The shape of attractors for 3-D dissipative dynamical systems

S. Neukirch*[,1] and H. Giacomini†[,1,2]
(1) C.N.R.S. Laboratoire de Mathématiques et Physique Théorique.
Faculté des Sciences et Techniques. Parc de Grandmont.
Université François Rabelais, 37200 Tours FRANCE.
(2) F.C.B.F. - Area de Fisica, Universidad nacional de Rosario,
2000 Rosario, Argentina

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Abstract

We introduce a new method to bound attractors of dissipative dynamical systems in phase and parameters spaces. The method is based on the determination of families of transversal surfaces (surfaces crossed by the flow in only one direction). This technique yields very restrictive geometric bounds in phase space for the attractors. It also gives ranges of parameters of the system for which no chaotic behaviour is possible. We illustrate our method on different 3-D dissipative systems.

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1 Introduction

We shall consider ordinary differential equations defining time evolution of 3-D dissipative dynamical systems :

$$\dot{x} = P(x,y,z)\,,\ \dot{y} = Q(x,y,z)\,,\ \dot{z} = R(x,y,z)\,, \tag{1}$$

with $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} < 0 \ \ \forall\,(x,y,z)$. Usually the functions P, Q and R are simple polynomials. These types of systems are dissipative : volumes in phase space contract under the flow because

*email: seb@celfi.phys.univ-tours.fr
†email: giacomin@celfi.phys.univ-tours.fr

the divergence of the vector field (P, Q, R) is always negative. Hence, the attractor of the system is necessarily of dimension less than three (it may be an equilibrium point, a limit cycle or a chaotic attractor). In this paper we are interested in the approximate location in phase space of the global attractor of the system, which contains all the dynamics evolving from all initial conditions. The global attractor is the set of points in phase space that can be reached from some initial condition set at an arbitrary long time in the past. The two fundamental properties of a global attractor are :

i- it is invariant under evolution,

ii- the distance of any solution from it vanishes as $t \to +\infty$.

This last property may simply be interpreted thus : if the solution starts outside the global attractor, then it is attracted into it as $t \to +\infty$ and once inside it cannot escape. Whereas if the solution starts inside the global attractor then it stays inside. The global attractor contains all the asymptotic motion for the dynamical system. It is common to talk of multiple attractors for a dynamical system and each of them may in its own right be considered as the attractor for initial conditions within its own basin of attraction. The notion of global attractor corresponds to the union of all such dynamically invariant attracting sets possible. In particular, it contains all possible structures such as equilibrium points, limit cycles, etc. The global attractor is sometimes contained in an absorbing ball in phase space and we want to obtain analytic estimates about its geometric shape. Moreover, this would enable us to find an upper bound for its Lyapunov dimension [2, 3].

Until very recently, approximated locations of attractors in phase space have been obtained by the method of Lyapunov functions. The latter is a smooth positive definite function that decreases along trajectories. This type of function is a generalisation of the energy function for mechanical systems : in the presence of friction or other dissipation, the energy decreases monotonically and the system stabilises on an equilibrium state where the energy is minimal.

Let us consider, as an example, the Lorenz system [5] defined by :

$$\dot{x} = \sigma\,(y - x)\;, \quad \dot{y} = r\,x - y - x\,z\;, \quad \dot{z} = x\,y - b\,z, \tag{2}$$

where σ, r, b are positive parameters. For $r < 1$ and σ and b arbitrary, every trajectory approaches the origin as $t \to +\infty$: the origin is globally stable. Hence there can be no limit cycle nor chaos for $r < 1$. The proof of this important result can be obtained by constructing an adequate Lyapunov function. There is no systematic way to construct these Lyapunov functions but often it is wise to try expressions involving sums of squares. Here we consider $V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$. The surfaces of constant V are concentric ellipsoïds about the origin. The idea is to show that if $r < 1$ and $(x, y, z) \neq (0, 0, 0)$, then $\dot{V} < 0$ along all trajectories. This would imply that each trajectory keeps moving to lower V and hence penetrates smaller and smaller ellipsoïds as $t \to +\infty$. But $V(x, y, z)$ is bounded below by 0, so $V(x(t), y(t), z(t)) \to 0$ and hence $(x(t), y(t), z(t)) \to 0$, as desired. Now we calculate :

$$\begin{aligned} \dot{V} &= 2\left(\frac{1}{\sigma}\, x\,\dot{x} + y\,\dot{y} + z\,\dot{z}\right) \\ &= 2(r+1)x\,y - 2\,x^2 - 2\,y^2 - 2\,b\,z^2 \\ &= -2\left((x - \frac{r+1}{2}\,y)^2 + (1 - \left(\frac{r+1}{2}\right)^2)\,y^2 + b\,z^2\right). \end{aligned} \tag{3}$$

This last quantity is strictly negative if $r < 1$ and $(x, y, z) \neq (0, 0, 0)$. It is easy to see that the condition $\dot{V}(x, y, z) = 0$ implies $(x, y, z) = (0, 0, 0)$. Therefore the origin is globally stable for $r < 1$.

The powerful aspect of this method is that one does not need to integrate the equations to determine the qualitative behavior of the trajectories. On the other hand, the difficult feature of this technique is that there is no general way to find adequate expressions for $V(x, y, z)$, as said above. No general ansatz is known for this function.

More than proving the stability of the equilibrium point, this method also provides us with its basin of attraction. But when the system exhibits another type of attractor (limit cycle or chaotic attractor) the situation becomes more complicated. First, the position of the attractor cannot be determined as easily as in the case of an equilibrium point (where we only had to solve $P = Q = R = 0$). We would like to use a method similar to the Lyapunov theorem to determine (at least roughly) the location of the attractor in phase space. Let us call A the set of points defining the global attractor. This attractor has an extension in phase space (A is bigger than the origin $O(0, 0, 0)$ which was the attractor in the former example). In the general case, it will not be possible to find a function $V(x, y, z)$ such that $\dot{V}(x, y, z) < 0$ for $(x, y, z) \in \mathbb{R}^3 \setminus \mathrm{A}$. In fact $\dot{V}$ is going to change sign and there will be a set of points for which $\dot{V}(x, y, z) \geq 0$. A first (naïve) assumption is to say that the attractor is included in the region where $\dot{V}(x, y, z) \geq 0$ since V decreases outside. As mentioned in [4] and as we shall see below, this argument is not correct. To fully understand what happens here, one has to see things geometrically, defining regions in phase space which are globaly attracting.

2 Geometric point of view

Let us consider the level surfaces of the function $V(x, y, z)$ defined by $V(x, y, z) = K$ in phase space. The quantity $\dot{V}$ defined by :

$$\dot{V} = \frac{\partial V}{\partial x}\,P + \frac{\partial V}{\partial y}\,Q + \frac{\partial V}{\partial z}\,R \tag{4}$$

is the scalar product between the vector (P, Q, R) tangent to the trajectory at the point (x, y, z) and the vector $(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z})$ normal to the surface at this point. Hence, in the region where $\dot{V}$ is of constant sign the level surfaces of $V(x, y, z)$ are crossed by the flow in only one direction. If $\dot{V}$ is of constant sign on the whole surface $V(x, y, z) = K$ we call this surface a tranversal or a semi-permeable surface.

