vibrations of an elastic rod clamped at both extremities are studied. The rod is modeled as an extensible, shearable, planar Kirchhoff elastic rod under large displacements and rotations, and the vibration frequencies are computed both analytically and numerically as a function of the loading. Of particular interest is the variation of mode frequencies as the load is increased through the buckling threshold. While for some modes there are no qualitative changes in the mode frequencies, other frequencies experience rapid variations after the buckling threshold, the thinner the rod, the more abrupt the variations. Eventually, a mismatch for half of the frequencies at buckling arises between the zero thickness limit of the extensible model and the inextensible model.

Keywords: Vibrations, Kirchhoff elastic rods, Buckling, Bifurcation.

1. Introduction

1 The first step in the study of vibrating elastic structures [1] focuses on the dynamical re-
2 sponse of the system around its unstressed configuration. In vibration analysis, the dynamics of
3 infinitely small-amplitude disturbances around the fundamental equilibrium state are generally
4 first considered, leading to a linear problem. However, nonlinear effects are known to play a key
5 role in many elastic systems, and in the context of vibrations, nonlinearities can be included by
6 studying large-amplitude oscillations around the fundamental state [2]. The second step in the
7 analysis of vibrations is to study the effect of external loads. They have a direct influence on the
8 dynamical response of the system, as easily demonstrated by tuning the natural frequencies of
9 a string by putting it under tension. Similarly, in compression, the natural frequencies of a rod
10 decrease. Again nonlinear effects become important when external loads not only change the
11 vibration response of the rod but also alter its overall stability through buckling. Several stud-
12 ies have investigated dynamical responses of post-buckled elastic rods around their post-buckled
13 state, see e.g. Chap. 12 of [1] and [3, 4, 5]. The present work also focuses on the problem
14 of small-amplitude vibrations around a pre-strained deformed nonlinear elastic rod, and shows
15 that vibration frequencies behave singularly at buckling. This problem is relevant for a number
16 of applications including the manufacturing of piano (or violin) soundboards where the wooden
17 board is bent before being clamped in the rigid metal frame [6, 7, 8]. Other systems where pre-
18 stress and/or pre-strain play an important role for the vibration response are gongs, cymbals, and
19 steel drums where plastic deformations of the metal plates are used to introduce separation of
20 response modes [9].
21

22 Here, we consider the problem of in-plane vibrations of a pre- and post-buckled Kirchhoff
 23 extensible shearable elastic rod. First, we study the equilibrium configurations of a clamped-
 24 clamped rod as the ends are gradually moved together. The rod has a straight natural shape and
 25 for small axial displacements the rod remains straight until a critical displacement is reached
 26 where the rod buckles in the plane. For each value of the axial displacement we study the small-
 27 amplitude vibrations around the equilibrium state and we follow how frequencies of the natural
 28 modes change as the load is increased. For both equilibrium and vibrations we enforce displacement
 29 control boundary conditions, that is, the axial displacement rather than the axial load is
 30 imposed.

31 The paper is organized as follows: In Section 2 we present Kirchhoff model for elastic rods,
 32 in Section 3 we derive the equations for the small-amplitude vibrations of a rod around its post-
 33 buckled equilibrium. We compute equilibriums in Section 4, vibrations in the extensible case
 34 in Section 5 and in the extensible case in Section 6. We then compare the two cases in Section
 35 7 and a show a discrepancy between them, which is further analyzed analytically in Section 8.
 36 Discussion (Section 9) and conclusion (Section 10) follow.

37 2. Model

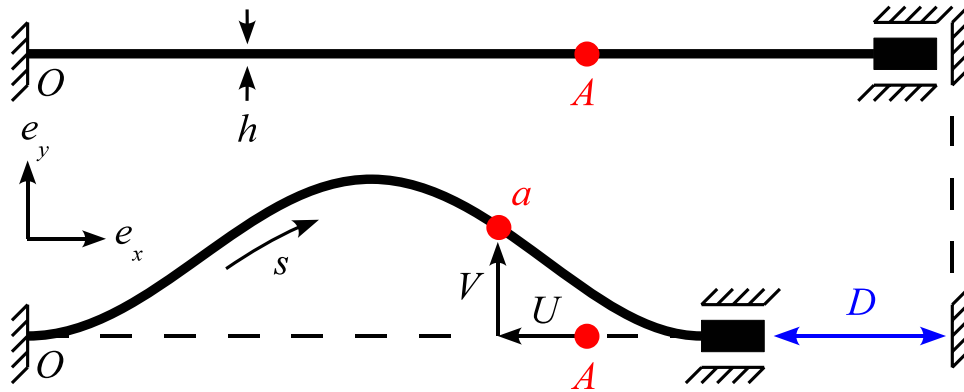


Figure 1: Clamped-clamped rod buckled in the (x,y) plane. The end-shortening D is controlled. The point A in the reference configuration moves to point a in the deformed configuration, introducing horizontal $U \leq 0$ and vertical V displacements. The origin is taken at the fixed point O at the left end of the rod.

38 We consider an elastic rod with a rectangular cross-section of width b and thickness h , total
 39 length L and arc length S in its unstressed reference state. In this state the rod lies along the e_x
 40 axis, from the origin $O = (0, 0, 0)$ to the point at $(L, 0, 0)$. The position vector of the center of the
 41 rod cross-section is noted $\mathbf{R}(S)$ and we have $\mathbf{R}(0) = (0, 0, 0)$ and $\mathbf{R}(L) = (L, 0, 0)$ in the reference
 42 state.

43 Kinematics

44 We use the special Cosserat theory of rods [10] where the rod can suffer bending, extension,
 45 and shear deformations. We work under the assumption that the rod cross-section remains planar
 46 (and rectangular) as the rod deforms and use a set of three Cosserat directors $(\mathbf{d}_1(S), \mathbf{d}_2(S), \mathbf{d}_3(S))$

47 embedded in each cross-section: \mathbf{d}_1 is perpendicular to the section plane, \mathbf{d}_2 is along the small
 48 span (of length h) of the section, and \mathbf{d}_3 is along the wide span (of length b) of the section. In the
 49 undeformed state, $\mathbf{d}_1(S) \equiv \mathbf{e}_x$, $\mathbf{d}_2(S) \equiv \mathbf{e}_y$, and $\mathbf{d}_3(S) \equiv \mathbf{e}_z$. We only consider deformed states
 50 that are (i) planar (where the rod center line $\mathbf{R}(S)$ lies in the (x, y) plane, the rod being bent along
 51 its small span h), and (ii) twist-less (where the director $\mathbf{d}_3(S) \equiv \mathbf{e}_z$). Note that in the presence of
 52 extension and shear, S may no longer be the arc-length of the curve $\mathbf{R}(S)$ in the deformed state.
 53 We introduce extension and shear strains, e_1 and e_2 , such that:

$$\mathbf{R}'(S) \stackrel{\text{def}}{=} d\mathbf{R}/dS = (1 + e_1) \mathbf{d}_1 + e_2 \mathbf{d}_2. \quad (1)$$

54 In the absence of extension ($e_1 = 0$) and shear ($e_2 = 0$), the director \mathbf{d}_1 is the unit tangent to the
 55 centerline $\mathbf{R}(S) = (X(S), Y(S), Z(S))$. We introduce the angle $\theta(S)$ to parametrize the rotation of
 56 the $(\mathbf{d}_1, \mathbf{d}_2)$ frame around the $\mathbf{e}_z = \mathbf{d}_3$ axis:

$$\mathbf{d}_1(S) = \begin{pmatrix} \cos \theta(S) \\ \sin \theta(S) \\ 0 \end{pmatrix}_{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z} \quad \text{and} \quad \mathbf{d}_2(S) = \begin{pmatrix} -\sin \theta(S) \\ \cos \theta(S) \\ 0 \end{pmatrix}_{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z}. \quad (2)$$

57 *Dynamics*

58 We use the Kirchhoff dynamical equations for elastic rods [10], where the stresses in the
 59 section are averaged to yield an internal force $\mathbf{N}(S)$ and an internal moment $\mathbf{M}(S)$. These internal
 60 forces and moments are the loads exerted on the section at S by the part of the rod at $\bar{S} > S$. In
 61 the absence of body force and couple, the linear and angular momentum balance then read

$$\mathbf{N}'(S, T) = \rho h b \dot{\mathbf{R}}(S, T), \quad (3)$$

$$\mathbf{M}'(S, T) + \mathbf{R}'(S, T) \times \mathbf{N}(S, T) = \rho I \ddot{\theta}(S, T), \quad (4)$$

62 where $(\cdot)' \stackrel{\text{def}}{=} \partial/\partial S$, $(\dot{\cdot}) \stackrel{\text{def}}{=} \partial/\partial T$, T is time, ρ the mass per unit volume of the material, and I the
 63 second moment of area of the cross-section (in the present case $I = h^3 b/12$).

64 *Constitutive law*

65 We use the standard linear constitutive relationship relating the bending strain $\kappa(S) \stackrel{\text{def}}{=} \theta'(S)$
 66 to the bending moment $M_3 \stackrel{\text{def}}{=} \mathbf{M} \cdot \mathbf{d}_3$:

$$M_3 = E I \kappa, \quad (5)$$

67 where E is Young's modulus. Note that κ is not the curvature in general. In a similar way, the
 68 tension $N_1 \stackrel{\text{def}}{=} \mathbf{N} \cdot \mathbf{d}_1$ and the shear force $N_2 \stackrel{\text{def}}{=} \mathbf{N} \cdot \mathbf{d}_2$ are linked to the extension e_1 and shear
 69 strains e_2 through

$$N_1 = E h b e_1, \quad (6)$$

$$N_2 = G h b e_2. \quad (7)$$

70 where G is the shear modulus.