Let us consider, as an example, the case of a two-dimensional dynamical system. Here we must study the level curves $V(x, y) = K$, associated to a given function $V(x, y)$. Suppose that the level curves of V are all closed and that the value of V is increasing with the distance from the origin (in other words, V is a sink centered on the origin). Suppose now that $\dot{V}$ is negative for points far from the origin and positive for points near the origin. The level curves $V = K$ with large K are crossed inwards by the flow. If we reduce the value of K, these curves will still be crossed inwards by the flow as long as each one lies entirely in the region where $\dot{V} < 0$. In figure (1) we have drawn the set of points where $\dot{V} = 0$. The smallest curve to be entirely crossed inwards is the curve tangent to this set, $V = K_1$. Symmetricaly, we have drawn the biggest level curve to be entirely crossed outwards by the flow, the curve $V = K_2$ which is also tangent to the set $\dot{V} = 0$.

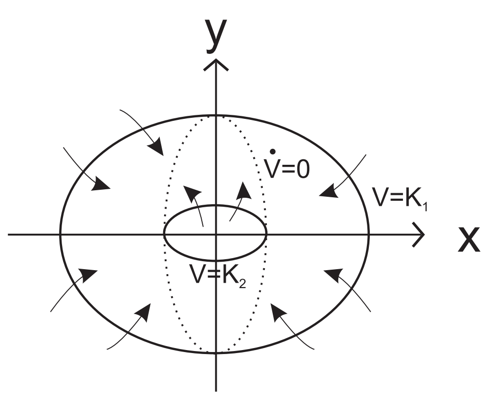

Figure 1: Two level curves of a function V(x, y) in a 2D phase space. $V(x, y) = K_1$ is the lowest curve which is crossed by the flow inwards. $V(x, y) = K_2$ is the upper curve which is crossed by the flow outwards. The global attractor of the system lies between these two curves.

The time evolution of the $V\ (x(t), y(t))$ function for an initial condition far from the origin is shown in figure (2). The global attractor of the system is included in the region of phase space defined by $K_1 < V(x, y) < K_2$. And if this region has no equilibrium point we know, thanks to the Bendixon-Poincaré theorem [1], that this attractor is a limit cycle. An analogous region for a 3-D system may contain limit cycles and/or chaotic attractors.

The region defined by $K_1 < V(x, y) < K_2$ is an overestimation of the global attractor of the system. The method tells us where the attractor is but not what the attractor is ! This

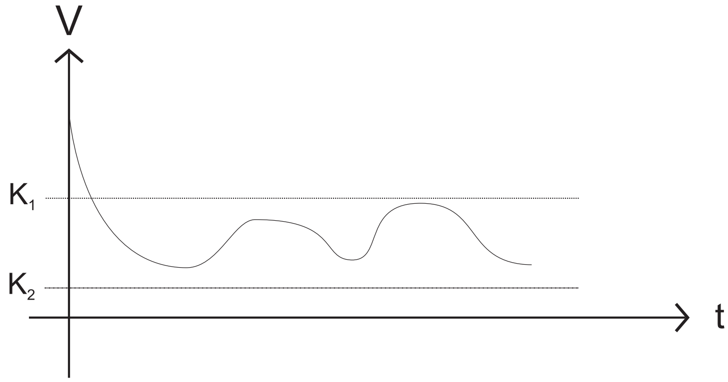

Figure 2: The time evolution of the function $V\,(x(t), y(t))$ considered in figure (1). For some initial condition far from the origin, V(t) is decreasing at least until $V(x, y) = K_1$. Then V remains in the region $K_1 < V(x, y) < K_2$.

means that the region $K_1 < V(x, y) < K_2$ contains points which lie on the attractor but also points which are not on the attractor. If we were more clever (or equivalently if the attractor was not so complicated) we would find a better V function fitting the attractor more tightly. These considerations will be developped in the last section.

In order to find the last entering curve $V(x, y) = K_1$ which will be the upper bound for the attractor, authors usually try to find K_1 with the help of Lagrange multipliers [2, 6] : they find the extrema of V on $\dot{V} = 0$ by introducing the function $V - k\,\dot{V}$ (k is the Lagrange multiplier). This boils down to finding the points in phase space where the gradients of V and $\dot{V}$ are colinear. This trick only works when the problem is simple because there are cases where tangency does not mean one way crossing. The level curve (or surface in 3-D systems) may be tangent at some point but may cross the curve $\dot{V} = 0$ at some other point(s), see figure (3).

Besides, for 3-D systems, this method is more difficult to apply because it is then necessary to study the sign of the function $\dot{V}(x, y, z)$ which depends on three variables. It is relatively easy to find subsets of positive and negative sign for the $\dot{V}$ function, but it is rather difficult to find the level curves of V which are entirely included in each subset since the parameters of the vector field are included in $\dot{V}$, together with the parameters of the function V. Hence, even if $V(x, y, z)$ is a polynomial, the problem is quite difficult.

3 Semi-permeable surfaces method

If we exploit further the geometric aspect of the problem, we notice that it is necessary for the function $\dot{V}(x, y, z)$ to be of constant sign only on the level set $V(x, y, z) = K$ and not

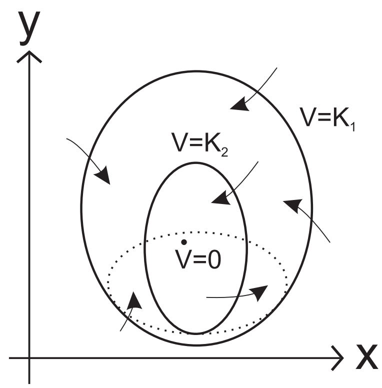

Figure 3: The method of colinear gradients may sometimes be misleading. Here the two level curves $V = K_1$ and $V = K_2$ are tangent to the curve $\dot{V} = 0$, but only the first one is semi-permeable. Hence, tangency does not mean one way crossing.

in an entire space subset. It means that we have to study $\dot{V}|_{V=K}$ instead of $\dot{V}$ in the entire phase space. Thanks to the equality $V = K$, which permits to replace one of the variables by the others, this new function will have only two variables. Of course, we restrict our studies to problems where this replacement is possible [1]. Then the analysis is much easier : we only have to study the sign of a two variables function when the variables vary on the entire surface (which means generally that we study the sign in all $\mathbb{R}^2$).

The semi-permeable surfaces introduced in this context must have the two following properties :

- each surface (or, if it is not connected, each piece of the surface) must divide the phase space in two disconnected regions D_1 and D_2; either the surface is closed and then we can define an interior (D_1) and an exterior (D_2), or the surface is infinite, i.e. it separates also two regions D_1 and D_2 in phase space.

- each surface must be oriented; this means that the gradient must point toward the same region (D_1 or D_2) on the whole surface.

Following is an example which illustrates the superiority of the semi-permeable surfaces method over the Lyapunov function method. Let us consider again the Lorenz system (2). In [2] the following surface is introduced :

$$-\frac{r}{\sigma}x^2 + y^2 + z^2 = 0, \tag{5}$$

[1]for quadratic V functions, it has been shown in [7] that the study of $\dot{V}|_{V=K}$ was equivalent to the study of $\dot{V}$ in the entire phase space.

which represents a certain bound (a double cone) for the attractor and which is calculated by means of the Lyapunov function method. In [8], the following family of surfaces is introduced :

$$a\,x^2 + y^2 + z^2 = R \quad \text{with} \;\; R \leq 0,\; a < 0. \tag{6}$$

For :

$$\frac{-2\sigma r - (\sigma-1)^2 - \sqrt{(\sigma-1)^4 + 4\sigma r(\sigma-1)^2}}{2\sigma^2} \leq a \leq \frac{-2\sigma r - (\sigma-1)^2 + \sqrt{(\sigma-1)^4 + 4\sigma r(\sigma-1)^2}}{2\sigma^2}, \tag{7}$$

the surfaces (6) are semi-permeable and they define a better bound than surfaces (5) for the attractor of (2).

4 Methods to obtain semi-permeable surfaces

As said in the former section, it is easier to check whether a surface is semi-permeable or not than to insure that a function possesses the Lyapunov property. But there is no general method to obtain these surfaces. In [8] we have determined several families of semi-permeable surfaces for the Lorenz system guided by the time-dependent integrals of motion that exist for special values of the parameters of the system. Let us give an example of the application of this method : the Lorenz system (2) has the first integral $I(x,y,z,t) = (x^2 - 2\sigma z)\,e^{2\sigma t}$ when $b = 2\sigma$ and σ and r are arbitrary (an easy calculation shows that $\frac{dI}{dt} \equiv 0$). Let us now consider the family of surfaces :

$$V(x,z) = x^2 - 2\sigma z = K\ , \tag{8}$$

where K is an arbitrary constant. It is easy to show that $\dot{V} = -b\,K$. Hence, each surface of the family is transversal. The direction of crossing depends on the sign of the constant K. A particular surface of the family is obtained for $K = 0$, and is invariant : an initial condition on this surface determines a trajectory that remains on the surface for all time. Besides, all the trajectories of the system are attracted by this invariant surface, as can be seen in figure (4). It is clear that the existence of these families of surfaces gives a lot of information about the dynamics of the system. The behaviour of trajectories is extremely simple in all the phase space with the exception of the invariant surface $x^2 = 2\sigma\,z$. This surface contains the global attractor of the system for the case $b = 2\sigma$.

The family of surfaces (8) obtained above enables us to characterise in a simple way this global attractor. The determination of this family of surfaces follows immediately from the existence of the integral of motion when $b = 2\sigma$.

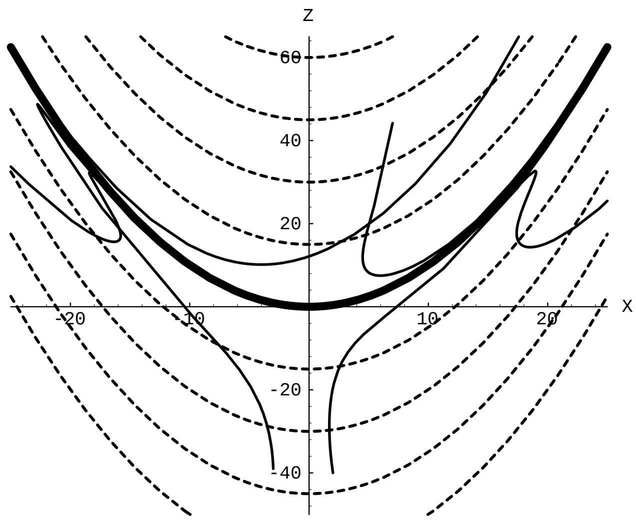

Figure 4: The dashed curves represents semi-permeables surfaces (8) in the case $b = 2\sigma$ for system (2). The bold curve is the invariant surface defined by eq. (8) with $K = 0$. We also show some trajectories of the system.

Now the natural question is : when $b \neq 2\sigma$, is it still possible to find similar families of surfaces that the flow crosses in only one direction ? In this case we shall no longer have at our disposal an integral of motion, and these semi-permeable surfaces will not fill the phase space because in the general case the global attractor is not contained in a two dimensional set. In order to find semi-permeable surfaces in the general case, when $b \neq 2\sigma$, we proceed as follows : we first propose a surface of the same mathematical form as the integral of motion, but with arbitrary coefficients, i.e.

$$V(x, z) = a_1\, x^2 + a_2\, z + a_3. \tag{9}$$

Then we calculate $\dot{V}$ on the surface and obtain : $\dot{V}_{|V=0} = (2a_1\sigma + a_2)xy + a_1(b - 2\sigma)x^2 + b\, a_3$. We now have an expression that depends only on two variables x and y. We must determine the coefficients a_1, a_2 and a_3 in such a way that this expression has the same sign for arbitrary values of x and y. We must hence set $a_2 = -2\sigma a_1$ which yields :

$$\dot{V}_{|V=0} = a_1(b - 2\sigma)x^2 + b\, a_3. \tag{10}$$

As a_1 must be non zero we can take $a_1 = 1$ without loss of generality. If we consider $b > 2\sigma$, we must set $a_3 > 0$ to get a first family of semi-permeable surfaces and if we consider $b < 2\sigma$, we must set $a_3 < 0$ to get a second family of semi-permeable surfaces. We show the latter family as well as some trajectories of the system in figure (5).

As we can see from figure (5), in the region filled by the surfaces the dynamics of the system is very simple. The complex behaviour can only occur in the region of phase space

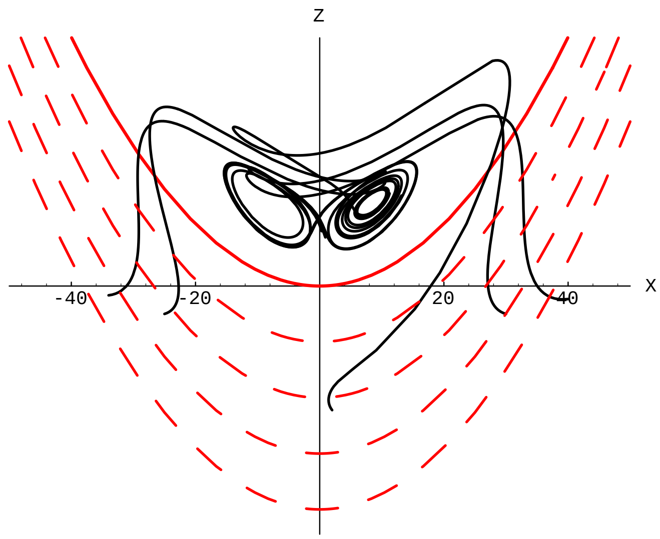

Figure 5: Semi-permeable surfaces (9) with $a_3 < 0$, $b < 2\sigma$, $a_2 = -2\sigma\, a_1$, $a_1 = 1$ for the system (2) together with its chaotic attractor.

that is not occupied by these surfaces. The global attractor of the system must be located in the region $z > 2\sigma x^2$.