71 *Equations in component form*

72 In the planar case considered here, we have $Z(S, T) \equiv 0$, $N_z(S, T) \equiv 0$, $M_x(S, T) \equiv 0$, and
73 $M_y(S, T) \equiv 0$, $\forall(S, T)$ so that the equations for the six remaining unknowns are

$$X' = (1 + e_1) \cos \theta - e_2 \sin \theta, \quad (8a)$$

$$Y' = (1 + e_1) \sin \theta + e_2 \cos \theta, \quad (8b)$$

$$\theta' = M/(EI), \quad (8c)$$

$$M' = e_2 N_1 - (1 + e_1) N_2 + \rho I \ddot{\theta}, \quad (8d)$$

$$N'_x = \rho h b \ddot{X}, \quad (8e)$$

$$N'_y = \rho h b \ddot{Y}, \quad (8f)$$

74 where $M = M_z = M_3$, $N_1 = N_x \cos \theta + N_y \sin \theta$, and $N_2 = -N_x \sin \theta + N_y \cos \theta$. The strains (e_1, e_2)
75 are given by Eqs (6) and (7) as functions of N_x and N_y .

76 *Dimensionless variables*

77 We scale all lengths with L , time with $\tau \stackrel{\text{def}}{=} L^2 \sqrt{\rho h b / (EI)}$, forces with EI/L^2 , and moments
78 with EI/L . This naturally introduces a parameter

$$\eta \stackrel{\text{def}}{=} \frac{I}{h b L^2} = \frac{1}{12} \left(\frac{h}{L} \right)^2, \quad (9)$$

79 which takes small values in the present case of slender rods. Dimensionless variables will be
80 written lowercase, e.g. $x \stackrel{\text{def}}{=} X/L$, or $m \stackrel{\text{def}}{=} ML/(EI)$. The constitutive relations (6) and (7) read:
81 $e_1 = \eta n_1$ and $e_2 = 2(1 + \nu) \eta n_2$, where the Poisson ratio ν arises from the relation $E = 2(1 + \nu) G$.

82 Rods with $\eta > 0$ are extensible, shearable rods for which the rotational inertia is accounted
83 for. They will be simply called *extensible* rods. Rods with $\eta = 0$ will be simply called *inextensi-*
84 *ble* rods, although they really are inextensible, unshearable rods for which the rotational inertia
85 is ignored. Such rods are frequently called elastica [16, 17].

86 3. Small-amplitude vibrations around pre- and post-buckled equilibrium

87 The systems of equations (8) in dimensionless form reads

$$x'(s, t) = \cos \theta + \eta (n_1 \cos \theta - 2(1 + \nu) n_2 \sin \theta), \quad (10a)$$

$$y'(s, t) = \sin \theta + \eta (n_1 \sin \theta + 2(1 + \nu) n_2 \cos \theta), \quad (10b)$$

$$\theta'(s, t) = m, \quad (10c)$$

$$m'(s, t) = -n_2 + \eta ((1 + 2\nu) n_1 n_2 + \ddot{\theta}), \quad (10d)$$

$$n'_x(s, t) = \ddot{x}, \quad (10e)$$

$$n'_y(s, t) = \ddot{y}, \quad (10f)$$

88 with $n_1 = n_x \cos \theta + n_y \sin \theta$ and $n_2 = -n_x \sin \theta + n_y \cos \theta$. In our problem, we consider a clamped-
89 clamped rod and control the end-shortening $d \stackrel{\text{def}}{=} 1 - (x(1, t) - x(0, t))$. This setup implies the
90 following boundary conditions:

$$x(0, t) = 0 \quad x(1, t) = 1 - d, \quad (11a)$$

$$y(0, t) = 0 \quad y(1, t) = 0, \quad (11b)$$

$$\theta(0, t) = 0 \quad \theta(1, t) = 0. \quad (11c)$$

91 For each given value of the end-shortening d , we find the equilibrium configuration $(x_e, y_e, \theta_e,$
 92 $m_e, n_{xe}, n_{ye})$ by solving system (10) with $\ddot{x}_e = 0$ and $\ddot{y}_e = 0$. Then we look for small-amplitude
 93 vibrations around this equilibrium configuration, that is, we set

$$x(s, t) = x_e(s) + \delta \bar{x}(s) e^{i\omega t}, \quad (12a)$$

$$y(s, t) = y_e(s) + \delta \bar{y}(s) e^{i\omega t}, \quad (12b)$$

$$\theta(s, t) = \theta_e(s) + \delta \bar{\theta}(s) e^{i\omega t}, \quad (12c)$$

$$m(s, t) = m_e(s) + \delta \bar{m}(s) e^{i\omega t}, \quad (12d)$$

$$n_x(s, t) = n_{xe}(s) + \delta \bar{n}_x(s) e^{i\omega t}, \quad (12e)$$

$$n_y(s, t) = n_{ye}(s) + \delta \bar{n}_y(s) e^{i\omega t}, \quad (12f)$$

94 where $\delta \ll 1$ is a small parameter, and ω is the frequency of the vibration. Inserting (12) into (10)
 95 and keeping only linear terms in δ , we obtain equations for the spatial modes $(\bar{x}, \bar{y}, \bar{\theta}, \bar{m}, \bar{n}_x, \bar{n}_y)$:

$$\begin{aligned} \bar{x}'(s) &= -\bar{\theta} \sin \theta_e + \eta (\bar{n}_1 \cos \theta_e - 2(1 + \nu) \bar{n}_2 \sin \theta_e) + \\ &\quad \eta \bar{\theta} (-n_{1e} \sin \theta_e - 2(1 + \nu) n_{2e} \cos \theta_e), \end{aligned} \quad (13a)$$

$$\begin{aligned} \bar{y}'(s) &= \bar{\theta} \cos \theta_e + \eta (\bar{n}_1 \sin \theta_e + 2(1 + \nu) \bar{n}_2 \cos \theta_e) + \\ &\quad \eta \bar{\theta} (n_{1e} \cos \theta_e - 2(1 + \nu) n_{2e} \sin \theta_e), \end{aligned} \quad (13b)$$

$$\bar{\theta}'(s) = \bar{m}, \quad (13c)$$

$$\bar{m}'(s) = -\bar{n}_2 + \eta \left((1 + 2\nu)(\bar{n}_1 n_{2e} + n_{1e} \bar{n}_2) - \omega^2 \bar{\theta} \right), \quad (13d)$$

$$\bar{n}'_x(s) = -\omega^2 \bar{x}, \quad (13e)$$

$$\bar{n}'_y(s) = -\omega^2 \bar{y}, \quad (13f)$$

96 with $\bar{n}_1 = \bar{n}_x \cos \theta_e + \bar{n}_y \sin \theta_e + \bar{\theta} (-n_{xe} \sin \theta_e + n_{ye} \cos \theta_e)$ and $\bar{n}_2 = -\bar{n}_x \sin \theta_e + \bar{n}_y \cos \theta_e +$
 97 $\bar{\theta} (-n_{xe} \cos \theta_e - n_{ye} \sin \theta_e)$. The boundary conditions on the spatial modes are

$$\bar{x}(0) = 0 \quad \bar{x}(1) = 0, \quad (14a)$$

$$\bar{y}(0) = 0 \quad \bar{y}(1) = 0, \quad (14b)$$

$$\bar{\theta}(0) = 0 \quad \bar{\theta}(1) = 0. \quad (14c)$$

98 For given parameters η and ν and given end-shortening d , the equilibrium $(x_e, y_e, \theta_e, m_e, n_{xe}, n_{ye})$
 99 is first computed from (10) with $\ddot{x}_e = 0$ and $\ddot{y}_e = 0$. Then the 6D system (13) with the six bound-
 100 ary conditions (14) is a well-defined boundary value problem, but with the additional unknown
 101 ω . For computational purpose, we normalize the linear solution of this problem by imposing the
 102 condition

$$\bar{m}^2(0) + \bar{n}_x^2(0) + \bar{n}_y^2(0) = 1. \quad (15)$$

103 4. Equilibrium

104 We use a 'home-made' predictor-corrector path-following code to address the problem nu-
 105 merically. For each value of $p \stackrel{\text{def}}{=} -n_{xe} = -N_{xe} L^2 / (EI)$ in the interval $(0; 8\pi^2)$, we compute
 106 the equilibrium solution $(x_e, y_e, \theta_e, m_e, n_{xe}, n_{ye})$ satisfying clamped-clamped boundary conditions
 107 (11).

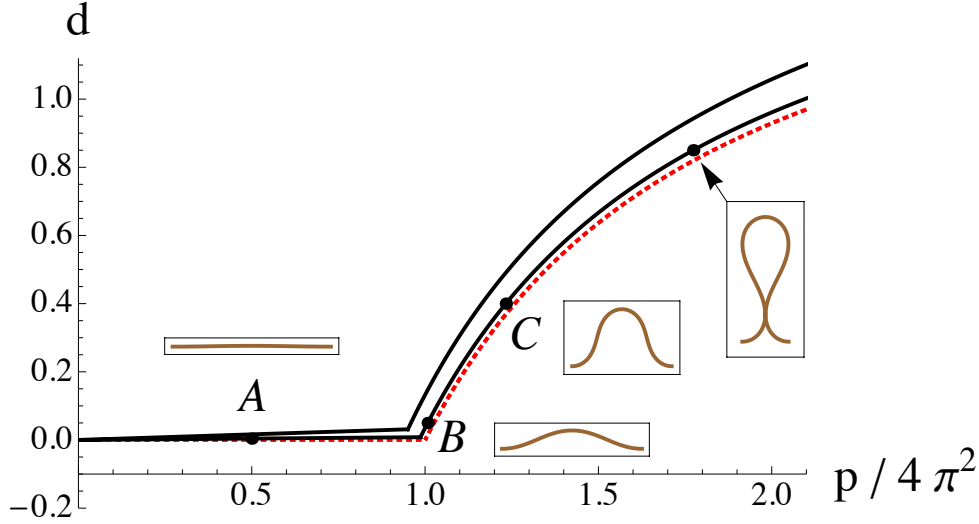


Figure 2: Fundamental and post-buckled equilibrium path of a clamped-clamped rod with $\eta = 1/1200$, $\eta = 1/4800$, and $\eta = 0$ (top to bottom). As the controlled end-shortening $d = D/L$ is gradually raised, an increasing axial load $p = -N_{xe}L^2/(EI)$ is recorded.