Using the method explained above, we have determined, from the other known integrals of motion of the Lorenz system, several other families of semi-permeable surfaces [8]. In the chaotic regime, only a bounded region of the phase space is not filled by these surfaces and the global attractor of the system must be contained in this region. In this way, we have obtained some information on the shape and location of the global attractor. These results are more restrictive than similar previous bounds that have been found by other authors thanks to the method of Lyapunov functions [2].

The integrals of motion give us a hint that is of fundamental importance for obtaining semi-permeable surfaces. In fact, when looking for this type of surfaces without having a previous idea of their mathematical expression we are faced with high algebraic difficulties. Nevertheless, some systems do not have integrals of motion, or at least, sufficiently simple integrals of motion to be found with the standard methods. In this paper we present an alternative method for determining semi-permeables surfaces. This new method is a variation of a technique introduced in [12] for finding integrals of motion. It can be applied to polynomial systems i.e. systems where P, Q, R are polynomials in the three variables x, y, z.

We shall introduce the new method by analysing a concrete example : the Lorenz system. We shall obtain again the semi-permeable surfaces determined above thanks to the new method. As the Lorenz system is linear with respect to each one of the three variables x, y, z, we propose a function $V(x,y,z)$ linear in z :

$$V = h_1(x,y)\, z + h_0(x,y), \tag{11}$$

where $h_0(x,y)$ and $h_1(x,y)$ are arbitrary functions of x and y. We define a function $M(x,y,z)$ as follows :

$$M(x,y,z) = \dot{V} + L(x,y,z)\, V, \tag{12}$$

where $L(x,y,z)$ is a polynomial of degree $n-1$ (n is the maximum degree of the polynomials P, Q, R). Since for the Lorenz system $n = 2$, $L(x,y,z)$ will be of the form $L(x,y,z) = \alpha_0 + \alpha_1\, x + \alpha_2\, y + \alpha_3\, z$, where the α_i are arbitrary parameters. The sign of $\dot{V}_{|V=0}$ is given by the sign of $M_{|V=0}$. In order to simplify the study of the sign of M in phase space, we shall impose conditions on the functions $h_0(x,y)$, $h_1(x,y)$ and on the parameters α_i. We shall obtain these conditions by imposing that M must be a function of only one variable, for instance the variable x. The explicit expression of $M(x,y,z)$ is :

$$\begin{aligned} M(x,y,z) \;=\; & \left(\alpha_3\, h_1(x,y) - x\frac{\partial h_1}{\partial y}\right) z^2 + \\ & \left(\alpha_3\, h_0(x,y) + (\alpha_0 - b + \alpha_1\, x + \alpha_2\, y)\, h_1(x,y) - x\frac{\partial h_0}{\partial y} + (rx-y)\frac{\partial h_1}{\partial y} + \right. \\ & \left. \sigma(y-x)\frac{\partial h_1}{\partial x}\right) z + (\alpha_0 + \alpha_1\, x + \alpha_2\, y)\, h_0(x,y) + x\, y\, h_1(x,y) + \\ & (rx-y)\frac{\partial h_0}{\partial y} + \sigma(y-x)\,\frac{\partial h_0}{\partial x}. \end{aligned} \tag{13}$$

As the coefficient of z^2 must be zero we obtain the following expression for $h_1(x,y)$: $h_1(x,y) = g_1(x)\exp(\alpha_3 \frac{y}{x})$ where $g_1(x)$ is an arbitrary function of x. Because we want to obtain a function V defined in all phase space we take $\alpha_3 = 0$. The coefficient of z in the expression (13) must also be zero. This condition leads to the following equation : $(\alpha_0 - b + \alpha_1\, x + \alpha_2\, y)\, g_1(x) + \sigma(y-x)g_1'(x) = x\frac{\partial h_0}{\partial y}$. The general solution of this equation is : $h_0(x,y) = \frac{1}{2x}(2x\, g_0(x) + y(-2b + 2\alpha_0 + 2\alpha_1\, x + \alpha_2\, y)\, g_1(x) + \sigma\, y(-2x+y)\, g_1'(x))$, where $g_0(x)$ is an arbitrary function of x. Now the resulting expression of M is a function of x and y. We do not want to obtain the more general semi-permeable surface of the form (11). Our aim is to give an example of the method explaining the differents steps of the algorithm. Hence, it is sufficient to consider $g_1(x) \equiv 1$, which yields :

$$\begin{aligned} M(x,y) \;=\; & \frac{\alpha_2}{2x^2}(\alpha_2 x - \sigma)y^3 + \\ & \left(2b\sigma - 2\alpha_0\sigma + (-2\alpha_2(b+1) + 3\alpha_0\alpha_2 + \alpha_2\sigma)x + 3\alpha_1\alpha_2 x^2\right)\frac{1}{2x^2}y^2 + \\ & \Big(b - \alpha_0 - b\alpha_0 + \alpha_0^2 - b\sigma + \alpha_0\sigma + (\alpha_1(2\alpha_0 - b - 1) + \alpha_2\, r)x + \end{aligned}$$

$$(1+\alpha_1^2)x^2+\alpha_2 x\, g_0(x)+\sigma\, x\, g_0'(x)\Big)\, \frac{y}{x}+$$
$$\alpha_0\, r - b\, r + \alpha_1\, r\, x + \alpha_0\, g_0(x) + \alpha_1\, x\, g_0(x) - \sigma\, x\, g_0'(x). \tag{14}$$

We want to obtain a function only of the variable x, so we take : $\alpha_0 = b$, $\alpha_2 = 0$ and $g_0(x) = -\frac{\alpha_1}{\sigma}(b-1)x - (1+\alpha_1^2)\frac{x^2}{2\,\sigma} + K_0$, where K_0 is an arbitrary constant. The resulting expression for M is :

$$\begin{aligned} M(x) &= K_0\, b + \alpha_1 \left((b-1)(1-\frac{b}{\sigma}) + k_0 + r \right) x + \\ &\left(1+\alpha_1^2 - \frac{b}{2\sigma} + \frac{\alpha_1^2}{\sigma} - \frac{3b\alpha_1^2}{2\sigma} \right) x^2 - \frac{\alpha_1}{2\sigma}\left(1+\alpha_1^2\right) x^3. \end{aligned} \tag{15}$$