108 In Fig. 2 equilibrium paths are given for both inextensible (i.e. $\eta = 0$) and extensible ($\eta > 0$)
 109 rods. In the latter case, we see that the displacement d starts to increase as soon the curve leaves
 110 the origin: due to extensibility the rod shorten before buckling. Buckling happens at $p = 4\pi^2$
 111 for the inextensible case and at lower values for extensible cases. These equilibrium paths show that
 112 extension and shear play a minor role in the buckling load and that the inextensible rod solution
 113 is obtained in the limit $\eta \rightarrow 0$ of the extensible rod solution, as expected. Closed-form formula
 114 for the dotted (red) curve is given in Appendix A.

115 5. Vibrations in the extensible case

116 *Vibrations around the straight state*

117 We first consider the equilibrium solution where the axially loaded rod is straight, but compressed:
 118

$$y_e(s) = 0, \quad x_e(s) = (1 - \eta p) s, \quad n_{ye}(s) = 0, \quad n_{xe}(s) = -p, \quad \theta_e(s) = 0, \quad m_e(s) = 0. \quad (16)$$

119 The vibrations around this straight equilibrium are either extensional or flexural. Extensional
 120 vibrations modes are solutions of:

$$\bar{x}'(s) = \eta \bar{n}_x \quad (17a)$$

$$\bar{n}'_x(s) = -\omega^2 \bar{x} \quad (17b)$$

121 with boundary conditions $\bar{x}(0) = 0 = \bar{x}(1)$. This yields vibrations frequencies of the form
 122 $\omega = j\pi/\sqrt{\eta}$ (with $j = 1, 2, \dots$), which do not vary with the load p . The upper curve, before
 123 buckling of Fig. 3-(a) is such a solution. Flexural vibrations modes are solutions of:

$$\bar{y}'''' + \mu_p \bar{y}'' - \mu_\omega^2 \bar{y} = 0, \quad (18)$$

124 with boundary conditions $\bar{y}(0) = 0 = \bar{y}(1)$ and $(\eta_\star^2 \omega^2 + [1 - p(\eta - \eta_\star)]^2) \bar{y}'(0, 1) + \eta_\star \bar{y}'''(0, 1) =$
 125 0 . We introduced the notations $\eta_\star = 2\eta(1 + \nu)$, $\mu_p = (1 - p(\eta - \eta_\star))p + \omega^2(\eta + \eta_\star)$, and
 126 $\mu_\omega^2 = \omega^2([1 - p(\eta - \eta_\star)]^2 - [1 - p(\eta - \eta_\star)]p\eta_\star - \eta\eta_\star\omega^2)$. The general solution is of the form:

$$\bar{y}(s) = A \cos k^+ s + B \cosh k^- s + C \sin k^+ s + D \sinh k^- s \quad (19)$$

127 where we used the two wave numbers $k^\pm = (1/\sqrt{2}) \sqrt{\mu_p^2 + 4\mu_\omega^2 \pm \mu_p}$. The boundary condi-
 128 tions require that

$$2a^+ a^- k^+ k^- (\cos k^+ \cosh k^- - 1) + ((a^+ k^+)^2 - (a^- k^-)^2) \sin k^+ \sinh k^- = 0, \quad (20)$$

129 with $a^\pm = [1 - p(\eta - \eta_\star)]^2 + \eta_\star (\mp(k^\pm)^2 + \eta_\star \omega^2)$, which is an equation for ω . The analytical
 130 solutions $\omega(p)$ given by (20) match numerical solutions given in Fig. 3-(a,b,c). Buckling occurs
 131 when $\omega = 0 = \mu_\omega$, i.e. for $k^+ = \sqrt{\mu_p} = 2\pi$ and $k^- = 0$. This yields:

$$p = \frac{\sqrt{1 + 16\pi^2(\eta_\star - \eta)} - 1}{2(\eta_\star - \eta)} = 4\pi^2 - 16\pi^4(\eta_\star - \eta) + O(\eta^2) \quad (21)$$

132 Vibrations around the buckled state

133 Once the equilibrium solution is known, we solve the boundary value problem (13)-(14)
 134 numerically with a shooting method: (i) we first use a guess for the unknown parameters $\chi =$
 135 $(\bar{m}(0), \bar{n}_x(0), \bar{n}_y(0), \omega)$ and we integrate the system (13) up to $s = 1$; (ii) we then check if the
 136 boundary conditions (14)-(15) are satisfied. If not, we change the guess χ accordingly (using a
 Newton-Raphson scheme) until the boundary conditions at $s = 1$ are satisfied.

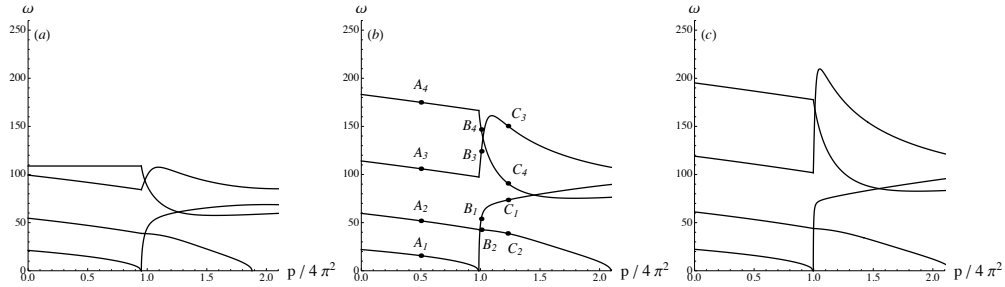


Figure 3: Frequencies for the lowest four vibration modes of a clamped-clamped rod around its fundamental and post-buckled equilibrium configurations in the extensible case: (a) for $\eta = 1/1200$, (b) for $\eta = 1/4800$, (c) for $\eta = 1/19200$ (plain). The labels A_i, B_i, C_i with $i = 1, 2, 3, 4$ correspond to the shapes given in Figs. 4, 5, 6.

137 Once a solution χ_i is found for a given $p = p_i$, we set $p = p_{i+1}$ and use the value χ_i as
 138 starting guess (predictor step) for the shooting method at $p = p_{i+1}$ (corrector step). In this
 139 setup, each curve $\omega = \omega(p)$ represents a path in the numerical bifurcation diagram, and we have
 140 numerically computed the four first paths (i.e. lowest four curves $\omega = \omega(p)$) for several values
 141 of the parameter $\eta = I/(hwL^2)$.
 142

143 In Fig. 3, frequencies for the first four modes are given as a function of the parameter p , for
 144 different values of η . It should be noted that the computations performed here are for a displace-
 145 ment control experiment, that is d (and not p) is controlled. Nevertheless, as there is a one-to-one

146 correspondence between d and p , for each value of the applied longitudinal displacement d , the
 147 equilibrium axial load p is read from Fig. 2 and then the frequency is computed and plotted in
 148 Fig. 3. We see in Fig. 3 that as $\eta \rightarrow 0$, frequencies globally increase and tend toward limiting
 149 curves. Finally we note that every curve $\omega(p)$ is continuous, but that the curves for the odd
 150 modes experience a rapid increase just after buckling. As $\eta \rightarrow 0$ this rapid increase becomes
 151 more abrupt to eventually turn into a discontinuity in the case $\eta = 0$, see Section 6. In each of
 152 Figs. 4, 5, and 6, dynamical shapes $(x(s, t) = x_e(s) + \bar{x}(s) \cos \omega t, y(s, t) = y_e(s) + \bar{y}(s) \cos \omega t)$ of
 the vibrating rod are plotted.

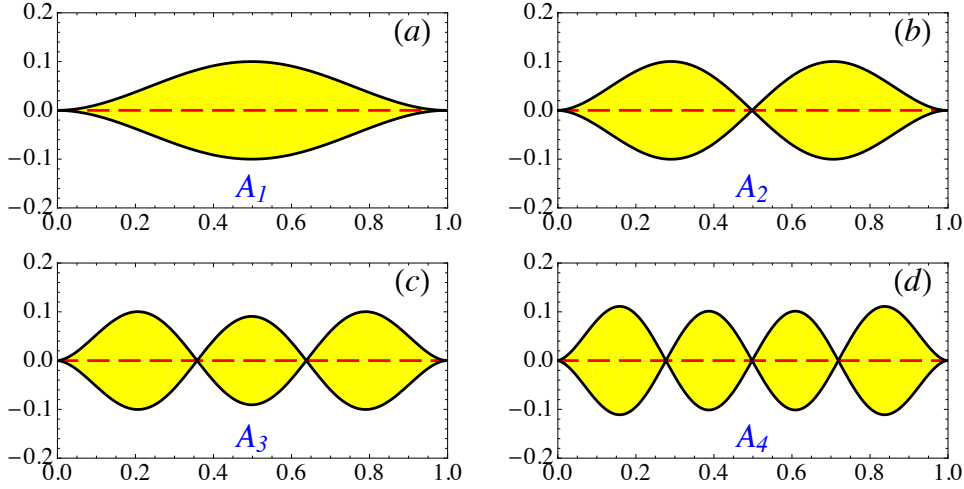


Figure 4: First four modes at $p/(4\pi^2) = 0.5$, for the extensible case with $\eta = 1/4800$: configurations A_1 (a), A_2 (b), A_3 (c), and A_4 (d) in Figure 3-(b).