For our purposes $M(x)$ must be of definite sign for arbitrary values of x, so we must take $\alpha_1 = 0$ and $M(x)$ becomes :

$$M(x) = b\, k_0 + (1-\frac{b}{2\sigma})x^2. \tag{16}$$

The resulting expression for the function V is : $V(x,z) = K_0 - \frac{x^2}{2\sigma} + z$. The family of surfaces $V = 0$ is semi-permeable for $K_0 > 0$ if $b < 2\sigma$ and for $K_0 < 0$ if $b > 2\sigma$. If $b = 2\sigma$, for $K_0 = 0$ we obtain the known invariant surface. In this way we arrive again at the results obtained with the help of an integral of motion of the Lorenz system. In the next section we shall make use of both methods to find semi-permeable surfaces. The first method has already been succesfully employed in [3, 8, 9] for the study of the Rabinovich, Lorenz and Rikitake systems. For systems where we do not know any integral of motion, we shall use the new method. Both methods will yield bounds for the attractors in phase space, range of values of the parameters for which no chaotic behaviour is possible and make out part of the basin of attraction of the equilibrium points.

5 Results on particular systems

We shall first consider the system :

$$\dot{x} = -s\,(x+y)\,,\ \dot{y} = -y - s\,x\,z\,,\ \dot{z} = v + s\,x\,y\,, \tag{17}$$

where s and v are positive parameters. This system has been introduced in the context of the qualitative study of the Lorenz attractor [10]. The divergence of the vector field is : $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = -s - 1 < 0$. Hence, this system contracts volumes in all phase space. No integral of motion is known in the litterature for this model. By appling the Painlevé method [11], we find that the quantity :

$$I(x,y,z,t) = (x^4 + 8\,x\,y - 4\,y^2 + 4\,x^2\,z)\, e^{\frac{4}{3}t} \tag{18}$$

is an integral of motion for the case $s = \frac{1}{3}$ and $v = 0$. Using the method described in the previous section for the Lorenz model, we propose a family of surfaces of the form :

$$V = d + c\,x^4 + b\,y^2 + a\,x\,y + ex^2\,z = 0. \qquad (19)$$

The expression of $\dot{V}$ on the surface $V = 0$ is given by :

$$\begin{aligned} \dot{V}_{|V=0} \;=\; & -x(ds(a+2e)x + v\,e^2\,x^3 + cs(a-2e)x^5 + 2ds(b+e)y + a(-e+as+es)x^2\,y + \\ & s(2bc - 2ce + e^2)x^4\,y + (-2be + 3abs + aes + 2bes)x\,y^2 + 2bs(b+e)y^3). \qquad (20) \end{aligned}$$

Owing to the factor x in the above expression, we have to set $b = -e$ in order to obtain a function that does not change sign. After that, $\dot{V}_{|V=0}$ contains a common factor x^2 multiplied by a second degree polynomial in y^2. The discriminant of this polynomial is a polynomial of degree 6 in x which must be negative for all x. Since the coefficient $(e-4c)^2 s^2$ of x^6 is positive, we must take $c = \frac{e}{4}$. After that, the discriminant is given by :

$$\Delta = (as + es - e)(8des(a+2e) + (-a^2 e + a^3 s + a^2 es + 8e^3 v)x^2 + 2e^2 s(a-2e)x^4). \qquad (21)$$

If we set $e = 0$ this expression cannot be negative. Therefore, without loss of generality we can set $e = -1$. So, the family of surfaces becomes :

$$V = d - \frac{x^4}{4} + a\,x\;y + y^2 - x^2\,z = 0. \qquad (22)$$

Moreover, $\dot{V}|_{V=0}$ is given by :

$$\dot{V}|_{V=0} = x^2 \left(2(s(1-a)-1)\,y^2 + a(s(1-a)-1)xy + s(a+2)\frac{x^4}{4} - vx^2 + ds(2-a) \right) \qquad (23)$$

and the discriminant Δ is :

$$\Delta = -(s(1-a)-1)\left(2s(a+2)x^4 - (8v + a^2(s(1-a)-1))x^2 + 8ds(2-a)\right). \qquad (24)$$

Since Δ must be negative for all x, the coefficient $-2\,(s(1-a)-1)s(a+2)$ of x^4 must be negative, which is satisfied in each of the three following cases :

i : $-2 < a < 1$ and $s > \frac{1}{1-a} > 0$

2i : $a < -2$ and $0 < s < \frac{1}{1-a}$

3i : $-2 < a$ and $\frac{1}{1-a} < s < 0$.

After that, we must impose that Δ has no real root, which is satisfied in each one of the two following cases :

4i : $d(a^2-4)s^2 + \left(v + \frac{1}{8}a^2(s(1-a)-1)\right)^2 < 0$

5i : $d(4-a^2) > 0$ and $s(a+2)\left(v + \frac{1}{8}a^2(s(1-a)-1)\right) < 0$.

Therefore, in order to have Δ negative for all x, we may combine anyone of the three cases $i, 2i, 3i$ with anyone of the two cases $4i, 5i$. We then have six different cases to consider. Two cases are particulary interesting : $(i,\, 4i)$ and $(2i,\, 5i)$. In the case $(i,\, 4i)$ the chaotic attractor is bounded by the semi-permeable surfaces (22) as shown in figure (6). In the case $(2i,\, 5i)$,

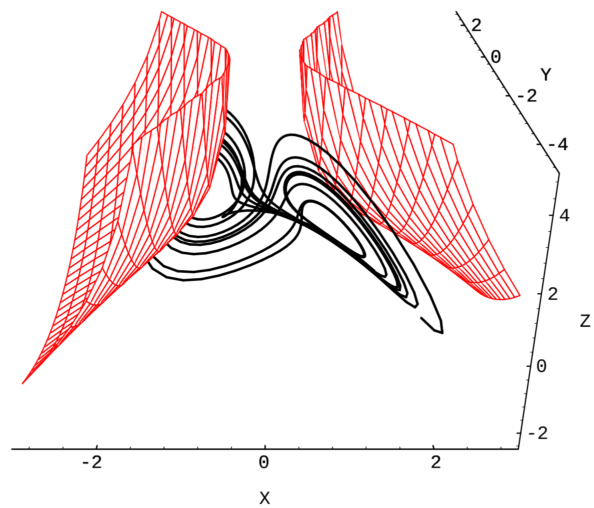

Figure 6: Chaotic attractor of system (17) with $v = 5/2, s = 3$. The attractor is bounded by surface (22) with $a = -1$ and $d = 1$ (condition $(i,\, 4i)$).

the semi-permeable surfaces are crossed by the flow in the upper direction. If we set $d = 0$ then the surfaces divide the phase space in three disconnected regions and the two equilibrium points (which are attracting here) are separated by these surfaces. This means that, for values of s and v that satisfy $(2i,\, 5i)$, we know a part of the basin of attraction of each one of the two points. This also means that trajectories cannot wander from one equilibrium point to another and hence that there is no chaos for these values of the parameters (see figure (7)).