153 We now focus on the first mode, which emerges from $\omega = 0$ at buckling. In Fig. 7-(a), we
 154 plot ω as function of the rise of the buckled rod at its mid-point: $y_e(1/2) = Y_e(L/2)/L$. Each
 155 curve corresponds to a different value of the parameter η , from $\eta = 1/1200$ (i.e. $L = 10h$)
 156 to $\eta = 1/480000$ (i.e. $L = 200h$). We see that all curves emerge from $\omega = 0$ at buckling
 157 ($y_e(1/2) = 0$) and asymptotically tend to the curve computed in the inextensible case (see Section
 158 6) when $y_e(1/2)$ becomes large. For very small η values, curves rise sharply from $\omega = 0$ and
 159 quickly approaches the inextensible asymptote. As a matter of fact these curves can be made to
 160 almost collapse on a master curve if the horizontal axis is plotted in unit of the rod thickness h :
 161 in Fig. 7-(b) we plot the frequency ω as function of $Y_e(L/2)/h$, for the same set of η values. All
 162 curves nearly collapse on a master curve which has a (numerically determined) slope $\simeq 28$ at the
 163 origin. Using $h = \sqrt{12\eta} L$ we obtain:
 164

$$\omega \simeq \frac{8.1}{\sqrt{\eta}} \frac{Y_e(1/2)}{L}, \quad (22)$$

165 which means that the limit $\eta \rightarrow 0$ is singular for this first mode. The same phenomenon happens
 166 for all odd modes.

167 In dimensional form (i.e. $\Omega = \omega \sqrt{EI/(\rho hb)}/L^2$) we have the following expression for the

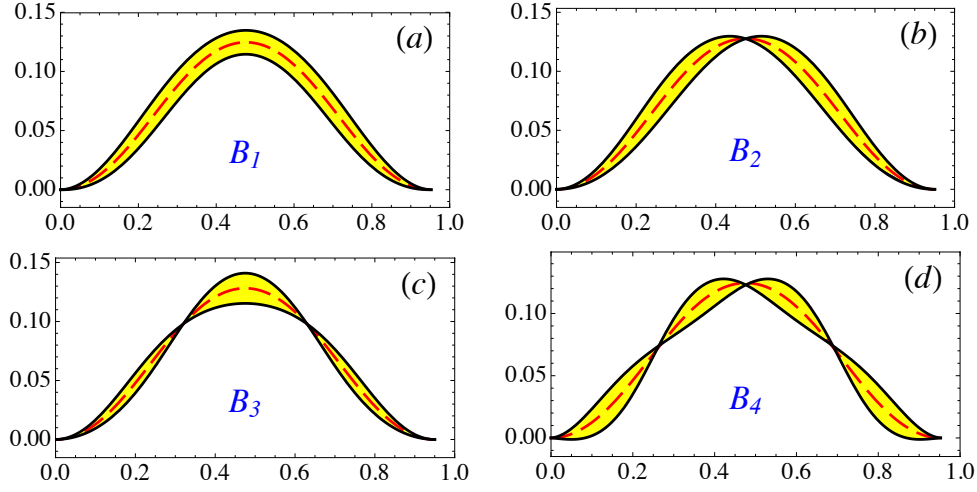


Figure 5: First four modes at $d = 0.05$, for $\eta = 1/4800$: configurations D_1 (a), D_2 (b), D_3 (c), and D_4 (d) in Figure 3.

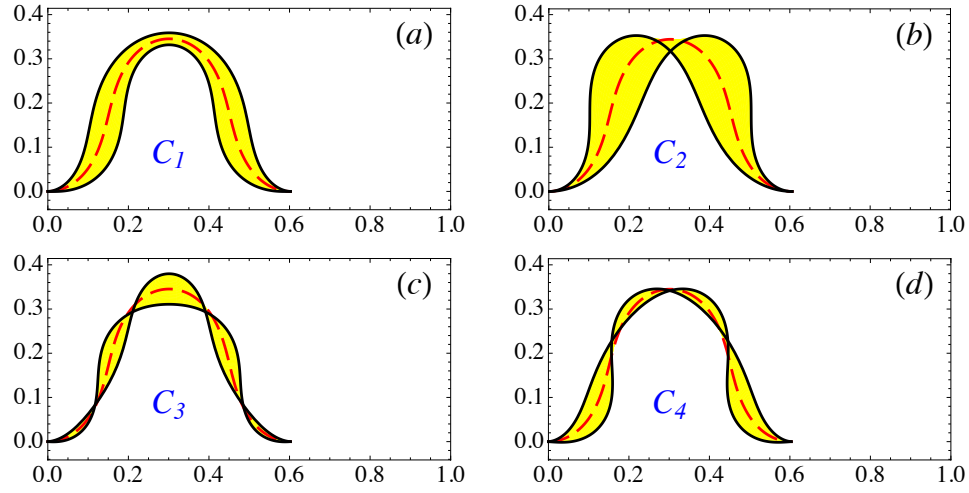


Figure 6: First four modes at $d = 0.4$, for $\eta = 1/4800$: configurations E_1 (a), E_2 (b), E_3 (c), and E_4 (d) in Figure 3.

168 frequency Ω (in rad/s):

$$\Omega \simeq 8.1 \frac{Y_e(L/2)}{L^2} \sqrt{\frac{E}{\rho}}, \quad \text{for } Y_e(L/2) \lesssim 2h. \quad (23)$$

169 The presence of the celerity $c = \sqrt{E/\rho}$ of compression elastic waves in this expression shows
 170 that directly after buckling and for a short loading interval (i.e. $Y_e(L/2) = 0$ to $Y_e(L/2) \simeq 2h$),
 171 the lowest mode of vibration of a buckled rod is of extension-compression type. We also see in
 172 Fig. 7-(b) that for $Y_e(L/2) \gtrsim 8h$ the behavior is of flexural type (i.e. curves have reached the
 173 inextensible asymptote). This separation between two different behaviors in the elastic response

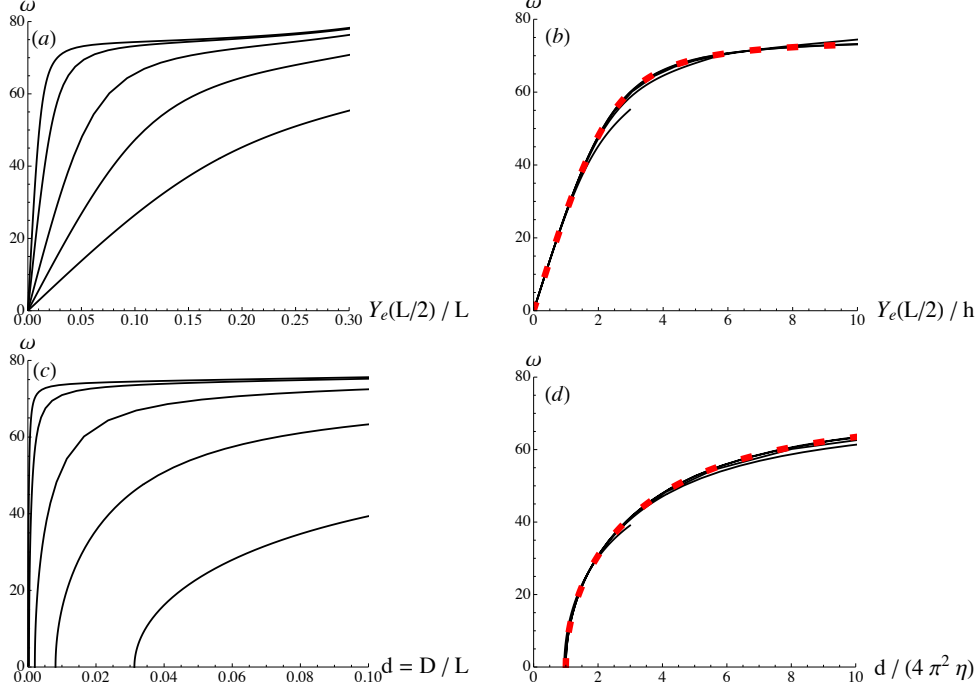


Figure 7: Post-buckled frequency of the mode emerging from $\omega = 0$ at buckling (for $\eta > 0$). (a) From bottom to top: $1/\eta = 1200, 4800, 19200, 120000, 480000$. (b) Same data but with horizontal axis rescaled with h ; The curves nearly collapse on a master curve whose slope at the origin is ≈ 28 . The dashed curve (red online), whose slope at the origin is $2\sqrt{2}\pi^2 \approx 27.9$ and which is hardly distinguishable from the previous ones, is the first mode solution of (25). (c) Same data as in (a) but with $d = D/L$ on the horizontal axis. (d) Same data but with the horizontal axis rescaled with $(4\pi^2 \eta)$; As in (b) the curves nearly collapse on a master curve. The dashed curve (red online), which is hardly distinguishable from the previous ones, is the first mode solution of (25).

174 of the rod could be a way to define the notion of a shallow (resp. deep) buckled equilibrium
 175 shape: a shallow (resp. deep) equilibrium shape has a vibrational response that is primarily
 176 extensional (resp. flexural).

177 We now focus on the second mode and show that it does not suffer the same singularity as the
 178 first one. The mode frequency emerges from a finite value $\omega_b = \omega_b(\eta)$ at buckling, and we see in
 179 Fig. 3 that its variation (with p) after buckling is much slower than for the first mode. In Figure 8
 180 the variation of the second mode frequency is plotted as a function of $y_e(1/2) = Y_e(L/2)/L$. We
 181 numerically extract the approximation:

$$\omega \simeq \omega_b(\eta) - (9 - 5400\eta) \left(\frac{Y_e(L/2)}{L} \right)^2 \quad (24)$$

182 In conclusion, the limit $\eta \rightarrow 0$ is smooth for the second mode (and in fact for all even modes).

183 An alternative equation is used in (for example) [12, 13, 14, 15] for the vibrations of post-
 184 buckled extensible rods. The transverse displacement $Y(S, T)$ is there solution of the equation:

$$EIY'''' + \rho h b \ddot{Y} + PY'' = 0 \quad (25a)$$

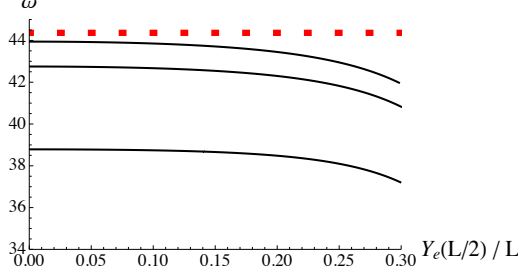


Figure 8: Post-buckled frequency of the second mode (for $\eta > 0$). Continuous (black) curves are, from bottom to top, for $1/\eta = 1200, 4800, 19200$. The (red) dashed curve is the second mode solution of (25).

$$\text{with } P = \frac{Ehb}{L} \left(D - \frac{1}{2} \int_0^L Y'^2 dS \right). \quad (25b)$$

186 In this model a certain number of assumptions on the nonlinear terms are made, see e.g. Eq. (B.3).
 187 Calculating the frequency of the first mode just after buckling, we find:

$$\Omega \approx \sqrt{2/3} \pi^2 \frac{Y_e(L/2)}{L^2} \sqrt{\frac{E}{\rho}} \quad (26)$$

188 where $\sqrt{2/3} \pi^2 \approx 8.06$, in agreement with our result (23). We plot in Fig. 7-(b) and (d) the first
 189 mode solution of Eq. (25), and we note that this solution is independent of η , which is not the
 190 case for Kirchhoff equations. Moreover the second mode of (25) is found to be independent of
 191 $Y_e(L/2)$ (see Fig. 8), in contradiction with (24). A more comprehensive comparison of the two
 192 models is the subject of a forthcoming paper.

193 6. Vibrations in the inextensible case

194 *Vibrations around the straight state*

195 We consider an inextensible unshearable rod of length L , strongly held at both sides by
 196 clamps separated by a distance L . The rod is held straight and hence no flexural dynamics at
 197 all can take place, i.e. flexural vibrations (that would be given by the $\eta = 0$ version of Eq. (18))
 198 are impossible here as they would require shortening of the ends.

199 Axial vibrations are given by:

$$\bar{x}'(s) = 0 \quad (27a)$$

$$\bar{n}'_x(s) = -\omega^2 \bar{x} \quad (27b)$$

200 with boundary conditions $\bar{x}(0) = 0 = \bar{x}(1)$. The solution is $\bar{x}(s) \equiv 0$, and $\bar{n}_x(s) \equiv \text{const.}$, where
 201 we see that any ω is admissible. Even if no extensional deformation is present, the load in this
 202 statically indeterminate problem can fluctuate with any frequency. This is illustrated in Fig. 9 by
 203 the (red) shaded region in the interval $0 \leq p \leq 4\pi^2$.

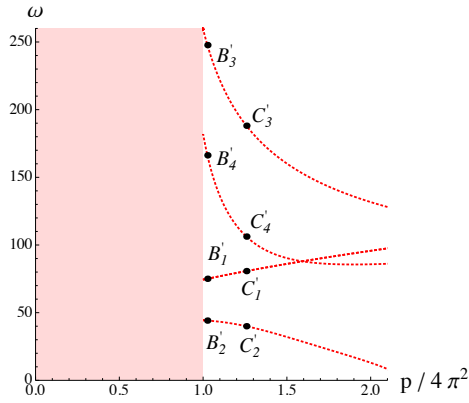


Figure 9: Frequencies for the lowest four vibration modes of a clamped-clamped rod around its fundamental and post-buckled equilibrium configurations in the inextensible case $\eta = 0$. The labels B'_i, C'_i with $i = 1, 2, 3, 4$ correspond to the shapes given in Figs. 10, 11.

204 *Vibrations around the buckled state*

205 As in the extensible case we solve the boundary value problem (13)-(14) numerically with a
 206 shooting method. The results are shown in Fig. 9 and in particular we see that at the buckling
 207 threshold, the first four frequencies are: $\omega \simeq 44.36, 74.4, 182.1,$ and 259.4 . Only one out of two
 208 of these frequencies is close to what was found in Fig. 3-(c).

209 Dynamical shapes ($x(s, t) = x_e(s) + \bar{x}(s) \cos \omega t, y(s, t) = y_e(s) + \bar{y}(s) \cos \omega t$) of the vibrating
 rod are plotted in Figs. 10, 11.

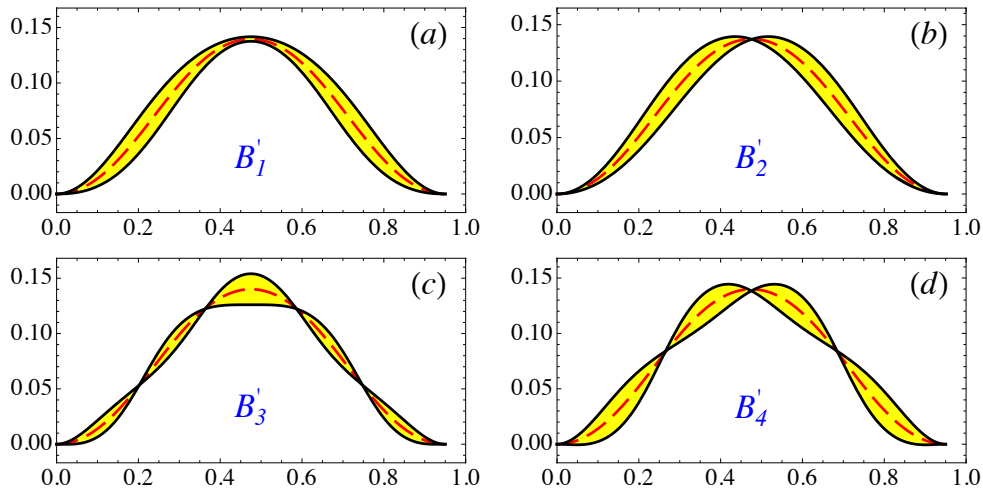


Figure 10: First four modes at $d = 0.05$, in the inextensional case: configurations B_1 (a), B_2 (b), B_3 (c), and B_4 (d) in Figure 3.

210

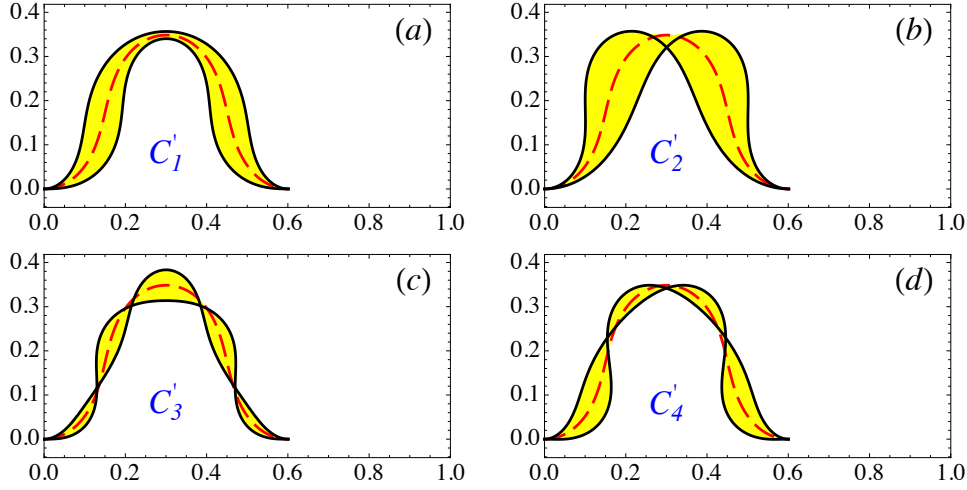


Figure 11: First four modes at $d = 0.4$, in the inextensible case: configurations C_1 (a), C_2 (b), C_3 (c), and C_4 (d) in Figure 3.

211 7. Comparison of the extensible and inextensible results

212 We here compare the post-buckled vibrations frequencies obtained numerically for the extensible ($\eta > 0$) and inextensible ($\eta = 0$) cases. On the one hand, we see in Fig. 9 that in the

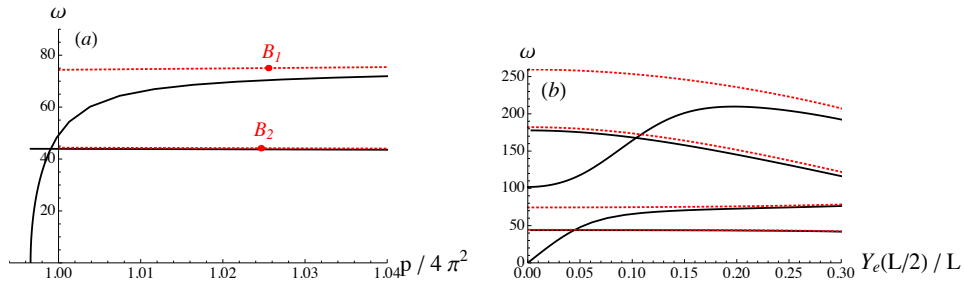


Figure 12: Frequencies of a clamped-clamped rod around its post-buckled equilibrium configurations. Comparison of the extensible case ($\eta = 1/19200$, plain curve) with the inextensible case (dotted, red) (a) for lowest two vibration modes, and (b) for lowest four vibration modes and plotted as a function of $Y_e(L/2)$, the rise of the buckled equilibrium shape at its middle point.

213 inextensible case there is no curve emerging from $\omega = 0$ at buckling, whereas there always is
 214 one for all η values in the extensible case (Fig. 3). On the other hand, in the inextensible case
 215 there is a curve emerging from $\omega \approx 74.4$, nowhere near any of the extensible curves. We plot
 216 in Fig. 12-(a) a comparison of the inextensible $\eta = 0$ and extensible $\eta = 1/19200$ cases, for the
 217 first two modes. We see that the second extensible mode (emerging from $\omega \approx 44$ at buckling)
 218 is always very close to its inextensible counterpart, but that the first extensible mode (emerging
 219 from $\omega = 0$ at buckling) is first very far from its inextensible counterpart and only approaches it
 220 later in the post-buckling regime. The consequence is that for a short interval after buckling the
 221

222 inextensible model wrongly predicts a fundamental mode at $\omega \simeq 44$ whereas for any ‘real life’
 223 elastic rod there is a vibration mode with lower frequency.