Continuing the study of system (17), we have looked for new integrals of motion but we have not been able to find any. In consequence, we have applied the new method introduced in section 4. We have obtained the following family of surfaces with this method :

$$V = a_1\, x^2 + y^2 + (z + a_1)^2 - a_4 = 0. \tag{25}$$

The scalar product on the surface is given by :

$$\dot{V}|_{V=0} = (s-1)\, y^2 + s\, z^2 + z\,(v + 2s\, a_1) + a_1^2 s + a_1 v - a_4\, s \tag{26}$$

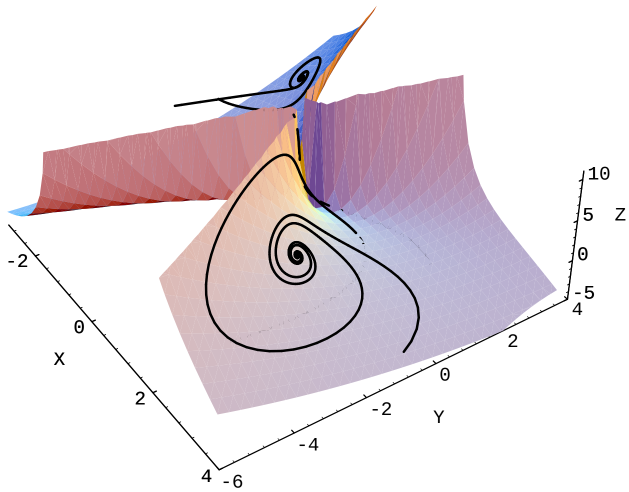

Figure 7: System (17) with $v = 1, s = 1/4$. The surface (22) with $d = 0, a = -5/2$ reveals part of the basin of attraction of each one of the equilibrium points. No chaotic behaviour is possible in this case.

and it is of definite sign when the following conditions are satisfied :

$$s > 1\,,\; 4\,a_4\,s^2 + v^2 \leq 0\,,\; a_1 < 0. \tag{27}$$

We have a family of x-axis hyperboloïd of revolution and each surface consists of two separated pieces. In figure (8) we see the chaotic attractor of system (17) and one of the semi-permeable surfaces. For each negative value of a_1, the optimal surface is obtained for $a_4 = -\frac{v^2}{4\,s^2}$. For this value of a_4, varying a_1 within negative values, we have a monoparametric family of semi-permeable surfaces. The optimal surface is the envolvent of the family, defined by

$$V = 0\,,\; \frac{\partial V}{\partial a_1} = 0, \tag{28}$$

i.e.

$$\frac{v^2}{4\,s^2} - \frac{x^4}{4} + y^2 - x^2\,z = 0. \tag{29}$$

It is remarkable that this last surface is a particular case of the family (22) with $a = 0$ and $d = \frac{v^2}{4\,s^2}$, satisfying conditions $(i,\, 4i)$.

We now consider the system :

$$\dot{x} = y\,,\; \dot{y} = z\,,\; \dot{z} = -A\,z + y^2 - x\,, \tag{30}$$

where A is a constant parameter. This system has been recently introduced [13] as the simplest system (since it has only one non-linear quadratic term in the vector field) exhibiting chaotic

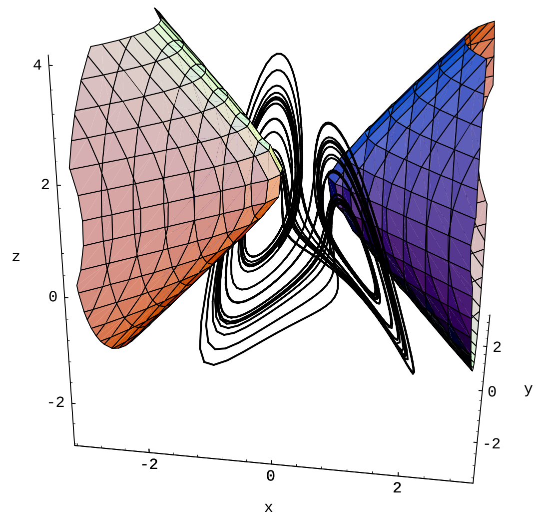

Figure 8: Chaotic attractor of system (17) with $v = 5/2$ and $s = 3$ and a semi-permeable surface (25) with $a_1 = -1$ and $a_4 = -\frac{v^2}{4\,s^2}$.

behaviour (for $A \simeq 2$). By applying different methods, we have not been able to find integrals of motion for this system. As the system is linear with respect to the z variable, we propose for the family of surfaces a function V linear in z, of the form :

$$V = g_1(x,y)\,z + g_0(x,y)\,, \tag{31}$$

where $g_0(x,y)$ and $g_1(x,y)$ are arbitrary functions of x and y. Following the method introduced in the previous section for the Lorenz system, we find that the following family of surfaces :

$$V = z - a\,x + (A + \frac{1}{a})\,y - d = 0, \tag{32}$$

with the scalar product on the surface given by :

$$\dot{V}|_{V=0} = a^2\,y^2 - y\,(1 + a\,A + a^3) + a\,d \tag{33}$$

is semi-permeable if :

$$\Delta = (1 + a\,A + a^3)^2 - 4\,a^3\,d < 0. \tag{34}$$

This condition yields two different cases :

- $A \leq -\left(\frac{27}{4}\right)^{1/3} \simeq -1.88$
  Here there exist values of a for which $(1{+}a\,A{+}a^3) = 0$ and for these values there are semi-permeable planes $\forall d$. The z-axis is surrounded by these planes. The chaotic attractor, when it exists, turns around this axis. Now, the semi-permeable planes prevent this situation from occuring, so the chaotic attractor cannot exist in this case.

- $A > -\left(\frac{27}{4}\right)^{1/3}$
  the chaotic attractor may exist in this case and when it exists, it is stuck in between two families of semi-permeable planes, one above it ($d > 0$) and one below ($d < 0$) (see figure (9)).

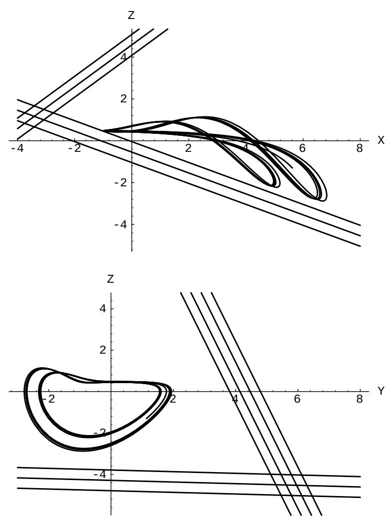

Figure 9: The chaotic attractor of system (30) with $A = 2.04$ contained in between planes ($a = 1$ and $d > 4.08$) and ($a = -\frac{1}{2}$ and $d < -0.04$). Note that the chaotic attractor is winding around the z-axis.