224 A full comparison between extensible and inextensible cases shows that all extensible odd
 225 modes experiences a similar mismatch with their inextensible counterpart, see Fig. 12-(b) for
 226 the first four modes. For the solution at buckling, with p given by (21), we further investigate
 227 numerically the variation with η of the frequencies of the first four modes in the extensible case
 and compare them with the first four frequency of the inextensible case $\eta = 0$. In Fig. 13, we

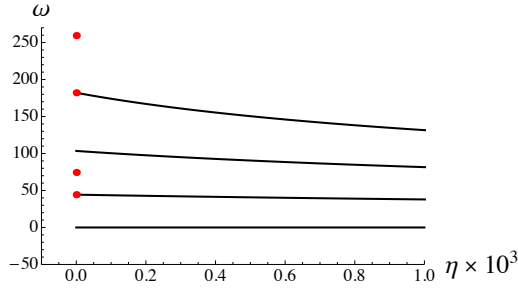


Figure 13: The curves show the variation with η of the frequencies of the first four modes in the extensible case, at the buckling threshold. The (red) dots at $\eta = 0$ show the frequency values of the first four modes of the inextensible case, at the buckling threshold. A mismatch for half of the modes exists.

228
 229 see that as $\eta \rightarrow 0$ the even extensible modes are converging to solutions of the inextensible case,
 230 but that the limit $\eta \rightarrow 0$ of the odd extensible modes does not correspond to the frequency values
 231 found in the inextensible case.

232 8. Analytical study for the inextensible case

233 In order to understand the mismatch of the odd modes’ frequencies at buckling between the
 234 extensible and inextensible cases, we look at the problem analytically. In the inextensible case,
 235 buckling happens for $p = 4\pi^2$. We look for the frequencies that emerge from the continuum
 236 present for $p \leq 4\pi^2$. If the $\eta \rightarrow 0$ limit was not singular we would just look for the solutions of
 237 (20) for $\eta = 0$ and $p = 4\pi^2$, that is:

$$2k^+k^- (\cos k^+ \cosh k^- - 1) + 4\pi^2 \sin k^+ \sinh k^- \quad (28)$$

238 with $k^\pm = \sqrt{\sqrt{4\pi^4 + \omega^2} \pm 2\pi^2}$. The solutions are listed in Table 1 and we see that, for the odd
 239 modes, they do not match what is found numerically.

240 To investigate matter further we set $\eta = 0$ in Eqs. (10), and look for the frequency values that
 241 exist just after buckling (i.e. around post-buckled configurations). In the post-buckled configura-
 242 tion, the equilibrium equations and the boundary conditions are given by

$$n_{ye} = \text{const.} \quad (29a)$$

$$n_{xe} = -p \quad (29b)$$

$$\theta_e'' = -p \sin \theta_e - n_{ye} \cos \theta_e \quad \text{with } \theta_e(0) = 0 = \theta_e(1) \quad (29c)$$

$$x_e' = \cos \theta_e \quad \text{with } x_e(1) - x_e(0) = 1 - d \quad (29d)$$

$$y_e' = \sin \theta_e \quad \text{with } y_e(0) = y_e(1), \quad (29e)$$

i	1	2	3	4	5	6	7	8
ω	0	44.36	103.5	182.1	280.6	398.8	536.8	694.6
k^+	2π	8.26	11.18	14.25	17.35	20.5	23.6	26.7
$(k^+ \bmod 2\pi)/(\pi/2)$		1.26	3.12	1.07	3.04	1.03	3.02	1.02
k^-	0	5.37	9.25	12.8	16.2	19.5	22.7	25.98

Table 1: Frequencies and wave numbers for the lowest eight modes of vibration as given by Eq. (20) for $p = 4\pi^2$ and $\eta = 0$. This also corresponds to the first eight solutions of $P_1(\omega_0) = 0$ (see Eq. (38)).

243 and without loss of generality, we choose $x_e(0) = y_e(0) = 0$. For equilibrium modes whose
244 shapes are invariant when reflected along the line parallel to the e_y axis and containing the point
245 $(x(1/2), 0)$, we have $n_{ye} = \text{const.} = 0$ in Eq. (29a). The first bifurcated equilibrium mode,
246 represented in Fig. 1 and on which we focus, is such a mode. We address the behavior of the
247 solutions after but close to buckling. Therefore, we expand the variables $\theta_e(s)$ and $y_e(s)$ in powers
248 of ϵ , a small parameter measuring the distance from buckling:

$$\theta_e(s) = \epsilon\theta_1(s) + \epsilon^2\theta_2(s) + \epsilon^3\theta_3(s) + O(\epsilon^4) \quad (30a)$$

$$x_e(s) = \epsilon x_1(s) + \epsilon^2 x_2(s) + \epsilon^3 x_3(s) + O(\epsilon^4) \quad (30b)$$

$$y_e(s) = \epsilon y_1(s) + \epsilon^2 y_2(s) + \epsilon^3 y_3(s) + O(\epsilon^4) \quad (30c)$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + O(\epsilon^4) \quad (30d)$$

249 We substitute these expansions in the equilibrium equations (29), which have to be satisfied to
250 all orders in ϵ . The solution up to order 3 reads:

$$\theta_e(s) = \epsilon \sin 2\pi s + \frac{\epsilon^3}{48} \cos^2(2\pi s) \sin(2\pi s) + O(\epsilon^4) \quad (31a)$$

$$x_e(s) = s + \frac{\epsilon^2}{16\pi} (\sin 4\pi s - 4\pi s) + O(\epsilon^4) \quad (31b)$$

$$y_e(s) = \frac{\epsilon}{2\pi} (1 - \cos 2\pi s) + \frac{\epsilon^3}{384\pi} (-20 + 23 \cos(2\pi s) - 3 \cos(6\pi s)) + O(\epsilon^4) \quad (31c)$$

$$p = 4\pi^2 + \epsilon^2 \pi^2 / 2 + O(\epsilon^4) \quad (31d)$$

251 In order to relate ϵ to the control parameter d and the amplitude after bifurcation, we compute
252 the end-shortening

$$d = 1 - (x_e(1) - x_e(0)) = \epsilon^2/4 + O(\epsilon^4) = 2 \left(\frac{p}{4\pi^2} - 1 \right) + O(\epsilon^4), \quad (32)$$

253 and the rod maximum deflection

$$y_e(1/2) = \frac{\epsilon}{\pi} \left(1 - \frac{5}{48} \epsilon^2 \right) + O(\epsilon^4). \quad (33)$$

254 *Vibration around the post-buckled equilibrium*

255 We expand all modal variables $(\bar{x}, \bar{y}, \bar{\theta}, \bar{m}, \bar{n}_x, \bar{n}_y)$ and the frequency ω in powers of ϵ . For
256 instance, we have $\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 + O(\epsilon^4)$, and so on. We can now solve equations

257 (13) (with $\eta = 0$) with boundary conditions (14), using the equilibrium solution (31). To order
 258 ϵ^0 we have:

$$\bar{x}'_0 = 0 \text{ with } \bar{x}_0(0) = 0 = \bar{x}_0(1) \quad (34a)$$

$$\bar{n}'_{x0} = -\omega_0^2 \bar{x}_0 \quad (34b)$$

$$\bar{y}_0'''' + 4\pi^2 \bar{y}_0'' - \omega_0^2 \bar{y}_0 = 0 \text{ with } \bar{y}_0(0) = \bar{y}_0(1) = \bar{y}'_0(0) = \bar{y}'_0(1) = 0 \quad (34c)$$

259 The first two equations describe the longitudinal mode and are decoupled from the third one
 260 which is associated with the transverse mode. More precisely, the longitudinal mode is given by

$$\bar{x}_0(s) = 0 \text{ and } \bar{n}_{x0}(s) \text{ constant.} \quad (35)$$

261 whereas for the transverse mode, the solution $\bar{y}_0(s)$ is

$$\bar{y}_0(s) = A_0 \left(\frac{\cos k_0^+ s - \cosh k_0^- s}{\cos k_0^+ - \cosh k_0^-} - \frac{k_0^- \sin k_0^+ s - k_0^+ \sinh k_0^- s}{k_0^- \sin k_0^+ - k_0^+ \sinh k_0^-} \right), \quad (36)$$

262 with $k_0^\pm \stackrel{\text{def}}{=} \sqrt{\sqrt{4\pi^4 + \omega_0^2} \pm 2\pi^2}$. The boundary conditions impose that

$$A_0 P_1(\omega_0) = 0 \text{ with} \quad (37)$$

$$P_1(\omega_0) \stackrel{\text{def}}{=} 2 k_0^+ k_0^- (\cos k_0^+ \cosh k_0^- - 1) + 4\pi^2 \sin k_0^+ \sinh k_0^- \quad (38)$$

263 which is an equation for ω_0 . The first eight solutions of $P_1(\omega_0) = 0$ are listed in Table 1. Note
 264 that to order ϵ^0 there is no frequency jump. To order ϵ^1 we have

$$\bar{x}'_1 = \bar{\theta}_0 \sin 2\pi s \text{ with } \bar{x}_1(0) = 0 = \bar{x}_1(1) \quad (39a)$$

$$\bar{n}'_{x1} = -\omega_0^2 \bar{x}_1 - 2 \omega_0 \omega_1 \bar{x}_0 \quad (39b)$$

$$\bar{y}_1'''' + 4\pi^2 \bar{y}_1'' - \omega_0^2 \bar{y}_1 = 2 \omega_0 \omega_1 \bar{y}_0 + 2\pi \bar{n}_{x0} \cos 2\pi s \quad (39c)$$

$$\text{with } \bar{y}_1(0) = \bar{y}_1(1) = \bar{y}'_1(0) = \bar{y}'_1(1) = 0. \quad (39d)$$

265 We start by solving equation (39a). The boundary condition $\bar{x}_1(1) = 0$ implies

$$A_0 P_2(\omega_0) = 0 \text{ with} \quad (40)$$

$$P_2(\omega_0) \stackrel{\text{def}}{=} k_0^+ k_0^- \left((k_0^{+2} - 2\pi^2)(\cos k_0^+ - \cosh k_0^-) + 2\pi^2(\cos k_0^+ \cosh k_0^- - 1) \right) \\ + (k_0^{+2} k_0^{-2} + 8\pi^4) \sin k_0^+ \sinh k_0^-. \quad (41)$$

266 The solutions have to satisfy Eqs. (37) and (40), which are transcendental equations for ω_0 . A
 267 numerical root finding analysis reveals that $P_1(\omega_0) = 0$ and $P_2(\omega_0) = 0$ share half of their roots,
 268 see Table 2 where columns with an even index correspond to common roots and match numerical
 269 values at $p = 4\pi^2$ for the continuous curves plotted in Fig. 3-(d). In the case of a common
 270 root, Eqs. (37) and (40) are fulfilled for non-vanishing A_0 , and the corresponding modes have
 271 frequencies that are continuous in the control parameter close to buckling.

272 In the case of distinct roots, we are compelled to set $A_0 = 0$. In this case where $P_1(\omega_0) \neq 0$,
 273 we solve Eqs. (39a) and (39b) to obtain $\bar{x}_1(s) = 0$ and $\bar{n}_{x1}(s)$ constant. The general solution of
 274 Eqs. (39c), (39d) is then

$$\bar{y}_1(s) = \frac{2\pi \bar{n}_{x0}}{k_0^{+2} k_0^{-2} P_1(\omega_0)} \left[c_1 k_0^+ \cos k_0^+ s - c_1 k_0^+ \cosh k_0^- s - c_2 k_0^- \sin k_0^+ s + c_2 k_0^+ \sinh k_0^- s \right. \\ \left. + P_1(\omega_0)(\cos k_0^+ s - \cos 2\pi s) \right] \quad (42)$$

i	1	2	3	4	5	6	7	8
ω_0	0	44.36	169.4	182.1	390.6	398.8	688.5	694.6
k_0^+	2π	8.26	13.8	14.25	20.3	20.5	26.6	26.7
$(k_0^+ \bmod 2\pi)/(\pi/2)$		1.26	0.78	1.07	0.90	1.03	0.95	1.02
k_0^-	0	5.37	12.3	12.8	19.3	19.5	25.8	25.98

Table 2: First eight solutions of $P_2(\omega_0) = 0$ (see Eq. (41)).

275 where

$$c_1 = k_0^-(\cos k_0^+ - 1)(\cosh k_0^- + 1) + k_0^+ \sin k_0^+ \sinh k_0^- \quad (43)$$

$$c_2 = k_0^+ \sin k_0^+ (\cosh k_0^- - 1) + k_0^-(\cos k_0^+ - 1) \sinh k_0^- \quad (44)$$

276 In order to select a mode, we need to proceed to order ϵ^2 , which reads

$$\bar{x}_2'(s) = -\bar{y}_1'(s) \sin 2\pi s \text{ with } \bar{x}_2(0) = 0 = \bar{x}_2(1) \quad (45)$$

277 The boundary conditions at $s = 1$ impose $P_3(\omega_0)/P_1(\omega_0) = 0$ where

$$\begin{aligned} P_3(\omega_0) = & 2k_0^{-2}k_0^{+2}(\cosh k_0^- \cos k_0^+ - 1) - 4k_0^+(k_0^{-2} + k_0^{+2}) \sin k_0^+ (\cosh k_0^- - 1) \\ & + (4k_0^-(k_0^{-2} + k_0^{+2})(1 - \cos k_0^+) + 4\pi^2 k_0^- k_0^+ \sin k_0^+) \sinh k_0^- \end{aligned} \quad (46)$$

278 Oddly enough this function $P_3(\omega_0)$ has the same set of common roots as $P_1(\omega_0)$ and $P_2(\omega_0)$, see
279 Table 3. The other roots correspond to the frequency values at which the (discontinuous) odd
280 mode curves emerge from $p = 4\pi^2$ in Fig. 3-(d). Moreover one can verify that

$$P_3(\omega_0) = k_0^+ k_0^- P_1(\omega_0) - 8(k_0^{+2} + k_0^{-2}) \left(k_0^+ \cos \frac{k_0^+}{2} \tanh \frac{k_0^-}{2} - k_0^- \sin \frac{k_0^+}{2} \right) \sinh k_0^- \sin \frac{k_0^+}{2} \quad (47)$$

281 which implies that the common roots must verify:

$$\frac{k_0^+}{2} \tanh \frac{k_0^-}{2} = \frac{k_0^-}{2} \tan \frac{k_0^+}{2} \quad (48)$$

282 These roots correspond to frequencies that do not vary abruptly after buckling has occurred. An
283 approximate formula is $k_0^+ \simeq \pi/2 + 2j\pi$ (with positive integers j), which yields $\omega_0 \simeq (\pi/2)^2(4j +$
284 $1) \sqrt{(4j - 3)(4j + 5)}$ (corresponding to columns with $i > 1$ even in Table 3).

285 Formulas for the roots of the three functions P_1 , P_2 , and P_3 in the limit of large k_0^+ are given
286 in Appendix C. In particular it is shown that the set of roots of P_3 which is not in common with
287 P_1 and P_2 is such that $k_0^+ \simeq 3\pi/2 + 2j\pi$ (with positive integers j). This implies that the frequencies
288 emerging from buckling are such that $\omega_0 \simeq (\pi/2)^2(4j + 3) \sqrt{(4j + 7)(4j - 1)}$ (corresponding to
289 columns with $i > 1$ odd in Table 3).

290 9. Discussion

291 In [11] it is shown that, under certain hypotheses, in the zero thickness limit a slender cylinder
292 behaves either as a flexible rod or as an extensible string and that one does not need to consider

i	1	2	3	4	5	6	7	8
ω_0	0	44.36	74.4	182.1	259.4	398.8	517.4	694.6
k_0^+	2π	8.26	9.83	14.25	16.7	20.5	23.2	26.7
$(k_0^+ \bmod 2\pi)/(\pi/2)$		1.26	2.26	1.07	2.65	1.03	2.76	1.02
k_0^-	0	5.37	7.56	12.8	15.5	19.5	22.3	25.98

Table 3: First eight solutions of $P_3(\omega_0) = 0$ (see Eq. (46)).

293 both bending and extension in a slender rod problem. Marigo and Meunier [11] result was obtain
294 for the statics of a clamped-free rod subjected to body and concentrated forces. We have showed
295 here that for displacement control clamped-clamped boundary conditions a model including both
296 bending and extension is necessary to correctly describe the vibrations of the rod. One could then
297 ask which constraint should be relaxed in order to fall back into the results of [11]. We simply
298 remark that in the load control case, see Appendix D, the inextensible limit is smooth and hence
299 a model including bending alone correctly captures the vibrational behavior of the rod.

300 From a theoretical point of view, the singular inextensible limit in the displacement control
301 case is rather surprising and unexpected. Indeed, Equations (13) can be easily recast in the form
302 of a classical eigenvalue problem

$$\mathcal{L}\mathbf{X} = \omega^2 \mathbf{M}\mathbf{X} \quad (49)$$

303 where \mathbf{X} is the six-dimensional vector build from the six normal mode variables, \mathcal{L} is a first-
304 order linear operator in $L^2([0, 1])$, (the set of square integrable functions on the unit interval)
305 and \mathbf{M} is an inertia matrix. Therefore, a naive application of the classical theory of perturbation
306 of eigenvalues for linear operators would suggest that once the eigenvalues have been found
307 for a value of the parameter η , they can be locally continued in this parameter. That is we
308 would expect the curves $\omega(p)$ of Fig. 3 to vary continuously as the parameter η is decreased to
309 zero. While it is true for some curves, other curves behaves singularly in the $\eta \rightarrow 0$ limit. The
310 fundamental mathematical reason for this phenomenon is that both the null spaces of the inertia
311 matrix \mathbf{M} and the linear operator \mathcal{L} have a non-empty intersection in the inextensible ($\eta = 0$)
312 case. Mathematically, the classical theory does not apply and new conditions for the analytic
313 continuation of frequencies with respect to the parameters emerge. Some frequencies satisfy
314 these relations (and hence can be analytically continued) while others do not. Finally, also due
315 to the joint degeneracy of \mathbf{M} and \mathcal{L} , there exists for $\eta = 0$ a continuous family of solutions at
316 the critical load, given by an arbitrary increase along \bar{n}_x (i.e. solutions of Eq. (27)). For larger
317 values of the load but still at $\eta = 0$, some of these solutions are selected and emerge, apparently
318 out of the blue. The mathematical structure of these linear problems and how they are related to
319 various limits (nearly inextensible rods) deserves further attention.

320 A tentative classification for the modes is given by the number of nodes present in $0 < s < 1$,
321 a mode i having $i - 1$ nodes and i antinodes. We see that this classification works for the modes
322 in the pre-buckling regime (Fig. 4), but fails for the third mode in the inextensible ($\eta = 0$) case
323 (see shape B'_3 in Fig. 10 and C'_3 in Fig. 11), as well as in the extensible rod ($\eta > 0$) case provided
324 we are far enough in the post-buckling regime (see shape C_3 in Fig. 6).

325 An interesting situation arises when the first mode intersects with the second mode. This can
326 easily be obtained by computing an approximation of the buckled rod rise $Y_e(L/2)$ for which the
327 first mode frequency $\omega^{(1)} \simeq 28 Y_e(L/2)/h$ meets the second mode frequency $\omega^{(2)} \simeq 44$. This
328 happens for $Y_e(L/2) \simeq 1.6h$. At this height the shape of the fundamental mode changes from a

329 single bump wave to a double bump wave. Anecdotically, we note that piano soundboards are
 330 precisely tuned in this parameter range (e.g. $h = 1$ cm, $L = 2$ m, and $Y_e(L/2) = 1$ cm). To
 331 which extent this toy model is relevant for the real problem of the piano soundboard and whether
 332 piano manufacturers are using the distinction between shallow shapes (where extension prevails)
 333 and deep buckled shapes (where bending prevails) to enrich the sound is a tantalizing idea that
 334 deserves further attention. We also note that in the case of the piano, the soundboard is a plate
 335 and not a rod, and that furthermore the soundboard is linked to the piano strings by a bridge, so
 336 really the whole system should be studied.