Now we consider once again the classical Lorenz system (2) for which several families of semi-permeable surfaces have been found in [8]. By using the new method, we have found an interesting family of surfaces that gives important information about the behaviour of the orbits on the chaotic attractor. We propose the following form for the family of surfaces :

$$V(x, y, z) = g_1(x, z)\, y + g_0(x, z) = 0, \tag{35}$$

where $g_1(x, z)$ and $g_0(x, z)$ are arbitrary functions of x and z. Following the method employed above we find :

$$g_1(x, z) \equiv 1 \text{ and } g_0(x, z) = a_1\, x^3 - 2\, a_1\, \sigma x\, z + a_2\, x, \tag{36}$$

which yields :

$$V(x, y, z) = y + a_1 x^3 - 2\, a_1 \sigma\, x\, z + a_2\, x. \tag{37}$$

If we write these surfaces as :

$$z = \frac{1}{2 a_1 \sigma}\left(a_1 x^2 + a_2 - \frac{y}{x}\right), \tag{38}$$

the scalar product in this case could be of constant sign, but the surfaces (which are disconnected) are not oriented : the gradient vector does not point toward the same space subset for $x > 0$ and for $x < 0$ (this is due to the $-\frac{y}{x}$ term). Whereas if we write V as :

$$y = -x\,(a_1\,x^2 - 2\,a_1\,\sigma\,z + a_2), \tag{39}$$

the surfaces are connected and oriented and the scalar product on the surfaces is :

$$\dot{V}|_{V=0} = x\,\left(-4\,a_1^2\,\sigma^3\,z^2 + z f(x) + g(x)\right), \tag{40}$$

where

$$\begin{aligned} f(x) &= -1 - 2a_1\sigma + 2a_1 b\sigma + 2a_1\sigma^2 + 4a_1a_2\sigma^2 + 4a_1^2\sigma^2x^2 \\ g(x) &= a_2 + r - a_2\sigma - a_2^2\sigma + (a_1 - 3a_1\sigma - 2a_1a_2\sigma)x^2 - a_1^2\sigma x^4. \end{aligned} \tag{41}$$

We see that (40) change sign at $x = 0$. Therefore this family of surfaces is not strictly semi-permeable. Nevertheless, we shall obtain some important information from it. Hence, we shall study the cases in which the function $-4\,a_1^2\,\sigma^3\,z^2 + z f(x) + g(x)$ holds the same sign $\forall (x, z) \in \mathbb{R}^2$. This happens when the two following conditions are satisfied :

$$2\,a_1\sigma(2\sigma - b) + 1 \geq 0 \tag{42}$$

$$\begin{aligned} 1 + 4\,a_1\sigma(1 - b - \sigma) + 8\,a_1a_2\sigma^2(2\,ba_1\sigma - 1) + \\ 4\,a_1^2\sigma^2\left((b-1)^2 + \sigma(4r - 2 + \sigma + 2b)\right) &< 0. \end{aligned} \tag{43}$$

When $x < 0$ the surfaces (37) are crossed by trajectories in one way and when $x > 0$ they are crossed by trajectories in the opposite way. Hence, these surfaces do not represent an external bound for the chaotic attractor when it exists.

We recall that the Lorenz attractor is formed by the addition of two wings, each wing lying around the equilibrium points C^+ and C^-, respectively. Therefore each surface of the family separates the attractor in two winding regions. One region in contained in $x > 0$ and the other one is contained in $x < 0$.

Let us study the behaviour of a trajectory around the 'positive' wing (around the equilibrium point C^+). The trajectory wanders around C^+ until it 'decides' to cross the $x = 0$ plane and goes wander around the other equilibrium point. All the essence of complexity in the system comes from the fact that we do not know when the trajectory 'decides' to jump to the other side of the plane $x = 0$. Here comes the interesting feature of this family of surfaces (37). The $x > 0$ side of this surface is placed between the $x = 0$ plane and the 'positive' wing of the attractor. Because (37) is semi-permeable in the $x > 0$ half space, once the trajectory

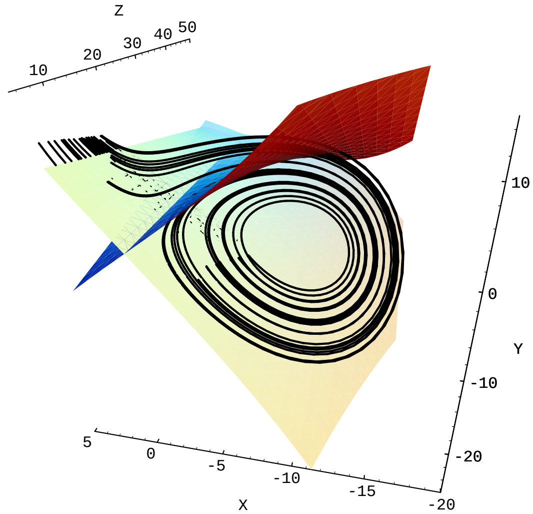

Figure 10: Two surfaces (37) with ($a_1 = -1/500; a_2 = -1109/498$) and ($a_1 = 1/500; a_2 = 292/201$) represented for negative x. The wing of the attractor around the equilibrium point C^- is restricted in the region between the two surfaces. The trajectories that cross the above surface (a_1 and a_3 positive) from right to left are the trajectories that go to the other wing (in $x > 0$).

has crossed this surface, it cannot go on wandering around the point C^+ and it is compelled to go winding around the other equilibrium point C^-. We may consider such surfaces as a separation between the two wings of the attractor. Besides, surfaces (37) give a bound in phase space for the period-1 limit cycles around each equilibrium point.

The last example we shall consider is the classical Rössler system :

$$\dot{x} = -y - z\,,\ \dot{y} = x + ay\,,\ \dot{z} = b + z(x - c)\,, \tag{44}$$

where a, b, c are positive parameters. For certain values of these parameters, this system has a chaotic attractor. Moreover, it has two equilibrium points when $c^2 \geq 4ab$. One of the points (P_{in}) is nested inside the chaotic attractor and the other one (P_{out}) is outside the chaotic region. This system has a non-constant divergence and there are no known integrals of motion for it. Nevertheless, using the new method, we find the following family of semi-permeable surfaces :

$$V = y + k_1 + (a + k_2)x - (1 + ak_2 + k_2^2)\,\mathrm{Log}|z| = 0. \tag{45}$$

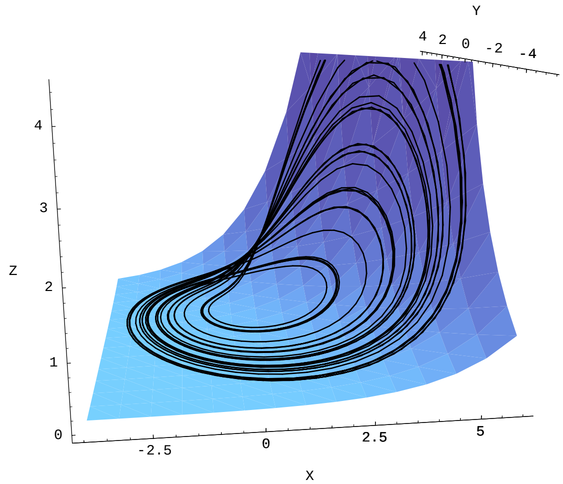

Figure 11: Chaotic attractor of system (44) with $a = \frac{2}{5}$, $b = 2$, $c = 4$ and the $z > 0$ sheet of the semi-permeable surface (45) with $k_1 \simeq -3.5$, and $k_2 \simeq 1$.