337 10. Conclusion

338 We have studied the in-plane vibrations of a slender extensible and shearable elastic rod
 339 around its post-buckled equilibrium configuration, in the displacement control case. We have
 340 shown that after buckling there is a narrow window in the loading parameter values in which
 341 half of the vibration frequencies vary abruptly. As the thickness of the rod is decreased, the
 342 vibration frequencies tend toward limiting values which do not correspond to what is found with
 343 a fully inextensible, unshearable model (i.e. an elastica). This mismatch present for half of the
 344 vibration frequencies has been found numerically and proved analytically. We conclude that we
 345 have identified a loading setup where the elastica model for slender rods fails to give the correct
 346 answer.

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351 Appendix A. Closed form solution for the planar elastica

352 In the inextensible, unshearable case, the solution of system (10) with $\ddot{x}_e = 0$ and $\ddot{y}_e = 0$
 353 corresponds to the equilibrium of the planar elastica, first studied by Euler [16] (see [17] for an
 354 historical account).

355 Closed-form solutions of these equations can be written in terms of elliptic functions, see e.g.
 356 [18]. In particular the dotted (red) curve in Fig. 2 has the parametric expression:

$$p = 16 K(\lambda)^2 \quad (\text{A.1})$$

$$d = 2 \left(1 - \frac{E(\lambda)}{K(\lambda)} \right)^2 \quad (\text{A.2})$$

357 with $\lambda \in [0, 1)$. The elliptic integrals are defined as $K(\lambda) = \int_0^{\pi/2} (1 - \lambda \sin^2 \theta)^{-1/2} d\theta$ and $E(\lambda) =$
 358 $\int_0^{\pi/2} (1 - \lambda \sin^2 \theta)^{1/2} d\theta$. Developing Eqs. (A.1) and (A.2) for $\lambda \ll 1$ yields Eqs. (31d) and (32)
 359 with $\epsilon^2 = 4\lambda$.

360 **Appendix B. Von Karman kinematics and ‘Strength of materials’ notations**

361 In section 2, we have introduced the Cosserat-Kirchhoff notations where the normal force N_1
 362 is related to the extension e_1 through the constitutive law (6) and where the current position (X, Y)
 363 of the central axis of the rod is given by Eqs. (8a) and (8b), all these quantities being functions
 364 of the arc-length S of the reference configuration $(X_{\text{ref}}, Y_{\text{ref}}) = (S, 0)$. In ‘Strength of materials’
 365 notations one uses the displacements (see Fig.1):

$$U = X - S, \quad V = Y, \quad (\text{B.1})$$

366 also functions of the arc-length S of the reference configuration. In the current configuration the
 367 derivative of the current position with regard to S will not yield a unit vector if extension occurs.
 368 If we restrict to the case where shear is not present (i.e. $e_2 = 0$), we have $X'(S)^2 + Y'(S)^2 =$
 369 $(1 + e_1)^2$ which yields:

$$U' + \frac{1}{2}V'^2 - e_1 = \frac{1}{2}(e_1^2 - U'^2). \quad (\text{B.2})$$

370 Usually the right-hand side is neglected and the following von Karman approximation is used:

$$U' + \frac{1}{2}V'^2 - e_1 \simeq 0. \quad (\text{B.3})$$

371 **Appendix C. Analytical formulas for the roots of the functions P_1 , P_2 , and P_3**

372 In the limit of large k_0^+ , that is in the limit of large $k_0^- = \sqrt{k_0^{+2} - 4\pi^2}$ and high frequencies
 373 $\omega_0 = k_0^+ k_0^-$, we have:

$$k_0^- \simeq k_0^+ - \frac{2\pi^2}{k_0^+} - \frac{2\pi^4}{k_0^{+3}} + O\left(\frac{1}{k_0^{+5}}\right). \quad (\text{C.1})$$

374 Function $P_1(\omega_0)$ (see Eq. (38)) then reads:

$$P_1(\omega_0) \simeq e^{k_0^-} \left[k_0^{+2} \cos k_0^+ + 2\pi^2 (\sin k_0^+ - \cos k_0^+) \right], \quad (\text{C.2})$$

375 and the solutions to $P_1(\omega_0) = 0$, for large k_0^+ , are:

$$k_0^+ \simeq \frac{\pi}{2} + \frac{1}{2j^2} + 2j\pi \quad \text{with } i = 2j \quad (\text{C.3})$$

$$k_0^+ \simeq \frac{3\pi}{2} + \frac{1}{2j^2} + 2j\pi \quad \text{with } i = 2j + 1 \quad (\text{C.4})$$

376 where j is a large integer, and i is the mode number (i.e. column number in Table 1). In the same
 377 limit of high frequencies, function $P_2(\omega_0)$ (see Eq. (41)) reads:

$$P_2(\omega_0) \simeq \frac{1}{2} e^{k_0^-} \left[k_0^{+4} (\sin k_0^+ - 1) + 2\pi^2 k_0^{+2} \cos k_0^+ + 6\pi^4 \right], \quad (\text{C.5})$$

378 and the solutions to $P_2(\omega_0) = 0$, for large k_0^+ , are:

$$k_0^+ \simeq \frac{\pi}{2} - \frac{3}{2j^2} + 2j\pi \quad \text{with } i = 2j - 1 \quad (\text{C.6})$$

$$k_0^+ \simeq \frac{\pi}{2} + \frac{1}{2j^2} + 2j\pi \quad \text{with } i = 2j \quad (\text{C.7})$$

379 where j is a large integer, and i is the mode number (i.e. column number in Table 2). In the same
 380 limit of high frequencies, function $P_3(\omega_0)$ (see Eq. (46)) reads:

$$P_3(\omega_0) \simeq e^{k_0^-} \left[4k_0^{+3} (1 - \sin k_0^+) + k_0^{+4} \cos k_0^+ + 2\pi^2 k_0^{+2} \right], \quad (\text{C.8})$$

381 and the solutions to $P_3(\omega_0) = 0$, for large k_0^+ , are:

$$k_0^+ \simeq \frac{\pi}{2} + \frac{1}{2j^2} + 2j\pi \quad \text{with } i = 2j \quad (\text{C.9})$$

$$k_0^+ \simeq \frac{3\pi}{2} - \frac{4}{\pi j} + 2j\pi \quad \text{with } i = 2j + 1 \quad (\text{C.10})$$

382 where j is a large integer, and i is the mode number (i.e. column number in Table 3). We see that
 383 in this limit the three functions P_1 , P_2 , and P_3 share half of their roots, namely those given by
 384 (C.3), (C.7), or (C.9).

385 Appendix D. Smooth inextensible limit in the load control case

386 We here show that in the case where the axial load p is controlled, the inextensible limit
 387 ($\eta \rightarrow 0$) is no longer singular. The boundary conditions for this case read:

$$x(0, t) = 0 \quad n_x(1, t) = -p, \quad (\text{D.1a})$$

$$y(0, t) = 0 \quad y(1, t) = 0, \quad (\text{D.1b})$$

$$\theta(0, t) = 0 \quad \theta(1, t) = 0. \quad (\text{D.1c})$$

388 Vibrations of an inextensible rod are then possible before buckling as the right end is allowed to
 389 move axially. We see in Fig. D.14 that the inextensible case $\eta = 0$ is obtain as the smooth limit
 390 $\eta \rightarrow 0$ of extensible case, and that no mismatch is present (in particular there is a mode starting
 391 from $\omega = 0$ after buckling in the inextensible case).

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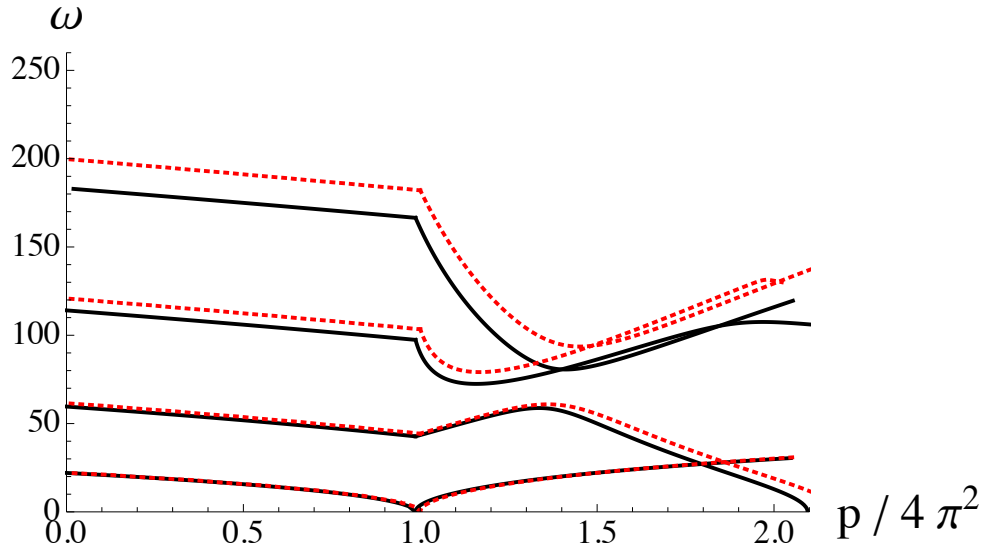


Figure D.14: Frequencies for the lowest four vibration modes of a clamped-clamped rod around its fundamental and post-buckled equilibrium configurations, *in load-control boundary conditions*. Plain curves are for the extensible case with $\eta = 1/4800$, and dotted (red) curves are for the inextensible case ($\eta = 0$).

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