Each one of these surfaces consists of two disconnected sheets. One sheet lies entirely in $z > 0$ and the other one in $z < 0$. The two sheets are obtained from the expression :

$$z = \pm \exp\left(\frac{y + k_1 + (a + k_2)x}{1 + ak_2 + k_2^2}\right). \tag{46}$$

The scalar product on the surface is given by :

$$\begin{aligned}\dot{V}|_{V=0} &= -b(1 + ak_2 + k_2^2)\frac{1}{z} + \left(k_1\, k_2 + c(1 + ak_2 + k_2^2)\right) - (a + k_2)z - \\ &\quad k_2(1 + ak_2 + k_2^2)\mathrm{Log}|z| \\ &\stackrel{\text{def}}{=} f(z).\end{aligned}$$

The function $f(z)$ must be of constant sign on each sheet (46), i.e. for $z > 0$ and for $z < 0$, respectively. From the study of this one variable function, we find that the necessary and sufficient conditions for each sheet to be semi-permeable are :

$$\begin{aligned} f(z_i^\star)\, z_i^\star\,(a + k_2) &< 0 \quad \text{with } i = 1, 2 \\ b\,(1 + ak_2 + k_2^2)(a + k_2) &> 0, \end{aligned} \tag{47}$$

with

$$z_1^\star = -\frac{k_2(1 + ak_2 + k_2^2) - \sqrt{k_2^2(1 + ak_2 + k_2^2)^2 + 4b(1 + ak_2 + k_2^2)(a + k_2)}}{2(a + k_2)} \tag{48}$$

$$z_2^\star = -\frac{k_2(1 + ak_2 + k_2^2) + \sqrt{k_2^2(1 + ak_2 + k_2^2)^2 + 4b(1 + ak_2 + k_2^2)(a + k_2)}}{2(a + k_2)}. \tag{49}$$

These conditions can be satisfied when the chaotic attractor exists (for example when $a = \frac{2}{5}, c = 4, b = 2$). In fact the $z < 0$ half space is filled by semi-permeable surfaces crossed by the flow upward. Hence, this proves that for such values of the parameters, the asymptotic ($t \to +\infty$) behaviour takes place in the $z > 0$ half space, where the chaotic attractor must lie entirely (see figure 12). Moreover, there are also semi-permeable surfaces lying in the $z > 0$ half space, bounding the chaotic attractor quite more tightly (see figure 11).

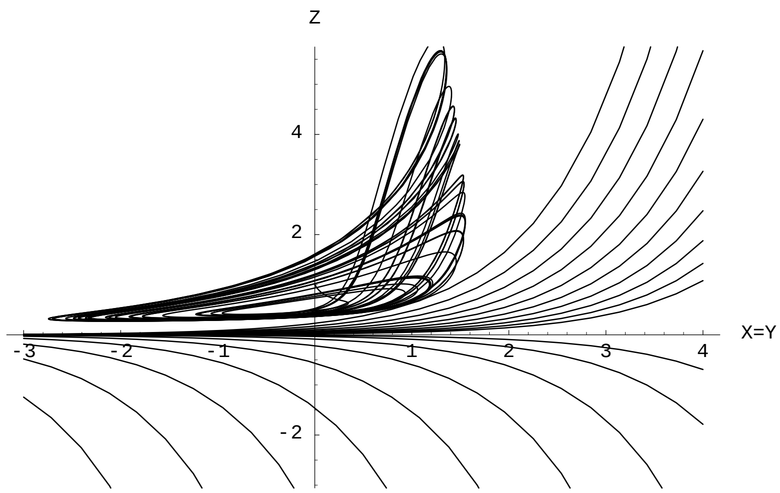

Figure 12: Semi-permeable surfaces in the $z < 0$ half space ($k_1 \in [-12.7; 10],\ k_2 = 1$) and in the $z > 0$ half space ($k_1 \in [-10; -3.5],\ k_2 \simeq -1$) together with a projection of the chaotic attractor ($a = \frac{2}{5},\ b = 2,\ c = 4$) on the plane $x = y$. We see that all the trajectories initially in the $z < 0$ half space eventually cross the $z = 0$ plane. Hence, the asymptotic behaviour takes place in the $z > 0$ half plane.

6 Getting closer to the attractor

So far we have introduced a method to get geometric bounds on the attractors of dissipative systems. These bounds are sometimes tight and sometimes loose. The natural question arising is : can we get closer to the attractor ?

- If we consider each point on a semi-permeable surface surrounding an attractor as an initial condition ($t = 0$) and integrate numerically, the set of points at $t > 0$ will define another semi-permeable surface (with a different shape). As $t \to +\infty$, the surface will merge with the attractor (for a simple example see [14] p. 42).

- Some authors define entering regions as a combination of different functions [15]. The surface surrounding the attractor is then defined by multiple equations, each one valid for a precise region in phase space. This is a way to tackle the complexity of the attractor. As regards our method, when more than one surrounding surface is known, one has to consider the composition of the different surfaces. This yields a tighter bound for the attractor. It is likely that by considering many more equations of surfaces, we could get even nearer to the attractor. To stick to the attractor (chaotic or limit cycle), we should consider an infinite combination of surfaces.

- If we want to bound the attractor with only one type of equation and we want this bound to get tighter and tighter, we shall have to refine the equation of the surface at each step. This is what is done in [16, 17] for the van der Pol system, where the attractor is a limit cycle (which equation is given by an unknown transcendental function). At each step of the procedure, the curve bounding the limit cycle is defined by a particular level curve of a polynomial function in two variables, of increasing degree. This curve (which is semi-permeable) is getting closer and closer to the limit cycle. Taking the limit, the curve (defined by an infinite series in two variables) seems to merge with the limit cycle.

7 Conclusions

We have shown that the method introduced in [8] for the Lorenz system works for other 3-D chaotic dynamical systems. We have also introduced a new method to find semi-permeable surfaces and applied it to several chaotic dynamical systems, showing that semi-permeable surfaces enable to bound the chaotic attractor in phase space or reveal ranges of parameters' values for which no chaotic behaviour is possible in these dissipative systems. This last aspect of the method represents an important theoretical progress in the study of 3-D dissipative dynamical systems.

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